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Symmetry Axioms and Perceived Ambiguity*

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Abstract

Since at least de Finetti [7], preference symmetry assumptions have played an important role in models of decision making under uncertainty. In the current paper, we explore (1) the relationship between the symmetry assumption of Klibanoff, Mukerji and Seo (KMS) [21] and alternative symmetry assumptions in the literature, and (2) assuming symmetry, the relationship between the set of *relevant measures*, shown by KMS [21] to reflect only perceived ambiguity, and the set of measures (which we will refer to as the *Bewley set*) developed by Ghirardato, Maccheroni and Marinacci [14], Nehring [24, 25] and Ghirardato and Siniscalchi [15, 16]. This Bewley set is the main alternative offered in the literature as possibly representing perceived ambiguity. Regarding symmetry assumptions, we show that, under relatively mild conditions, a variety of preference symmetry conditions from the literature (including that in KMS [21]) are equivalent. In KMS [21], we showed that, under symmetry, the Bewley set and the set of relevant measures are not always the same. Here, we establish a preference condition, No Half Measures, that is necessary and sufficient for the two to be same under symmetry. This condition is rather stringent. Only when it is satisfied may the Bewley set be interpreted as reflecting only perceived ambiguity and not also taste aspects such as ambiguity aversion.

Keywords: Symmetry, beliefs, ambiguity, ambiguity aversion, model uncertainty, Ellsberg

JEL codes: D01, D80, D81, D83

1 Introduction

Recent literature on ambiguity (meaning subjective uncertainty about probabilities over states) or model uncertainty (see e.g., surveys by Gilboa and Marinacci [17] and

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Marinacci [22]), makes heavy use of models of decision making under uncertainty. In this literature, a concern relevant to applying these models is how to connect the representation of preferences to perceived ambiguity or the models over which you are uncertain. In a recent contribution, Klibanoff, Mukerji and Seo (henceforth KMS) [21], we proposed an answer to this question for a broad class of symmetric preferences. In particular, we defined a notion of relevant measures and showed that these measures reflect only perceived ambiguity and not ambiguity aversion or risk aversion.

Since at least de Finetti [7], preference symmetry assumptions have played an important role in models of decision making under uncertainty. In the current paper we explore (1) the relationship between the symmetry assumption of KMS [21] and alternative symmetry assumptions in the literature, and (2) assuming symmetry, the relationship between the set of *relevant measures* developed by KMS [21] and the set of measures (which we will refer to as the *Bewley set*) developed by Ghirardato, Maccheroni and Marinacci [14], Nehring [24, 25] and Ghirardato and Siniscalchi [15, 16]. This latter relationship is of particular interest because this Bewley set is the main alternative offered in the literature as possibly representing perceived ambiguity. Regarding symmetry assumptions, we show that, under relatively mild conditions, a variety of preference symmetry conditions from the literature (including that in KMS [21]) are equivalent.

In KMS [21], we showed that, under symmetry, the Bewley set and the set of relevant measures are not always the same. This is important because, when they differ, the Bewley set is affected by changes in ambiguity aversion, and thus *cannot* be interpreted as reflecting only perceived ambiguity. In the present paper, we establish a preference condition, No Half Measures, that is necessary and sufficient for the two to be same for symmetric preferences. In contrast to the mild conditions used for the symmetry equivalence result, No Half Measures is rather stringent. Yet our results, combined with those in KMS [21], imply that it is only under these stringent conditions that the Bewley set may be interpreted as reflecting only perceived ambiguity and not also taste aspects such as ambiguity aversion.

When incorporating considerations of ambiguity or model uncertainty in finance models it is natural for the modeler to want to impose constraints on the preferences of the decision maker(s) in the model so that they reflect some type of calibration of perceived ambiguity to external data. For example, in an asset pricing model, the modeler may want to impose the restriction that an investor seeks to make her portfolio robust against only a certain limited set of stochastic processes or probabilistic forecasts or different forecasting models (when incorporating model uncertainty) because these processes or forecasts pass certain tests of inference on past data. The theory developed in KMS [21] gives reasons why, if such constraints were to be imposed one should do so through constraints on the relevant measures. KMS [21] shows where the relevant measures appear in representation functionals for symmetric versions of a variety of popular ambiguity models. For instance, in the α -MEU model (see e.g., Ghirardato, Maccheroni and Marinacci [14]) and the smooth ambiguity model (see e.g., Klibanoff, Marinacci and Mukerji [20], Nau [23], Seo [26]) and the extended MEU with contraction

model (see e.g., Gajdos et. al. [13]), the set of probabilistic forecast models entertained by the decision maker should be the set of measures that appear in the functional representations. Thus, reassuringly, the theory of relevant measures provides foundations to the common and intuitive practice in the way these ambiguity models are used.

However, KMS's theory [21] is predicated on a specification of symmetry. It is common to invoke some notion of symmetry/exchangeability, at least implicitly, to underpin statistical inference (e.g., of the kind used to justify restrictions on the set of forecasting models considered when incorporating model uncertainty). Our results in Section 3 demonstrate that KMS's specification of symmetry is not special and in fact equivalent to notions invoked elsewhere in the literature under conditions that would be met in most settings.

Our study of the No Half Measures condition, in Section 4, shows that the possibilities of establishing foundations for practices concerning the way constraints on model uncertainty are incorporated in ambiguity models via theories invoking the Bewley set are limited. This is particularly true in the case of functional forms which allow for non-extreme attitudes to ambiguity that may be varied parametrically in the functional representation (such as in the α -MEU model, the extended MEU with contraction model, and the smooth ambiguity model). For instance, in the α -MEU model, the Bewley set can be identified with the set of measures appearing in the model only if α is 0 or 1.

2 Setting and Preferences

In this section, we describe aspects of the formal setting, notation and preferences from KMS [21] that will be useful here.

Let S be a compact metric space and $\Omega = S^\infty$ the state space with generic element $\omega = (\omega_1, \omega_2, \dots)$. The state space Ω is also compact metric (Aliprantis and Border [1, Theorems 2.61 and 3.36]). Denote by Σ_i the Borel σ -algebra on the i -th copy of S , and by Σ the product σ -algebra on S^∞ . An act is a simple Anscombe-Aumann act, a measurable $f : S^\infty \rightarrow X$ having finite range (i.e., $f(S^\infty)$ is finite) where X is the set of lotteries (i.e., finite support probability measures on an outcome space Z). The set of acts is denoted by \mathcal{F} , and \succsim is a binary relation on $\mathcal{F} \times \mathcal{F}$. As usual, we identify a constant act (an act yielding the same element of X on all of S^∞) with the element of X it yields.

Denote by Π the set of all finite permutations on $\{1, 2, \dots\}$ i.e., all one-to-one and onto functions $\pi : \{1, 2, \dots\} \rightarrow \{1, 2, \dots\}$ such that $\pi(i) = i$ for all but finitely many $i \in \{1, 2, \dots\}$. For $\pi \in \Pi$, let $\pi\omega = (\omega_{\pi(1)}, \omega_{\pi(2)}, \dots)$ and $(\pi f)(\omega) = f(\pi\omega)$.

For any topological space Y , $\Delta(Y)$ denotes the set of (countably additive) Borel probability measures on Y . Unless stated otherwise, a measure is understood as a countably additive Borel measure. For later use, $ba(Y)$ is the set of finitely additive bounded real-valued set functions on Y , and $ba_+^1(Y)$ the set of non-negative probability charges in $ba(Y)$. A measure $p \in \Delta(S^\infty)$ is called symmetric if the order doesn't

matter, i.e., $p(A) = p(\pi A)$ for all $\pi \in \Pi$, where $\pi A = \{\pi\omega : \omega \in A\}$. Denote by ℓ^∞ the i.i.d. measure with the marginal $\ell \in \Delta(S)$. Define $\int_{S^\infty} f dp \in X$ by $(\int_{S^\infty} f dp)(B) = (\int_{S^\infty} f(\omega)(B) dp(\omega))$. (Since f is simple, this is well-defined.)

Fix $x_*, x^* \in X$ such that $x^* \succ x_*$. For any event $A \in \Sigma$, 1_A denotes the act giving x^* on A and x_* otherwise. Informally, this is a bet on A . More generally, for $x, y \in X$, xAy denotes the act giving x on A and y otherwise. A finite cylinder event $A \in \Sigma$ is any event of the form $\{\omega : \omega_i \in A_i \text{ for } i = 1, \dots, n\}$ for $A_i \in \Sigma_i$ and some finite n .

Endow $\Delta(S)$, $\Delta(\Delta(S))$ and $\Delta(S^\infty)$ with the relative weak* topology. To see what this is, consider, for example, $\Delta(S)$. The relative weak* topology on $\Delta(S)$ is the collection of sets $V \cap \Delta(S)$ for weak* open $V \subseteq ba(S)$, where the weak* topology on $ba(S)$ is the weakest topology for which all functions $\ell \mapsto \int \psi d\ell$ are continuous for all bounded measurable ψ on S . Also note that a net $\ell_\alpha \in ba(S)$ converges to $\ell \in ba(S)$ under the weak* topology if and only if $\int \psi d\ell_\alpha \rightarrow \int \psi d\ell$ for all bounded measurable ψ on S . For a set $D \subseteq \Delta(S)$, denote the closure of D in the relative weak* topology by \overline{D} .

The support of a probability measure $m \in \Delta(\Delta(S))$, denoted $\text{supp } m$, is a relative weak* closed set such that $m((\text{supp } m)^c) = 0$ and if $L \cap \text{supp } m \neq \emptyset$ for relative weak* open L , $m(L \cap \text{supp } m) > 0$. (See e.g., Aliprantis and Border [1, p.441].)

Let $\Psi_n(\omega) \in \Delta(S)$ denote the empirical frequency operator $\Psi_n(\omega)(A) = \frac{1}{n} \sum_{t=1}^n I(\omega_t \in A)$ for each event A in S . Define the limiting frequency operator Ψ by $\Psi(\omega)(A) = \lim_n \Psi_n(\omega)(A)$ if the limit exists and 0 otherwise. Also, to map given limiting frequencies or sets of limiting frequencies to events in S^∞ , we consider the natural inverses $\Psi^{-1}(\ell) = \{\omega : \Psi(\omega) = \ell\}$ and $\Psi^{-1}(L) = \{\omega : \Psi(\omega) \in L\}$ for $\ell \in \Delta(S)$ and $L \subseteq \Delta(S)$.

The following axioms are used in describing assumptions on preferences \succsim at various points in this paper. For remarks and interpretation of these axioms see KMS [21].

Axiom 1 (C-complete Preorder). *\succsim is reflexive, transitive and the restriction of \succsim to X is complete.*

Axiom 2 (Monotonicity). *If $f(\omega) \succsim g(\omega)$ for all $\omega \in S^\infty$, $f \succsim g$.*

Axiom 3 (Risk Independence). *For all $x, x', x'' \in X$ and $\alpha \in (0, 1)$, $x \succsim x'$ if and only if $\alpha x + (1 - \alpha)x'' \succsim \alpha x' + (1 - \alpha)x''$.*

Axiom 4 (Non-triviality). *There exist $x, y \in X$ such that $x \succ y$.*

The key axiom is Event Symmetry which implies that the ordinates of S^∞ are viewed as interchangeable.

Axiom 5 (Event Symmetry). *For all finite cylinder events $A \in \Sigma$ and finite permutations $\pi \in \Pi$,*

$$\alpha 1_A + (1 - \alpha)h \sim \alpha 1_{\pi A} + (1 - \alpha)h \text{ for all } \alpha \in [0, 1] \text{ and all acts } h \in \mathcal{F}. \quad (2.1)$$

We now state three forms of continuity that are used in the paper.

Axiom 6 (Mixture Continuity). *For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0, 1] : \lambda f + (1 - \lambda)g \succsim h\}$ and $\{\lambda \in [0, 1] : h \succsim \lambda f + (1 - \lambda)g\}$ are closed in $[0, 1]$.*

To describe our second and third continuity axioms, it is notationally convenient to introduce the following binary relation \succsim^* (see e.g., Ghirardato, Maccheroni and Marinacci [14]) derived from \succsim :

$$f \succsim^* g \text{ if } \alpha f + (1 - \alpha)h \succsim \alpha g + (1 - \alpha)h \text{ for all } \alpha \in [0, 1] \text{ and } h \in \mathcal{F}. \quad (2.2)$$

A weakening of Mixture Continuity of \succsim that is sufficient for some of our results is

Axiom 7 (Mixture Continuity of \succsim^*). *For all $f, g, h \in \mathcal{F}$, the sets $\{\lambda \in [0, 1] : \lambda f + (1 - \lambda)g \succsim^* h\}$ and $\{\lambda \in [0, 1] : h \succsim^* \lambda f + (1 - \lambda)g\}$ are closed in $[0, 1]$.*

Finally, borrowed from Ghirardato, Maccheroni and Marinacci [14], we have the following adaptation of Arrow's [3] monotone continuity:

Axiom 8 (Monotone Continuity of \succsim^*). *For all $x, x', x'' \in X$, if $A_n \searrow \emptyset$ and $x' \succ x''$, then $x' \succsim^* x A_n x''$ for some n .*

Definition 2.1. \succsim satisfies *Weak Continuous Symmetry* if it satisfies all of the above axioms except possibly Mixture Continuity.

When we say that \succsim is Weak Continuous Symmetric, we mean that it satisfies Weak Continuous Symmetry.

3 Relating Event Symmetry to the literature

A key axiom in our approach is Event Symmetry. We now show that this condition relates quite closely to a variety of other conditions from the literature, including strengthenings of de Finetti [7]'s Exchangeability, of Hewitt and Savage [19]'s Symmetry, of Seo [26]'s Dominance and of Klibanoff, Marinacci and Mukerji [20]'s Consistency. One of those conditions (condition (viii) below) requires some additional definitions which we state next.

Definition 3.1. For $f \in \mathcal{F}$, f^Ψ is the (not necessarily simple) act uniquely defined as follows:

$$f^\Psi(\omega) = \begin{cases} \int_{S^\infty} f d\ell^\infty & \text{if } \ell = \Psi(\omega) \in \Delta(S) \\ \delta_{x^*} & \{\omega : \Psi(\omega) \text{ is not defined}\} \end{cases}.$$

Note this definition associates with each act f an act f^Ψ that, for each event $\{\omega : \Psi(\omega) = \ell\}$ corresponding to the limiting frequencies generated by ℓ , yields the lottery generated by f under the assumption that the i.i.d. process ℓ^∞ governs the realization of the state.

Since f^Ψ need not be simple, but is an element of the space $\hat{\mathcal{F}}$ of all bounded and measurable functions from Ω to X , it is necessary to consider extending \succsim to $\hat{\mathcal{F}}$. In

particular, we consider extensions continuous in the following sense: $\hat{\succsim}$ on $\hat{\mathcal{F}}$ satisfies *Norm Continuity* if $f \hat{\succsim} g$ whenever $f_k \hat{\succsim} g_k$ for all $k = 1, 2, \dots$ and f_k and g_k norm-converge to f and g respectively.

We now show that, under mild conditions, Event Symmetry is in fact equivalent to a variety of other conditions appearing in the literature. A first contribution, then, is to show that despite the seeming variety of preference formalizations of symmetry in the ambiguity literature, reassuringly their differences will not matter in most settings. Secondly, this also shows that results from KMS [21] derived assuming Event Symmetry are robust in the sense that they would continue to hold under alternative symmetry formulations.

Theorem 3.1. *The following conditions are equivalent under the assumption that \succsim is reflexive, transitive and satisfies Mixture Continuity:*

- (i) for every $f \in \mathcal{F}$ and $\pi \in \Pi$, $f \sim \frac{1}{2}f + \frac{1}{2}\pi f$,
- (ii) for every $f \in \mathcal{F}$, $\pi \in \Pi$ and $\alpha \in [0, 1]$, $f \sim \alpha\pi f + (1 - \alpha)f$,
- (iii) for every $f \in \mathcal{F}$ and $\pi_i \in \Pi$, $f \sim \frac{1}{n} \sum_{i=1}^n \pi_i f$,
- (iv) for every $f \in \mathcal{F}$, $\pi_i \in \Pi$ and $\alpha_i \in [0, 1]$ with $\sum_{i=1}^n \alpha_i = 1$, $f \sim \sum_{i=1}^n \alpha_i \pi_i f$, and
- (v) for every $f \in \mathcal{F}$ and $\pi \in \Pi$, $f \sim^* \pi f$.

Moreover, the above are equivalent to each of the following under the assumption that \succsim satisfies C-complete Preorder, Mixture Continuity, Monotonicity, Risk Independence, Non-triviality and Monotone Continuity of \succsim^* :

- (vi) Event Symmetry,
- (vii) for every $f, g \in \mathcal{F}$, if $\int f dp \succsim \int g dp$ for all symmetric $p \in \Delta(S^\infty)$, then $f \succsim g$.

Finally, if, in addition, there exists an extension of \succsim to $\hat{\mathcal{F}}$ that is reflexive, transitive and satisfies Norm Continuity, then the following is equivalent to all of the above:

- (viii) for $f, g \in \mathcal{F}$, $f \succsim g$ if and only if $f^\Psi \hat{\succsim} g^\Psi$, if $\hat{\succsim}$ is any such extension.

All of these conditions are strengthenings of Hewitt and Savage [19]’s symmetry: given $p \in \Delta(S^\infty)$, $p(A) = p(\pi A)$ for all finite cylinder events $A \in \Sigma$. In terms of preference, this translates into $1_A \sim 1_{\pi A}$ for all such events and all permutations $\pi \in \Pi$.¹ Event Symmetry strengthens this by requiring the indifference to be preserved under mixture with any common third act. Under reflexivity, transitivity and Mixture Continuity of \succsim , conditions (i)-(v) each imply Event Symmetry.

Condition (ii) is closely related to Epstein and Seo [10]’s Strong Exchangeability, the first behavioral axiom in the literature that captures the idea that the agent views all experiments as identical, i.e., i.i.d. (See Epstein and Seo [10] for a behavioral interpretation of condition (ii). A similar interpretation applies to (i), (iii) and (iv).) Their axiom states that condition (ii) holds when f depends only on a finite number of experiments. However, under their regularity axiom, their Strong Exchangeability extends to every act f and hence is equivalent to condition (ii).

¹Epstein and Seo [11] assume $f \sim \pi f$ for all permutations π and acts f . They show that there is a modeling trade-off between this symmetry axiom, a type of dynamic consistency and ambiguity.

Epstein and Seo [10], under MEU, characterize two models of preferences where the decision maker is indifferent to permutations – one that restricts to identical experiments (no ambiguity on idiosyncratic factors) and permits ambiguity about parameters, and the other that allows ambiguity on idiosyncratic factors but rules out ambiguity about parameters. Only the former of the two models satisfies (ii). Epstein and Seo [12] go further and characterize a single model within MEU that reflects ambiguity on both at the same time. The unifying framework permits a behavioral distinction between idiosyncratic factors and parameters as two sources of ambiguity.

Condition (i) is a special case of condition (ii) when $\alpha = \frac{1}{2}$. Since Epstein and Seo [10] consider MEU models, they could restrict to the case $\alpha = \frac{1}{2}$. The above theorem shows that $\alpha = \frac{1}{2}$ is sufficient to capture the same idea in general as long as reflexivity, transitivity and Mixture Continuity of \succsim hold.

De Castro and Al-Najjar ([5], [2]) use condition (iii) and its generalization to collections of transformations Γ other than the finite permutations Π . They provide conditions on Γ under which complete, transitive, monotonic, continuous and risk independent preferences satisfying (iii) with respect to Γ are such that each act is indifferent to an associated act based on limiting frequencies (where the notion of limiting frequencies uses the given Γ) where the associated act is constructed much like f^Ψ above with Γ -ergodic measures replacing the i.i.d. measures as parameters. They also prove a representation theorem for a utility function similar to (4.2) using an expected utility assumption on parameter-based acts added to the conditions on preferences and on Γ mentioned in the previous sentence.

Condition (iv) strengthens condition (iii).

Condition (v) is stronger than Event Symmetry in that f is not necessarily a binary act.

Condition (vii) is analogous to Seo [26]’s Dominance and Cerreia-Vioglio et. al.[6]’s Consistency, and condition (viii) to Klibanoff, Marinacci and Mukerji [20]’s Consistency. Seo’s Dominance is stated with lotteries over acts, objects that are not available in the domain of this paper, and condition (vii) restricts $p \in \Delta(S^\infty)$ to be symmetric while p is unrestricted in Seo’s Dominance. Thus, the difference between the two is that Seo’s Dominance requires f to induce a better lottery than g under all processes, not just symmetric ones. The reason for the additional restriction here is to reflect the fact that the experiments are symmetric. We want to include as reflecting dominance, for example, the following case:

$$f(\omega) = x^* H_1 x_* \text{ and } g(\omega) = \left(\frac{1}{2}x^* + \frac{1}{2}x_*\right) H_2 x_*$$

where $H_i = \{\omega \in S^\infty : \omega_i = H\}$ and $S = \{H, T\}$. The act f is a bet that the first coin comes up heads, and the act g is a bet that the second coin comes up heads but with a less valuable reward for winning. Under symmetry of the experiments, it is intuitively clear that f is better than g . Condition (vii) indeed implies that $f \succsim g$, while Seo’s Dominance would not – for example, when $p = \delta_{THTT\dots}$, $\int g dp \succ \int f dp$.

Cerreia-Vioglio et. al. [6]’s Consistency is similar to condition (vii), but instead of considering transformations (e.g. permutations) and/or all symmetric measures, they

assume a family of objectively rational probability measures on a general state space and require $\int f dp \succsim \int g dp$ for all measures p in that family. Taking the family to be all symmetric measures makes Cerreia-Vioglio et. al.'s Consistency exactly condition (vii). With their Consistency they provide representation theorems for Bewley preference, Choquet expected utility, variational preferences, uncertainty averse preferences and, as was mentioned earlier, the smooth ambiguity model. Also, they prove that the generalization of condition (v) to other collections of transformations Γ together with some (mild) axioms imply their Consistency when the family of objectively rational measures is the Γ -invariant measures.

The content of condition (viii) is that given the “knowledge” that everything is driven by some (as yet unknown) i.i.d. process, it seems reasonable that when evaluating an act, the individual would ultimately care only about the induced mapping from the space of i.i.d. processes to the lotteries generated under each process. Klibanoff, Marinacci and Mukerji [20]’s Consistency assumption says that when evaluating an act, an individual cares only about the induced mapping from probability measures on the state space to lotteries. Their assumption was stated in terms of acts and “second order acts” (maps from probability measures on the state space to outcomes). The latter objects do not appear as such in the present paper, but their role is played by the subset of acts measurable with respect to limiting frequency events, the acts $f^\Psi \in \hat{\mathcal{F}}$. The identification (in terms of preference) of f with f^Ψ stated in condition (viii) is analogous to the identification of f with an “associated second order act” in Klibanoff, Marinacci and Mukerji [20]’s Consistency with the qualification that f^Ψ only induces the same mapping from probability measures to lotteries as f for i.i.d. probability measures. Thus, condition (viii) strengthens Klibanoff, Marinacci and Mukerji [20]’s Consistency to incorporate the known symmetry of the ordinates in the same way that condition (vii) strengthened Seo [26]’s Dominance. Theorem 3.1 says that, under our other axioms, each of these strengthenings is equivalent to Event Symmetry. One can view the equivalence of condition (iii) and condition (viii) as following from the i.i.d. case of the sufficient statistic result in Al-Najjar and de Castro [2].

4 When do Relevant measures and the Bewley set agree?

We begin by recalling definitions of these sets and some key results from KMS [21] that we build upon.

KMS [21] defined what it means for a marginal $\ell \in \Delta(S)$ to be a relevant measure. For notational convenience, let \mathcal{O}_ℓ be the collection of open subsets of $\Delta(S)$ that contains ℓ . That is, for $\ell \in \Delta(S)$, $\mathcal{O}_\ell = \{L \subseteq \Delta(S) : L \text{ is open, } \ell \in L\}$.

Definition 4.1. A measure $\ell \in \Delta(S)$ is *relevant* (according to preferences \succsim) if, for any $L \in \mathcal{O}_\ell$, there are $f, g \in \mathcal{F}$ such that $f \approx g$ and $\int f d\hat{\ell}^\infty = \int g d\hat{\ell}^\infty$ for all $\hat{\ell} \in \Delta(S) \setminus L$.

In words, ℓ is relevant if it satisfies the following property: For each open set containing ℓ , there are acts that are not indifferent despite generating identical induced distributions over outcomes when any measure outside this set governs the independent realization of each ordinate S . From KMS [21], we know that the set of relevant measures, R , is closed and that, a measure is not relevant if and only if the limiting frequency event generated by some open neighborhood of that measure is null according to the preferences. Thus, behavior is as if relevant measures were the possible resolutions of ambiguity. In this sense, the relevant measures reflect perceived ambiguity. Furthermore KMS [21, Theorem 3.4] showed that, under mild conditions, the set of relevant measures reflects *only* perceived ambiguity. Notably, the set is not affected by increases or decreases in ambiguity aversion.

The Bewley set, C , is defined using a Bewley-style (Bewley [4]) representation of the induced relation \succsim^* (defined by (2.2)). For Weak Continuous Symmetric preferences violating Anscombe-Aumann Independence, \succsim^* is incomplete relative to \succsim . In the context of Weak Continuous Symmetric preferences, KMS [21, Theorem 4.5] provided a Bewley-style (Bewley [4]) representation result for \succsim^* , identifying both the Bewley set C and the set of relevant measures R . It is of interest to note that this result is a generalization of de Finetti's theorem. We restate it here, and will make use of it in proving several of the results of this section:

Theorem 4.1. *\succsim is Weak Continuous Symmetric if and only if \succsim is transitive and there exist a non-empty compact convex set $M \subseteq \Delta(\Delta(S))$ and a non-constant vNM utility function u such that*

$$f \succsim^* g \text{ if and only if } \int u(f) dp \geq \int u(g) dp \text{ for all } p \in C, \quad (4.1)$$

where $C = \{\int \ell^\infty dm(\ell) : m \in M\}$. Furthermore $R = \overline{\bigcup_{m \in M} \text{supp } m}$ and M is unique.

The main contribution of this section is to show that preference condition No Half Measures, defined below, is equivalent to the set of relevant measures fully determining the Bewley set, and thus necessary and sufficient for the Bewley set to inherit the interpretation of reflecting only perceived ambiguity from the set of relevant measures.

First a preliminary definition: Let Σ^Ψ be the σ -algebra generated by the sets

$$\Psi^{-1}(\ell) \text{ for } \ell \in \Delta(S).$$

This algebra contains events based on limiting frequencies.

Condition (No Half Measures). For $A \in \Sigma^\Psi$, at least one of the following holds:

- (i) If $1_{S^\infty} \succ x \in X$, then $\alpha x + (1-\alpha)f \not\prec \alpha 1_A + (1-\alpha)f$ for some $f \in \mathcal{F}$ and $\alpha \in (0, 1]$;
- (ii) For all $f \in \mathcal{F}$ and $\alpha \in (0, 1]$, $\alpha 1_\emptyset + (1-\alpha)f \sim \alpha 1_A + (1-\alpha)f$.

To understand what this condition says, fix any bet on an event $A \in \Sigma^\Psi$ and let $x^* \in X$ be the lottery you get if you win and x_* the lottery if you lose (with $x^* \succ x_*$). Imagine agreeing to lose for sure (i.e., having the bet shrunk from A to the empty set) in return for improvement in the losing lottery (call the new losing lottery w , with $x^* \succ w \succeq x_*$). Notice that this is a trade-off between utility on A and A^c – on A , x^* is lowered to w , while on A^c , x_* is raised to w . No Half Measures says that either there is always some mixture within which you do not prefer to make this trade-off (i.e., sometimes the increase on A^c is worth less than *any* reduction on A), or, for every mixture, you are willing to agree to substitute a sure loss without any improvement in the losing stakes (i.e., you require *no* compensation for the reduced payoff on A). When this is true, then identifying the measures ℓ that are relevant is the same as identifying the measures that sometimes have an infinite marginal rate of substitution (i.e., that get assigned full belief²). Thus, under this condition, classifying measures into those that sometimes get positive weight and those that do not is the same as classifying measures according to the range of marginal rates of substitution they sometimes receive. When this condition fails, the latter classification will make use of variations in ranges of marginal rates of substitution, not just zero vs. positive. Intuitively, these intermediate marginal rates of substitution depend not only on perceived ambiguity, but also on taste aspects of preferences such as ambiguity aversion.

We now show the connection between No Half Measures and the relation between the Bewley set and the set of relevant measures.

Theorem 4.2. *Suppose \succeq is Weak Continuous Symmetric. Then \succeq satisfies No Half Measures if and only if $C = \{\int \ell^\infty dm(\ell) : m \in \Delta(R)\}$ in (4.1).*

Recall that, without No Half Measures, $C = \{\int \ell^\infty dm(\ell) : m \in M\}$ where $M \subseteq \Delta(R)$, so that knowing R need not pin down C . Our theorem shows that the No Half Measures condition is identifying those cases in which C reflects *only* the set R of processes considered relevant. In this sense, No Half Measures is required for the approach in Ghirardato, Maccheroni and Marinacci [14], Nehring [24] and Ghirardato and Siniscalchi [15, 16] based on C to agree with the relevant measures and thus be interpretable as reflecting *only* perceived ambiguity and not perceived ambiguity intertwined with ambiguity aversion.

How restrictive is the No Half Measures condition?

One restriction is that No Half Measures implies there is only a finite set of relevant measures, and so only a finite number of possible resolutions of ambiguity.

Theorem 4.3. *If a Weak Continuous Symmetric \succeq satisfies No Half Measures then R is finite.*

What are the further implications of No Half Measures in the context of specific decision models? We begin with the Maxmin expected utility (MEU) model (Gilboa

²A measure that always gets less than full belief will not be relevant, which motivates the name, No Half Measures.

and Schmeidler [18]). For this model, No Half Measures implies the set of measures in the representation are the i.i.d. products generated by the measures in R . We then examine the α -MEU model and show there, when R is non-singleton No Half Measures is only compatible with extreme values of α (α equals 0 or 1). Similarly, we conclude this subsection with an examination of the smooth ambiguity model – when R is non-singleton we show No Half Measures places non-trivial restrictions on the function ϕ .³

4.1 MEU Model

Theorem 4.4. *Suppose there is a non-constant vN-M utility function u , and a non-empty compact convex set $M \subseteq \Delta(\Delta(S))$ such that, for $F \equiv \{\int \ell^\infty dm(\ell) : m \in M\}$,*

$$V(f) \equiv \min_{p \in F} \int u(f) dp$$

represents \succsim . Then \succsim satisfies No Half Measures if and only if

$$V(f) = \min_{\ell \in R} \int_{S^\infty} u(f) d\ell^\infty$$

where R is finite.

The contribution of Theorem 4.4 is that in this symmetric MEU context, No Half Measures is equivalent to limiting the set of measures to the i.i.d. products generated by a finite set of relevant measures, R . The proof uses Ghirardato, Maccheroni and Marinacci [14, Theorem 14] to show that F is the Bewley set, and then applies Theorems 4.1, 4.2 and 4.3 to obtain the result. Note that the supposition in the first sentence of the theorem could equivalently be replaced by assuming \succsim satisfies Weak Continuous Symmetry and the axioms in Gilboa and Schmeidler [18].

We can apply Theorem 4.4 to the Extended MEU with contraction model to get the following:

Corollary 4.1. *Suppose there is a non-constant vN-M utility function u , a $\beta \in [0, 1]$, a finite set $D \subseteq \Delta(S)$ and a probability measure $q = \int \ell^\infty dm(\ell)$ for an $m \in \Delta(D)$, such that,*

$$\begin{aligned} V(f) &\equiv \beta \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f) dp + (1 - \beta) \int u(f) dq \\ &= \min_{p \in \{\beta \ell^\infty + (1 - \beta)q : \ell \in D\}} \int u(f) dp \end{aligned}$$

represents \succsim . Then \succsim satisfies No Half Measures if and only if ($\beta = 1$ or ($q \in \{\ell^\infty : \ell \in D\}$ and ($D = \{q\}$ or $\beta = 0$))).

³Similarly, though we do not fully investigate it, No Half Measures imposed on the Vector expected utility model (Siniscalchi [28]) will imply non-trivial restrictions on the adjustment function A and the adjustment factors ζ_i appearing in that model.

To see that this is a strong condition, observe that the restriction ($\beta = 1$ or ($q \in \{\ell^\infty : \ell \in D\}$ and ($D = \{q\}$ or $\beta = 0$))) is equivalent to MEU without contraction ($\beta = 1$) or expected utility with an i.i.d. prior, ruling out all intermediate, non-extreme, cases of ambiguity attitude.

4.2 α -MEU Model

According to KMS [21, footnote 9] (and the example below), the Bewley set, C , generally will depend on α as well as the set of measures appearing in the representation. Since No Half Measures ensures C is fully determined by R , it must eliminate the dependence of C on α . When does this dependence occur? Our next example (see also Eichberger, Grant and Kelsey [8, Section 4] and Eichberger et. al. [9, section 3.2.2]), suggests that $\alpha \in (0, 1)$ creates dependence.

Example 4.1. Let $S = \{H, T\}$ and denote $\ell \in \Delta(S)$ by the corresponding probability of H . Consider the \succsim represented by

$$\frac{3}{4} \min_{p \in co\{\frac{2}{3}^\infty, \frac{1}{4}^\infty\}} \int u(f) dp + \frac{1}{4} \max_{p \in co\{\frac{2}{3}^\infty, \frac{1}{4}^\infty\}} \int u(f) dp.$$

[21, Theorem 4.1] implies that $R = \{\frac{2}{3}, \frac{1}{4}\}$. Using the results of Siniscalchi ([27], [29]), the Bewley set C is given by $C = co\{\frac{3}{4}\frac{2}{3}^\infty + \frac{1}{4}\frac{1}{4}^\infty, \frac{1}{4}\frac{2}{3}^\infty + \frac{3}{4}\frac{1}{4}^\infty\}$, the convex hull of *some strict* convex combinations of $\{\frac{2}{3}^\infty, \frac{1}{4}^\infty\}$ (i.e., convex combinations that are not themselves i.i.d.) and is a proper subset of $co\{\ell^\infty : \ell \in R\}$ (and, *which* proper subset is determined by α in this case where $\alpha = 1/4$. In case $\alpha = 0$ or 1 , R fully determines C since $C = co\{\frac{2}{3}^\infty, \frac{1}{4}^\infty\}$. In other cases, when $\alpha \in (0, 1)$, the Bewley set C is influenced by α .

Our next result shows that the troublesome case in the example (i.e., $\alpha \in (0, 1)$ and R non-singleton) is exactly what is ruled out by No Half Measures. In this sense, No Half Measures turns out to be extremely restrictive in the context of α -MEU:

Theorem 4.5. *Suppose there is a non-constant vN-M utility function u , an $\alpha \in [0, 1]$ and a finite set $D \subseteq \Delta(S)$ such that*

$$V(f) \equiv \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(f) dp + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(f) dp$$

represents \succsim . Then No Half Measures is equivalent to ($\alpha = 0$ or $\alpha = 1$ or D singleton).

Eichberger et. al. [9, Theorem 2] prove a related result – for any finite state space, when C is non-singleton, any \succsim having an α -MEU representation with Bewley set C as the set of measures has α equal to 0 or 1. Since S itself, let alone S^∞ , need not be finite our result requires different arguments. In fact, Eichberger et. al. [9] provide an example with an infinite compact metric state space showing that their result may fail in such settings. In light of this, our theorem shows that Event Symmetry and our continuity together with the product structure of the state space is sufficient to extend their finite state space conclusion.

4.3 Smooth Ambiguity Model

Theorem 4.6. *Suppose there is a non-constant vN-M utility function u , a strictly increasing continuously differentiable function $\phi : u(X) \rightarrow \mathbb{R}$ and a Borel probability measure $\mu \in \Delta(\Delta(S))$ with $\text{supp } \mu$ finite such that*

$$U(f) \equiv \int_{\Delta(S)} \phi \left(\int_{S^\infty} u(f) d\ell^\infty \right) d\mu(\ell) \quad (4.2)$$

represents \succsim . Then No Half Measures is equivalent to $(\sup \bigcup_{r,t \in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$ or $\text{supp } \mu$ singleton).

Some examples violating the condition $\sup \bigcup_{r,t \in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$ include $\phi(x) = x$ or $\phi(x) = 2x - \cos(x)$ with $u(X) = \mathbb{R}_{++}$ or \mathbb{R}_+ , and $\phi(x) = x + \ln(1+x)$ with $u(X) = \mathbb{R}_{++}$ or \mathbb{R}_+ . When utility is bounded both above and below (e.g., there is a best and a worst outcome), it is easy to find examples violating the condition (see Section 4.3 in KMS [21]). Examples satisfying the condition in the theorem include, $\phi(u) = -\exp(-u/\theta)$ or $\phi(x) = x - \exp(-x/\theta)$ with $u(X) = \mathbb{R}$ and $\phi(x) = \frac{x^{1-\gamma}}{1-\gamma}$ or $\phi(x) = x + \frac{x^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$ with $u(X) = \mathbb{R}_{++}$ or \mathbb{R}_+ .

Following the statement of the No Half Measures condition, we explained that this condition fails when those measures that sometimes get positive weight differ from those that sometimes get arbitrarily close to full weight (where “weight” is measured, as explained earlier, by utility trade-offs between the limiting frequency event corresponding to the measure and its complementary event). By characterizing No Half Measures in the context of the Extended MEU with contraction, α -MEU and the smooth ambiguity models, we have seen three concrete illustrations of when the Bewley set, based on (magnitudes of) these positive weights or marginal rates of substitution, will involve parameters usually thought of as tastes (e.g., β , α and ϕ) and when it will not. The cases where it will involve these parameters are extensive: For Extended MEU with contraction, No Half Measures fails except for MEU with *no* contraction or expected utility with an i.i.d. prior. For α -MEU, No Half Measures fails in essentially all the cases beyond MEU and max-max EU. For the smooth ambiguity model, No Half Measures fails whenever the variation in the slope of ϕ is bounded and μ is non-degenerate.

5 Appendix: Proofs

Denote by $B(S)$ the set of bounded measurable functions on S . Similarly for $B(\Delta(S))$ and $B(S^\infty)$. When B is replaced by B_0 this denotes the restriction to simple measurable functions. Similarly, when B_0 is replaced by B_0^l (resp. B_0^u) this denotes the restriction to lower (resp. upper) semicontinuous simple functions. For $b \in B(S^\infty)$, we write $\|b\|$ for the sup-norm of b (i.e., $\sup_\omega |b(\omega)|$).

5.1 Proof of Theorem 3.1

i) \Rightarrow v): Let $f, h \in \mathcal{F}$ and $\pi \in \Pi$. Since π is a finite permutation, π^N is the identity for some N . Thus, for any $g \in \mathcal{F}$,

$$g^*(\omega) \equiv \lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=1}^k \pi^{i-1} g(\omega) = \frac{1}{N} \sum_{i=1}^N \pi^{i-1} g(\omega)$$

is well defined everywhere.

Step 1. $g \sim g^*$.

By repeatedly applying i),

$$\begin{aligned} g &\sim \frac{1}{2}g + \frac{1}{2}\pi g \sim \frac{1}{2} \left(\frac{1}{2}g + \frac{1}{2}\pi g \right) + \frac{1}{2}\pi^2 \left(\frac{1}{2}g + \frac{1}{2}\pi g \right) \\ &= \frac{1}{2^2} \sum_{i=1}^{2^2} \pi^{i-1} g \sim \frac{1}{2^k} \sum_{i=1}^{2^k} \pi^{i-1} g, \end{aligned}$$

for all k .

For any positive integer n , let $q(n)$ be the quotient when we divide n by N , and $r(n)$ the remainder. (That is, $n = q(n) \cdot N + r(n)$.) Then,

$$\begin{aligned} \frac{1}{2^k} \sum_{i=1}^{2^k} \pi^{i-1} g &= \frac{q(2^k)}{2^k} \sum_{i=1}^N \pi^{i-1} g + \frac{1}{2^k} \sum_{i=1}^{r(2^k)} \pi^{i-1} g \\ &= \frac{q(2^k)N}{2^k} g^* + \frac{1}{2^k} \sum_{i=1}^{r(2^k)} \pi^{i-1} g. \end{aligned}$$

(Here $\sum_{i=1}^0 a_i$ is understood to be 0.) Since $r(2^k)$ can take at most finite number of integers, we can find a subsequence $2^{j(k)}$ of 2^k such that there is $\bar{K} = r(2^{j(k)})$ for all k . Thus,

$$g \sim \alpha_k g^* + (1 - \alpha_k) \left(\frac{1}{\bar{K}} \sum_{i=1}^{\bar{K}} \pi^{i-1} g \right) \text{ for all } k,$$

where $\alpha_k = \frac{q(2^{j(k)})N}{2^{j(k)}}$. (Note that $1 - \alpha_k = \frac{\bar{K}}{2^{j(k)}}$.) Since $\alpha_k \rightarrow 1$, Mixture Continuity implies $g \sim g^*$.

Step 2. $\alpha f + (1 - \alpha) h \sim \alpha \pi f + (1 - \alpha) h$.

By Step 1,

$$\alpha f + (1 - \alpha) h \sim (\alpha f + (1 - \alpha) h)^* = \alpha f^* + (1 - \alpha) h^*.$$

Similarly,

$$\alpha \pi f + (1 - \alpha) h \sim \alpha (\pi f)^* + (1 - \alpha) h^* = \alpha f^* + (1 - \alpha) h^*.$$

By Preorder, this step is proved.

v) \Rightarrow iv): By v),

$$\alpha f + (1 - \alpha) h \sim \alpha \pi f + (1 - \alpha) h \quad (5.1)$$

for all $\alpha \in [0, 1]$ and $h \in \mathcal{F}$. Note that $f \sim \pi f$ when $\alpha = 1$.

Setting $h = f$ gives,

$$f \sim \alpha \pi f + (1 - \alpha) f \text{ for all } \alpha \in [0, 1]. \quad (5.2)$$

Moreover, setting $h = \beta f + (1 - \beta) \pi' f$ leads to

$$\begin{aligned} f &\sim (\alpha + (1 - \alpha) \beta) f + (1 - \alpha) (1 - \beta) \pi' f \quad (\text{by (5.2)}) \\ &= \alpha f + (1 - \alpha) (\beta f + (1 - \beta) \pi' f) \\ &\sim \alpha \pi f + (1 - \alpha) (\beta f + (1 - \beta) \pi' f) \quad (\text{by (5.1)}) \\ &= (1 - \alpha) \beta f + \alpha \pi f + (1 - \alpha) (1 - \beta) \pi' f. \end{aligned}$$

Since α and β can be taken arbitrarily, $f \sim \alpha f + \beta \pi f + (1 - \alpha - \beta) \pi' f$ for all $\alpha, \beta \in [0, 1]$. Since $f \sim \pi'' f$ for $\pi'' \in \Pi$,

$$f \sim \pi'' f \sim \alpha \pi'' f + \beta \pi \pi'' f + (1 - \alpha - \beta) \pi' \pi'' f.$$

For any $\pi_1, \pi_2, \pi_3 \in \Pi$, take $\pi = \pi_2 (\pi_1)^{-1}$, $\pi' = \pi_3 (\pi_1)^{-1}$ and $\pi'' = \pi_1$. Then, we get iv) for $n = 3$. Repeat the argument to show iv) for any n .

Clearly, iv) implies ii) and iii). Moreover, ii) implies i) and so does iii). Thus, we have shown that i)-v) are equivalent.

Assume the additional axioms to show equivalence of i)-vii).

v) \Leftrightarrow vi): Clearly, v) implies vi). By Theorem 4.1, the converse holds.

vi) \Leftrightarrow vii): vi) implies vii) by Theorem 4.1. The converse holds since

$$\int (\lambda f + (1 - \lambda) h) dp = \int (\lambda \pi f + (1 - \lambda) h) dp \text{ for all symmetric } p \in \Delta(S^\infty).$$

Assume the existence of an extension $\hat{\succsim}$ satisfying Preorder and Norm Continuity to show the equivalence of i)-viii).

vi) \Leftrightarrow viii): The easy direction is viii) implies vi): Note that

$$(\lambda f + (1 - \lambda) h)^\Psi = (\lambda \pi f + (1 - \lambda) h)^\Psi$$

for all $f, h \in \mathcal{F}$ and $\lambda \in [0, 1]$. Thus, viii) implies $\lambda f + (1 - \lambda) h \sim \lambda \pi f + (1 - \lambda) h$ for all $f, h \in \mathcal{F}$ and $\lambda \in [0, 1]$, which implies Event Symmetry.

We now turn to the other direction. Let $\hat{\succsim}$ be any extension of \succsim to $\hat{\mathcal{F}}$ satisfying Preorder and Norm Continuity. Define $\hat{\succsim}^*$ on $\hat{\mathcal{F}}$ by

$$f \hat{\succsim}^* g \text{ if } \alpha f + (1 - \alpha) h \hat{\succsim} \alpha g + (1 - \alpha) h \text{ for all } \alpha \in [0, 1] \text{ and } h \in \hat{\mathcal{F}}. \quad (5.3)$$

Note that \succsim , $\hat{\succsim}$, \succsim^* and $\hat{\succsim}^*$ on X all agree and can be represented by some non-constant mixture linear function u on X .

Our argument proceeds by steps along the following lines: In steps 1-3, given an arbitrary $h' \in \hat{\mathcal{F}}$, we construct sequences of simple acts that norm-converge to h' from above and from below. In steps 4-7, we show that $\hat{\succsim}^*$ is the unique extension of \succsim^* satisfying C-complete Preorder, Norm Continuity, Monotonicity, Independence and Non-triviality. In step 8, we use the representation of \succsim^* in Theorem 4.1 to construct a representation for $\hat{\succsim}^*$, and use that representation to show vi) implies viii).

Step 1. For any $h' \in \hat{\mathcal{F}}$, and $x, y \in X$ with $x \succ y$, there is $h \in \mathcal{F}$ such that $\frac{1}{2}h(\omega) + \frac{1}{2}y \prec \frac{1}{2}h'(\omega) + \frac{1}{2}x$ and $h(\omega) \succsim h'(\omega)$ for all $\omega \in \Omega$: Take $\dots, x_{-1}, x_0, x_1, \dots \in X$ such that $x_0 = x$, $x_{-1} = y$ and $u(x_i) - u(x_{i-1}) = u(x) - u(y)$. For each $\omega \in \Omega$, set $h(\omega) = x_i$ if $x_{i-1} \prec h'(\omega) \preceq x_i$. Notice that $h \in \mathcal{F}$ since h' is bounded above and below and $u(x) - u(y) > 0$. Then h does the job since $h(\omega) \succsim h'(\omega)$ by construction and

$$\begin{aligned} \frac{1}{2}u(h(\omega)) + \frac{1}{2}u(y) &= \frac{1}{2}u(h(\omega)) - \frac{1}{2}u(x_i) + \frac{1}{2}u(x_{i-1}) + \frac{1}{2}u(x) \\ &< \frac{1}{2}u(h'(\omega)) + \frac{1}{2}u(x). \end{aligned}$$

Step 2. For any $h' \in \hat{\mathcal{F}}$, there is $h_k \in \mathcal{F}$ such that h_k norm-converges to h' and $h_k(\omega) \succsim h'(\omega)$ for all $\omega \in \Omega$: Take $z_k, z \in X$ such that $u(z_k) \searrow u(z)$. This is possible by Mixture Continuity. By Step 1, for each k , since $z_k \succ z$, there is $h_k \in \mathcal{F}$ such that $h_k(\omega) \succsim h'(\omega)$ and $\frac{1}{2}h_k(\omega) + \frac{1}{2}z \prec \frac{1}{2}h'(\omega) + \frac{1}{2}z_k$ for all $\omega \in \Omega$. Thus, $\sup_{\omega \in \Omega} |u(h_k(\omega)) - u(h'(\omega))| \leq u(z_k) - u(z)$. For any $x \succ y$, there exists K such that for all $k > K$, $u(z_k) - u(z) \leq u(x) - u(y)$. Thus, h_k norm-converges to h' .

Step 3. For any $h' \in \hat{\mathcal{F}}$, there is $h_k \in \mathcal{F}$ such that h_k norm-converges to h' and $h'(\omega) \succsim h_k(\omega)$ for all $\omega \in \Omega$: Slightly change Steps 1 and 2.

Step 4. $\hat{\succsim}$ satisfies Monotonicity: Take $f, g \in \hat{\mathcal{F}}$ such that $f(\omega) \hat{\succsim} g(\omega)$ for all $\omega \in \Omega$. By Steps 2 and 3, there are $f_k, g_k \in \mathcal{F}$ such that f_k, g_k norm-converge to f, g respectively, and $f_k(\omega) \hat{\succsim} f(\omega)$ and $g(\omega) \hat{\succsim} g_k(\omega)$ for all $\omega \in \Omega$. Then, $f_k(\omega) \succsim g_k(\omega)$ for all $\omega \in \Omega$, and Monotonicity of \succsim implies $f_k \succsim g_k$, hence $f_k \hat{\succsim} g_k$. Norm Continuity guarantees $f \hat{\succsim} g$.

Step 5. $\hat{\succsim}^*$ satisfies C-complete Preorder, Norm Continuity, Monotonicity, Independence and Non-triviality: $\hat{\succsim}^*$ inherits C-complete Preorder, Norm Continuity, Monotonicity and Non-triviality from the corresponding properties of $\hat{\succsim}$ and satisfies Independence by (5.3).

Step 6. $\hat{\succsim}^*$ extends \succsim^* : Take $f, g \in \mathcal{F}$ such that $f \succsim^* g$, that is, $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$ for all $\alpha \in [0, 1]$ and $h \in \mathcal{F}$ (and thus $\alpha f + (1 - \alpha)h \hat{\succsim} \alpha g + (1 - \alpha)h$ for all $\alpha \in [0, 1]$ and $h \in \mathcal{F}$ since $\hat{\succsim}$ is an extension of \succsim). Now fix $h' \in \hat{\mathcal{F}}$. By Step 2, there is $h_k \in \mathcal{F}$ norm-converging to h' . Moreover, by the mixture linearity of u , $\alpha f + (1 - \alpha)h_k$ and $\alpha g + (1 - \alpha)h_k$ norm-converge to $\alpha f + (1 - \alpha)h'$ and $\alpha g + (1 - \alpha)h'$ respectively. Since $\alpha f + (1 - \alpha)h_k \hat{\succsim} \alpha g + (1 - \alpha)h_k$ for all $\alpha \in [0, 1]$ and $k = 1, 2, \dots$, Norm Continuity implies $\alpha f + (1 - \alpha)h' \hat{\succsim} \alpha g + (1 - \alpha)h'$ for all $\alpha \in [0, 1]$. Since h' is arbitrary, $f \hat{\succsim}^* g$.

Step 7. All extensions of \succsim^* satisfying the axioms in Step 5 are the same: Assume $\succsim_1^*, \succsim_2^*$ on $\hat{\mathcal{F}}$ are two such extensions. It is enough to show that $f \succsim_1^* g$ for $f, g \in \hat{\mathcal{F}}$ implies $f \succsim_2^* g$ since the labeling of the extensions is arbitrary. By Steps 2 and 3, there are $f_k, g_k \in \mathcal{F}$ such that f_k and g_k norm-converge to f and g respectively and $f_k(\omega) \succsim_1^* f(\omega)$, $g(\omega) \succsim_1^* g_k(\omega)$ for all $\omega \in \Omega$. Thus, $f_k \succsim_1^* f \succsim_1^* g \succsim_1^* g_k$. Since $\succsim_1^*, \succsim_2^*$ coincide on \mathcal{F} , $f_k \succsim_2^* g_k$. Norm Continuity of \succsim_2^* implies $f \succsim_2^* g$.

Step 8. vi) implies viii): Theorem 4.1, there is $M \subseteq \Delta(\Delta(S))$ such that for all $f, g \in \mathcal{F}$, $f \succsim^* g$ iff $\int u(f) dp \geq \int u(g) dp$ for all $p \in \left\{ \int \ell^\infty dm(\ell) : m \in M \right\}$. Define an extension \succsim_1^* of \succsim^* to $\hat{\mathcal{F}}$ by

$$f \succsim_1^* g \text{ iff } \int u(f) dp \geq \int u(g) dp \text{ for all } p \in \left\{ \int \ell^\infty dm(\ell) : m \in M \right\}.$$

Then, one can check that \succsim_1^* satisfies all the axioms in Step 5. By Step 7, $\succsim_1^* = \hat{\succsim}^*$. Therefore, $f \hat{\succsim}^* f^\Psi$ and $g \hat{\succsim}^* g^\Psi$ for any $f, g \in \mathcal{F}$ and hence $f \hat{\succsim} f^\Psi$ and $g \hat{\succsim} g^\Psi$. Transitivity of $\hat{\succsim}$ implies viii).

5.2 Proofs for Section 4

It is sometimes convenient in the proofs to use an alternative formulation of No Half Measures. The following lemma gives the alternative formulation and shows that it is equivalent to No Half Measures. We may rely on this lemma without reference to it.

Lemma 5.1. *The following are equivalent:*

- (a) *No Half Measures;*
- (b) *For $A \in \Sigma^\Psi$, if there exists an $x \in X$ such that $1_{S^\infty} \succ x \succsim^* 1_A$, then $1_A \sim^* 1_\emptyset$.*

Proof of Lemma 5.1

(a) \Rightarrow (b): Fix $A \in \Sigma^\Psi$. Suppose for each $x \in X$ with $1_{S^\infty} \succ x$, there exists $f \in \mathcal{F}$ and $\alpha \in (0, 1]$ such that $\alpha x + (1 - \alpha)f \not\prec \alpha 1_A + (1 - \alpha)f$. Then $x \not\prec^* 1_A$ and (b) is vacuously satisfied. Suppose instead that for all $f \in \mathcal{F}$ and $\alpha \in (0, 1]$, $\alpha 1_\emptyset + (1 - \alpha)f \sim \alpha 1_A + (1 - \alpha)f$. Then $1_A \sim^* 1_\emptyset$ and (b) is satisfied.

(b) \Rightarrow (a): Fix $A \in \Sigma^\Psi$. Suppose (b) holds. Can it be that neither of the possibilities in (a) hold? If neither holds, $1_A \not\sim^* 1_\emptyset$ and for some $x \in X$ with $1_{S^\infty} \succ x$, for all $f \in \mathcal{F}$ and $\alpha \in (0, 1]$, $\alpha x + (1 - \alpha)f \succ \alpha 1_A + (1 - \alpha)f$. But then, $1_{S^\infty} \succ x \succsim^* 1_A$ and $1_A \not\sim^* 1_\emptyset$ contradicting (b).

5.2.1 Proof of Theorem 4.2

Normalize u so that $u(x^*) = 1$ and $u(x_*) = 0$.

(if): We show No Half Measures. Fix any $A \in \Sigma^\Psi$ and assume $1_{S^\infty} \succ x \succsim^* 1_A$. Then, $u(x) \geq \int u(1_A) d\ell^\infty = \ell^\infty(A)$ for all $\ell \in R$. Note that $\ell^\infty(A) \in \{0, 1\}$ for every $A \in \Sigma^\Psi$. Since $1_{S^\infty} \succ x$, $1 > u(x)$ and thus $\ell^\infty(A) = 0$ for all $\ell \in R$. Then, $1_A \sim^* 0$. By Lemma 5.1, No Half Measures holds.

(only if): Since \succsim satisfies Weak Continuous Symmetry, Theorem 4.1 delivers sets C , R and M . By definition, $C \subseteq \left\{ \int \ell^\infty dm(\ell) : m \in \Delta(R) \right\}$. For the other inclusion, follow the steps.

Step 1. For $\hat{\ell} \in \Delta(S)$, if $\delta_{\hat{\ell}} \notin M$, then $\max_{m \in M} m(L) < 1$ for some $L \in \mathcal{O}_{\hat{\ell}}$. We show this by contradiction. Suppose that, for every $L \in \mathcal{O}_{\hat{\ell}}$, there is $m_L \in M$ such that $m_L(L) = 1$. We can view $\mathcal{O}_{\hat{\ell}}$ as a directed set with the inverse inclusion. Then, $\{m_L\}$ is a net and it has a limit point in M because M is compact. The limit point has to be $\delta_{\hat{\ell}}$, a contradiction.

Step 2. $\hat{\ell}^\infty \in C$ for every $\hat{\ell} \in R$. To see this, again use proof by contradiction. Suppose not. Then, $\delta_{\hat{\ell}} \notin M$ for some $\hat{\ell} \in R$. By Step 1, $a = \max_{m \in M} m(L) < 1$ for some $L \in \mathcal{O}_{\hat{\ell}}$. Let $A = \Psi^{-1}(L)$ and $x = ax^* + (1-a)x_*$. Then, $u(x) = a < 1$ and, for any $m \in M$,

$$\begin{aligned} u(x) &= a \geq m(L) \\ &= \int_L \ell^\infty(A) dm + \int_{\Delta(S) \setminus L} \ell^\infty(A) dm \quad (\text{since } \ell^\infty(A) = 1 \text{ for } \ell \in L, \text{ and } 0 \text{ otherwise}) \\ &= \int_L \int_{S^\infty} u(1_A) d\ell^\infty dm + \int_{\Delta(S) \setminus L} \int_{S^\infty} u(1_A) d\ell^\infty dm = \int_{\Delta(S)} \int_{S^\infty} u(1_A) d\ell^\infty dm(\ell). \end{aligned}$$

Thus, $1 \succ x \succsim^* 1_A$. By No Half Measures, $m(L) = \int \int u(1_A) d\ell^\infty dm(\ell) = 0$ for every $m \in M$. But since $\hat{\ell} \in L$ and L is open, this implies $\hat{\ell} \notin \text{supp } m$ for every $m \in M$ and thus $\hat{\ell} \notin R$. This is a contradiction.

Step 3. $C \supseteq \left\{ \int \ell^\infty dm(\ell) : m \in \Delta(R) \right\}$. Denote by $\overline{co}^{w*}(\cdot)$ the weak* closed convex hull in $ba(S^\infty)$. Observe that

$$\min_{p \in \overline{co}^{w*}(\{\ell^\infty : \ell \in R\})} \int \psi dp = \inf_{p \in \{\ell^\infty : \ell \in R\}} \int \psi dp = \inf_{p \in \left\{ \int \ell^\infty dm(\ell) : m \in \Delta(R) \right\}} \int \psi dp$$

for any bounded measurable ψ on S^∞ . Since the set of measures in the MEU functional is unique up to the weak* closed convex hull (Gilboa and Schmeidler [18]),

$$\overline{co}^{w*}(\{\ell^\infty : \ell \in R\}) \supseteq \left\{ \int \ell^\infty dm(\ell) : m \in \Delta(R) \right\}.$$

By this inclusion, Step 2 and weak* closedness and convexity of C , we have

$$C \supseteq \overline{co}^{w*}(\{\ell^\infty : \ell \in R\}) \supseteq \left\{ \int \ell^\infty dm(\ell) : m \in \Delta(R) \right\}.$$

This completes the proof.

5.2.2 Proof of Theorem 4.3

Suppose R is not finite. Then, we can take distinct $\ell_n \in R$ for each n . Let $A_n = \bigcup_{k \geq n} \Psi^{-1}(\ell_k)$. Then, $A_n \searrow \emptyset$. To try to verify Monotone Continuity of \succsim^* , take

three constant acts $x \succ x' \succ x''$. Let $p_n = \ell_n^\infty$. Then, $p_n(A_n) = 1$ and $u(x') < p_n(A_n)u(x) + (1 - p_n(A_n))u(x'')$. By Theorem 4.2, $p_n \in C$ and Theorem 4.1 implies that it is not possible that $x' \succsim^* x A_n x''$ for any n . This contradicts Monotone Continuity of \succsim^* .

5.2.3 Proof of Theorem 4.4

Fix the representation V .

(if) If \succsim is represented by $\min_{\ell \in R} \int_{S^\infty} u(f) d\ell^\infty$ then the fact that \succsim satisfies No Half Measures follows from one direction of Theorem 4.5.

(only if) By Ghirardato, Maccheroni and Marinacci [14, Theorem 14], the Bewley set, C , associated with \succsim is F . By Theorem 4.1, \succsim satisfies Weak Continuous Symmetry. Suppose No Half Measures. Then, by Theorems 4.2 and 4.3, $C = \{\int \ell^\infty dm(\ell) : m \in \Delta(R)\}$ and R is finite. Thus,

$$V(f) = \min_{p \in F} \int u(f) dp = \min_{p \in \{\int \ell^\infty dm(\ell) : m \in \Delta(R)\}} \int u(f) dp = \min_{\ell \in R} \int_{S^\infty} u(f) d\ell^\infty.$$

5.2.4 Proof of Theorem 4.5

We first show ($\alpha = 0$ or $\alpha = 1$ or D singleton) implies No Half Measures. Recall that No Half Measures concerns only events $A \in \Sigma^\Psi$. If D singleton, then $1_{S^\infty} \succ x \succ 1_A$ implies $\ell^\infty(A) = 0$ for the $\ell \in D$, in which case $1_A \sim^* 1_\emptyset$ is obvious. To see the other cases, observe that $1_{S^\infty} \succ x$ if and only if $u(x) < 1$ and $x \succsim^* 1_A$ if and only if

$$\begin{aligned} & \lambda u(x) + (1 - \lambda) \left[\alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \right] \\ & \geq \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} (\lambda p(A) + (1 - \lambda) \int u(h) dp) + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} (\lambda p(A) + (1 - \lambda) \int u(h) dp) \end{aligned} \quad (5.4)$$

for all $\lambda \in [0, 1]$ and all acts h . Furthermore, $1_A \sim^* 1_\emptyset$ if and only if

$$\begin{aligned} & \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} (\lambda p(A) + (1 - \lambda) \int u(h) dp) \\ & + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} (\lambda p(A) + (1 - \lambda) \int u(h) dp) \\ & = \alpha \min_{p \in \{\ell^\infty : \ell \in D\}} ((1 - \lambda) \int u(h) dp) + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} ((1 - \lambda) \int u(h) dp) \end{aligned} \quad (5.5)$$

for all $\lambda \in [0, 1]$ and all acts h . In fact, $1_A \sim^* 1_\emptyset$ if and only if $p(A) = 0$ for all $p \in \{\ell^\infty : \ell \in D\}$. To see this last statement, note that the “if” direction is immediate, the “only if” direction follows for $\alpha \neq 1$ by setting $\lambda = 1$ in (5.5), and if $\alpha = 1$, it follows by setting $\lambda = \frac{1}{3}$ and $h = 1_{A^c}$ in (5.5). Therefore, in the presence of the other requirements in the theorem, No Half Measures is equivalent to the statement: For all

$A \in \Sigma^\Psi$, $p(A) = 1$ for some $p \in \{\ell^\infty : \ell \in D\}$ implies there does not exist an $x \in X$ such that $u(x) < 1$ and (5.4) satisfied. Now consider the case $\alpha = 0$ and let $h = 1_A$. Then the right-hand side of (5.4) becomes 1 and therefore greater than any $u(x) < 1$ violating (5.4). Finally consider $\alpha = 1$ and let $\lambda = \frac{1}{3}$ and $h = 1_{A^c}$ in (5.4). Simplifying yields

$$\begin{aligned} & \frac{1}{3}u(x) + \frac{2}{3} \left[\min_{p \in \{\ell^\infty : \ell \in D\}} (1 - p(A)) \right] \\ & \geq \min_{p \in \{\ell^\infty : \ell \in D\}} \left(\frac{1}{3}p(A) + \frac{2}{3}(1 - p(A)) \right) \end{aligned}$$

which simplifies to $u(x) \geq 1$, thus showing No Half Measures holds.

We next show the other direction – that No Half Measures implies ($\alpha = 0$ or $\alpha = 1$ or D singleton). To see this, we assume D non-singleton and $\alpha \in (0, 1)$ and show that No Half Measures is violated.

No Half Measures violated is equivalent, by the arguments above, to, for some $A \in \Sigma^\Psi$, $p(A) = 1$ for some $p \in \{\ell^\infty : \ell \in D\}$ while (5.4) is satisfied for some $u(x) < 1$. To this end, consider $A = \Psi^{-1}(\hat{\ell})$ for some $\hat{p} = \left(\hat{\ell}\right)^\infty \in \{\ell^\infty : \ell \in D\}$. Observe that $\hat{p}(A) = 1$, $p(A) = 0$ for all $p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}$ and $\{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}$ is non-empty (since D non-singleton). For this event A , we will show that (5.4) is satisfied for $u(x) = \frac{1 + \max[\alpha, 1 - \alpha]}{2}$. Note that $\alpha \in (0, 1)$ implies $\max[\alpha, 1 - \alpha] < u(x) < 1$. Rewriting (5.4) using this A and $u(x)$, we want to show

$$\begin{aligned} & \alpha \min \left[\lambda + (1 - \lambda) \int u(h) d\hat{p}, \min_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} (1 - \lambda) \int u(h) dp \right] \\ & + (1 - \alpha) \max \left[\lambda + (1 - \lambda) \int u(h) d\hat{p}, \max_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} (1 - \lambda) \int u(h) dp \right] \\ & \leq \lambda \frac{1 + \max[\alpha, 1 - \alpha]}{2} + (1 - \lambda) \left[\alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \right] \end{aligned} \quad (5.6)$$

holds for all acts h and all $\lambda \in [0, 1]$. If $\lambda = 0$, (5.6) holds with equality. If $\lambda = 1$, (5.6) simplifies to $1 - \alpha \leq \frac{1 + \max[\alpha, 1 - \alpha]}{2}$, which is true. For the remainder of the argument we assume $\lambda \in (0, 1)$. We proceed by considering three mutually exclusive and exhaustive cases: (1) $\hat{p} \notin \arg \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$, (2) $\hat{p} \in \arg \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$ and $\frac{\lambda}{1 - \lambda} + \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \leq \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$ and (3) $\hat{p} \in \arg \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$ and $\frac{\lambda}{1 - \lambda} + \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp > \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$. For each case we show (5.6) holds.

Case (1): If $\hat{p} \notin \arg \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$, then $p(A)$ and $\int u(h) dp$ have a common

minimizer located in $\{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}$ and

$$\begin{aligned}
& \alpha \min \left[\lambda + (1 - \lambda) \int u(h) d\hat{p}, \min_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} (1 - \lambda) \int u(h) dp \right] \\
& + (1 - \alpha) \max \left[\lambda + (1 - \lambda) \int u(h) d\hat{p}, \max_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} (1 - \lambda) \int u(h) dp \right] \\
& = \alpha(1 - \lambda) \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \left(\lambda p(A) + (1 - \lambda) \int u(h) dp \right) \\
& \leq \lambda(1 - \alpha) + (1 - \lambda) \left[\alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \right] \\
& < \lambda \frac{1 + \max[\alpha, 1 - \alpha]}{2} + (1 - \lambda) \left[\alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \right].
\end{aligned}$$

Case (2): If $\hat{p} \in \arg \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$ and $\frac{\lambda}{1 - \lambda} + \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \leq \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$, then, noting that

$$\max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp = \max_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} \int u(h) dp,$$

we have

$$\begin{aligned}
& \alpha \min \left[\lambda + (1 - \lambda) \int u(h) d\hat{p}, \min_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} (1 - \lambda) \int u(h) dp \right] \\
& + (1 - \alpha) \max \left[\lambda + (1 - \lambda) \int u(h) d\hat{p}, \max_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} (1 - \lambda) \int u(h) dp \right] \\
& = \alpha \min \left[\lambda + (1 - \lambda) \int u(h) d\hat{p}, \min_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} (1 - \lambda) \int u(h) dp \right] \\
& + (1 - \alpha)(1 - \lambda) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \\
& \leq \lambda\alpha + (1 - \lambda) \left[\alpha \int u(h) d\hat{p} + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \right] \\
& < \lambda \frac{1 + \max[\alpha, 1 - \alpha]}{2} + (1 - \lambda) \left[\alpha \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp + (1 - \alpha) \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \right].
\end{aligned}$$

Case (3): If $\hat{p} \in \arg \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$ and $\frac{\lambda}{1 - \lambda} + \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp > \max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp$ then

$$\begin{aligned}
& \alpha \min \left[\lambda + (1 - \lambda) \int u(h) d\hat{p}, \min_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} (1 - \lambda) \int u(h) dp \right] \\
& + (1 - \alpha) \max \left[\lambda + (1 - \lambda) \int u(h) d\hat{p}, \max_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} (1 - \lambda) \int u(h) dp \right] \\
& = \alpha(1 - \lambda) \min_{p \in \{\ell^\infty : \ell \in D\} \setminus \{\hat{p}\}} \int u(h) dp + (1 - \alpha) \left(\lambda + (1 - \lambda) \int u(h) d\hat{p} \right)
\end{aligned}$$

$$\begin{aligned}
&= \lambda(1 - \alpha) + (1 - \lambda) \left[\alpha \left(\min_{p \in \{\ell^\infty: \ell \in D\} \setminus \{\hat{p}\}} \int u(h) dp \right. \right. \\
&\quad + \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp - \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \Big) \\
&\quad + (1 - \alpha) \left(\min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right. \\
&\quad \left. \left. + \max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp - \max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right) \right] \\
&= \lambda(1 - \alpha) \\
&\quad + (1 - \lambda) \left[\alpha \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp + (1 - \alpha) \max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right] \\
&\quad + (1 - \lambda) \left[(\alpha - 1) \left(\max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp - \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right) \right. \\
&\quad \left. + \alpha \left(\min_{p \in \{\ell^\infty: \ell \in D \setminus \{\hat{p}\}\}} \int u(h) dp - \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right) \right] \\
&\leq \lambda(1 - \alpha) + (1 - \lambda) \left[\alpha \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp + (1 - \alpha) \max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right] \\
&\quad + (1 - \lambda)(2\alpha - 1) \left(\max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp - \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right) \\
&= \lambda \frac{1 + \max[\alpha, 1 - \alpha]}{2} \\
&\quad + (1 - \lambda) \left[\alpha \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp + (1 - \alpha) \max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right] \\
&\quad + \lambda \left(1 - \alpha - \frac{1 + \max[\alpha, 1 - \alpha]}{2} \right) \\
&\quad + (1 - \lambda)(2\alpha - 1) \left(\max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp - \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right).
\end{aligned}$$

To complete Case (3), we must show that the last two terms in the final expression above have a non-positive sum. Observe that the hypothesis of Case (3) implies

$$0 \leq \left(\max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp - \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp \right) < \frac{\lambda}{1 - \lambda}.$$

Since $(\max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp - \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp)$ enters linearly, the sum of the last two terms is bounded above by the maximum of the sum when substituting 0 for $(\max_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp - \min_{p \in \{\ell^\infty: \ell \in D\}} \int u(h) dp)$ and the sum when substituting

$\frac{\lambda}{1-\lambda}$. Carrying out these substitutions and simplifying yields

$$\begin{aligned}
& \lambda \left(1 - \alpha - \frac{1 + \max[\alpha, 1 - \alpha]}{2} \right) \\
& + (1 - \lambda)(2\alpha - 1) \left(\max_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp - \min_{p \in \{\ell^\infty : \ell \in D\}} \int u(h) dp \right) \\
& \leq \max \left[\lambda \left(1 - \alpha - \frac{1 + \max[\alpha, 1 - \alpha]}{2} \right), \lambda \left(\alpha - \frac{1 + \max[\alpha, 1 - \alpha]}{2} \right) \right] \\
& = \lambda \left(\frac{\max[\alpha, 1 - \alpha] - 1}{2} \right) < 0.
\end{aligned}$$

5.2.5 Proof of Theorem 4.6

Normalize $u(x^*) = 1$ and $u(x_*) = 0$.

(if): Suppose $\sup \bigcup_{r, t \in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$. No Half Measures says that for $A \in \Sigma^\Psi$, if there exists an $x \in X$ such that $1_{S^\infty} \succ x \succ^* 1_A$, then $1_A \sim^* 1_\emptyset$. We prove this by showing that, for $A \in \Sigma^\Psi$, $1_A \approx^* 1_\emptyset$ implies there does not exist an $x \in X$ such that $1_{S^\infty} \succ x \succ^* 1_A$. Suppose $1_A \approx^* 1_\emptyset$. In terms of the representation, using the definition of \succ^* , this nonindifference may be written as follows:

$$\int_{\Delta(S)} \phi \left(\lambda \ell^\infty(A) + (1 - \lambda) \int_{S^\infty} u(h) d\ell^\infty \right) d\mu(\ell) \neq \int_{\Delta(S)} \phi \left((1 - \lambda) \int_{S^\infty} u(h) d\ell^\infty \right) d\mu(\ell)$$

for some $\lambda \in [0, 1]$ and act h . From the facts that ϕ is strictly increasing and $\ell^\infty(A) \in \{0, 1\}$, this is equivalent to the requirement that $\mu(B) > 0$ for the set $B \equiv \{\ell \in \Delta(S) : \ell^\infty(A) = 1\}$. Now we show that $1_{S^\infty} \succ x \succ^* 1_A$ is impossible. Observe that $1_{S^\infty} \succ x$ means simply $u(x) < 1$. Similarly, $x \succ^* 1_A$ if and only if

$$\int_{\Delta(S)} \phi \left(\lambda u(x) + (1 - \lambda) \int_{S^\infty} u(h) d\ell^\infty \right) d\mu(\ell) \geq \int_{\Delta(S)} \phi \left(\lambda \ell^\infty(A) + (1 - \lambda) \int_{S^\infty} u(h) d\ell^\infty \right) d\mu(\ell) \quad (5.7)$$

for all $\lambda \in [0, 1]$ and all acts h . Notice that when $\lambda = 0$, (5.7) is automatically satisfied with equality. Therefore a necessary condition for $x \succ^* 1_A$ to hold is that when both sides of (5.7) are differentiated with respect to λ and evaluated at $\lambda = 0$ the derivative on the left-hand side is weakly larger than that on the right-hand side. Doing this differentiation and evaluating at $\lambda = 0$ yields

$$\begin{aligned}
& \int_{\Delta(S)} \left(u(x) - \int_{S^\infty} u(h) d\ell^\infty \right) \phi' \left(\int_{S^\infty} u(h) d\ell^\infty \right) d\mu(\ell) \\
& \geq \int_{\Delta(S)} \left(\ell^\infty(A) - \int_{S^\infty} u(h) d\ell^\infty \right) \phi' \left(\int_{S^\infty} u(h) d\ell^\infty \right) d\mu(\ell),
\end{aligned}$$

which simplifies to

$$u(x) \int_{\Delta(S)} \phi' \left(\int_{S^\infty} u(h) d\ell^\infty \right) d\mu(\ell) \geq \int_{\Delta(S)} \ell^\infty(A) \phi' \left(\int_{S^\infty} u(h) d\ell^\infty \right) d\mu(\ell). \quad (5.8)$$

Consider acts h that are bets on A (or A^c) such that $u(h) = a_A b$ for $a, b \in u(X)$. Such acts exist for any $a, b \in u(X)$. For such an h , (5.8) simplifies to

$$u(x) [\phi'(a) \mu(B) + \phi'(b) (1 - \mu(B))] \geq \phi'(a) \mu(B). \quad (5.9)$$

Since $\mu(B) > 0$ and $\sup \bigcup_{r, t \in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$, we can choose a and b to make $\frac{\phi'(a)}{\phi'(b)}$ as large as desired, leading the right-hand side of (5.9) to be as close as desired to 1 and violating the inequality (since $u(x) < 1$). This completes the “if” direction of the proof.

(only if): We now show the converse under the assumption that $\text{supp } \mu$ is non-singleton (if $\text{supp } \mu$ is a singleton, then properties of ϕ beyond the fact that it is strictly increasing cannot affect preferences). By the proof of KMS [21, Theorem 4.3] and the fact that $\text{supp } \mu$ is finite, \succsim satisfies Symmetry, Mixture Continuity of \succsim^* and Monotone Continuity of \succsim^* . Suppose No Half Measures holds. By Theorem 4.2, $C = \{\int \ell^\infty d\mu(\ell) : \ell \in \Delta(R)\}$. By KMS [21, Theorem 4.3], $R = \text{supp } \mu$. Therefore $C = \{\int \ell^\infty d\mu(\ell) : \ell \in \Delta(\text{supp } \mu)\}$. Fix any $\ell_1 \in \text{supp } \mu$. Note that $\ell_1^\infty \in C$ and is an extreme point of C . By the characterization of C from Proposition 17 of Ghirardato and Siniscalchi [15], simplified using the i.i.d. structure and the continuous differentiability of ϕ ,

$$C = \overline{co} \left(\left\{ \frac{\int \phi'(\int e d\ell^\infty) \ell^\infty d\mu(\ell)}{\int \phi'(\int e d\ell^\infty) d\mu(\ell)} : \ell \in \Delta(S), e \in \text{int } B_b(\Sigma, u(X)) \right\} \right),$$

where $\text{int } B_b(\Sigma, u(X))$ is the interior of the set of all Σ -measurable functions $a : S^\infty \rightarrow \mathbb{R}$ for which there exist $\alpha, \beta \in u(X)$ satisfying $\alpha \geq a(\omega) \geq \beta$ for all $\omega \in S^\infty$ (i.e., informally, the interior of the set of bounded utility-acts). Since $\ell_1^\infty \in C$ and is an extreme point of C , it must be that there exists a sequence of measures in $\left\{ \frac{\int \phi'(\int e d\ell^\infty) \ell^\infty d\mu(\ell)}{\int \phi'(\int e d\ell^\infty) d\mu(\ell)} : \ell \in \Delta(S), e \in \text{int } B_b(\Sigma, u(X)) \right\}$ converging to ℓ_1^∞ . If $L \subseteq \Delta(S)$ is an open set containing ℓ_1 , $\mu(L) > 0$. Since $\text{supp } \mu$ is non-singleton, there exist open sets containing ℓ_1 such that $\mu(L) < 1$. Let \hat{L} be such a set and consider the event $A = \Psi^{-1}(\hat{L})$. Suppose $\sup \bigcup_{r, t \in u(X)} \frac{\phi'(t)}{\phi'(r)} = K < +\infty$. Then

$$\begin{aligned} & \sup \left\{ \frac{\int \phi'(\int e d\ell^\infty) \ell^\infty(A) d\mu(\ell)}{\int \phi'(\int e d\ell^\infty) d\mu(\ell)} : \ell \in \Delta(S), e \in \text{int } B_b(\Sigma, u(X)) \right\} \\ &= \sup \left\{ \frac{\int_{\hat{L}} \phi'(\int e d\ell^\infty) d\mu(\ell)}{\int_{\hat{L}} \phi'(\int e d\ell^\infty) d\mu(\ell) + \int_{\Delta(S) \setminus \hat{L}} \phi'(\int e d\ell^\infty) d\mu(\ell)} : \ell \in \Delta(S), e \in \text{int } B_b(\Sigma, u(X)) \right\} \\ &\leq \frac{K}{K+1} < 1 = \ell_1^\infty(A) \end{aligned}$$

so that no sequence in $\left\{ \frac{\int \phi'(\int e d\ell^\infty) \ell^\infty(A) d\mu(\ell)}{\int \phi'(\int e d\ell^\infty) d\mu(\ell)} : \ell \in \Delta(S), e \in \text{int } B_b(\Sigma, u(X)) \right\}$ can converge to ℓ_1^∞ , a contradiction. Thus, $\sup \bigcup_{r, t \in u(X)} \frac{\phi'(t)}{\phi'(r)} = +\infty$, completing the proof.

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