Iacone, Fabrizio; Lazarova, Stepana

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Fabrizio Iacone and Stepana Lazarova

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Fabrizio Iacone*
Department of Economics and Related Studies
University of York, Heslington, York YO10 5DD

Štěpána Lazarová
Department of Economics, Queen Mary, University of London
Mile End Road, London E1 4NS

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Abstract

We consider changes in the degree of persistence of a process when the degree of persistence is characterized as the order of integration of a strongly dependent process. To avoid the risk of incorrectly specifing the data generating process we employ local Whittle estimates which uses only frequencies local to zero. The limit distribution of the test statistic under the null is not standard but it is well known in the literature. A Monte Carlo study shows that this inference procedure performs well in finite samples.

Keywords: Long memory, persistence, break, local Whittle estimate.

JEL classification: C22.

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1 Introduction

We consider changes in the degree of persistence of a time series. In our case we characterize the degree of persistence as the order of integration δ of a strongly dependent process. Changes in the order of integration have been documented, sometimes heuristically, in a number of macroeconomic variables, such as output (De Long and Summers, 1988), the budget deficit (Hakkio and Rush, 1991), inflation (Halunga, Osborn, and Sensier, 2009; Kumar and Okimoto, 2007; Hassler and Meller, 2014), financial markets bubbles (Sollis, 2006), among many others. Interest in the characterization of the degree of persistence and in its potential instability is particularly strong in the evaluation of macroeconomic policies such as inflation targeting because *ceteris paribus* a reduction of the order indicates a tighter control of the variable of interest (provided that the process is mean reverting, at least after the change). By the same argument, periods associated to $\delta = 1$ indicate lack of control.

In some of the applied work it is assumed that δ is limited to integer numbers only (typically, $\delta = 0$ or $\delta = 1$). Tests to detect changes between these two states were developed by Kim (2000), Kim, Belaire-Franch and Badilli-Amador (2002), Busetti and Taylor (2004), Harvey, Leybourne and Taylor (2006), Leybourne, Taylor and Kim (2007) among others. In all these cases, the test statistics are based on ratios of partial sums and it is possible to detect a change in the order of integration because the limit distributions are well behaved under the null.

However, the assumption of integer δ seems particularly restrictive in the context of testing for a change in persistence because it leaves no alternative between fast reversion to the mean ($\delta = 0$) and no reversion at all ($\delta = 1$). Important variations in the long term dynamics may be represented with relatively fractional changes in δ. This approach was advocated by Beran and Terrin (1996) who recommended testing for a change in this parameter in the context of a fully parametric model, and discussed by Horváth and Shao (1999). This is appealing because of good asymptotic properties of the maximum likelihood estimators but the requirement that the user specifies the correct model for the data generating process may be inconvenient, especially when a large number of

parameters has to be considered, as the uncertainty about the model may adversely reflect on the result of the procedure.

The case for semiparametric estimation of δ is even more compelling in case the process is subject to a break. The uncertainty about the possibility of a break should make the researcher even less confident when formulating a fully parametric model because the model selection procedure must be designed to deliver the correct model even under the alternative hypothesis that a break has indeed taken place.

A modified approach to testing for a change in persistence has been followed by Sibbertsen and Kruse (2009) who simulated appropriate critical values for the test statistic in Leybourne *et at.* (2007). Their critical values depend on δ. A non-parametric approach was adopted by Lavancier, Leipus, Philippe and Surgailis (2013) who also proposed a modification of the test statistic in Kim (2000) and related statistics. Semiparametric detection of a break is considered in Shimotsu (2006), where however the potential breakpoint is fixed in advance. Our choice is closer to the latter in the sense of being semiparametric. Like in the parametric test in Horváth and Shao (1999), we estimate δ before and after a potential break point, and compute a Wald type statistic for the difference between the two estimators. Since the potential break-point is in fact unknown we derive the limit distribution of the supremum of the Wald type statistic. However, unlike in Horváth and Shao (1999), we estimate δ by local Whittle estimator, so our procedure does not require us to specify a complete parametric model and it is therefore robust to that type of misspecification. We find that under the null the limit distribution is well known and does not depend on δ.

The structure of the paper is as follows. In Section 2 we present the relevant asymptotic theory and in Section 3 we analyse the small sample properties with a Monte Carlo exercise. We present an application in Section 4 and we conclude in Section 5. The proofs of the theorems are to be found in the Appendix.

2 Testing for a change in the order of integration

To establish notation we first introduce the model for the case of a stationary process. Our model is similar to the model of Robinson (1995). For a stationary process x_t with covariance $\gamma_s = E\left[\left(x_t - Ex_0\right)\left(x_{t+s} - E\left(x_0\right)\right)\right]$ and spectral density $f\left(\lambda\right)$ such that $\gamma_s = \int_{-\pi}^{\pi} f\left(\lambda\right) e^{i\lambda s} d\lambda$ we consider a process as integrated of order δ, denoted $x_t \in I\left(\delta\right)$, if there is $\delta < 1/2$ and $G \in (0, \infty)$ such that

$$f\left(\lambda\right) \sim G\lambda^{-2\delta} \text{ as } \lambda \to 0^{+} \tag{1}$$

where notation $a \sim b$ is used to indicate that the ratio a/b tends to 1. In model (1) the order of integration δ is usually the parameter of interest and G depends on all the other parameters of the spectral density.

For example, if x_t is ARFIMA(p, δ, q), $\Phi\left(L\right)\Delta^{\delta} x_t = \Theta\left(L\right)\varepsilon_t$, for ε_t independent and identically distributed with $E\left(\varepsilon_t\right) = 0$, $E\left(\varepsilon_t^2\right) = \sigma_\varepsilon^2$, then $2\pi G = \Phi\left(1\right)^{-1}\Theta\left(1\right)\sigma_\varepsilon^2$. In comparison with such full parametric specification, the model in (1) is usually considered semiparametric.

We now introduce the local Whittle estimator and discuss how to use it to test for a change in δ when the process is subject to a break in the order of integration. For a generic time series x_t observed at times $t = 1, ..., T$, define the Fourier transform $w\left(\lambda\right) = \frac{1}{\sqrt{2\pi}} \sum_{t=1}^{T} x_t e^{-i\lambda t}$ and the periodogram $I\left(\lambda\right) = \left|w\left(\lambda\right)\right|^2$. The local Whittle estimator is computed by minimizing with respect to d the loss function

$$R(d) = \ln\left(\frac{1}{m}\sum_{j=1}^{m} \lambda_j^{2d} I(\lambda_j)\right) - 2d\frac{1}{m}\sum_{j=1}^{m} \ln(\lambda_j), \tag{2}$$

where $\lambda_j = \frac{2\pi j}{T}$, for integers $j = 1, ..., m$, are Fourier frequencies and m is a user-chosen parameter. This loss function is discussed by Robinson (1995).

For a stationary process x_t, parameter δ in (1) does not depend on time. In practice, the persistence of a process may be subject to change over time. We consider a situation where the persistence measure δ can change at a certain point in time. We assume that there exists a break fraction τ^* with $0 < \tau^* < 1$ such that for $t < \left\lfloor \tau^* T \right\rfloor$, x_t is drawn

from an $I(\delta_1)$ process, and for $t \geq \lfloor \tau^* T \rfloor$, x_t is a realization of an $I(\delta_2)$ process with $\delta_1 \neq \delta_2$. That is, at different points in time the series x_t is observed from two possibly different processes, x_{1t} which is $I(\delta_1)$ and x_{2t} which is $I(\delta_2)$, with $x_t = x_{1t}$ if $t < \lfloor \tau^* T \rfloor$ and $x_t = x_{2t}$ if $t \geq \lfloor \tau^* T \rfloor$. If $\delta_1 = \delta_2$, it is possible that x_{1t} and x_{2t} are generated the same process. We wish to test the hypothesis of stability of the persistence. Our hypotheses of interest are therefore

$$\begin{aligned} H_0 &: \delta_1 = \delta_2, \\ H_A &: \delta_1 \neq \delta_2. \end{aligned}$$

In order to test whether the parameter δ remained stable over the sample period, we estimate δ on two subsamples and compare the two estimators. For a time series sample x_t observed at times $t = 1$, ..., T, and an interval $[\sigma, \tau] \subset [0, 1]$, we define the Fourier transform and the periodogram of series $0, \ldots, 0, x_{\lfloor \sigma T \rfloor + 1}, \ldots, x_{\lfloor \tau T \rfloor + 1}, 0, \ldots, 0$ as

$$w_{\sigma\tau}(\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{t=\lfloor \sigma T \rfloor + 1}^{\lfloor \tau T \rfloor} x_t e^{-i\lambda t} \qquad \text{and} \qquad I_{\sigma\tau}(\lambda) = \left| w_{\sigma\tau}(\lambda) \right|^2$$

and the related local Whittle loss function as

$$R(d; \sigma, \tau) = \ln \left(\frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d} I_{\sigma\tau}(\lambda_j) \right) - 2d \frac{1}{m} \sum_{j=1}^{m} \ln(\lambda_j). \tag{3}$$

We select τ in $(0, 1)$ and estimate parameter δ on for intervals $[0, \tau]$ and $[\tau, 1]$. Let

$$\widehat{\delta}_1(\tau) = \arg \min_{d \in [-1/2, 1/2]} R(d; 0, \tau), \tag{4}$$

$$\widehat{\delta}_2(\tau) = \arg \min_{d \in [-1/2, 1/2]} R(d; \tau, 1), \tag{5}$$

so $\widehat{\delta}_1(\tau)$ and $\widehat{\delta}_2(\tau)$ are the estimators computed using only the first or the second part of the sample for a given τ. Given the estimators $\widehat{\delta}_1(\tau)$ and $\widehat{\delta}_2(\tau)$ of δ on the two subsamples, we can base a test statistic for the test of stability of δ on the normalized

difference of the two estimators. We define the test statistic as

$$\widehat{t}(\tau) = \sqrt{4\tau(1-\tau)m}\left(\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)\right). \tag{6}$$

For any given $\tau \in (0,1)$, it can be showed that under regularity conditions, as $T \to \infty$, test statistic $\widehat{t}(\tau)$ converges in distribution to a standard normal,

$$\widehat{t}(\tau) \to_d N(0,1). \tag{7}$$

As the potential location $\lfloor \tau T \rfloor$ of the break is usually unknown, we consider $\widehat{t}(\tau)$ for all τ in a closed subset $[\tau_l, \tau_h]$ of $(0,1)$. Following Andrews (1993) we introduce

$$\widehat{t}^2 = \sup_{\tau \in [\tau_l, \tau_h] \subset (0,1)} \widehat{t}(\tau)^2. \tag{8}$$

We establish weak convergence of $\widehat{t}(\tau)$ to a tight limit. This convergence together with the continuous mapping theorem then gives us the distribution of the $\widehat{t}^2$ test statistic under the null hypothesis.

Our analysis proceeds under the following assumptions.

Assumption 1 *The processes x_{1t} and x_{2t} have linear representation*

$$x_{\ell t} - E(x_{\ell t}) = \textstyle\sum_{j=0}^{\infty} \alpha_{\ell j} \varepsilon_{t-j}, \qquad \ell = 1, 2,$$

where $\sum_{j=0}^{\infty} \alpha_{\ell j}^2 < \infty$ and, for $p = 1, ..., 8$,

$$E(\varepsilon_t^p | F_{t-1}) = \omega_p < \infty \quad a.s., \qquad t = 0, \pm 1, \ldots,$$

$\omega_1 = 0$ and $\omega_2 = \sigma_\varepsilon^2$.

Let $\mathcal{F}_t$ be the σ-algebras of events generated by ε_s, $s \leq t$.

Assumption 2 *In a neighbourhood $(0, \varepsilon)$ of the origin, $A_\ell(\lambda) = \sum_{j=0}^{\infty} \alpha_{\ell j} e^{ij\lambda}$ are dif-*

ferentiable for $\ell = 1, 2$ *and*

$$\frac{d}{d\lambda} A_\ell(\lambda) = O\left(\frac{|A_\ell(\lambda)|}{\lambda}\right) \qquad \text{as } \lambda \to 0+ .$$

Assumption 3 *For some* $\beta \in (0, 2]$*, the spectral densities* f_1 *and* f_2 *satisfy*

$$\begin{aligned} f_1(\lambda) &\sim G\lambda^{\delta_1}\left(1 + O\left(\lambda^\beta\right)\right) \qquad \text{as } \lambda \to 0+, \\ f_2(\lambda) &\sim G\lambda^{\delta_2}\left(1 + O\left(\lambda^\beta\right)\right) \qquad \text{as } \lambda \to 0+, \end{aligned}$$

where $G \in (0, \infty)$ *and* $\delta_1, \delta_2 \in [\Delta_1, \Delta_2] \subset [-1/2, 1/2]$.

Assumption 4 *As* $n \to \infty$,

$$\frac{1}{m} + \frac{m^{1+2\beta}(\log m)^2}{n^{2\beta}} \to 0.$$

Let $B(\tau)$ be a standard Brownian motion process on $[0, 1]$ and let "$\Rightarrow$" denote weak convergence in the Skorohod topology. We obtain the following theorem.

Theorem 1 *Under Assumptions 1–4 and under the null hypothesis, for* $[\tau_l, \tau_h] \subset (0, 1)$,

$$\widehat{t}^2 \Rightarrow \sup_{\tau \in [\tau_l, \tau_h]} \frac{(B(\tau) - \tau B(1))^2}{4\tau(1-\tau)} \tag{9}$$

as $T \to \infty$.

Proof of Theorem 1 is provided in Section 6.2 of the appendix.

The limit process $\sup_{\tau \in [\tau_l, \tau_h]} \frac{(B(\tau) - \tau B(1))^2}{4\tau(1-\tau)}$ is the supremum over $[\tau_l, \tau_h]$ of the square of a standardized tied down Bessel process. The distribution of the test statistic is identical to the distribution obtained by Andrews (1993) who also discusses what happens when $[\tau_l, \tau_h] = [0, 1]$. Andrews (1993) provides tables of various quantiles for the distribution. The upper 5% quantile is 8.85 when $[\tau_l, \tau_h] = [0.15, 0.85]$ and 9.31 when $[\tau_l, \tau_h] = [0.1, 0.9]$.

We can test $H_0 : \delta_1 = \delta_2$ against $H_A : \delta_1 \neq \delta_2$ at α size by computing the $\widehat{t}^2$ statistic and comparing its value with the upper α% quantile. A value of the $\widehat{t}^2$ statistic in excess

of the critical value leads to a rejection of H_0.

The following theorem shows that the test is consistent. With increasing sample size, the power of the test approaches 1 in probability.

Theorem 2 *Under Assumptions 1–4 and under the alternative hypothesis, for* $[\tau_l, \tau_h] \subset (0,1)$,

$$\widehat{t}^2 \xrightarrow{p} \infty$$

as $T \to \infty$.

Proofs of Theorem 2 can be found in Section 6.3 of the appendix.

Remark 1. Assumptions 1–4 are based on the assumptions of Robinson (1995) who uses them to establish consistency and limit normality of the local Whittle estimator. The most notable difference is that in our case finite moments up to the eight order are needed instead of Robinson's fourth moments. This is because of the additional requirement of establishing tightness in the context of our problem of interest.

Remark 2. The statistic $\widehat{t}^2$ is related to the test statistic of Horváth and Shao (1999), where however δ is estimated within a fully parametric model.

Remark 3. If the location of the breakpoint is known in advance it seems natural to test for a break using the statistic $\widehat{t}^2(\tau^*)$ using critical values from the χ^2_1 distribution. When the potential breakpoint is not known then the statistic $\widehat{t}^2(\tau)$ for a user chosen point may be considered. This is similar to the test advocated by Shimotsu (2006) who suggests to divide $[\tau_l, \tau_h]$ in equally spaced intervals. However, testing using the statistic $\widehat{t}^2(\tau)$ may result in low power when compared to testing using the $\widehat{t}^2$ statistic. To understand why, consider the case $\delta_1 > \delta_2$ and $\tau < \tau^*$. Then observations $x_1, \ldots, x_{\lfloor \tau T \rfloor}$ are obtained from a $I(\delta_1)$ process whereas a part of observations $x_{\lfloor \tau T \rfloor + 1}, \ldots, x_T$ comes from a $I(\delta_1)$ and a part from a $I(\delta_2)$ process. Therefore the periodogram of series $0, \ldots, 0, x_{\lfloor \tau T \rfloor + 1}, \ldots, x_T$ has features similar to those of the periodogram of a signal plus noise process with signal $I(\delta_1)$ and noise $I(\delta_2)$. Dalla, Giraitis and Hidalgo (2006) have shown that in this case $\widehat{\delta}_2(\tau) \to_p \delta_1$. In their Theorem 3, these authors discuss the conditions under which the

estimator $\widehat{\delta}_2(\tau)$ of δ_1 may be subject to a lower order bias: this would at least warrant some power to a test using the $\widehat{t}^2(\tau)$ statistic.

3 A Monte Carlo exercise

The results of Section 2 are asymptotic. We therefore examine performance of the proposed test procedure in finite samples. In the first exercise, summarized in Table 1, we study the size of the test under a range of data generating processes (DGP) and bandwidths.

We consider model

$$(1-\phi L)\Delta^{\delta}x_t = \varepsilon_t,$$

where ε_t is independently normally distributed with $E(\varepsilon_t) = 0$ and $E(\varepsilon_t^2) = 1$. We consider three cases. First, $\phi = 0$, $\delta = 0$, so that x_t is an iid process. Second, $\phi = 0$, $\delta = 0.4$, so x_t is a fractional noise. Third, $\phi = 0.5$, $\delta = 0$, so that x_t is an AR(1) process. We simulate the fractional noise process using the Cholesky decomposition of the covariance matrix. The empirical size of the tests is measured by how often the test statistic exceeds the 5% critical value.

We use $T = 128$, 256, 512, 1024 and consider four settings for m, $m = \lfloor T^{0.5} \rfloor$, $\lfloor T^{0.65} \rfloor$, $\lfloor T^{0.79} \rfloor$ and $\lfloor T^{0.9} \rfloor$. In cases in which m exceeds $T/2-1$, we set $m = T/2-1$. For example, when $T = 128$ then $m = \lfloor 128^{0.9} \rfloor = 78$ and we set $m = 63$. For each case, we run 10000 repetitions and we record three results: how often the test statistic $\widehat{t}^2$ exceeds the critical value of 8.85 for $[\tau_l, \tau_h] = [0.15, 0.85]$, how often the test statistic $\widehat{t}^2(1/2)$ exceeds the critical value 3.84 and how often the Wald statistic $W\left(\widehat{\delta}\right) = 4m\left(\widehat{\delta} - \delta\right)^2$ exceeds the critical value of 3.84. Notice that the Wald statistic $W\left(\widehat{\delta}\right)$ is computed using the whole sample to estimate δ.

We study performance of the various bandwidths. Assumption 4 requires that $m/T^{0.8} \to 0$ so $m = \lfloor T^{0.79} \rfloor$ is the largest bandwidth consistent with this assumption among the ones we considered. MSE-optimal bandwidths for the estimator $\widehat{\delta}$ are of the type $m = \lfloor \alpha T^{0.8} \rfloor$ where α depends on the curvature of the spectrum of $\Delta^{\delta}x_t$, see for example Henry (2001).

Two sources of lower order bias can affect $\widehat{\delta}$. The first one is due to the curvature of the spectrum of $\Delta^{\delta}x_t$ so that, for example, if $\phi > 0$ then the estimator is subject to a positive bias which for given m is stronger the larger is ϕ. The second source of bias is due to the approximation of $\left|1-e^{-i\lambda}\right|^{2\delta}$ by $\lambda^{2\delta}$ in the loss function. For given δ and ϕ, both biases become stronger the larger m is for a given T.

The variance can be also underestimated due to the fact that m is finite. Although $\frac{1}{m}\sum_{j=1}^{m}\nu_j^2 \to 1$ as $m \to \infty$, with $T = 128$, for example, $\frac{1}{m}\sum_{j=1}^{m}\nu_j^2 \approx 0.5$ if $m = \lfloor T^{0.5} \rfloor$ and $\frac{1}{m}\sum_{j=1}^{m}\nu_j^2 \approx 0.8$ if $m = \lfloor T^{0.9} \rfloor$ (recall the requirement that $m < T/2 - 1$). Thus the variance is more severely underestimated the smaller the bandwidth is.

The three models we consider illustrate the potential distortions discussed above. In the iid case no bias should occur and the best estimate of δ in the MSE sense should be obtained for the largest bandwidth. Regarding the other two models, in the AR(1) case the estimate is subject to a bias due to the curvature of the spectrum of x_t whereas in the fractional noise case the estimate is subject to an error due to the approximation of $\left|1-e^{-i\lambda}\right|^{2\delta}$ by $\lambda^{2\delta}$. The bandwidths $m = \lfloor T^{0.5} \rfloor$ and $m = \lfloor T^{0.65} \rfloor$ have emerged as popular among practitioners. In particular, Abadir, Distaso and Giraitis (2007) have found that the latter bandwidth gives a good MSE performance in a range of situations. Our choice of bandwidths is based partly on the choice already considered in the literature. It should however be noted that our problem is different. We are not interested in minimum MSE estimation of δ but rather in correct size when testing the null hypothesis of no break using the $\widehat{t}^2$ and $\widehat{t}^2(\tau)$ tests.

We first comment briefly on the size of the Wald test since the Wald test statistic has been widely used in applied work. For the iid model, the limit χ_1^2 distribution seems acceptable for all the bandwidths. Therefore the iid data generating process case seems to be a benchmark for the most favorable situation. It is worth noting that the 5% nominal level is best approximated for larger bandwidths, perhaps because $\frac{1}{m}\sum_{j=1}^{m}\nu_j^2$ is closer to 1 for larger values of m. For the fractional noise, the χ_1^2 limit is also fairly appropriate except for the extreme case of $m = \lfloor T^{0.9} \rfloor$. This case is not covered by Robinson's theory because $\widehat{\delta}$ is severely biased due to the approximation of $\left|1-e^{-i\lambda}\right|^{2\delta}$ by $\lambda^{2\delta}$. Note that the χ_1^2 approximation seems to work better with bandwidth $\lfloor T^{0.65} \rfloor$

rather than $\lfloor T^{0.79} \rfloor$. Finally, for the DGP is the AR(1) model, when $\widehat{\delta}$ is subject to a lower order bias that is stronger the larger is the bandwidth, $m = \lfloor T^{0.5} \rfloor$ seems the only safe solution though we still see some size distortion even in large samples.

We now comment on the performance of our statistics $\widehat{t}^2$ and $\widehat{t}^2(\tau)$. In terms of size, tests based on these two statistics perform best for the largest bandwidths, $m = \lfloor T^{0.79} \rfloor$ or $m = \lfloor T^{0.9} \rfloor$. This is a reversal of the conclusions for the estimator $\widehat{\delta}$ of the degree of persistence where the smallest bandwidths are the best choice. This may seem surprising in view of the size distortions for $W\left(\widehat{\delta}\right)$. However, since this size distortion is mostly generated by the bias in the estimation of δ, it is possible that the bias affecting $\widehat{\delta}_1(\tau)$ and $\widehat{\delta}_2(\tau)$ is similar, especially when τ divides $[0,1]$ approximately in half, and that this bias mostly cancels when we take the difference $\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)$. Simulation carried out by Shimotsu (2006) and his discussion of the results seem to support this conjecture.

Tests $\widehat{t}^2$ and $\widehat{t}^2(\tau)$ have similar size properties. Lack of knowledge of the location of the potential break fraction does not have an adverse effect on size. Both tests are also subject to size distortion, in particular when $\delta = 0$. In some cases the better size performance when $\delta = 0.4$ may be due to the fact that the estimation interval for $\widehat{\delta}_1(\tau)$ and $\widehat{\delta}_2(\tau)$ is restricted to $[-0.49, 0.49]$, a choice that may limit the range of $\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)$. Overall the size distortion is usually smaller than the distortion observed for widely used $W\left(\widehat{\delta}\right)$. In our opinion this potential size distortion is something that must be kept in mind but is not a reason to avoid using these tests.

In the second part of the Monte Carlo exercise we deal with detecting a break in δ and study the power of the test. The results of the power simulations are reported in Table 2. The tests are based on $\widehat{t}(\tau)^2$ and $\widehat{t}^2$, the null hypothesis of stability of δ being rejected if the test statistic exceeds the appropriate critical value with nominal size set at 5%. We use the same sample sizes and bandwidths as in the size exercise and we carry out 1000 repetitions for each experiment.

We consider three models, Model A with $\delta_1 = 0.4$, $\delta_2 = 0$ and $\tau^* = 1/2$, Model B with $\delta_1 = 0.2$, $\delta_2 = 0$ and $\tau^* = 1/2$, and Model C with $\delta_1 = 0.4$, $\delta_2 = 0$ and $\tau^* = 2/3$.

Table 1: Empirical size in the case of no breaks

	iid			AR(1)			ARFIMA(0,0.4,0)		
T	$\widehat{t}^2$	$\widehat{t}(\tau)^2$	$W\left(\widehat{\delta}\right)$	$\widehat{t}^2$	$\widehat{t}(\tau)^2$	$W\left(\widehat{\delta}\right)$	$\widehat{t}^2$	$\widehat{t}(\tau)^2$	$W\left(\widehat{\delta}\right)$
128	0.048	0.114	0.191	0.049	0.121	0.214	0.025	0.041	0.119
256	0.076	0.117	0.159	0.077	0.125	0.163	0.056	0.038	0.098
512	0.091	0.115	0.129	0.089	0.118	0.127	0.067	0.034	0.079
1024	0.104	0.097	0.116	0.101	0.099	0.113	0.062	0.031	0.068
128	0.081	0.105	0.135	0.109	0.104	0.507	0.063	0.032	0.088
256	0.104	0.097	0.111	0.119	0.107	0.414	0.053	0.030	0.071
512	0.115	0.081	0.089	0.121	0.087	0.321	0.052	0.032	0.055
1024	0.105	0.071	0.079	0.106	0.078	0.234	0.041	0.035	0.047
128	0.093	0.081	0.101	0.089	0.067	0.987	0.051	0.035	0.099
256	0.094	0.078	0.083	0.099	0.094	0.998	0.040	0.035	0.075
512	0.084	0.064	0.071	0.109	0.092	1.000	0.042	0.040	0.084
1024	0.074	0.061	0.060	0.106	0.085	1.000	0.038	0.045	0.071
128	0.092	0.076	0.090	0.074	0.054	0.999	0.050	0.043	0.145
256	0.083	0.067	0.072	0.079	0.075	1.000	0.043	0.047	0.181
512	0.065	0.061	0.063	0.090	0.091	1.000	0.041	0.052	0.260
1024	0.057	0.055	0.055	0.106	0.093	1.000	0.048	0.055	0.419

" report the empirical size of the test with statistic $\widehat{t}^2 = \sup_{\tau\in[0.15,0.85]} \widehat{t}(\tau)^2$.

$(\tau)^2$" report the empirical size of the test with statistic $\widehat{t}(\tau)^2$, $\tau = 1/2$.

$V\left(\widehat{\delta}\right)$" report the empirical size of the test with the Wald statistic.

Table 2: Empirical power in the case of one break

$\delta_1 = 0.4$, $\delta_1 = 0$, $\tau^* = 1/2$			$\delta_1 = 0.2$, $\delta_1 = 0$, $\tau^* = 1/2$			$\delta_1 = 0.4$, $\delta_1 = 0$, $\tau^* = 2/3$		
$\widehat{t}^2$	$\widehat{t}(1/2)^2$	$\widehat{t}(1/3)^2$	$\widehat{t}^2$	$\widehat{t}(1/2)^2$	$\widehat{t}(1/3)^2$	$\widehat{t}^2$	$\widehat{t}(1/2)^2$	$\widehat{t}(1/3)^2$
0.171	0.271	0.077	0.081	0.158	0.110	0.144	0.085	0.053
0.318	0.401	0.100	0.145	0.213	0.137	0.311	0.121	0.071
0.455	0.543	0.111	0.194	0.214	0.141	0.427	0.120	0.069
0.600	0.679	0.113	0.268	0.268	0.147	0.551	0.140	0.053
0.373	0.520	0.146	0.147	0.211	0.132	0.367	0.124	0.058
0.590	0.728	0.202	0.233	0.279	0.178	0.534	0.192	0.086
0.801	0.880	0.212	0.293	0.348	0.169	0.750	0.172	0.073
0.938	0.970	0.245	0.386	0.484	0.223	0.906	0.216	0.080
0.626	0.800	0.242	0.202	0.267	0.159	0.570	0.246	0.091
0.868	0.947	0.430	0.328	0.423	0.240	0.808	0.351	0.171
0.982	0.995	0.566	0.476	0.625	0.290	0.958	0.456	0.216
0.997	1.000	0.719	0.735	0.848	0.459	1.000	0.593	0.335
0.733	0.859	0.350	0.222	0.314	0.174	0.655	0.331	0.141
0.945	0.982	0.639	0.402	0.544	0.293	0.918	0.538	0.305
1.000	1.000	0.829	0.655	0.813	0.447	1.000	0.714	0.464
1.000	1.000	0.974	0.939	0.981	0.709	1.000	0.912	0.705

" report the empirical power of the test with statistic $\widehat{t}^2 = \sup_{\tau \in [0.15, 0.85]} \widehat{t}(\tau)^2$.

$1/2)^2$" report the empirical power of the test with statistic $\widehat{t}(\tau)^2$ with $\tau = 1/2$.

$1/3)^2$" report the empirical power of the test with statistic $\widehat{t}(\tau)^2$ with $\tau = 1/3$.

By comparing Models A and B we examine the effect of altering the dimension of the break, while a comparison of Models A and C allows us to concentrate on the effect of altering the location of the break. In all cases we assume that both before and after the break, the process is a fractional Gaussian noise, with different order of integration in the two subsamples.

In all tests, for a given sample size and bandwidth rule we observe that the power is higher the larger is the break and the closer it is to the middle of the sample. We also observe that for a given bandwidth rule and model the power increases with the sample size and that for a given sample size and model the power increases with the bandwidth.

Finally, we compare the power of the tests using the statistics $\widehat{t}^2$ and $\widehat{t}(\tau)^2$. We find that if the potential breakpoint is chosen correctly in the $\widehat{t}(\tau)^2$, so that $\widehat{t}(\tau^*)^2$ is used, then $\widehat{t}(\tau^*)^2$ has more power than $\widehat{t}^2$. Otherwise the power of the $\widehat{t}(\tau)^2$ may be quite limited, especially when $\tau^* = 2/3$. An interesting finding is that when $\tau^* = 2/3$ then the test based on $\widehat{t}(1/2)^2$ has more power than the test based on $\widehat{t}(1/3)^2$ and thus the larger error in choosing τ compared to τ^* is penalized with a more relevant loss of power.

4 Empirical application: From the Bundesbank to the ECB

We use our semiparametric test for persistence stability to analyse the inflation rate in Germany for the period 1997-2017. Interest in this inflation persistence is motivated by the fact that stabilizing inflation is a key monetary policy target. This is sometimes recognized explicitly in a formal inflation target, for example in Germany (until 1999) and the Euro area (after 1999), or in the United Kingdom, Canada, New Zealand, and other countries. Even in cases in which a formal inflation targeting commitment may be missing, such as for the US, inflation stabilization is still relevant. In practice it is of course impossible to maintain inflation constantly on the target but it is at least important that deviations from the targets are not too extreme and not too strongly persistent because such deviations would signal long term imbalances.

The order of integration provides an intuitive and simple measure of persistence that can be given an easy economic interpretation. A low level persistence can be associated with tighter inflation control. Conversely, a large degree of the persistence index signals a situation in which the central bank does not or cannot control inflation. A test for a break, and possibly a comparative study of the estimators before and after the break, would also reveal if a structural change, either in the management of monetary policy, or in the structure of the economy, or both, has taken place. There is therefore a wide range of empirical work dedicated to the estimation of the order of integration and testing for a change in this order. In studies on US data, Kumar and Okimoto (2007), Sibbertsen and Kruse (2009) and Martins and Rodrigues (2012) have all found that inflation persistence declined since 1982. On the other hand, Hassler and Meller (2014) have found that inflation persistence has increased since 1973, a second break in 1980 not being significant. With integer orders only, Halunga, Osborn and Sensier (2009) have concluded that inflation persistence increased in the early 1970s and returned under control in the early 1980s.

Germany has received comparatively less attention, featuring occasionally in wider studies for a range of countries such as in a study by Martins and Rodrigues (2012). A dedicated study of the case of Germany seems of particular interest because of the history of its central bank's monetary policy. The Bundesbank was committed to the monetary policy target of price stability which was formally implemented with an intermediate target in form of monetary aggregate. However, the Bundesbank also announced an inflation projection for the medium term which was set as 2% since 1986 (with a band 1.5% - 2% in 1997-1998). Although the Bundesbank was formally committed to a monetary target, Bernanke and Mihov (1996) showed that "the Bundesbank is much better described as an inflation targeter that as a money targeter". The same inflation target for monetary policy was officially adopted by the European Central Bank (ECB) for the Euro area, although with the slightly different statement of "below but close to 2%".

Broadly speaking, therefore, the ECB targeted the same inflation rate as the Bundesbank did and indeed there is evidence that inflation has been on average on target during

the ECB mandate, see for example Hualde and Iacone (2017). However as the ECB is a different institution from the Bundesbank and as its mandate is for the euro area, rather than for Germany only, it is important to check if the change in the monetary authority resulted in an increase or decrease of persistence. This experiment is particularly interesting because it is sometimes difficult to identify if a change in inflation persistence is due to a change in the structure of the economy, rather than on the attitude of the central bank. However, the introduction of the euro provides us with a natural experiment to compare the attitude of the ECB against the Bundesbank. Of course, the fact that the euro was introduced in January 1999 also provides us with an additional piece of information and we could also test for a break with known breakpoint. This would be advantageous because if the choice of a breakpoint is correct the test has more power. However, as the sample spans several other periods of interest, including the effects of the German reunification and the financial crisis, testing over the whole sample offers a wider picture of the inflation dynamics.

In our empirical analysis, we use CPI data from Datastream, series code BDCONPRCF. The monthly time series spans the period from January 1986 to April 2017. We obtain inflation as $\ln(cpi_t) - \ln(cpi_{t-1})$ and then compute the test statistics $\widehat{t}^2$ for trimming region $[\tau_l, \tau_h] = [0.15, 0.85]$, and $\widehat{t}(\tau)^2$ with τ set so that $\lfloor \tau T \rfloor$ corresponds to January 1999. Setting the bandwidth $m = \lfloor T^{0.79} \rfloor = 108$, we computed $\widehat{t}^2 = 5.43$ and $\widehat{t}(\tau)^2 = 5.26$, the whole sample estimate for the order of integration being $\widehat{\delta} = 0.02$. With bandwidth $m = \lfloor T^{0.65} \rfloor = 47$ we compute $\widehat{t}^2 = 1.36$ and $\widehat{t}(\tau)^2 = 0.79$ and $\widehat{\delta} = 0.08$ (for January 1999 we estimate $\widehat{\delta}_1 = 0.09$, $\widehat{\delta}_2 = -0.04$). As the frequency of the observations is monthly, it is possible that the estimators of δ (and of δ_1 and δ_2) are subject to a lower order bias. With period $\frac{2\pi}{\lambda} = 12$ we have $\lambda = \frac{2\pi}{12} = 0.52$ and $\frac{2\pi m}{T} = \frac{2\pi}{12}$ so $m = \frac{T}{2\pi}\frac{2\pi}{12} = \frac{376}{12} = 31.333$. Thus, we repeat the exercise with $m = 29$ and compute $\widehat{t}^2 = 1.70$, $\widehat{t}(\tau)^2 = 0.33$, $\widehat{\delta} = 0.33$ ($\widehat{\delta}_1 = 0.35$, $\widehat{\delta}_2 = 0.24$ for January 1999), and with $m = 27$ we obtain $\widehat{t} = 2.17$, $\widehat{t}(\tau)^2 = 1.30$, $\widehat{\delta} = 0.39$ ($\widehat{\delta}_1 = 0.48$, $\widehat{\delta}_2 = 0.26$). We find again that the persistence across the two samples is slightly higher during the Bundesbank tenure than afterwards but not significantly so. Except for the test $\widehat{t}(\tau)^2$ with largest bandwidth, no test leads to the rejection of the null hypothesis. Overall we interpret

these results as evidence that inflation persistence for Germany did not increase with the change of the monetary authority and that at most it declined in the second part of the sample. On balance we conclude that the German inflation was not subject to major instability over these years.

5 Conclusions

We study the local Whittle estimator of the memory parameter in the presence of a structural break in the stochastic component. We find that when the location of the break is unknown the consistency of the test based on $\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)$ may rest on a lower order bias only and a test based on $\sup_\tau \left(\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)\right)$, $\tau \in [\tau_l, \tau_h] \subset (0,1)$, seems advisable. A Monte Carlo exercise supports this conjecture. We also find that in some circumstances the size of the test may be incorrect but that this effect is mitigated as the sample gets larger and if larger bandwidths are used. We apply the test to study the persistence of inflation in Germany over the period 1986–2014. We find that the persistence did not change and that we can conclude that the transition from the Bundesbank to the Eurosystem did not deteriorate the measure of the inflation control.

6 Appendix

In this Appendix we present the technical results together with their proofs and auxiliary lemmas.

6.1 Consistency of estimators of δ

Proposition 1 *Under Assumptions 1–4 and under the null hypothesis with* $\delta_1 = \delta_2 = \delta$,

$$\sup_{\tau_\ell \leq \tau \leq 1} \left|\widehat{\delta}_1(\tau) - \delta\right| \xrightarrow{p} 0 \qquad as\ T \to \infty,$$

$$\sup_{0 \leq \tau \leq \tau_h} \left|\widehat{\delta}_2(\tau) - \delta\right| \xrightarrow{p} 0 \qquad as\ T \to \infty,$$

so that

$$\sup_{\tau_\ell \leq \tau \leq \tau_h} \left|\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)\right| \xrightarrow{p} 0 \qquad as\ T \to \infty.$$

Let $A_j = A(\lambda_j)$, $u_{\tau,j} = \frac{w_{0\tau}(\lambda_j)}{A_j}$, $v_{\tau,j} = \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{\lfloor \tau T \rfloor} \varepsilon_t e^{i\lambda_j t}$, $u_{\sigma\tau,j} = u_{\tau,j} - u_{\sigma,j}$ and $v_{\sigma\tau,j} = v_{\tau,j} - v_{\sigma,j}$.

Lemma 1 *Under Assumptions 1–4, for any integers $1 \leq j_s \leq m$ with $1 \leq s \leq p$ where $p = 2, \ldots, 6$ and 8, and for σ and τ such that $[\sigma, \tau] \subset [0, \tau^*]$ or $[\sigma, \tau] \subset [\tau^*, 1]$, there is a finite constant C such that*

$$\operatorname{cum}\left(u_{\sigma\tau,j_1} - v_{\sigma\tau,j_1}, \ldots, u_{\sigma\tau,j_p} - v_{\sigma\tau,j_p}\right) \leq C(\tau - \sigma)^{\frac{p}{2}} (j_1 \cdots j_p)^{-\frac{1}{2}}. \tag{10}$$

Proof of Lemma 1. When $p = 8$, using formulas (2.6.3) and (2.10.3) of Brillinger (1981), the cumulant on the left of (10) can be written as

$$\begin{aligned}
&\frac{\kappa_8}{T^4} \frac{1}{(2\pi)^8} \underbrace{\int \cdots \int_{-\pi}^{\pi}}_{7\times} \left(\frac{A(\omega_1)}{A(\lambda_{j_1})} - 1\right) \left(\frac{A(\omega_2)}{A(\lambda_{j_2})} - 1\right) \left(\frac{A(\omega_3)}{A(\lambda_{j_3})} - 1\right) \\
&\times \left(\frac{A(\omega_4)}{A(\lambda_{j_4})} - 1\right) \left(\frac{A(\omega_5)}{A(\lambda_{j_5})} - 1\right) \left(\frac{A(\omega_6)}{A(\lambda_{j_6})} - 1\right) \left(\frac{A(\omega_7)}{A(\lambda_{j_7})} - 1\right) \\
&\times \left(\frac{A(-\omega_1 - \ldots - \omega_7)}{A(\lambda_{j_8})} - 1\right) D_{\sigma\tau}(\omega_1 + \lambda_{j_1}) D_{\sigma\tau}(\omega_2 + \lambda_{j_2}) D_{\sigma\tau}(\omega_3 + \lambda_{j_3}) \\
&\times D_{\sigma\tau}(\omega_4 + \lambda_{j_4}) D_{\sigma\tau}(\omega_5 + \lambda_{j_5}) D_{\sigma\tau}(\omega_6 + \lambda_{j_6}) D_{\sigma\tau}(\omega_7 + \lambda_{j_7}) \\
&\times D_{\sigma\tau}(-\omega_1 - \ldots - \omega_7 + \lambda_{j_8})\, d\omega_1 \ldots d\omega_7,
\end{aligned}$$

where $\kappa_8 = \operatorname{cum}(\varepsilon_t, \ldots, \varepsilon_t)$ is the eighth cumulant of ε_t and

$$D_{\sigma\tau}(\lambda) = \sum_{t=\lfloor \sigma T \rfloor + 1}^{\lfloor \tau T \rfloor} e^{it\lambda}.$$

It follows from the Schwarz inequality and periodicity that this is bounded by

$$\frac{\kappa_8}{(2\pi)^4} \left(P_{\sigma\tau,j_1} \cdots P_{\sigma\tau,j_8}\right)^{\frac{1}{2}}$$

where

$$P_{\sigma\tau,j} = \int_{-\pi}^{\pi} \left|\frac{A(\omega)}{A(\lambda_j)} - 1\right|^2 K_{\sigma\tau}(\omega - \lambda_j)\, d\omega$$

and

$$K_{\sigma\tau}(\lambda) = K_{\sigma\tau,T}(\lambda) = \frac{|D_{\sigma\tau}(\lambda)|^2}{2\pi T}.$$

Noting that $K_{\sigma\tau,T}(\lambda) = \frac{\lfloor \tau T \rfloor - \lfloor \sigma T \rfloor}{T} K_{01,\lfloor \tau T \rfloor - \lfloor \sigma T \rfloor}(\lambda)$, we write

$$P_{\sigma\tau,j} = \frac{\lfloor \tau T \rfloor - \lfloor \sigma T \rfloor}{T} \int_{-\pi}^{\pi} \left|\frac{A(\omega)}{A(\lambda_j)} - 1\right|^2 K_{01,\lfloor \tau T \rfloor - \lfloor \sigma T \rfloor}(\omega - \lambda_j)\, d\omega. \tag{11}$$

The kernel $K_{01,\lfloor\tau T\rfloor-\lfloor\sigma T\rfloor}$ has the property

$$K_{01,\lfloor\tau T\rfloor-\lfloor\sigma T\rfloor}\ (\lambda) \leq \frac{C}{|\lambda|}, \qquad 0<\lambda\leq\pi, \quad T\geq 1$$

by Lemma 1 of Lazarová (2005). Using the arguments in the proof of Lemma 3 of Robinson (1995), the integral in (11) can be seen to be $O\left(j^{-1}\right)$ uniformly over integers $1\leq j\leq[T/2]$. Therefore

$$P_{\sigma\tau,j}\leq C\left(\tau-\sigma\right)j^{-1} \tag{12}$$

uniformly over integers $1\leq j\leq[T/2]$ and bound (10) holds for $p=8$. A similar approach yields proof of bound (10) for $p=2,\ldots,6$. ■

Lemma 2 *Under Assumptions 1–4, for $1\leq j\leq m$, and for σ and τ such that $\tau-\sigma\geq 1/T$ and $[\sigma,\tau]\subset[0,\tau^*]$ or $[\sigma,\tau]\subset[\tau^*,1]$, there is a finite constant C such that*

$$\begin{array}{lrcl} \textit{(a)} & E\left|u_{\sigma\tau,j}-v_{\sigma\tau,j}\right|^8 & \leq & C\left(\tau-\sigma\right)^4 j^{-4}, \\ \textit{(b)} & E\left|v_{\sigma\tau,j}\right|^8 & \leq & C\left(\tau-\sigma\right)^4. \end{array}$$

Proof of Lemma 2. (a) Using formula (2.8) of McCullagh (1987), we have that for random variables $Y_1,\ldots,Y_r$,

$$E\left(Y_1\cdots Y_r\right)=\sum_{\pi}\prod_{B\in\pi}\operatorname{cum}\left(Y_i:i\in B\right), \tag{13}$$

where π runs through the list of all partitions of $\{1,\ldots,r\}$ and B runs through the list of all blocks of the partition π. Since $E\left(u_{\sigma\tau,j}-v_{\sigma\tau,j}\right)=0$, part (a) is implied by Lemma 1.

(b) We have

$$\begin{aligned} E\left|v_{\sigma\tau,j}\right|^8 &= \frac{1}{\left(2\pi T\right)^4}\sum_{t,s,r,v,t_1,s_1,r_1,v_1=\lfloor\sigma T\rfloor+1}^{\lfloor\tau T\rfloor} E\varepsilon_t\varepsilon_s\varepsilon_r\varepsilon_v\varepsilon_{t_1}\varepsilon_{s_1}\varepsilon_{r_1}\varepsilon_{v_1} \\ &\quad\times e^{i(t-s)\lambda_j}e^{i(r-v)\lambda_j}e^{i(t_1-s_1)\lambda_j}e^{i(r_1-v_1)\lambda_j} \\ &\leq \frac{1}{\left(2\pi T\right)^4}\sum_{t,s,r,v,t_1,s_1,r_1,v_1=\lfloor\sigma T\rfloor+1}^{\lfloor\tau T\rfloor}\left|E\varepsilon_t\varepsilon_s\varepsilon_r\varepsilon_v\varepsilon_{t_1}\varepsilon_{s_1}\varepsilon_{r_1}\varepsilon_{v_1}\right|. \end{aligned}$$

Using (13) it can be seen that $E\left|v_{\sigma\tau,j}\right|^8$ is bounded by

$$\frac{C}{T^4}\left(\sum_{t,s,r,v}\kappa_2^4+\sum_{t,s,r}\left(\kappa_2^2\kappa_4+\kappa_2\kappa_3^2\right)+\sum_{t,s}\left(\kappa_2\kappa_6+\kappa_3\kappa_5+\kappa_4^2\right)+\sum_{t}\kappa_8\right),$$

where $\kappa_p = \operatorname{cum}(\varepsilon_t, \ldots, \varepsilon_t)$ is the p-th cumulant of ε_t and where the sums run from $\lfloor \sigma T \rfloor + 1$ to $\lfloor \tau T \rfloor$. This is bounded by

$$C\left(\frac{(\tau-\sigma)^4 T^4}{T^4} + \frac{(\tau-\sigma)^3 T^3}{T^4} + \frac{(\tau-\sigma)^2 T^2}{T^4} + \frac{(\tau-\sigma) T}{T^4}\right) \leq C(\tau-\sigma)^4$$

since $\frac{1}{T} \leq \tau - \sigma$. ■

Let

$$\nu_j = \log j - \frac{1}{m}\sum_{k=1}^{m} \log k = \log\left(\frac{j}{m}\right) - \frac{1}{m}\sum_{k=1}^{m} \log\left(\frac{k}{m}\right). \tag{14}$$

Let $D_k(\lambda) = \sum_{t=\lfloor \sigma T \rfloor + 1}^{\lfloor \tau T \rfloor} e^{it\lambda}$ and $K_k = \frac{1}{2\pi T} |D_k(\lambda)|^2$.

Lemma 3 *For k such that $k/T \sim a$ with $0 < a < 1$ as $T \to \infty$, we have*

$$\begin{aligned} &(a) \qquad \frac{2\pi}{mT}\sum_{j=1}^{m}\sum_{\ell=1}^{m} \nu_j \nu_\ell K_k(\lambda_j - \lambda_\ell) &&= \frac{k}{T} + o(1), \\ &(b) \qquad \frac{2\pi}{mT}\sum_{j=1}^{m}\sum_{\ell=1}^{m} \nu_j \nu_\ell K_k(\lambda_j + \lambda_\ell) &&= o(1). \end{aligned}$$

Proof of Lemma 3. (a) We have

$$\begin{aligned} &\frac{2\pi}{mT}\sum_{j=1}^{m}\sum_{\ell=1}^{m} \nu_j \nu_\ell K_k(\lambda_j - \lambda_\ell) \\ = \;& \frac{2\pi}{mT}\sum_{j=1}^{m} \nu_j^2 \sum_{\ell=1}^{T} K_k(\lambda_j - \lambda_\ell) - \frac{2\pi}{mT}\sum_{j=1}^{m} \nu_j^2 \sum_{\ell=m+1}^{T} K_k(\lambda_j - \lambda_\ell) \\ &+ \frac{2\pi}{mT}\sum_{j=1}^{m} \nu_j \sum_{\ell=1}^{m} (\nu_\ell - \nu_j) K_k(\lambda_j - \lambda_\ell). \end{aligned} \tag{15}$$

The first term on the right of (15) is equal to

$$\frac{k}{T}\frac{1}{m}\sum_{j=1}^{m} \nu_j^2 = \frac{k}{T}(1 + o(1))$$

because $m^{-1}\sum_{j=1}^{m} \nu_j^2 = 1 + O\left(m^{-1}\log^2 m\right)$. Kernel K has the following properties:

$$\begin{aligned} K_k(\lambda) &\leq \frac{k^2}{2\pi T} \qquad \lambda \in [0, 2\pi], \\ K_k(\lambda) &\leq \frac{\pi}{2T\lambda^2} \qquad \lambda \in (0, \pi]. \end{aligned}$$

For $|\ell| \leq T/(2k)$ the first bound is at least as good as the second bound.

The second term on the right of (15) is bounded by

$$\frac{C\log^2 m}{mT}\sum_{j=1}^{m}\sum_{\ell=m-j+1}^{T-j}K_k\left(\lambda_\ell\right)\leq\frac{C\log^2 m}{mT}\sum_{j=1}^{m}\left(\sum_{\ell=m-j+1}^{T/2}+\sum_{\ell=j}^{T/2}\right)\frac{1}{T\lambda_\ell^2}$$

$$\begin{aligned}&=\frac{C\log^2 m}{m}\sum_{j=1}^{m}\left(\sum_{\ell=m-j+1}^{T/2}+\sum_{\ell=j}^{T/2}\right)\frac{1}{\ell^2}\leq\frac{C\log^2 m}{m}\sum_{j=1}^{m}\left(\frac{1}{m-j+1}+\frac{1}{j}\right)\\&\leq\frac{C\log^3 m}{m}\end{aligned}$$

because $\nu_j=O\left(\log m\right)$ and because kernel K is symmetric.

Let $a_k=[T/\left(2k\right)]$. For sufficiently large T it is $m>a_k$ and the third term on the right of (15) is bounded in absolute value by

$$\begin{aligned}&\frac{C\log m}{mT}\sum_{\ell=1}^{m-1}\sum_{j=\ell+1}^{m}\left|\nu_{j-\ell}-\nu_j\right|K_k\left(\lambda_\ell\right)\\&\leq\frac{C\log m}{mT}\left(\sum_{\ell=1}^{a_k}\sum_{j=\ell+1}^{m}\left|\nu_{j-\ell}-\nu_j\right|\frac{k^2}{T}+\sum_{\ell=a_k+1}^{m-1}\sum_{j=\ell+1}^{m}\left|\nu_{j-\ell}-\nu_j\right|\frac{T}{\ell^2}\right).\end{aligned}\tag{16}$$

By the mean value theorem,

$$\nu_{j-\ell}-\nu_j=\log\left(j-\ell\right)-\log j=-\frac{\ell}{\xi}\qquad\qquad j-\ell\leq\xi\leq j,$$

so that

$$\left|\nu_{j-\ell}-\nu_j\right|\leq\frac{\ell}{j-\ell}\qquad\qquad 1\leq\ell\leq m,\ \ell+1\leq j\leq m.$$

Therefore (16) is bounded by

$$\begin{aligned}&\frac{C\log m}{mT}\left(\frac{k^2}{T}\sum_{\ell=1}^{a_k}\ell\sum_{j=\ell+1}^{m}\frac{1}{j-\ell}+T\sum_{\ell=a_k+1}^{m-1}\sum_{j=\ell+1}^{m}\frac{1}{\left(j-\ell\right)\ell}\right)\\&=\frac{C\log m}{m}\left(\sum_{\ell=1}^{a_k}\ell\sum_{j=1}^{m-\ell}\frac{1}{j}+\sum_{\ell=a_k+1}^{m-1}\frac{1}{\ell}\sum_{j=1}^{m-\ell}\frac{1}{j}\right)\\&\leq\frac{C\log m}{m}\left(\log m\sum_{\ell=1}^{a_k}\ell+\log^2 m\right)\leq\frac{C\log^3 m}{m}=o\left(1\right).\end{aligned}$$

Gathering results, we obtain

$$\frac{2\pi}{mT}\sum_{j=1}^{m}\sum_{\ell=1}^{m}\nu_j\nu_\ell K_k\left(\lambda_j-\lambda_\ell\right)=\frac{k}{T}+o\left(1\right).$$

(b) We have that

$$\frac{2\pi}{mT}\sum_{j=1}^{m}\sum_{\ell=1}^{m}\nu_j\nu_\ell K_k\left(\lambda_j+\lambda_\ell\right)=\frac{2\pi}{mT}\sum_{j=1}^{m}\sum_{\ell=1+j}^{m+j}\nu_j\nu_{\ell-j}K_k\left(\lambda_\ell\right)$$

which is bounded in absolute value by

$$\frac{C\log^2 m}{mT}\sum_{j=1}^{m}\sum_{\ell=1+j}^{m+j}\frac{T}{\ell^2}\leq\frac{C\log^2 m}{m}\sum_{j=1}^{m}\frac{1}{j+1}\leq C\frac{\log^3 m}{m}$$

so that indeed $2\pi m^{-1}T^{-1}\sum_{j=1}^{m}\sum_{\ell=1}^{m}\nu_j\nu_\ell K_k\left(\lambda_j+\lambda_\ell\right)=o\left(1\right)$. ■

For a triangular array $\left\{\mu_{j,m}\left(a\right),1\leq j\leq m\right\}_{m=1}^{\infty}$, let $\mu\left(a\right)=\lim_{m\to\infty}\frac{1}{m}\sum_{j=1}^{m}\mu_{j,m}\left(a\right)$. For simplicity, we drop the reference to m and a in what follows and write μ_j for $\mu_{j,m}\left(a\right)$ and μ for $\mu\left(a\right)$.

Lemma 4 *Under Assumptions 1–4,*

(a) $$\frac{1}{m}\sum_{j=1}^{m}\mu_j\frac{I_{0\tau,j}}{G\lambda_j^{-2\delta_1}}-\tau\mu\Longrightarrow 0\qquad\tau\in\left[0,\tau^*\right],$$

(b) $$\frac{1}{m}\sum_{j=1}^{m}\mu_j\frac{I_{\tau^*\tau,j}}{G\lambda_j^{-2\delta_2}}-\left(\tau-\tau^*\right)\mu\Longrightarrow 0\qquad\tau\in\left[\tau^*,1\right],$$

(c) $$\frac{1}{m}\sum_{j=1}^{m}\mu_j\frac{I_{\tau\tau^*,j}}{G\lambda_j^{-2\delta_1}}-\left(\tau^*-\tau\right)\mu\Longrightarrow 0\qquad\tau\in\left[0,\tau^*\right],$$

(d) $$\frac{1}{m}\sum_{j=1}^{m}\mu_j\frac{I_{\tau1,j}}{G\lambda_j^{-2\delta_2}}-\left(1-\tau\right)\mu\Longrightarrow 0\qquad\tau\in\left[\tau^*,1\right],$$

(e) $$\frac{1}{m}\sum_{j=1}^{m}\mu_j\frac{w_{0\tau^*,j}}{G^{\frac{1}{2}}\lambda_j^{-\delta_1}}\frac{\overline{w}_{\tau^*\tau,j}}{G^{\frac{1}{2}}\lambda_j^{-\delta_2}}\Longrightarrow 0\qquad\tau\in\left[\tau^*,1\right],$$

(f) $$\frac{1}{m}\sum_{j=1}^{m}\mu_j\frac{w_{\tau\tau^*,j}}{G^{\frac{1}{2}}\lambda_j^{-\delta_1}}\frac{\overline{w}_{\tau^*1,j}}{G^{\frac{1}{2}}\lambda_j^{-\delta_2}}\Longrightarrow 0\qquad\tau\in\left[0,\tau^*\right],$$

where $\mu_j = \mu_{j,m}$ assumes either of the following values for all j:

$$\mu_j = \left(\frac{j}{m}\right)^a \qquad \text{with } \mu = \frac{1}{1+a}, \qquad a > -1, \tag{17}$$

$$\mu_j = \log\left(\frac{j}{m}\right)\left(\frac{j}{m}\right)^a \nu_j \qquad \text{with } \mu = \frac{1-a}{(1+a)^3}, \qquad a > -1. \tag{18}$$

Moreover, for any $\varepsilon > 0$, the convergence in parts (a) to (f) holds uniformly over $a \geq -1+\varepsilon$ in the sense that

$$\sup_{a \geq -1+\varepsilon} \left| \frac{1}{m} \sum_{j=1}^{m} \mu_j \frac{I_{0\tau,j}}{G\lambda_j^{-2\delta_1}} - \tau\mu \right| \Longrightarrow 0 \qquad \tau \in [0, \tau^*]$$

in part (a) and similarly in parts (b)-(f).

Proof of Lemma 4. (a) Denote $g_j = G\lambda_j^{-2\delta_1}$ and let

$$Y(\tau) = \frac{1}{m} \sum_{j=1}^{m} \mu_j \frac{I_{0\tau,j}}{g_j} - \tau\mu.$$

We need to prove that $Y(\tau) = o_p(1)$ for any $\tau \in [0, \tau^*]$ and that the process Y is tight. Denote $f_j = f_1(\lambda_j)$, $I_{\varepsilon 0\tau,j} = |v_{\tau,j}|^2 = \left| \frac{1}{\sqrt{2\pi T}} \sum_{t=1}^{\lfloor \tau T \rfloor} \varepsilon_t e^{i\lambda_j t} \right|^2$ and write $Y(\tau)$ as

$$\begin{aligned} &\frac{1}{m} \sum_{j=1}^{m} \mu_j \left(1 - \frac{g_j}{f_j}\right) \frac{I_{0\tau,j}}{g_j} + \frac{1}{m} \sum_{j=1}^{m} \mu_j \left(\frac{I_{0\tau,j}}{f_j} - \frac{2\pi}{\sigma^2} I_{\varepsilon 0\tau,j} \right) \\ &+ \frac{1}{m} \sum_{j=1}^{m} \mu_j \left(\frac{2\pi}{\sigma^2} I_{\varepsilon 0\tau,j} - \frac{\lfloor \tau T \rfloor}{T} \right) + \left(\frac{\lfloor \tau T \rfloor}{T} \frac{1}{m} \sum_{j=1}^{m} \mu_j - \tau\mu \right). \end{aligned} \tag{19}$$

The first moment of the absolute value the first term of (19) is bounded by

$$\max_{1 \leq j \leq m} \left| 1 - \frac{g_j}{f_j} \right| \frac{1}{m} \sum_{j=1}^{m} \left|\mu_j\right| E \left| \frac{I_{0\tau,j}}{g_j} \right|$$

which is $o(1)$ as $T \to \infty$ because

$$\max_{1 \leq j \leq m} \left| 1 - \frac{g_j}{f_j} \right| = o(1)$$

by Assumption 1,

$$E \left| \frac{I_{0\tau,j}}{g_j} \right| \leq C \qquad j = 1, \ldots, m$$

for T sufficiently large by Assumptions 1, 2 and 4 and by Lemma 3 of Lazarová (2005), and because for μ_j defined in (17) and (18), $\frac{1}{m} \sum_{j=1}^{m} \left|\mu_j\right| < \infty$.

By summation by parts, expectation of the absolute value of the second term of (19) can be bounded by

$$\frac{1}{m}\sum_{k=1}^{m-1}\left|\mu_k-\mu_{k+1}\right|E\left|\sum_{j=1}^{k}\left(\frac{I_{0\tau,j}}{f_j}-\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}\right)\right|+\left|\mu_m\right|E\left|\frac{1}{m}\sum_{j=1}^{m}\left(\frac{I_{0\tau,j}}{f_j}-\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}\right)\right|.$$

Proceeding as Robinson (1995) did in bounding his expression (3.17), p. 1637, and employing Lemma 3 of Lazarová (2005), we obtain

$$E\left|\frac{I_{0\tau,j}}{f_j}-\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}\right|\leq C\frac{\log^{\frac{1}{2}}j}{j^{\frac{1}{2}}},$$

so

$$E\left|\sum_{j=1}^{k}\left(\frac{I_{0\tau,j}}{f_j}-\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}\right)\right|\leq Ck^{\frac{1}{2}}\log^{\frac{1}{2}}k.$$

When $\mu_j=\log\left(\frac{j}{m}\right)\left(\frac{j}{m}\right)^a\nu_j$, we have that by the mean value theorem,

$$\left|\mu_j-\mu_{j+1}\right|\leq C\xi^{a-1}m^{-a}\left|\nu_j\right|\leq Cj^{a-1}m^{-a}\log m$$

for $j<\xi<j+1$ because $\left|\nu_j\right|=O\left(\log m\right)$ and $\xi^{a-1}\leq Cj^{a-1}$. Similarly, when $\mu_j=\left(\frac{j}{m}\right)^a$, we have $\left|\mu_j-\mu_{j+1}\right|\leq Cj^{a-1}m^{-a}$. Therefore the first absolute moment of the second term of (19) is bounded by

$$Cm^{-1-a}\log^{\frac{3}{2}}m\sum_{k=1}^{m-1}k^{a-\frac{1}{2}}$$

which is $o\left(1\right)$ for $a>-1$.

Using summation by parts, the third term of (19) can be bounded by

$$\begin{aligned}&\frac{1}{m}\sum_{k=1}^{m-1}\left|\mu_k-\mu_{k+1}\right|\left|\sum_{j=1}^{k}\left(\frac{1}{T\sigma_\varepsilon^2}\sum_{t=1}^{\lfloor\tau T\rfloor}\sum_{s=1}^{\lfloor\tau T\rfloor}\varepsilon_t\varepsilon_s e^{i(t-s)\lambda_j}-\frac{\lfloor\tau T\rfloor}{T}\right)\right|\\&+\frac{\left|\mu_m\right|}{m}\left|\sum_{j=1}^{m}\left(\frac{1}{T\sigma_\varepsilon^2}\sum_{t=1}^{\lfloor\tau T\rfloor}\sum_{s=1}^{\lfloor\tau T\rfloor}\varepsilon_t\varepsilon_s e^{i(t-s)\lambda_j}-\frac{\lfloor\tau T\rfloor}{T}\right)\right|.\end{aligned}\tag{20}$$

By Assumption 3,

$$\begin{aligned}
& E\left|\sum_{j=1}^{k}\left(\frac{1}{T\sigma_\varepsilon^2}\sum_{t=1}^{\lfloor\tau T\rfloor}\sum_{s=1}^{\lfloor\tau T\rfloor}\varepsilon_t\varepsilon_s e^{i(t-s)\lambda_j}-\frac{\lfloor\tau T\rfloor}{T}\right)\right|^2 \\
= & \frac{\omega_4}{\sigma_\varepsilon^2}\frac{\lfloor\tau T\rfloor}{T}\frac{r^2}{T}+C\frac{1}{T^2}\sum_{t=1}^{\lfloor\tau T\rfloor}\sum_{s=1}^{\lfloor\tau T\rfloor}\sum_{j=1}^{k}\sum_{\ell=1}^{k}\left(e^{i(t-s)(\lambda_j-\lambda_\ell)}+e^{i(t-s)(\lambda_j+\lambda_\ell)}\right) \\
= & O\left(k^2T^{-1}\right)+k\frac{2\pi}{kT}\sum_{j=1}^{k}\sum_{\ell=1}^{k}\left(\left|K_{\lfloor\tau T\rfloor}\left(\lambda_j-\lambda_\ell\right)\right|^2+\left|K_{\lfloor\tau T\rfloor}\left(\lambda_j+\lambda_\ell\right)\right|^2\right),
\end{aligned}$$

where the second term is $O(k)$ using arguments as in Lemma 3. The first term of (20) is therefore

$$\begin{aligned}
& O_p\left(\frac{1}{m}\sum_{k=1}^{m-1}k^{a-1}m^{-a}\log m\left(kT^{-1/2}+k^{1/2}\right)\right) \\
= & O_p\left(m^{-a-1}\log m\sum_{k=1}^{m-1}\left(k^aT^{-1/2}+k^{a-1/2}\right)\right)
\end{aligned}$$

$$\begin{aligned}
= & O_p\left(m^{-a-1}T^{-1/2}\log m\sum_{k=1}^{m-1}k^a+m^{-a-1}\log m\sum_{k=1}^{m-1}k^{a-1/2}\right) \\
= & O_p\left(T^{-1/2}\log m+m^{-1/2}\log m\right) \\
= & o_p(1).
\end{aligned}$$

In a similar way, the second term of (20) is

$$\frac{1}{m}\left(O\left(mT^{-1/2}\right)+O\left(m^{1/2}\right)\right)=o_p(1).$$

Finally, the last term of (19) is $o(1)$ by the definition of μ. Gathering results and using the Markov inequality, we can see that $Y(\tau)=o_p(1)$ for any $\tau\in[0,\tau^*]$.

To prove tightness of process Y, we write $Y(\tau)$ as

$$Y(\tau)=Y_1(\tau)+Y_2(\tau)-\mu\tau,$$

where

$$\begin{aligned}
Y_1(\tau) &= \frac{1}{m}\sum_{j=1}^{m}\mu_j\frac{f_j}{g_j}\left(\frac{I_{0\tau,j}}{f_j}-\frac{2\pi}{\sigma^2}I_{\varepsilon 0\tau,j}\right), \\
Y_2(\tau) &= \frac{2\pi}{\sigma_\varepsilon^2}\frac{1}{m}\sum_{j=1}^{m}\mu_j\frac{f_j}{g_j}I_{\varepsilon 0\tau,j}
\end{aligned}$$

with $I_{\varepsilon 0\tau,j}$ as defined above equation (19). Denoting

$$\pi_j = \mu_j f_j / g_j,$$

we obtain for $0 \leq \sigma \leq \tau \leq \tau^*$ that

$$Y_1(\tau) - Y_1(\sigma) = \frac{2\pi}{m} \sum_{j=1}^{m} \pi_j \left(a_{1j} + \ldots + a_{9j}\right)$$

where

$$\begin{aligned}
a_{1j} &= \left|u_{\sigma\tau,j} - v_{\sigma\tau,j}\right|^2, \qquad a_{2j} = \left(u_{\sigma\tau,j} - v_{\sigma\tau,j}\right)\bar{v}_{\sigma\tau,j}, \qquad a_{3j} = \bar{a}_{2j}, \\
a_{4j} &= \left(u_{\sigma\tau,j} - v_{\sigma\tau,j}\right)\left(\bar{u}_{\sigma,j} - \bar{v}_{\sigma,j}\right), \qquad a_{5j} = \left(u_{\sigma\tau,j} - v_{\sigma\tau,j}\right)\bar{v}_{\sigma,j}, \\
a_{6j} &= v_{\sigma\tau,j}\left(\bar{u}_{\sigma,j} - \bar{v}_{\sigma,j}\right), \qquad a_{7j} = \left(u_{\tau,j} - v_{\tau,j}\right)\left(\bar{u}_{\sigma\tau,j} - \bar{v}_{\sigma\tau,j}\right), \\
a_{8j} &= \left(u_{\tau,j} - v_{\tau,j}\right)\bar{v}_{\sigma\tau,j}, \qquad a_{9j} = v_{\tau,j}\left(\bar{u}_{\sigma\tau,j} - \bar{v}_{\sigma\tau,j}\right)
\end{aligned}$$

and where $u_{\tau,j}$, $v_{\tau,j}$, $u_{\sigma\tau,j}$ and $v_{\sigma\tau,j}$ were defined at the beginning of this section. It is

$$\begin{aligned}
E\left|Y_1(\tau) - Y_1(\sigma)\right|^4 &\leq CE\left|\frac{1}{m}\sum_{j=1}^{m}\pi_j a_{1j}\right|^4 + \ldots + CE\left|\frac{1}{m}\sum_{j=1}^{m}\pi_j a_{9j}\right|^4 \\
&= \frac{C}{m^4}\sum_{r=1}^{9}\sum_{j,k,\ell,p=1}^{m}\left|\pi_j\pi_k\pi_\ell\pi_p\right|\left|E a_{rj}\bar{a}_{rk}a_{r\ell}\bar{a}_{rp}\right| \\
&\leq \frac{C}{m^4}\sum_{r=1}^{9}\sum_{j,k,\ell,p=1}^{m}\left|\pi_j\pi_k\pi_\ell\pi_p\right|\left(E\left|a_{rj}\right|^4 E\left|a_{rk}\right|^4 E\left|a_{r\ell}\right|^4 E\left|a_{rp}\right|^4\right)^{\frac{1}{4}},
\end{aligned}$$

where the last inequality follows from the Schwarz inequality. When $r = 9$,

$$E\left|a_{9j}\right|^4 = E\left|v_{\tau,j}\left(\bar{u}_{\sigma\tau,j} - \bar{v}_{\sigma\tau,j}\right)\right|^4 \leq \left(E\left|v_{\tau,j}\right|^8 E\left|\bar{u}_{\sigma\tau,j} - \bar{v}_{\sigma\tau,j}\right|^8\right)^{\frac{1}{2}}.$$

By Lemma 2, the last displayed expression is $O\left(\left(\tau-\sigma\right)^2 j^{-2}\right)$. It can be shown in a similar way that for $1 \leq r \leq 8$, $E\left|a_{rj}\right|^4$ is also $O\left(\left(\tau-\sigma\right)^2 j^{-2}\right)$. Therefore

$$E\left|Y_1(\tau) - Y_1(\sigma)\right|^4 \leq C\left(\tau-\sigma\right)^2\left(\frac{1}{m}\sum_{j=1}^{m}\left|\mu_j\right|\frac{f_j}{g_j}j^{-\frac{1}{2}}\right)^4.$$

Since $\frac{1}{m}\sum_{j=1}^{m}\left|\mu_j\right| < \infty$ and $\max_{1\leq j\leq m}\frac{f_j}{g_j} = O(1)$ by Assumption 1, we have

$$\frac{1}{m}\sum_{j=1}^{m}\left|\mu_j\right|\frac{f_j}{g_j}j^{-\frac{1}{2}} \leq \max_{1\leq j\leq m}\frac{f_j}{g_j}\frac{1}{m}\sum_{j=1}^{m}\left|\mu_j\right| \leq C.$$

It follows that process $Y_1(\tau)$ is tight.

Regarding process Y_2, we note that

$$I_{\varepsilon 0\tau,j} - I_{\varepsilon 0\sigma,j} = \left|v_{\sigma\tau,j}\right|^2 + v_{\sigma\tau,j}\bar{v}_{\sigma,j} + v_{\sigma,j}\bar{v}_{\sigma\tau,j}$$

and obtain bound

$$E\left|Y_2(\tau) - Y_2(\sigma)\right|^4 \leq CE\left|\frac{1}{m}\sum_{j=1}^{m}\pi_j\left|v_{\sigma\tau,j}\right|^2\right|^4 + 2CE\left|\frac{1}{m}\sum_{j=1}^{m}\pi_j v_{\sigma\tau,j}\bar{v}_{\sigma,j}\right|^4.$$

The first term on the right is

$$\begin{aligned}
&\frac{C}{m^4}\sum_{j,k,\ell,p=1}^{m}\pi_j\pi_k\pi_\ell\pi_p E\left|v_{\sigma\tau,j}v_{\sigma\tau,k}v_{\sigma\tau,\ell}v_{\sigma\tau,p}\right|^2 \\
\leq\ &\frac{C}{m^4}\sum_{j,k,\ell,p=1}^{m}\left|\pi_j\pi_k\pi_\ell\pi_p\right|\left(E\left|v_{\sigma\tau,j}\right|^8 E\left|v_{\sigma\tau,\ell}\right|^8 E\left|v_{\sigma\tau,\ell}\right|^8 E\left|v_{\sigma\tau,p}\right|^8\right)^{\frac{1}{4}} \\
\leq\ &C\left(\frac{1}{m}\sum_{j=1}^{m}\left|\pi_j\right|\left(E\left|v_{\sigma\tau,j}\right|^8\right)^{\frac{1}{4}}\right)^4 \leq C(\tau-\sigma)^2\left(\frac{1}{m}\sum_{j=1}^{m}\left|\pi_j\right|\right)^4.
\end{aligned}$$

The second term on the right is

$$\begin{aligned}
&\frac{C}{m^4}\sum_{j,k,\ell,p=1}^{m}\pi_j\pi_k\pi_\ell\pi_p E v_{\sigma\tau,j}\bar{v}_{\sigma,j}\bar{v}_{\sigma\tau,k}v_{\sigma,k}v_{\sigma\tau,\ell}\bar{v}_{\sigma,\ell}\bar{v}_{\sigma\tau,p}v_{\sigma,p} \\
\leq\ &\frac{C}{m^4}\sum_{j,k,\ell,p=1}^{m}\left|\pi_j\pi_k\pi_\ell\pi_p\right| \\
&\times\left(E\left|v_{\sigma\tau,j}\right|^8 E\left|\bar{v}_{\sigma,j}\right|^8 E\left|\bar{v}_{\sigma\tau,k}\right|^8 E\left|v_{\sigma,k}\right|^8 E\left|v_{\sigma\tau,\ell}\right|^8 E\left|\bar{v}_{\sigma,\ell}\right|^8 E\left|\bar{v}_{\sigma\tau,p}\right|^8 E\left|v_{\sigma,p}\right|^8\right)^{\frac{1}{8}} \\
\leq\ &C(\tau-\sigma)^2\sigma^2\left(\frac{1}{m}\sum_{j=1}^{m}\left|\pi_j\right|\right)^4.
\end{aligned}$$

Now

$$\begin{aligned}
\frac{1}{m}\sum_{j=1}^{m}\left|\pi_j\right| &= \frac{1}{m}\sum_{j=1}^{m}\left|\mu_j\right|\frac{f_j}{g_j} \leq \max_{1\leq j\leq m}\frac{f_j}{g_j}\frac{1}{m}\sum_{j=1}^{m}\left|\mu_j\right| \\
&= O(1)\frac{1}{m}\sum_{j=1}^{m}\left|\mu_j\right| = O(1),
\end{aligned}$$

so $Y_2(\tau)$ is indeed tight. It follows that process Y is tight and that part (a) holds. The proof of parts (b), (c) and (d) is similar.

Examining the proofs, it can be seen that the convergence holds uniformly over $a \geq -1+\varepsilon$.

(e) Let $u_{0\tau^*,j} = \frac{w_{0\tau^*,j}}{G^{\frac{1}{2}}\lambda_j^{-d_1}}$, $u_{\tau^*\tau,j} = \frac{w_{\tau^*\tau,j}}{G^{\frac{1}{2}}\lambda_j^{-d_2}}$ and $Z(\tau) = \frac{1}{m}\sum_{j=1}^{m}\mu_j u_{0\tau^*,j}\bar{u}_{\tau^*\tau,j}$. We have

$$\begin{aligned} u_{0\tau^*,j}\bar{u}_{\tau^*\tau,j} &= (u_{0\tau^*,j} - v_{0\tau^*,j})(\bar{u}_{\tau^*\tau,j} - \bar{v}_{\tau^*\tau,j}) + (u_{0\tau^*,j} - v_{0\tau^*,j})\bar{v}_{\tau^*\tau,j} \\ &\quad + v_{0\tau^*,j}(\bar{u}_{\tau^*\tau,j} - \bar{v}_{\tau^*\tau,j}) + v_{0\tau^*,j}\bar{v}_{\tau^*\tau,j}. \end{aligned}$$

By Lemma 3 of Lazarová (2005),

$$E|u_{\sigma\tau,j} - v_{\sigma\tau,j}|^2 = O\left(\frac{\log j}{j}\right)$$

for $0 \le \sigma \le \tau \le 1$. Further, for $0 \le \sigma \le \tau \le 1$,

$$E|v_{\sigma\tau,j}|^2 = \frac{\sigma_\varepsilon^2}{2\pi}\frac{\lfloor \tau T \rfloor - \lfloor \sigma T \rfloor}{T}.$$

Using the Schwarz inequality, we obtain

$$E|u_{0\tau^*,j}\bar{u}_{\tau^*\tau,j}| = O\left(\frac{\log^{\frac{1}{2}} j}{j^{\frac{1}{2}}}\right)$$

and

$$E\left|\frac{1}{m}\sum_{j=1}^{m}\mu_j u_{0\tau^*,j}\bar{u}_{\tau^*\tau,j}\right| \le \frac{C}{m}\sum_{j=1}^{m}|\mu_j|\frac{\log^{\frac{1}{2}} j}{j^{\frac{1}{2}}} \le Cm^{-1}\log^{\frac{1}{2}} m\sum_{j=1}^{m}|\mu_j| j^{-\frac{1}{2}}.$$

We again employ the summation by parts. We have

$$\begin{aligned} E\left|\frac{1}{m}\sum_{j=1}^{m}\mu_j u_{0\tau^*,j}\bar{u}_{\tau^*\tau,j}\right| &\le \frac{1}{m}\sum_{k=1}^{m-1}|\mu_k - \mu_{k+1}| E\left|\sum_{j=1}^{k} u_{0\tau^*,j}\bar{u}_{\tau^*\tau,j}\right| \\ &\quad + |\mu_m| E\left|\frac{1}{m}\sum_{j=1}^{m} u_{0\tau^*,j}\bar{u}_{\tau^*\tau,j}\right| \end{aligned}$$

which is bounded by

$$\frac{1}{m}\sum_{k=1}^{m-1}|\mu_k - \mu_{k+1}| k^{\frac{1}{2}}\log^{\frac{1}{2}} k + Cm^{-1}m^{\frac{1}{2}}\log^{\frac{1}{2}} m.$$

Since $\left|\mu_k - \mu_{k+1}\right| \leq Ck^{a-1}m^{-a}\log m$, the first term is bounded by

$$\begin{aligned}\frac{C}{m}\sum_{k=1}^{m-1} m^{-a}k^{a-1}\log m k^{\frac{1}{2}}\log^{\frac{1}{2}} k &= Cm^{-a-1}\log^{\frac{3}{2}} m\sum_{k=1}^{m-1} k^{a-\frac{1}{2}} \\ &= \begin{cases} Cm^{-\frac{1}{2}}\log^{\frac{3}{2}} m & a > -\frac{1}{2}, \\ Cm^{-\frac{1}{2}}\log^{\frac{5}{2}} m & a = -\frac{1}{2}, \\ Cm^{-a-1}\log^{\frac{3}{2}} m & a < -\frac{1}{2}, \end{cases}\end{aligned}$$

which is $o(1)$ if $a > -1$. In total,

$$E\left|\frac{1}{m}\sum_{j=1}^{m}\mu_j u_{0\tau^*,j}\bar{u}_{\tau^*\tau,j}\right| = o(1)$$

and the pointwise convergence of $m^{-1}\sum_{j=1}^{m}\mu_j u_{0\tau^*,j}\bar{u}_{\tau^*\tau,j}$ in probability to zero is established.

To prove that process Z is tight, write

$$Z(\tau) - Z(\sigma) = \frac{1}{m}\sum_{j=1}^{m}\mu_j\left(a_{1j} + a_{2j} + a_{3j} + a_{4j}\right) \qquad \tau^* \leq \sigma \leq \tau \leq 1,$$

where

$$\begin{aligned} a_{1j} &= \left(u_{0\tau^*,j} - v_{0\tau^*,j}\right)\left(\bar{u}_{\sigma\tau,j} - \bar{v}_{\sigma\tau,j}\right), & a_{2j} &= \left(u_{0\tau^*,j} - v_{0\tau^*,j}\right)\bar{v}_{\sigma\tau,j}, \\ a_{3j} &= v_{0\tau^*,j}\left(\bar{u}_{\sigma\tau,j} - \bar{v}_{\sigma\tau,j}\right), & a_{4j} &= v_{0\tau^*,j}\bar{v}_{\sigma\tau,j}. \end{aligned}$$

We have

$$E\left|Z(\tau) - Z(\sigma)\right|^4 \leq CE\left|\frac{1}{m}\sum_{j=1}^{m}\mu_j a_{1j}\right|^4 + \cdots + CE\left|\frac{1}{m}\sum_{j=1}^{m}\mu_j a_{4j}\right|^4.$$

The first term is equal to

$$\frac{1}{m^4}\sum_{j,k,\ell,p=1}^{m}\mu_j\mu_k\mu_\ell\mu_p E a_{1j}\bar{a}_{1k}a_{1\ell}\bar{a}_{1p}$$

where

$$\left|E a_{1j}\bar{a}_{1k}a_{1\ell}\bar{a}_{1p}\right| \leq \left(E\left|a_{1j}\right|^4 E\left|a_{1k}\right|^4 E\left|a_{1\ell}\right|^4 E\left|a_{1p}\right|^4\right)^{\frac{1}{4}}.$$

By Lemma 2,

$$\begin{aligned} E\left|a_{1j}\right|^4 &\leq \left(\tau^*\right)^2(\tau-\sigma)^2 j^{-4}, & E\left|a_{2j}\right|^4 &\leq \left(\tau^*\right)^2(\tau-\sigma)^2 j^{-2}, \\ E\left|a_{3j}\right|^4 &\leq \left(\tau^*\right)^2(\tau-\sigma)^2 j^{-2} & E\left|a_{4j}\right|^4 &\leq \left(\tau^*\right)^2(\tau-\sigma)^2. \end{aligned}$$

Therefore

$$E\left|Z\left(\tau\right)-Z\left(\sigma\right)\right|^{4}\leq C\left(\tau-\sigma\right)^{2}\left(\frac{1}{m}\sum_{j=1}^{m}\left|\mu_{j}\right|\right)^{4}$$

and the process $Z\left(\tau\right)$ is tight.

Part (f) can be proved in a similar way. ■

Proof of Proposition 1. To prove that $\sup_{\tau\in[\tau_{\ell},1]}\left|\widehat{\delta}_{1}\left(\tau\right)-\delta\right|\xrightarrow{p}0$ under the null, it is sufficient to prove that for any $\zeta>0$ there exists $\eta>0$ such that

$$P\left(\inf_{\tau\in[\tau_{\ell},1]}\inf_{d\in\left[-\frac{1}{2},\frac{1}{2}\right],|d-\delta|\geq\eta}\left(R\left(d,I_{0\tau}\right)-R\left(\delta,I_{0\tau}\right)\right)\geq\zeta\right)\rightarrow1\qquad\text{as }T\rightarrow\infty,\tag{21}$$

where

$$R\left(d,I_{0\tau}\right)=\log\left(\frac{1}{m}\sum_{j=1}^{m}I_{0\tau,j}\left(\frac{j}{m}\right)^{2d}\right)-\frac{2d}{m}\sum_{j=1}^{m}\log\left(\frac{j}{m}\right).$$

Define

$$\ell\left(d,I_{0\tau}\right)=\frac{1}{m}\sum_{j=1}^{m}\frac{I_{0\tau}\left(\lambda_{j}\right)}{G\lambda_{j}^{-2\delta}}\left(\frac{j}{m}\right)^{2(d-\delta)}$$

and write

$$R\left(d,I_{0\tau}\right)-R\left(\delta,I_{0\tau}\right)=\log\ell\left(d,I_{0\tau}\right)-\log\ell\left(\delta,I_{0\tau}\right)-2\left(d-\delta\right)\frac{1}{m}\sum_{j=1}^{m}\log\left(\frac{j}{m}\right).$$

By Lemma 4, for any $\varepsilon>0$,

$$\sup_{\delta-\frac{1}{2}+\frac{\varepsilon}{2}\leq d\leq\frac{1}{2}}\left|\ell\left(d,I_{0\tau}\right)-\frac{\tau}{1+2\left(d-\delta\right)}\right|\Longrightarrow0.$$

Therefore uniformly in $\tau\in[\tau_{\ell},1]$ and $d\in\left[\delta-\frac{1}{2}+\frac{\varepsilon}{2},\frac{1}{2}\right]$, $|d-\delta|\geq\eta$, as $T\rightarrow\infty$,

$$\begin{aligned}&R\left(d,I_{0\tau}\right)-R\left(\delta,I_{0\tau}\right)\\=\;&\log\left(\frac{\tau}{1+2\left(d-\delta\right)}\right)-\log\left(\tau\right)-2\left(d-\delta\right)\frac{1}{m}\sum_{j=1}^{m}\log\left(\frac{j}{m}\right)+o_{p}\left(1\right)\\=\;&-\log\left(1+2\left(d-\delta\right)\right)+2\left(d-\delta\right)+o_{p}\left(1\right)\geq c+o_{p}\left(1\right)\end{aligned}\tag{22}$$

where $c>0$, because $m^{-1}\sum_{j=1}^{m}\log\left(j/m\right)=-1+o\left(1\right)$ and because $\log\left(1+x\right)<x$ for all $\left|x\right|>0$.

On the other hand, uniformly in $\tau \in [\tau_\ell, 1]$ and $d \in \left[-\frac{1}{2}, \delta - \frac{1}{2} + \frac{\varepsilon}{2}\right]$, as $T \to \infty$,

$$\begin{aligned} & R(d, I_{0\tau}) - R(\delta, I_{0\tau}) \\ \geq\ & \log \ell\left(\delta - \frac{1}{2} + \frac{\varepsilon}{2}, I_{0\tau}\right) - \log \ell(\delta, I_{0\tau}) + 2(d-\delta) + o_p(1) \\ =\ & \log\left(\frac{\tau}{1+(-1+\varepsilon)}\right) - \log(\tau) + o_p(1) + 2(d-\delta) + o(1) \\ =\ & -\log\varepsilon + 2(d-\delta) + o_p(1) \geq -\log\varepsilon - 1 + o_p(1) \geq c + o_p(1) \end{aligned} \tag{23}$$

when ε is small. Bounds (22) and (23) imply that condition (21) is satisfied.

The proof that $\sup_{\tau\in[0,\tau_h]} \left|\widehat{\delta}_2(\tau) - \delta\right| \xrightarrow{p} 0$ is similar. Finally,

$$\begin{aligned} & \sup_{\tau\in[\tau_l,\tau_h]} \left|\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)\right| = \sup_{\tau\in[\tau_l,\tau_h]} \left|\widehat{\delta}_1(\tau) - \delta + \delta - \widehat{\delta}_2(\tau)\right| \\ \leq\ & \sup_{\tau\in[\tau_l,\tau_h]} \left|\widehat{\delta}_1(\tau) - \delta\right| + \sup_{\tau\in[\tau_l,\tau_h]} \left|\widehat{\delta}_2(\tau) - \delta\right| \to_p 0. \end{aligned}$$

■

6.2 Asymptotic distribution of test statistic under the null

Proposition 2 *Under Assumptions 1–4 and under the null with $\delta_1 = \delta_2 = \delta$,*

$$2\sqrt{m}\begin{pmatrix} \widehat{\delta}_1(\tau) - \delta \\ \widehat{\delta}_2(\tau) - \delta \end{pmatrix} \Longrightarrow \begin{pmatrix} \frac{1}{\tau}B(\tau) \\ \frac{1}{1-\tau}(B(1) - B(\tau)) \end{pmatrix}$$

on $\tau \in [\tau_\ell, \tau_h]$, so that

$$2\sqrt{m}\left(\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)\right) \Longrightarrow \frac{B(\tau) - \tau B(1)}{\tau(1-\tau)}$$

and

$$4m\tau(1-\tau)\left(\widehat{\delta}_1(\tau) - \widehat{\delta}_2(\tau)\right)^2 \Longrightarrow \frac{(B(\tau) - \tau B(1))^2}{\tau(1-\tau)}.$$

Lemma 5 *Under Assumptions 1–4 and under the null with $\delta_1 = \delta_2 = \delta$,*

(a)
$$\widehat{\delta}_1(\tau) - \delta = -\frac{1}{2g_1(\tau)}\frac{1}{m}\sum_{j=1}^{m} \nu_j \frac{I_{0\tau,j}}{\lambda_j^{-2\delta}}(1 + o_p(1)),$$

where $o_p(1)$ is uniform over $\tau \in [\tau_\ell, 1]$,

(b)
$$\widehat{\delta}_2(\tau) - \delta = -\frac{1}{2g_2(\tau)}\frac{1}{m}\sum_{j=1}^{m} \nu_j \frac{I_{\tau 1,j}}{\lambda_j^{-2\delta}}(1 + o_p(1)),$$

where $o_p(1)$ *is uniform over* $\tau \in [0, \tau_h]$.

Proof of Lemma 5. (a) The proof is an extension of the proof of bound (11) in Theorem 1 of Dalla et al. (2006). Write

$$\frac{\partial R(d, I_{0\tau})}{\partial d} = \frac{T(d, I_{0\tau})}{V(d, I_{0\tau})}$$

where

$$\begin{aligned} T(d, I_{0\tau}) &= \frac{2}{m}\sum_{j=1}^{m} \frac{I_{0\tau}(\lambda_j)}{\lambda_j^{-2\delta}} \left(\frac{j}{m}\right)^{2(d-\delta)} \nu_j, \\ V(d, I_{0\tau}) &= \frac{1}{m}\sum_{j=1}^{m} \frac{I_{0\tau}(\lambda_j)}{\lambda_j^{-2\delta}} \left(\frac{j}{m}\right)^{2(d-\delta)}, \end{aligned}$$

where v_j is defined in (14). By the mean value theorem,

$$\widehat{\delta}_1(\tau) - \delta = \left(\frac{\partial T\left(\widetilde{\delta}(\tau), I_{0\tau}\right)}{\partial d}\right)^{-1} \left(T\left(\widehat{\delta}_1(\tau), I_{0\tau}\right) - T(\delta, I_{0\tau})\right),$$

where $\widetilde{\delta}_1(\tau)$ is an intermediate point between δ and $\widehat{\delta}_1(\tau)$. Since $\mathbb{I}\left(\sup_{\tau\in[\tau_\ell,1]} \left|\widehat{\delta}_1(\tau) - \delta\right| > \varepsilon\right) = o_p(1)$ for any $\varepsilon > 0$ by Proposition 1, we have

$$\widehat{\delta}_1(\tau) - \delta = \left(\widehat{\delta}_1(\tau) - \delta\right) \mathbb{I}\left(\sup_{\tau\in[\tau_\ell,1]} \left|\widehat{\delta}_1(\tau) - \delta\right| \le \varepsilon\right) (1 + o_p(1)).$$

Let $0 < \varepsilon < \min\left\{\frac{1}{2} - \delta, \frac{1}{2} + \delta\right\}$. When $\sup_{\tau\in[\tau_\ell,1]} \left|\widehat{\delta}_1(\tau) - \delta\right| \le \varepsilon$, Lemma 4 implies that

$$V\left(\widehat{\delta}_1(\tau), I_{0\tau}\right) \ge \frac{G}{m}\sum_{j=1}^{m} \left(\frac{j}{m}\right)^{\varepsilon} \frac{I_{0\tau}(\lambda_j)}{G\lambda_j^{-2\delta}} \Longrightarrow \frac{G\tau}{1+2\varepsilon} > 0 \qquad \text{for all } \tau_\ell \le \tau \le 1.$$

We also have $\sup_{\tau\in[\tau_\ell,1]} \left|\widehat{\delta}_1(\tau) - \delta\right| \in (-1/2, 1/2)$, therefore $\frac{\partial R}{\partial d}\left(\widehat{\delta}_1(\tau), I_{0\tau}\right) = 0$, $T\left(\widehat{\delta}_1(\tau), I_{0\tau}\right) = 0$ and

$$\left(\widehat{\delta}_1(\tau) - \delta\right) \mathbb{I}\left(\sup_{\tau\in[\tau_\ell,1]} \left|\widehat{\delta}_1(\tau) - \delta\right| \le \varepsilon\right) = -\left(\frac{\partial T\left(\widetilde{\delta}(\tau), I_{0\tau}\right)}{\partial d}\right)^{-1} T(\delta, I_{0\tau}). \qquad (24)$$

From Lemma 4 with $\mu_j = \log\left(\frac{j}{m}\right)\left(\frac{j}{m}\right)^a \nu_j$ and from Proposition 1 it follows that

$$\frac{\partial T\left(\widetilde{\delta}_1(\tau), I_{0\tau}\right)}{\partial d} = \frac{4G}{m}\sum_{j=1}^{m}\log\left(\frac{j}{m}\right)\left(\frac{j}{m}\right)^{2\left(\widetilde{\delta}_1(\tau)-\delta\right)}\nu_j \frac{I_{0\tau}(\lambda_j)}{G\lambda_j^{-2\delta}} \Longrightarrow 4G\tau.$$

because $2\left|\widetilde{\delta}_1(\tau)-\delta\right| \leq 2\left|\hat{\delta}_1(\tau)-\delta\right|$. Therefore (24) is equal to

$$-\left(4G\tau\left(1+o_p(1)\right)\right)^{-1} T\left(\delta, I_{0\tau}\right) = \frac{T\left(\delta, I_{0\tau}\right)}{4G\tau}\left(1+o_p(1)\right)$$

and part (a) is established.

Part (b) is proved similarly. ■

Lemma 6 *Under Assumptions 1–4,*

$$\textit{(a)} \qquad \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\frac{2\pi}{\sigma_\varepsilon^2}\left(I_{\varepsilon 0\tau,j}-\tau\right) \Longrightarrow B(\tau) \qquad \tau\in[0,1],$$

$$\textit{(b)} \qquad \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\frac{2\pi}{\sigma_\varepsilon^2}w_{\varepsilon\tau\tau^*,j}\overline{w}_{\varepsilon\tau^*1,j} \Longrightarrow 0 \qquad \tau\in[0,\tau^*],$$

$$\textit{(c)} \qquad \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\frac{2\pi}{\sigma_\varepsilon^2}w_{\varepsilon 0\tau^*,j}\overline{w}_{\varepsilon\tau^*\tau,j} \Longrightarrow 0 \qquad \tau\in[\tau^*,1].$$

Proof of Lemma 6. (a) Let

$$Y(\tau) = \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left(\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}-\tau\right).$$

We have

$$Y(\tau) = \sum_{t=2}^{\lfloor\tau T\rfloor}\varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s c_{t-s} = \sum_{t=1}^{\lfloor\tau T\rfloor} z_t$$

where

$$\begin{aligned} c_s &= 2m^{-\frac{1}{2}}T^{-1}1/\sigma_\varepsilon^2\sum_{j=1}^{m}\nu_j\cos(s\lambda_j), \\ \nu_j &= \ln j - \frac{1}{m}\sum_{k=1}^{m}\ln k, \end{aligned}$$

$z_1 = 0$ and $z_t = \varepsilon_t\sum_{s=1}^{t-1}\varepsilon_s c_{t-s}$ for $t \geq 2$.

The second moment of $Y(\tau)$ is

$$\begin{aligned} E\left|Y(\tau)\right|^2 &= \sigma_\varepsilon^4 \sum_{t=2}^{\lfloor \tau T \rfloor} \sum_{s=1}^{t-1} c_{t-s}^2 \\ &= \frac{1}{mT^2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} \nu_j \nu_\ell \sum_{t=1}^{\lfloor \tau T \rfloor} \sum_{s=1}^{\lfloor \tau T \rfloor} \left(e^{i(t-s)(\lambda_j - \lambda_k)} + e^{i(t-s)(\lambda_j + \lambda_k)} \right) \\ &= \frac{1}{mT^2} \sum_{j=1}^{m} \sum_{\ell=1}^{m} \nu_j \nu_\ell \left(\left| D_{\lfloor \tau T \rfloor}(\lambda_j - \lambda_k) \right|^2 + \left| D_{\lfloor \tau T \rfloor}(\lambda_j + \lambda_k) \right|^2 \right) \\ &= \frac{2\pi}{mT} \sum_{j=1}^{m} \sum_{\ell=1}^{m} \nu_j \nu_\ell \left(K_{\lfloor \tau T \rfloor}(\lambda_j - \lambda_k) + K_{\lfloor \tau T \rfloor}(\lambda_j + \lambda_k) \right) \end{aligned}$$

where $D_{\lfloor \tau T \rfloor}(\lambda) = \sum_{t=1}^{\lfloor \tau T \rfloor} e^{it\lambda}$ and $K_{\lfloor \tau T \rfloor}(\lambda) = (2\pi T)^{-1} \left| D_{\lfloor \tau T \rfloor}(\lambda) \right|^2$. By Lemma 3, $E\left|Y(\tau)\right|^2 \to \tau$ for $\tau \in [0,1]$. The central limit theorem follows as in Robinson (1995). This shows pointwise convergence.

We next prove tightness of process Y. First, we prove two results about the weights c_s. We can provide an alternative bound for (4.20) in Robinson (1995) because

$$c_s^2 \leq C m^{-1} T^{-2} \sum_{j=1}^{m} \nu_j^2 \sum_{j=1}^{m} \cos^2(s\lambda_j) \leq C T^{-2} m$$

using $\sum_{j=1}^{m} \nu_j^2 \leq Cm$ because $\sum_{j=1}^{m} \nu_j^2 = m + o(m)$. Moreover, notice that

$$\nu_m = \ln m - \frac{1}{m} \sum_{k=1}^{m} \ln k = -\frac{1}{m} \sum_{k=1}^{m} \ln(k/m) = 1 + O\left(\frac{\ln m}{m} \right),$$

where the last equality is from Abadir, Distaso and Giraitis (2007), page 1368, so $|\nu_m| < C$. Then, we can also bound c_s as

$$\begin{aligned} |c_s| &= \left| 2m^{-1/2} T^{-1} \sum_{j=1}^{m-1} (\nu_{j+1} - \nu_j) \sum_{k=1}^{j} \cos(s\lambda_k) + 2m^{-1/2} \sigma_\varepsilon^{-2} T^{-1} \nu_m \sum_{j=1}^{m} \cos(s\lambda_j) \right| \\ &\leq Cm^{-1/2} T^{-1} \left| \sum_{j=1}^{m-1} (\ln(j+1) - \ln j) \sum_{k=1}^{j} \cos(s\lambda_k) \right| + Cm^{-1/2} T^{-1} \left| \sum_{j=1}^{m} \cos(s\lambda_j) \right| \\ &\leq Cm^{-1/2} T^{-1} \left| \sum_{j=1}^{m-1} j_m^{-1} \sum_{k=1}^{j} \cos(s\lambda_k) \right| + Cm^{-1/2} T^{-1} \left| \sum_{j=1}^{m} \cos(s\lambda_j) \right|, \end{aligned}$$

where $j_m \in (j, j+1)$. Note that the second term is bounded by $Cm^{-1/2}s^{-1}$. As for the first term,

$$\left(\frac{T}{2\pi s}\right)^{-1} \sum_{j=1}^{T} j^{-1} \left(\sum_{k=1}^{j} \cos\left(s\frac{2\pi k}{T}\right)\right) \to \mathrm{Si}\left(2\pi s\right) = O\left(1\right)$$

where $\mathrm{Si}\left(x\right) = \int_0^x \frac{\sin t}{t} dt$ is the sine integral. This means that it also holds that

$$m^{-1/2}T^{-1}\left|\sum_{j=1}^{m-1} j_m^{-1} \sum_{k=1}^{j} \cos\left(s\lambda_k\right)\right| = O\left(m^{-1/2}s^{-1}\right).$$

Therefore

$$c_s^2 \leq C \min\left(T^{-2}m, m^{-1}s^{-2}\right). \tag{25}$$

We now look at

$$Y\left(\tau\right) - Y\left(\sigma\right) = \sum_{t=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} z_t$$

and in particular at

$$E\left(Y\left(\tau\right) - Y\left(\sigma\right)\right)^4 = \sum_{t_1=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} \sum_{t_2=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} \sum_{t_3=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} \sum_{t_4=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(z_{t_1} z_{t_2} z_{t_3} z_{t_4}\right). \tag{26}$$

If $t_1 \neq t_2 \neq t_3 \neq t_4$, consider $t_1 > \max\left(t_2, t_3, t_4\right)$ first. Then,

$$E\left(z_{t_1} z_{t_2} z_{t_3} z_{t_4} | \mathcal{F}_{t_1 - 1}\right) = z_{t_2} z_{t_3} z_{t_4} E\left(z_{t_1} | \mathcal{F}_{t_1 - 1}\right) = 0$$

and

$$E\left(z_{t_1} z_{t_2} z_{t_3} z_{t_4}\right) = E\left(E\left(z_{t_1} z_{t_2} z_{t_3} z_{t_4} | \mathcal{F}_{t_1 - 1}\right)\right) = 0,$$

so the expectation in (26) is 0 in this case. The other cases ($t_2 > \max\left(t_1, t_3, t_4\right)$ and other cases) may be treated in the same way.

Next, consider $t_1 = t_2 \neq t_3 \neq t_4$, so that (26) is

$$\sum_{t_1=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} \sum_{t_3=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} \sum_{t_4=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(z_{t_1}^2 z_{t_3} z_{t_4}\right).$$

Again consider $t_1 > t_3 > t_4$ first. Then we have

$$\begin{aligned} E\left(z_{t_1}^2 z_{t_3} z_{t_4}\right) &= E\left(E\left(z_{t_1}^2 z_{t_3} z_{t_4} | \mathcal{F}_{t_1 - 1}\right)\right) = E\left(z_{t_1}^2\right) E\left(z_{t_3} z_{t_4}\right) \\ &= E\left(z_{t_1}^2\right) E\left(E\left(z_{t_3} z_{t_4} | \mathcal{F}_{t_3 - 1}\right)\right) = E\left(z_{t_1}^2\right) E\left(z_{t_4} E\left(z_{t_3} | \mathcal{F}_{t_3 - 1}\right)\right) = 0 \end{aligned}$$

and cases $t_1 > t_4 > t_3$ can be treated in the same way. On the other hand, if, for example, $t_3 > t_1 > t_4$, then

$$E\left(z_{t_1}^2 z_{t_3} z_{t_4}\right) = E\left(E\left(z_{t_1}^2 z_{t_3} z_{t_4} | \mathcal{F}_{t_3-1}\right)\right) = E\left(z_{t_1}^2 z_{t_4} E\left(z_{t_3} | \mathcal{F}_{t_3-1}\right)\right) = 0.$$

Again, other cases can be treated in the same way.

Next, consider $t_1 = t_2 \neq t_3 = t_4$, so that (26) is

$$\sum_{t_1=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} \sum_{t_3=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(z_{t_1}^2 z_{t_3}^2\right) = \sum_{t_1=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(z_{t_1}^2\right) \sum_{t_3=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(z_{t_3}^2\right) \tag{27}$$

again using the law of iterated expectations. Then, note that

$$E\left(z_t^2\right) = E\left(\varepsilon_t^2 \sum_{s_1=1}^{t-1} \sum_{s_2=1}^{t-1} \varepsilon_{s_1} c_{t-s_1} \varepsilon_{s_2} c_{t-s_2}\right) = E\left(\varepsilon_t^2\right) \sum_{s_1=1}^{t-1} \sum_{s_2=1}^{t-1} E\left(\varepsilon_{s_1} \varepsilon_{s_2}\right) c_{t-s_1} c_{t-s_2}$$

using the law of iterated expectations. The last double sum is equal to

$$\begin{aligned} E\left(\sum_{s=1}^{t-1} \varepsilon_s c_{t-s}\right)^2 &= E \sum_{s=1}^{t-1} \left(\varepsilon_s c_{t-s}\right)^2 + 2E \sum_{s=1}^{t-1} \sum_{r=1}^{s-1} \left(\varepsilon_s c_{t-s} \varepsilon_r c_{t-r}\right) \\ &= \sum_{s=1}^{t-1} E\left(\varepsilon_s\right)^2 c_{t-s}^2 + 2 \sum_{s=1}^{t-1} \sum_{r=1}^{s-1} E\left(\varepsilon_s \varepsilon_r\right) c_{t-s} c_{t-r} \\ &= E\left(\varepsilon_t^2\right) \sum_{s=1}^{t-1} c_{t-s}^2 = E\left(\varepsilon_t^2\right) \sum_{s=1}^{t-1} c_s^2 \end{aligned}$$

where we used $E\left(\varepsilon_s \varepsilon_r\right) = 0$ for $r \neq s$. Using bound (25), we get

$$\sum_{s=1}^{T} c_s^2 \leq \sum_{s=1}^{\lfloor T/m \rfloor} c_s^2 + \sum_{s=1+\lfloor T/m \rfloor}^{T} c_s^2 \leq C \frac{T}{m} T^{-2} m + \sum_{s=1+T/m}^{T} m^{-1} \left(T/m\right)^{-1} = O\left(T^{-1}\right) \tag{28}$$

and

$$E\left(\sum_{t_1=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(z_{t_1}^2\right)\right) \leq CT\left(\tau - \sigma\right) \frac{1}{T} \leq C\left(\tau - \sigma\right)$$

so we can bound the expression (27) by $C\left(\tau - \sigma\right)^2$. Cases of type $t_1 = t_2 = t_3 \neq t_4$ also have expectations 0 using the law of iterated expectations.

Finally, for the cases with $t_1 = t_2 = t_3 = t_4$ the fourth moment in (26) is

$$\begin{aligned}\sum_{t=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(z_t^4\right) &= \sum_{t=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(\varepsilon_t \sum_{s=1}^{t-1} \varepsilon_s c_{t-s}\right)^4 \\ &= \sum_{t=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(\varepsilon_t^4 \sum_{s_1=1}^{t-1}\sum_{s_2=1}^{t-1}\sum_{s_3=1}^{t-1}\sum_{s_4=1}^{t-1} \varepsilon_{s_1} c_{t-s_1} \varepsilon_{s_2} c_{t-s_2} \varepsilon_{s_3} c_{t-s_3} \varepsilon_{s_4} c_{t-s_4}\right) \\ &= \sum_{t=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(\varepsilon_t^4\right) E\left(\sum_{s=1}^{t-1} \varepsilon_s c_{t-s}\right)^4.\end{aligned}$$

Consider

$$E\left(\sum_{s_1=1}^{t-1}\sum_{s_2=1}^{t-1}\sum_{s_3=1}^{t-1}\sum_{s_4=1}^{t-1} \varepsilon_{s_1} c_{t-s_1} \varepsilon_{s_2} c_{t-s_2} \varepsilon_{s_3} c_{t-s_3} \varepsilon_{s_4} c_{t-s_4}\right). \tag{29}$$

Proceeding as before, expression (29) vanishes if $s_1 \neq s_2 \neq s_3 \neq s_4$, if $s_1 = s_2 \neq s_3 \neq s_4$ (and similar cases) and if $s_1 = s_2 = s_3 \neq s_4$ (and similar cases).

On the other hand, when $s_1 = s_2 \neq s_3 = s_4$, (29) is equal to

$$\sum_{s_1=1}^{t-1} E\left(\varepsilon_{s_1}^2\right) c_{t-s_1}^2 \sum_{s_3=1}^{t-1} E\left(\varepsilon_{s_3}\right)^2 c_{t-s_3}^2,$$

so (26) is bounded by

$$\sum_{t=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(\varepsilon_t^4\right) \left(\sum_{s=1}^{t-1} E\left(\varepsilon_s c_{t-s}\right)^2\right)^2 \leq CT\left(\tau - \sigma\right) \times \left(\frac{1}{T}\right)^2 = C\left(\tau - \sigma\right) \times \frac{1}{T}.$$

Finally, cases having $s_1 = s_2 = s_3 = s_4$ yield that (29) is equal to

$$\sum_{t=1+\lfloor \sigma T \rfloor}^{\lfloor \tau T \rfloor} E\left(\varepsilon_t^4\right) \sum_{s=1}^{t-1} E\left(\varepsilon_s^4\right) c_{t-s}^4$$

and note that, proceeding as in Robinson (1995) for his expression (4.22), using bounds (25), we obtain

$$\begin{aligned}\sum_{s=1}^{t-1} c_s^4 &\leq \sum_{s=1}^{T/m} c_s^4 + \sum_{s=1+T/m}^{T} c_s^4 \\ &\leq C\left(\frac{T}{m}\right)\left(\frac{m^2}{T^4}\right) + C\left(\frac{m}{T}\right)^3\left(\frac{1}{m^2}\right) \leq C\frac{m}{T^3} \leq C\frac{1}{T^2}\end{aligned}$$

and conclude that (26) is bounded by

$$CT\left(\tau-\sigma\right)\times\frac{1}{T^{2}}=C\left(\tau-\sigma\right)\frac{1}{T}\leq C\left(\tau-\sigma\right).$$

The proof for the other two parts follow in the same way. ■

Lemma 7 *Under Assumptions 1–4,*

(a)

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_{j}\begin{pmatrix}\frac{I_{0\tau,j}}{G\lambda_{j}^{-2\delta}}-\tau\\ \frac{I_{\tau1,j}}{G\lambda_{j}^{-2\delta}}-(1-\tau)\\ \frac{w_{\tau\tau^{*},j}}{G^{\frac{1}{2}}\lambda_{j}^{-\delta}}\frac{\overline{w}_{\tau^{*}1,j}}{G^{\frac{1}{2}}\lambda_{j}^{-\delta}}\end{pmatrix}\Longrightarrow\begin{pmatrix}B\left(\tau\right)\\ B\left(\tau^{*}\right)-B\left(\tau\right)\\ 0\end{pmatrix}\qquad\tau\in\left[\tau_{l},\tau^{*}\right],$$

(b)

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_{j}\begin{pmatrix}\frac{I_{0\tau,j}}{G\lambda_{j}^{-2\delta}}-\tau\\ \frac{I_{\tau1,j}}{G\lambda_{j}^{-2\delta}}-(1-\tau)\\ \frac{w_{0\tau^{*},j}}{G^{\frac{1}{2}}\lambda_{j}^{-\delta}}\frac{\overline{w}_{\tau^{*}\tau,j}}{G^{\frac{1}{2}}\lambda_{j}^{-\delta}}\end{pmatrix}\Longrightarrow\begin{pmatrix}B\left(\tau\right)-B\left(\tau^{*}\right)\\ B\left(1\right)-B\left(\tau\right)\\ 0\end{pmatrix}\qquad\tau\in\left[\tau^{*},\tau_{h}\right].$$

Proof of Lemma 7. (a) By Lemma 6, it is sufficient to prove that for $\tau\in[\tau_l,\tau^*]$,

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_{j}\left(\frac{I_{0\tau,j}}{G\lambda_{j}^{-2\delta}}-\frac{2\pi}{\sigma_{\varepsilon}^{2}}I_{\varepsilon0\tau,j}\right)\Longrightarrow0,\tag{30}$$

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_{j}\left(\frac{I_{\tau1,j}}{G\lambda_{j}^{-2\delta}}-\frac{2\pi}{\sigma_{\varepsilon}^{2}}I_{\varepsilon\tau1,j}\right)\Longrightarrow0,\tag{31}$$

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_{j}\left(\frac{w_{\tau\tau^{*},j}}{G^{\frac{1}{2}}\lambda_{j}^{-\delta}}\frac{\overline{w}_{\tau^{*}1,j}}{G^{\frac{1}{2}}\lambda_{j}^{-\delta}}-\frac{2\pi}{\sigma_{\varepsilon}^{2}}w_{\varepsilon\tau\tau^{*},j}\overline{w}_{\varepsilon\tau^{*}1,j}\right)\Longrightarrow0.\tag{32}$$

Proceeding as in the proof of (4.8) of Robinson (1995) while employing Lemma 3 of Lazarová (2005) and referring to (12), we obtain that

$$\sum_{j=1}^{k}\left(\frac{I_{0\tau,j}}{G\lambda_{j}^{-2\delta}}-\frac{2\pi}{\sigma_{\varepsilon}^{2}}I_{\varepsilon0\tau,j}\right)=O_{p}\left(k^{\frac{1}{3}}\log^{\frac{2}{3}}k+k^{\beta+1}T^{-\beta}+k^{\frac{1}{2}}T^{-\frac{1}{4}}\right)$$

uniformly over $\tau\in[\tau_\ell,\tau^*]$ and $1\leq k\leq m$. Using summation by parts, we get

$$\begin{aligned}\left|\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_{j}\left(\frac{I_{0\tau,j}}{G\lambda_{j}^{-2\delta}}-\frac{2\pi}{\sigma_{\varepsilon}^{2}}I_{\varepsilon0\tau,j}\right)\right| &\leq \frac{1}{\sqrt{m}}\sum_{k=1}^{m-1}\left|\nu_{k}-\nu_{k+1}\right|\left|\sum_{j=1}^{k}\left(\frac{I_{0\tau,j}}{G\lambda_{j}^{-2\delta}}-\frac{2\pi}{\sigma_{\varepsilon}^{2}}I_{\varepsilon0\tau,j}\right)\right|\\ &\quad+\left|\nu_{m}\right|\left|\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\left(\frac{I_{0\tau,j}}{G\lambda_{j}^{-2\delta}}-\frac{2\pi}{\sigma_{\varepsilon}^{2}}I_{\varepsilon0\tau,j}\right)\right|.\end{aligned}$$

The first term is

$$\frac{1}{\sqrt{m}}\sum_{k=1}^{m-1}\frac{1}{k}O_p\left(k^{\frac{1}{3}}\log^{\frac{2}{3}}k+k^{\beta+1}T^{-\beta}+k^{\frac{1}{2}}T^{-\frac{1}{4}}\right)=o_p\left(1\right)$$

and the second term is

$$m^{-\frac{1}{2}}\log m O_p\left(m^{\frac{1}{3}}\log^{\frac{2}{3}}m+m^{\beta+1}T^{-\beta}+m^{\frac{1}{2}}T^{-\frac{1}{4}}\right)=o_p\left(1\right).$$

Therefore

$$\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left(\frac{I_{0\tau,j}}{f_j}-\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}\right)\xrightarrow{p}0$$

for $\tau\in[\tau_l,\tau^*]$.

Next we prove tightness of the process on the left hand side of (30). Write

$$\begin{aligned}\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left(\frac{I_{0\tau,j}}{G\lambda_j^{-2\delta}}-\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}\right) &= \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\frac{f_j}{g_j}\left(\frac{I_{0\tau,j}}{f_j}-\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}\right)\\ &\quad+\frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left(\frac{f_j}{g_j}-1\right)\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}. \qquad (33)\end{aligned}$$

Let

$$\begin{aligned}Y_1\left(\tau\right) &= \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\frac{f_j}{g_j}\left(\frac{I_{0\tau,j}}{f_j}-\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}\right),\\ Y_2\left(\tau\right) &= \frac{1}{\sqrt{m}}\sum_{j=1}^{m}\nu_j\left(\frac{f_j}{g_j}-1\right)\frac{2\pi}{\sigma_\varepsilon^2}I_{\varepsilon 0\tau,j}.\end{aligned}$$

From the proof of Lemma 4 we can see that

$$E\left|Y_1\left(\tau\right)-Y_1\left(\sigma\right)\right|^4\leq C\left(\tau-\sigma\right)^2\left(m^{-\frac{1}{2}}\sum_{j=1}^{m}\left|\nu_j\right|\frac{f_j}{g_j}j^{-\frac{1}{2}}\right)^4.$$

It is

$$\begin{aligned}m^{-\frac{1}{2}}\sum_{j=1}^{m}\left|\nu_j\right|\frac{f_j}{g_j}j^{-\frac{1}{2}} &= m^{-1}\sum_{j=1}^{m}\left|\nu_j\right|\frac{f_j}{g_j}\left(\frac{j}{m}\right)^{-\frac{1}{2}}\\ &\leq \max_{1\leq j\leq m}\frac{f_j}{g_j}m^{-1}\sum_{j=1}^{m}\left|\nu_j\right|\left(\frac{j}{m}\right)^{-\frac{1}{2}}\leq C.\end{aligned}$$

Therefore

$$E\left|Y_1\left(\tau\right)-Y_1\left(\sigma\right)\right|^4\leq C\left(\tau-\sigma\right)^2$$

and $Y_1(\tau)$ is tight. Further, proceeding as in the proof of Lemma 4 we obtain

$$E\left|Y_2(\tau) - Y_2(\sigma)\right|^4 \leq C(\tau-\sigma)^2 \left(m^{-\frac{1}{2}} \sum_{j=1}^{m} |\nu_j| \left| \frac{f_j}{g_j} - 1 \right| \right)^4 .$$

Now

$$|\nu_j| \leq \log m$$

and

$$\left| \frac{f_j}{g_j} - 1 \right| \leq C \left(\frac{j}{T} \right)^{\beta}$$

by Assumption 1, therefore

$$\begin{aligned} m^{-\frac{1}{2}} \sum_{j=1}^{m} |\nu_j| \left| \frac{f_j}{g_j} - 1 \right| &\leq C m^{-\frac{1}{2}} T^{-\beta} \log m \sum_{j=1}^{m} j^{\beta} \leq C m^{-\frac{1}{2}} m^{\beta+1} T^{-\beta} \log m \\ &= O\left(\frac{m^{\beta+\frac{1}{2}}}{T^{\beta}} \log m \right) = o(1) \end{aligned}$$

by Assumption 1, so $Y_2(\tau)$ is tight and (30) holds. The proofs of (31) and of (32) use similar arguments.

Part (b) is proved in a similar way. ■

Proof of Proposition 2. The proposition follows from Lemma 5 and Lemma 7. ■
Proof of Theorem 1. The theorem follows from Proposition 2 and from the continuous mapping theorem. ■

6.3 Power of test

Lemma 8 *Under Assumptions 1–4 and under the null hypothesis with $\delta_1 \neq \delta_2$, as $T \to \infty$,*

$$\widehat{\delta}_1(\tau^*) \xrightarrow{p} \delta_1 \qquad \text{and} \qquad \widehat{\delta}_2(\tau^*) \xrightarrow{p} \delta_2,$$

so that

$$\widehat{\delta}_1(\tau^*) - \widehat{\delta}_2(\tau^*) \xrightarrow{p} \delta_1 - \delta_2.$$

Proof of Lemma 8. The lemma can be proved using the same strategy as in the proof of Lemma 4. ■

Proof of Theorem 2. The theorem follows from Lemma 8. ■

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School of Economics and Finance

Queen Mary
University of London

This working paper has been produced by
the School of Economics and Finance at
Queen Mary University of London

School of Economics and Finance
Queen Mary University of London
Mile End Road
London E1 4NS
Tel: +44 (0)20 7882 7356
Fax: +44 (0)20 8983 3580
Web: www.econ.qmul.ac.uk/research/workingpapers/