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Macroeconomic and Stock Market Interactions with Endogenous Aggregate Sentiment Dynamics

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1 **Macroeconomic and Stock Market Interactions with Endogenous**
2 **Aggregate Sentiment Dynamics**

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12 **Abstract**

13 This paper studies the implications of heterogeneous capital gain expectations on output and
14 asset prices. We consider a disequilibrium macroeconomic model where agents' expectations on
15 future capital gains affect aggregate demand. Agents' beliefs take two forms – fundamentalist
16 and chartist – and the relative weight of the two types of agents is endogenously determined. We
17 show that there are two sources of instability arising from the interaction of the financial with the
18 real part of the economy, and from the heterogeneous opinion dynamics. Two main conclusions
19 are derived. On the one hand, perhaps surprisingly, the non-linearity embedded in the opinion
20 dynamics far from the steady state can play a stabilizing role by preventing the economy from
21 moving towards an explosive path. On the other hand, however, real-financial interactions and
22 sentiment dynamics do amplify exogenous shocks and tend to generate persistent fluctuations and
23 the associated welfare losses. We consider alternative policies to mitigate these effects.

24 **Keywords:** Real-financial interactions, heterogeneous expectations, aggregate sentiment dynam-
25 ics, macro-financial instability

26 **JEL classifications:** E12, E24, E32, E44.

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1 Introduction

The way in which the dynamic interaction between stock markets and the macroeconomy has been understood by the economics profession has evolved significantly over the last thirty years. As Shiller (2003) has argued, while the rational representative agent framework and the related Efficient Market Hypothesis represented the dominant theoretical modeling paradigm in financial economics during the 1970s, the behavioral finance approach has gained increasing ground within the economics community over the last two decades. The main reason for this significant paradigm shift is well known: following Shiller (1981) and LeRoy and Porter (1981), a large number of studies have documented various empirical regularities of financial markets – such as the excess volatility of stock prices – which are clearly inconsistent with the Efficient Market Hypothesis, see e.g. Frankel and Froot (1987, 1990), Shiller (1989), Allen and Taylor (1990), and Brock et al. (1992), among many others. During the 1990s several researchers like Day and Huang (1990), Chiarella (1992), Kirman (1993), Lux (1995) and Brock and Hommes (1998) have developed models of financial markets with heterogeneous agents following the seminal work by Beja and Goldman (1980) in order to explain such empirical regularities. Ever since, financial market models with heterogeneous agents using rule-of-thumb strategies have become central in the behavioral finance literature, see e.g. Chiarella and He (2001, 2003), De Grauwe and Grimaldi (2005), Chiarella et al. (2006), and Dieci and Westerhoff (2010).

The importance of different types of heterogeneity (regarding preferences, risk aversion or available information) and boundedly rational behavior at the micro level for the dynamics of the macroeconomy has also been increasingly acknowledged in macroeconomics (Akerlof, 2002, 2007). In this context, a particularly fruitful new strand of the literature has focused on the consequences of heterogeneous boundedly rational expectations for the dynamics of the macroeconomy and the conduct of economic policy, see e.g. Branch and McGough (2010), Branch and Evans (2011), De Grauwe (2011, 2012), Proaño (2011, 2013), among others. In these studies, the Brock and Hommes (1997) (BH) approach has been the preferred specification for the endogenous switch between alternative heuristics. In contrast, the development of macroeconomic models using the Weidlich-Haag-Lux (WHL) approach (see Weidlich and Haag, 1983 and Lux, 1995) is still in a nascent stage, with Franke (2012), Franke and Ghonghadze (2014), Flaschel et al. (2015), Chiarella et al. (2015) and Lojak (2016) as notable exceptions.

While the WHL and the BH approaches are quite similar in spirit – and similarly close to Keynes' (1936) and Simon's (1957) views on expectations under bounded rationality (see also Kahneman and Tversky, 1973 and Kahneman, 2003) – there is a fundamental difference between them: In the BH approach the variation in the share of agents using a particular heuristic depends on a measure of utility, or forecast accuracy, related to that particular rule of thumb which is thought to be relevant at the microeconomic level. In contrast, in the WHL approach the switch between different heuristics or attitudes, such as optimism or pessimism, is determined by an aggregate sentiment index composed

63 e.g. by macroeconomic variables describing the state of the economy in the business cycle, see also
64 Franke (2014). The WHL approach thus incorporates an additional link from the macroeconomic
65 environment to microeconomic decision-making based on psychological grounds and on Keynes' notion
66 that "Knowing that our own individual judgment is worthless, we endeavor to fall back on the judgment
67 of the rest of the world which is perhaps better informed. That is, we endeavor to conform with the
68 behavior of the majority or the average. The psychology of a society of individuals each of whom is
69 endeavoring to copy the others leads to what we may strictly term a *conventional* judgment." (Keynes,
70 1937, p. 114; his emphasis).¹

71 In this latter line of research the main contribution of this paper is to study the effects of aggregate
72 sentiments in stock markets on the real economy using the WHL approach to model the expectations
73 formation process in stock markets. More specifically, we incorporate aggregate sentiment dynamics
74 in a stock market populated by heterogeneous agents, and examine the effects of herding and spec-
75 ulative behavior in combination with real-financial market interactions. We adopt the distinction
76 between *chartists* and *fundamentalists* which may be a key ingredient to explain bubbles as argued
77 by Brunnermeier (2008). *Ceteris paribus*, chartists tend to exert a destabilizing influence on the price
78 of financial assets, whereas the presence of fundamentalists is stabilizing.

79 In spite of its simplicity, our model features a variety of interesting aspects. The presence of
80 self-reinforcing mechanisms in the aggregate dynamics allows for the existence of nontrivial multiple
81 equilibria. In the economy, there are two sources of instability deriving from the feedback effects
82 between real and financial markets via Tobin's q (as in Blanchard's 1981 seminal model) *and* from the
83 endogenous aggregate sentiment dynamics produced by the interaction of heterogeneous agents in the
84 stock markets. We prove that the dynamical system describing the evolution of the economy always has
85 either a single steady state (with uniformly distributed agents) or three steady states (the equilibrium
86 with uniformly distributed agents, one with a dominance of chartists and one where fundamentalists
87 dominate), but even though various subdynamics of the model can be stable (at either the uniform or
88 the fundamentalist of the three steady states), the complete system may be repelling around all of its
89 equilibria. Given the complexity of the 4D nonlinear system, we use numerical simulations to explore
90 the properties of the economy. Our results show that the dynamical system describing the economy
91 is generally bounded: all trajectories remain in an economically meaningful subset of the state space.
92 In this sense, unfettered markets with possibly accelerating real-financial feedback mechanisms may
93 have some in-built stabilizing mechanism (based on aggregate sentiment dynamics) that prevent the
94 economy from moving along an infeasible path. Nonetheless, real-financial interactions and sentiment
95 dynamics *do* amplify exogenous shocks and may generate persistent fluctuations and the associated
96 welfare losses. Indeed, despite the relatively simple behavior of the subsystem describing the evolution

¹Indeed, the central equation of the WHL approach which describes the dynamics of population shares might be provided from game theoretic foundations along the lines of Brock and Durlauf (2001), Blume and Durlauf (2003) and He et al. (2016). We are grateful to Tony He for pointing this link out to us.

97 of output without heterogeneous beliefs, the dynamics of the complete system can exhibit somewhat
98 irregular fluctuations.

99 Finally, it is worth stressing that, unlike in most of the current macroeconomic literature, our model
100 is based on a dynamic disequilibrium approach in which the evolution of the variables over time is
101 described by gradual adjustment processes, and no equilibrium condition is imposed a priori. This
102 dynamic disequilibrium approach – discussed in detail in Chiarella and Flaschel (2000) and Chiarella
103 et al. (2005) – seems like a natural complement to the behavioral WHL approach to expectation
104 formation, see also Chiarella et al. (2009).

105 The remainder of the paper is organized as follows. In section 2 we lay out the macroeconomic
106 framework. Section 3 derives the main analytical results concerning the dynamics of the economy.
107 Section 4 illustrates the properties of the model by means of numerical simulations. Section 5 analyzes
108 some policy measures. Section 6 concludes, and the proofs of all Propositions are in the Appendix.

109 2 The Model

110 2.1 Core Real-Financial Interactions

111 We consider a closed economy consisting of households, firms and a monetary authority. We assume
112 that households are the sole owners of the firms' stocks or equities E which represent claims on the
113 firms' physical capital stock K .

114 Unlike in Chiarella and Flaschel (2000) and Chiarella et al. (2005), we abstract from the “Met-
115 zlerian” inventory accelerator mechanism in the modeling of goods market dynamics² in order to
116 focus on the interaction emerging from a stock market driven by aggregate sentiment dynamics and
117 the macroeconomy. We assume instead that aggregate production evolves according to a dynamic
118 multiplier specification³

$$\dot{Y} = \beta_y(Y^d - Y), \quad (1)$$

119 where Y represents aggregate output, Y^d aggregate demand and $\beta_y > 0$ the speed of adjustment of
120 output to market disequilibrium as in the seminal paper by Blanchard (1981).

121 Let p_e denote the equity price, and p the price of capital goods. The Brainard and Tobin (1968)
122 q ratio is then given by

$$q = p_e E / p K. \quad (2)$$

123 Without loss of generality, we normalize the price of output to one, $p = 1$, and assume further that
124 the horizon of our analysis is sufficiently short as to guarantee that both E and K are constant

²These potentially destabilizing macroeconomic channels arising from the real side of the economy could be however
reincorporated in the present framework in a straightforward manner.

³For any dynamic variable z , \dot{z} denotes its time derivative, \hat{z} its growth rate and z_o its steady state value.

125 magnitudes. We normalize K assuming $K = 1$. As a result, changes in q are determined solely by
 126 changes in p_e . Further, we assume that financial markets dynamics affect the real economy via the
 127 impact of Tobin's q on aggregate demand. Hence, aggregate demand is given by:

$$Y^d = a_y Y + A + a_q(p_e - p_{eo})E, \quad (3)$$

128 where $a_y \in (0, 1)$ is the propensity to spend, A is autonomous expenditure, and $a_q > 0$ measures the
 129 responsiveness of output demand to the difference between the actual value of stocks and their steady
 130 state value p_{eo} . Inserting equation (3) into equation (1) yields

$$\dot{Y} = \beta_y [(a_y - 1)Y + a_q(p_e - p_{eo})E + A]. \quad (4)$$

131 In addition to E , we assume that there are two more financial assets, namely, as is customary,
 132 money M and short-term fix-price bonds B .⁴ For simplicity we assume that the monetary authorities
 133 fix the interest rate on the bonds B at the level r , accommodating the households' excess demand
 134 for money. This allows us to abstract from the traditional interest rate effect on aggregate output so
 135 central in New Neoclassical Consensus models (see e.g. Woodford, 2003) and focus in isolation on the
 136 stock price effects under aggregate sentiment dynamics, as discussed below.

137 Since in our economy profits are assumed to be entirely redistributed to firms' owners (households)
 138 as dividends, the expected return on equity ρ_e^e is

$$\rho_e^e = \frac{bY}{p_e E} + \pi_e^e. \quad (5)$$

139 where $b \geq 0$ is the profit share, $bY/(p_e E)$ is the dividend rate, and π_e^e represents the *average*, or *market*
 140 expectation of future capital gains $\pi_e = \dot{p}_e/p_e$, i.e., the growth rate of equity prices.

141 Finally, we assume that the equity market is imperfect due to information asymmetries, adjustment
 142 costs, and/or institutional restrictions, so that the equity price p_e does not move instantaneously to
 143 clear the market. More specifically, we assume that

$$\hat{p}_e = \beta_e(\rho_e^e - \rho_{eo}^e) = \beta_e \left(\frac{bY}{p_e E} + \pi_e^e - \rho_{eo}^e \right), \quad (6)$$

144 where β_e describes the adjustment speed at which the equity price reacts to discrepancies between the
 145 expected rate of return on equity and its steady state value, ρ_{eo}^e , which is assumed to be a given and
 146 strictly positive parameter in the model. As we will discuss below, while equation (6) seems rather
 147 stylized at first sight, it actually describes a complex mechanism due to the intrinsic nonlinearity of
 148 the dynamics of the capital gain expectations π_e^e .

⁴See Charpe et al. (2011) for an explicit analysis and also for a critique of allowing governments to issue a perfectly liquid asset B , with a given unit price.

149 **2.2 Aggregate Sentiment Dynamics**

150 Based on the empirical findings of Frankel and Froot (1987, 1990) and Allen and Taylor (1990), and
 151 the extensive literature they sparked, we assume that traders in financial markets use various types of
 152 heuristics when forming their expectations about future asset price developments. To be specific, we
 153 assume that traders in the stock market use either a *fundamentalist* rule (denoted by the superscript
 154 f) according to which they expect capital gains to converge back to their long-run-steady state value
 155 (assumed to be zero), i.e.

$$\dot{\pi}_e^{e,f} = \beta_{\pi_e^{e,f}}(0 - \pi_e^e), \quad (7)$$

156 or a *chartist* rule (denoted by c) given by

$$\dot{\pi}_e^{e,c} = \beta_{\pi_e^{e,c}}(\hat{p}_e - \pi_e^e), \quad (8)$$

157 where $\beta_{\pi_e^{e,f}}$ and $\beta_{\pi_e^{e,c}}$ are the speed of adjustment parameters of the two heuristics-based forecasting
 158 rules, respectively.

159 Suppose that at any given time a share $\nu_c \in [0, 1]$ of the population consists of financial market
 160 participants using the chartist rule and a share $\nu_f = 1 - \nu_c$ consists of traders using the fundamentalist
 161 rule. The law of motion of aggregate capital gain expectations can then be expressed as

$$\begin{aligned} \dot{\pi}_e^e &= \nu_c(\beta_{\pi_e^{e,c}}(\hat{p}_e - \pi_e^e)) + (1 - \nu_c)(\beta_{\pi_e^{e,f}}(0 - \pi_e^e)) \\ &= \nu_c\beta_{\pi_e^{e,c}}\hat{p}_e - (\nu_c\beta_{\pi_e^{e,c}} + (1 - \nu_c)\beta_{\pi_e^{e,f}})\pi_e^e. \end{aligned} \quad (9)$$

162 According to this equation the evolution of *aggregate, market-wide* expectations of future capital gains
 163 is given by the weighted average of the *change* of the expectations, or forecasts, resulting from the use
 164 of the fundamentalist or chartist forecasting rule. Further, as the interplay between fundamentalists
 165 and chartists is well understood in the literature (see e.g. Hommes, 2006), we assume in the following
 166 that $\beta_{\pi_e^{e,c}} = \beta_{\pi_e^{e,f}} = \beta_{\pi_e^e}$ for simplicity and in order to focus on other rather new channels which
 167 emerge from the aggregate sentiments dynamics.⁵ Then, the above equation becomes

$$\dot{\pi}_e^e = \beta_{\pi_e^e}(\nu_c\hat{p}_e - \pi_e^e). \quad (10)$$

168 Observe that in equations (7) and (8), both fundamentalists and chartists are assumed to use
 169 aggregate expectations π_e^e as the reference value for the updating of their own expectations. This
 170 specification is meant to reflect Keynes' (1936, p.156) famous view of the stock market as a process of
 171 choosing the most beautiful model in a beauty contest, where the winner is the one who has selected

⁵Further, by assuming that the two heuristics are updated with the same speed or frequency we are able to focus on the implications of the use of the different heuristics *per se*. We think that the latter are more relevant behaviorally and capture the most relevant part of heterogeneity in the stock market.

172 the model who is chosen as the most beautiful by the (relative) majority of players. Winning requires
 173 guessing the views of the other players.

174 We endogenize the variable ν_c by adopting the aggregate sentiment dynamics approach by Weidlich
 175 and Haag (1983) and Lux (1995) as recently reformulated in Franke (2012, 2014), which provides
 176 behavioral microfoundations to agents' attitudes in financial markets. Accordingly, agents decide
 177 whether to take either a chartist, or a fundamentalist stance depending on the current status of the
 178 economy (captured by the key variables Y , p_e), on expectations on the evolution of financial gains
 179 (π_e^e), and – crucially – on the current composition of the market (captured by the variable x , defined
 180 below).

181 Formally, suppose that there are $2N$ agents in the economy. Of these, N_c use the chartist forecasting
 182 rule and N_f use the fundamentalist rule, so that $N_c + N_f = 2N$. Following Franke (2012) we describe
 183 the distribution of chartists and fundamentalists in the market by focusing on the *difference* in the
 184 size of the two groups (normalized by $2N$). To be precise, we define

$$x \equiv \frac{N_c - N_f}{2N}. \quad (11)$$

185 Therefore $x \in [-1, +1]$, $\nu_c = N_c/N = \frac{1+x}{2}$ and $\nu_f = N_f/N = \frac{1-x}{2}$, and $x > 0$ indicates a dominance of
 186 chartists, while $x < 0$ implies a majority of fundamentalists at any given point in time.

187 Let $p^{f \rightarrow c}$ be the transition probability that a fundamentalist becomes a chartist, and likewise for
 188 $p^{c \rightarrow f}$. The change in x depends on the relative size of each population multiplied by the relevant
 189 transition probability. Given the continuous time setting of the present framework, we take the limit
 190 of \dot{x} as the population N becomes very large as in Franke (2012), so that the intrinsic noise from
 191 different realizations at the individual level can be neglected. Then:

$$\dot{x} = (1 - x)p^{f \rightarrow c} - (1 + x)p^{c \rightarrow f}. \quad (12)$$

192 The key behavioral assumption concerns the determinants of transition probabilities: we suppose
 193 that they are determined by a *switching index*, s , which captures the expectations of traders on
 194 market performance. An increase in s raises the probability of a fundamentalist becoming a chartist,
 195 and decreases the probability of a chartist becoming a fundamentalist. More precisely, assuming that
 196 the relative changes of $p^{c \rightarrow f}$ and $p^{f \rightarrow c}$ in response to changes in s are linear and symmetric:

$$p^{f \rightarrow c} = \beta_x \exp(a_x s), \quad (13)$$

$$p^{c \rightarrow f} = \beta_x \exp(-a_x s). \quad (14)$$

198 The switching index depends positively on market composition (capturing the herding component
199 of agents' behavior) and on economic activity; and negatively on deviation of the market value of the
200 capital stock and of the average capital gain expectations from their respective steady state values.
201 As in Franke and Westerhoff (2014), this can be written as:⁶

$$s = s_x x + s_y (Y - Y_o) - s_{p_e} (p_e - p_{eo})^2 - s_{\pi_e^e} (\pi_e^e)^2. \quad (15)$$

202 Deviations of share prices and capital gain expectations from their steady state values tend to
203 favor fundamentalist behavior as doubts concerning the macroeconomic situation become widespread.
204 This can be interpreted as a change in the state of confidence, whereby agents believe that increasing
205 deviations from the steady state eventually become unsustainable.

206 The economy is described by the 4D dynamical system consisting of equations (4), (6), (10), and
207 (12), where ν_c results from equation (11) and $p^{f \rightarrow c}$ and $p^{c \rightarrow f}$ are given by equations (13) and (14),
208 i.e.

$$\dot{Y} = \beta_y [(a_y - 1)Y + a_q (p_e - p_{eo})E + A], \quad (16)$$

$$\dot{p}_e = \beta_e \left(\frac{bY}{p_e E} + \pi_e^e - \rho_{eo}^e \right) p_e, \quad (17)$$

$$\dot{\pi}_e^e = \beta_{\pi_e^e} \left(\frac{1+x}{2} \beta_e \left(\frac{bY}{p_e E} + \pi_e^e - \rho_{eo}^e \right) - \pi_e^e \right), \quad (18)$$

$$\dot{s} = (1-x)\beta_x \exp(a_x s) - (1+x)\beta_x \exp(-a_x s). \quad (19)$$

209 and s is given by equation (15).

210 The model provides a simple but general framework to capture some key real-financial interactions,
211 and the feedback between economic variables and agents' attitudes and expectations.

212 3 Local Stability Analysis

213 Let $\mathbf{z} = (z_1, z_2, \dots, z_n)$. For any dynamical system $\dot{\mathbf{z}} = g(\mathbf{z})$, a steady state is defined as the state in
214 which $\dot{\mathbf{z}} = \mathbf{0}$. Then, it is straightforward to prove the following Lemma:⁷

⁶We adopt a quadratic specification only for the sake of simplicity and expositional clarity. All of our results can be extended to more general switching index functions $s = s(x, Y, p_e, \pi_e^e)$, with $s'_x > 0$, $s'_y > 0$, $s'_{p_e} < 0$, and $s'_{\pi_e^e} < 0$, where s'_i is the derivative of the function $s(\cdot)$ with respect to i .

⁷Recall that the steady state value of the expected return on equity, ρ_{eo}^e , is assumed to be a parameter of the model. Therefore Lemma 1 can be interpreted as identifying a one-parameter *family* of steady states.

215 **Lemma 1** *The dynamical system formed by of equations (16), (17), (18), and (19) always has the*
 216 *following steady state solution:*

$$Y_o = \frac{A}{1 - a_y}, \quad (20)$$

$$p_{eo} = \frac{bA}{(1 - a_y)\rho_{eo}^e E}, \quad (21)$$

$$\pi_{eo}^e = 0, \quad (22)$$

$$x_o = 0. \quad (23)$$

217 While Lemma 1 defines the unique steady state values of the variables Y , p_e and π_e^e , which will
 218 always exist independently of the steady state values of x , it does not rule out the existence of further
 219 steady states which however may arise solely due to the nonlinearity of the population dynamics.

220 In the following, we shall analyze the local stability of various subparts of the model separately.
 221 This exercise allows us to understand the sources of instability (and the stabilizing forces) in the
 222 economy before exploring the complete model by means of numerical simulations.

223 3.1 Core Real-Financial Interactions

224 We begin by analyzing the interaction between the macroeconomy and the stock market under the
 225 assumption of constant capital gains expectations $\pi_e^e = \bar{\pi}_e^e = 0$. This assumption reduces our macroe-
 226 conomic model to a 2D core system formed by equations (16) and (17).⁸

227 **Proposition 1** *The dynamical system formed by equations (16) and (17) has a unique steady state:*

228 $Y_o = \frac{A}{1 - a_y}$ and $p_{eo} = \frac{bA}{(1 - a_y)\rho_{eo}^e E}$ with the following stability conditions:⁹

229 (i) if $\frac{a_q b}{1 - a_y} < \rho_{eo}^e$, then the steady state is (asymptotically) stable;

230 (ii) if $\frac{a_q b}{1 - a_y} > \rho_{eo}^e$, then the steady state is an (unstable) saddle point.

231 In this model, Tobin's q plays a key role in breaking down the dichotomy between the real and
 232 financial components of the economy. An increase in p_e has a positive effect on the rate of change of
 233 output, but a negative effect on the expected return on equity. Similarly, real markets influence asset
 234 markets via the role of output as the main determinant of the rate of profit of firms, and thus of the

⁸The proofs of all Propositions can be found in Appendix A.

⁹Given the fact that this dynamical subsystem is linear, local stability implies also global stability.

235 rate of return on real capital. A higher output level has a positive effect on \hat{p}_e , but a negative effect
 236 on the rate of change of output.¹⁰

237 Proposition 1 concerns the interaction of real and financial adjustment processes and does not
 238 depend on the presence of capital gain expectations, which are introduced next.

239 3.2 Real-Financial Interactions with Constant Heterogeneous Beliefs

240 As a next step, we introduce heterogeneous expectations in the basic 2D macroeconomic model while
 241 assuming agents' attitudes, and thus ν_c , to be exogenously given. This allows us to analyze the
 242 effect of expectations on the dynamics of real financial interactions. Not surprisingly, introducing
 243 heterogeneity in agents' expectations, may play a destabilizing role in the economy.

244 The next Proposition characterizes the dynamics of the 3D model when $\beta_e < 1$.

245 **Proposition 2** Consider the dynamical system formed by equations (16), (17) and (18) and let $\beta_e <$
 246 1. For any $\nu_c \in [0, 1]$, at the steady state given by equations (20)-(22):

247 (i) if $a_q b / (1 - a_y) < \rho_{eo}^e$ then the system is locally (asymptotically) stable,

248 (ii) if $a_q b / (1 - a_y) > \rho_{eo}^e$ then the system is unstable.

249 Observe that Proposition 2 holds for any $\nu_c \in [0, 1]$, and so it provides some important insights
 250 on the dynamics of the system formed by equations (16), (17) and (18). Interestingly, as in the 2D
 251 system, the stability of the steady state depends on the relation between a_q , $b / (1 - a_y)$ and ρ_{eo}^e . In
 252 the case where $\beta_e < 1$ the introduction of heterogeneous expectations (chartist and fundamentalist)
 253 changes neither the number of steady states, nor their stability properties.

254 The validity of Proposition 2 (the irrelevance of the *exogenous* share of chartists and fundamen-
 255 talists in the markets for the stability of the system) depends of course on $\beta_e < 1$. The following
 256 Proposition applies for the case where $\beta_e > 1$:

257 **Proposition 3** Consider the dynamical system formed by equations (16), (17) and (18). Further, let

$$\nu_c^* = \frac{\beta_y(1 - a_y) + \beta_e \rho_{eo}^e + \beta_{\pi_e^e}}{\beta_{\pi_e^e} \beta_e} = \frac{\beta_y(1 - a_y)}{\beta_{\pi_e^e} \beta_e} + \frac{\rho_{eo}^e}{\beta_{\pi_e^e}} + \frac{1}{\beta_e}.$$

¹⁰It is also interesting to consider briefly the dynamics of the model under perfect foresight i.e. $\pi_e^e = \hat{p}_e$, see e.g. Turnovsky (1995). In this case, the population dynamics and a separate law of motion for share price expectations are redundant, and the law of motion of share prices is:

$$\hat{p}_e = \beta_e \left(\frac{bY}{p_e E} + \hat{p}_e - \rho_{eo}^e \right) \iff \hat{p}_e = \frac{\beta_e}{1 - \beta_e} \left(\frac{bY}{p_e E} - \rho_{eo}^e \right).$$

It is straightforward to confirm by a standard local stability analysis that if $\beta_e < 1$, the conditions for local stability of the steady state are the same as those postulated in Proposition 1.

258 Under the assumption that $\beta_e > 1$, if $\nu_c^* \in [0, 1]$ and $\nu_c > \nu_c^*$, then the steady state given by equations
259 (20)-(22) is unstable.

260 According to Proposition 3, if $\beta_e > 1$ and the share of chartists in the market ν_c is beyond the
261 endogenously determined threshold value ν_c^* , the destabilizing influence of the chartists will lead to
262 macroeconomic instability, as higher capital gains expectations will lead to higher share prices and
263 higher output which will in turn translate into higher capital gain expectations. Accordingly, ν_c^*
264 represents an endogenous upper bound on ν_c above which the system loses stability to exogenous
265 shocks. Higher values for β_{π_e} and/or β_e lower ν_c^* , making the whole system more prone to overall
266 instability.

267 The previous analysis has only described the dynamics of the economy in a neighborhood of the
268 steady state characterized by equations (20), (21) and (22). The introduction of aggregate sentiments,
269 and by extension of a varying influence of chartist expectations, is likely to lead to explosive dynam-
270 ics, for instance if either the speed of adjustment in financial markets β_e or the coefficient β_{π_e} are
271 sufficiently high. This explosiveness may be tamed far off the steady state through the activation of
272 nonlinear policy measures or, as we will discuss below, by intrinsic nonlinear changes in behavior, thus
273 ensuring that all trajectories remain within an economically meaningful bounded domain.

274 We will explore the global dynamics of the system with aggregate sentiment dynamics by numerical
275 simulations in section 4 below. In the next section, we explore the possibility that endogenous changes
276 in the agents' populations, ν_c , reduce the influence of chartists far off the steady state and thereby
277 create turning points in the evolution of capital gain expectations.

278 3.3 Real-Financial Interactions with Endogenous Aggregate Sentiments

279 As previously mentioned, while Lemma 1 characterizes a particular steady state solution that al-
280 ways exists, other steady states may also exist for particular parameter constellations. The following
281 proposition focuses on the role of the parameters s_x and a_x for the emergence of multiple steady
282 states.

283 **Proposition 4** Consider the dynamical system formed by equations (16)-(19). If $s_x \leq 1/a_x$ then the
284 steady state given by equations (20)-(23) is unique. If $s_x > 1/a_x$, then there are two additional steady
285 state values for x_o : one characterized by a dominance of fundamentalists, e_f , and one where chartists
286 dominate, e_c .

287 The intuition behind Proposition 4 is captured in Figure 1, which illustrates the number of steady
288 states of x for different values of a_x and s_x . While the steady state is unique if $s_x \leq 1/a_x$, there are
289 multiple steady states if $s_x > 1/a_x$. For example, for $s_x = 2/a_x$, there are three steady states: one

290 with a large prevalence of fundamentalists ($x \approx -1$), one with populations of equal size ($x = 0$), and
 291 one with a large prevalence of chartists ($x \approx 1$).

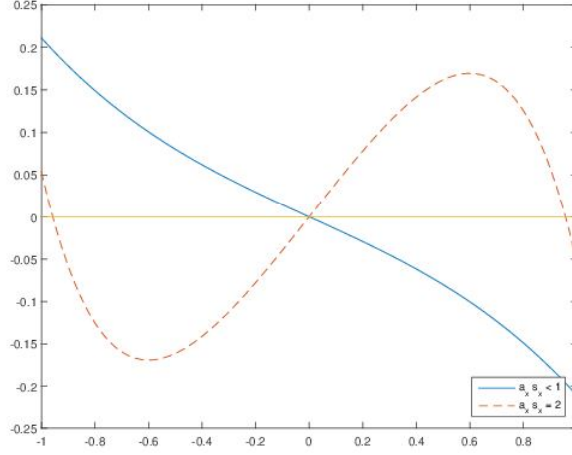


Figure 1: Steady states of population dynamics for different values of a_x and s_x

292 Before analyzing the dynamics of the complete system numerically in the next section, it is inter-
 293 esting to consider the properties of the opinion dynamics and the expectations part of the model in
 294 isolation. We thus assume that output and dividend payments are fixed at their steady state values
 295 Y_o and p_{eo} in the rest of this section. By inserting equations (20) and (21) into (18) we get

$$\dot{\pi}_e^e = \beta \pi_e^e \left[\beta_e \frac{1+x}{2} - 1 \right] \pi_e^e, \quad (24)$$

296 and from equation (15),

$$s = s_x x - s_{\pi_e^e} (\pi_e^e)^2. \quad (25)$$

297 Inserting this expression in equation (19) yields

$$\dot{x} = \beta_x \left[(1-x) \exp(a_x (s_x x - s_{\pi_e^e} (\pi_e^e)^2)) - (1+x) \exp(-a_x (s_x x - s_{\pi_e^e} (\pi_e^e)^2)) \right]. \quad (26)$$

298 A quick glance at equation (24) makes clear that the condition $\dot{\pi}_e^e = 0$ can be fulfilled either when
 299 $\pi_e^e = 0$, or when $\pi_e^e \neq 0$. This means that the multiplicity of steady states arises here not only through
 300 the nonlinear equation (26), as discussed in Proposition 4, but also through equation (24). The next
 301 two Propositions deal with the case with $\pi_{eo}^e = 0$.

302 **Proposition 5** Consider the dynamical system formed by equations (24) and (26). Then:

303 (i) if $s_x \in (0, 1/a_x)$, $e_o = (\pi_{eo}^e, x_o) = (0, 0)$ is the only steady state with $\pi_{eo}^e = 0$;

304 (ii) if $s_x > 1/a_x$, then two additional steady states exist, $e_f = (0, x_o^f)$ and $e_c = (0, x_o^c)$ with $x_o^f < 0$
 305 and $x_o^c > 0$, respectively.

306 In other words, if the aggregate sentiment dynamics display a strong self-reinforcing behavior,
 307 multiple equilibria emerge in which either fundamentalists or chartists dominate. The next Proposition
 308 describes some stability properties of the steady states identified in Proposition 5.

309 **Proposition 6** Consider the dynamical system formed by equations (24) and (26). Then:

310 (i) Let $s_x \in (0, 1/a_x)$. If $\beta_e > 2$, then $e_o = (\pi_{e_o}^e, x_o) = (0, 0)$ is an unstable saddle point. If $\beta_e < 2$,
 311 then e_o is locally asymptotically stable.

312 (ii) Let $s_x > 1/a_x$. The steady state $e_o = (0, 0)$ is unstable. The steady states $e_c = (0, x_o^c)$ and
 313 $e_f = (0, x_o^f)$ are locally asymptotically stable if and only if $(1 + x_o^c)\beta_e < 2$ and $(1 + x_o^f)\beta_e < 2$,
 314 respectively.

315 By Proposition 6, it follows that sentiment dynamics may lead to local instability. This raises
 316 the issue of the global viability of the dynamical system formed by equations (24) and (26). It is
 317 difficult to draw any definite analytical conclusions on this issue and we shall analyze it in detail
 318 by means of numerical methods in the next section. To be sure, opinion dynamics do incorporate
 319 a stabilizing mechanism far off the steady state(s), as x always points inwards at the border of the
 320 x -domain $[-1, 1]$. Yet the global viability of the system will ultimately depend on the properties of
 321 the *interaction* between market expectations and opinion dynamics.

322 Consider, for example, case (i) of Proposition 6 and suppose that $\beta_e > 2$, so that $e_o = (0, 0)$
 323 is unstable. It can be shown that there must be an upper and a lower turning point for π_e^e in the
 324 economically relevant phase space $[-1, 1] \times [-\infty, +\infty]$. For suppose, by way of contradiction, that π_e^e
 325 tends to infinity. By equation (26) it follows that \dot{x} becomes negative and approaches $-\infty$. But then as
 326 x approaches -1 , by equation (24) it follows that $\dot{\pi}_e^e$ becomes negative, which contradicts the starting
 327 assumption. A similar argument rules out the possibility that π_e^e becomes infinitely negative and
 328 therefore there must always be an upper or lower turning point for capital gain inflation or deflation.
 329 This implies that all trajectories stay within a compact subset of the phase space and the interaction
 330 between expectation dynamics and herding mechanism would thus be bounded, if taken by itself.¹¹

It is also worth noting that the dynamical system formed by equations (24) and (26) features two
 additional steady states for the case where $\pi_{e_o}^e \neq 0$, $e_+ = (\pi_{e_o}^+, x_o^+)$ and $e_- = (\pi_{e_o}^-, x_o^-)$, with

$$x_o = \frac{2}{\beta_e} - 1, \quad \text{and} \quad \pi_{e_o}^e = \pm \sqrt{\frac{s_x \left(\frac{2}{\beta_e} - 1 \right) - \ln \left(\frac{1}{\beta_e - 1} \right) / 2a_x}{s\pi_e^e}}.$$

¹¹Given the instability of the steady state, this suggests the existence of a limit cycle.

331 These steady states¹² are locally asymptotically stable if

$$a_x s_x < \frac{1}{1 - x_o^2}.$$

332 4 Numerical Simulations

333 This section examines the properties of the model using numerical simulations.¹³ We first illustrate
 334 the effects of capital gain expectations on the dynamics of Tobin's q using the 3D model comprising
 335 the output equation (16), the share price equation (17) and the capital gains equation (18) and then,
 336 in a second step, investigate the complete 4D dynamical system including the endogenous dynamics
 337 of aggregate sentiments.

Table 1: Baseline Parameter Calibration of the 2D model

Autonomous spending	A	0.128
Profit share	b	0.35
Elasticity of aggregate demand to income	a_y	0.8
Elasticity of aggregate demand to Tobin's q	a_q	0.05
Adjustment speed of Tobin's q	β_e	2
Adjustment speed of output	β_y	2
Parameter in population dynamics	a_x	0.8
Steady state capital stock	K_o	1
Steady state equity stock	E_o	1
Steady state population	x_o	0
Steady state expectations	π_{eo}^e	0
Steady state expected capital return	ρ_{eo}^e	0.14
Steady state output capital ratio	$\frac{Y_o}{K_o}$	0.64
Steady state share price	p_{eo}	1.6

338 The calibration of the 2D model is shown in Table 1. The profit share b is set at 0.35, in line with
 339 the long term average in Karabarounis and Neiman (2014). Based on Bloomberg data from 2000 to
 340 2013, the return on equity (adjusted for R&D spending) is on average 14 percent in the United States,
 341 so we set $\rho_{eo}^e = 0.14$. Brooks and Ueda (2011) argue that Tobin's q has been fluctuating between
 342 1.4 and 1.7 over the period 1990 to 2013. We set its steady state value within this range at 1.6. It
 343 follows that the steady state output capital ratio is $\frac{Y_o}{K_o}$ is 0.64. Mukherjee and Bhattacharya (2010)
 344 estimate that, in 18 OECD countries, the propensity to spend out of income fluctuates between 0.6
 345 and 1.2. We set a_y equal to 0.8. Therefore by equation (20) the autonomous spending component
 346 $A = Y_o(1 - a_y)$ equals 0.128.

¹²For these steady states to be economically meaningful the following conditions must hold: $x_o = \left[\frac{2}{\beta_e} - 1\right] \in [-1, 1]$
 and $2a_x s_x \left(\frac{2}{\beta_e} - 1\right) \geq \ln\left(\frac{1}{\beta_e - 1}\right)$.

¹³The numerical simulation are performed using the SND package (Chiarella et al., 2002).

347 The elasticity of aggregate demand to Tobin's q , a_q , is set equal to 0.05. The dynamic output
 348 multiplier, β_y , and the speed of adjustment of Tobin's q , β_e , are both set equal to 2. Unless otherwise
 349 stated, the experiment considered in this section is a 1 percent shock on output with no auto-regressive
 350 component. All diagrams reporting simulation results display the deviation of variables from their
 351 steady state value in percent, unless otherwise stated.

352 Figure 2 illustrates the dynamic adjustments of the 3D model consisting of the output equation
 353 (16), the share price equation (17) and the capital gains expectations equation (18) for $\beta_{\pi_e} = 0$,
 354 $\beta_{\pi_e} = 0.2$ and $\beta_{\pi_e} = 4$.¹⁴ In all cases, the parameter a_q is small enough (0.05) to ensure that the
 355 determinant is positive, and $\nu_c = 0.5$, which corresponds to $\nu_c = \frac{1+x}{2}$ with $x_o = 0$ in line with the 4D
 356 model calibration presented below.

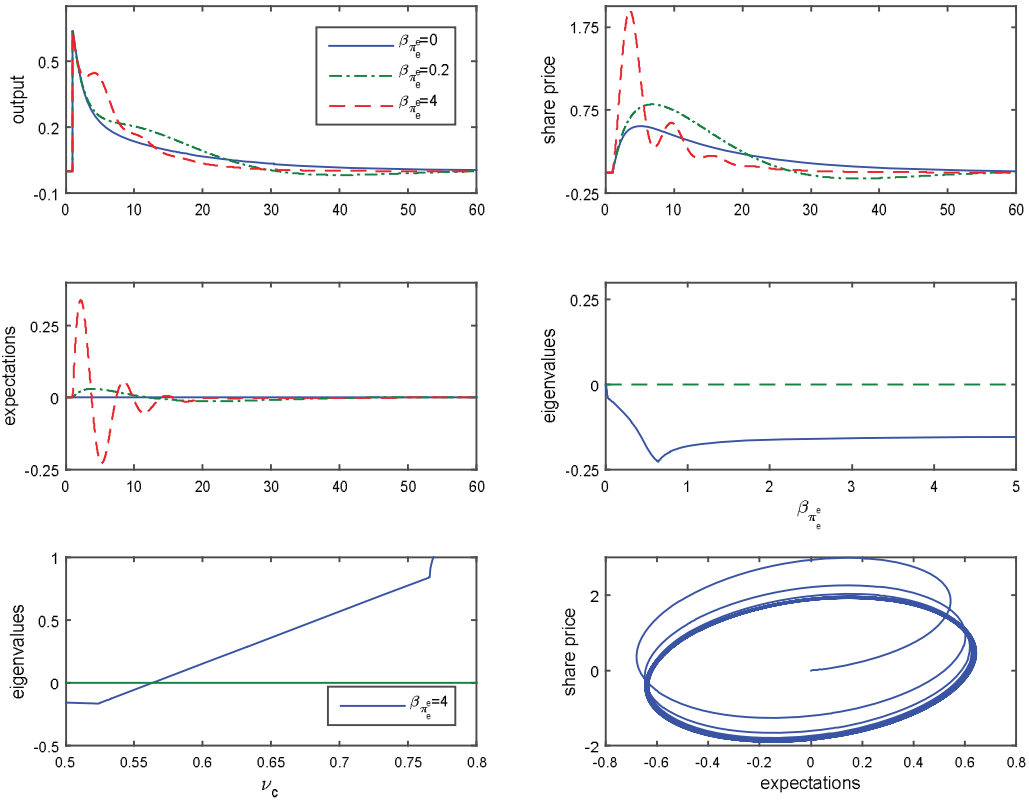


Figure 2: Dynamic responses following a positive one-percent output shock and maximum eigenvalues for the 3D model (Y, p_e, π_e^e) .

357 If $\beta_{\pi_e} = 0$ the dynamics of the system is rather simple: the positive shock on output is followed
 358 by an increase in share price p_e as the expected return on the capital stock ρ_e^e rises. The dynamics
 359 of p_e is hump-shaped as the increase in the share price is modest at the beginning and does not

¹⁴It is worth noting that the simulations based on $\beta_{\pi_e} = 0$ represent the dynamics of the 2D model and are thus related to the analytical stability conditions described in Proposition 1.

360 immediately reduce the return on capital. When the equity price rises enough to lower the return on
361 equity, the economy converges back to its steady state. If $\beta_{\pi_e} = 0.2$ the model displays an oscillatory
362 behavior after the aggregate demand shock due to the activated feedback channel between π_e^e and
363 p_e , as capital gains expectations amplify *both* the increase in the price of equity initiated by a higher
364 return on capital *and* the decline in the price of equity when the rate of return diminishes due to a fall
365 in the price of equities. As the share price p_e undershoots its steady state value it generates further
366 oscillations in aggregate output. These fluctuations are not, however, self-sustaining and the economy
367 returns to the steady state.

368 The dashed red line in Figure 2 corresponds to the case where the speed of adjustment in capital
369 gains expectations β_{π_e} is increased from 0.2 to 4 with $a_q = 0.05$, which implies that the stability
370 conditions in Proposition 2 continue to hold. As the (negative) trace of the corresponding Jacobian
371 matrix declines with β_{π_e} , the model is stable but displays oscillations around the trajectory converging
372 back to the steady state. As shown by the solid blue line in the second row, second column graph,
373 the maximum real part of the eigenvalues is always negative for all values of the speed of adjustment
374 of expectations, β_{π_e} . Raising β_{π_e} increases the amplitude of the fluctuations of the expectations but
375 β_{π_e} has a stabilizing effect on output. Adaptive expectations are inherently stable given the influence
376 of the equity price on the real return on equity. In contrast, the graphs in the third row of Figure 2
377 highlight the importance of the parameter ν_c for the stability of the 3D model (Y, p_e, π_e^e) as discussed
378 in Proposition 3. In the left panel of the third row, the maximum real part of the eigenvalues turns
379 positive for values of ν_c strictly larger than 0.56. Increasing the value of ν_c at 0.56 while keeping
380 $\beta_{\pi_e} = 4$ produces self-sustaining oscillations of the model, as shown in the right panel of this figure.¹⁵

381 Figure 3 illustrates the case of multiple steady states described at the end of section 3 for the
382 subsystem (π_e^e, x) where the steady state for expectations and population are different from zero. In
383 the upper two panels we set $\beta_e = 1.15$, $s_x = 1.5$ and $a_x = 1$ (so that $s_x > 1/a_x$), which implies
384 $x_o = \frac{2}{\beta_e} - 1 = 0.74$ and $\pi_{eo}^e = 0.57$. Following a positive shock on the population variable x , the
385 population dynamics fluctuates around its steady state value following dampening oscillations. In this
386 case, the prevalence of chartist expectations (as $x_o = 0.74 > 0$) does not lead to explosive dynamics
387 due to the relatively slow adjustment in the price of shares. On the contrary, as illustrated in the
388 two lower panels in Figure 3, increasing the speed at which the price of shares adjusts, $\beta_e = 1.5$,
389 makes the steady state $e_+ = (\pi_{eo}^+, x_o^+)$ locally unstable. Following the shock, the population features
390 an explosive oscillatory dynamic response until the excess volatility in the financial markets leads
391 agents to switch towards fundamentalist expectations. The economy then converges towards a stable
392 equilibrium dominated by fundamentalists where capital gains expectations are zero.

393 The next simulation in Figure 4 considers the influence of the aggregate sentiment dynamics on
394 the price of capital and the financial multiplier by setting $\beta_x = 0.75$. The choice of $a_x = 0.8$ and

¹⁵Given the parametrization of the model, while the value of ν_c^* is 0.585, the cut-off value for instability is 0.5635. These values corroborate Proposition 3 as identifying a *sufficient* condition for local instability.

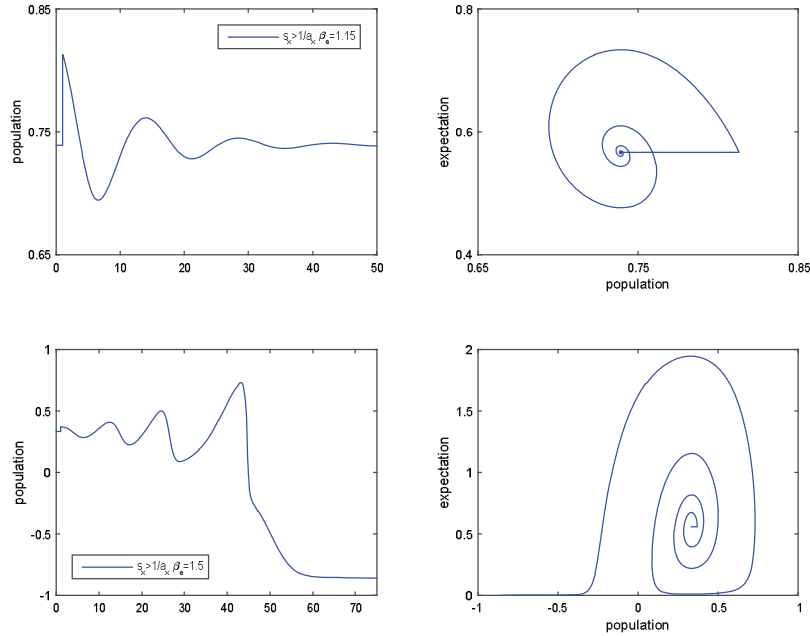


Figure 3: Dynamic response for the 2D model (π_e^e, x) following a positive shock on the population dynamics in the multiple (non-zero) steady state case.

395 $s_x = 0.8$ corresponds to the case of a unique steady state with $x_o = 0$ for the relative population of
 396 fundamentalists and chartists. We now set $s_y = 20$ in order to incorporate the impact of real economic
 397 activity on the aggregate sentiments of the agents. As a first step, we focus on a linear version of the
 398 opinion switching index abstracting from the influence of price and capital gains volatility by setting
 399 $s_{p_e} = s_{\pi_e} = 0$ (we analyze the general case with $s_{p_e} \neq 0$ and $s_{\pi_e} \neq 0$ in Figure 7 below). The rest of
 400 the parameters are similar to those of the dashed green line in Figure 2 ($\beta_{\pi_e} = 4$). Figure 4 compares
 401 the 3D model just discussed (solid blue line) with the 4D model (green line).

402 As Figure 4 clearly shows, the addition of the population dynamics generates larger fluctuations
 403 in output and equity prices. Following a positive output shock, the increase in chartist population
 404 further raises capital gain expectations, which further increases the expected returns on equity and
 405 the demand for equity. The dashed-dotted red line corresponds to the 4D model where the self-
 406 reference parameter s_x in the aggregate sentiment index is increased from 0.8 to 1. This value of s_x
 407 still generates a unique steady state ($x_o = 0$) of the population variable. But the population dynamics
 408 now exhibits larger fluctuations between -0.2 and 0.3. These larger fluctuations translate into wider
 409 oscillations in capital gains expectations, share prices, and economic activity, with the reversal of
 410 expectations towards fundamentalism generating a decline in output by 6 percent.

411 Given that the stability conditions cannot be derived analytically for the 4D model, the interpreta-
 412 tion of the numerical simulations is indicative only. In order to interpret them recall that Proposition 6

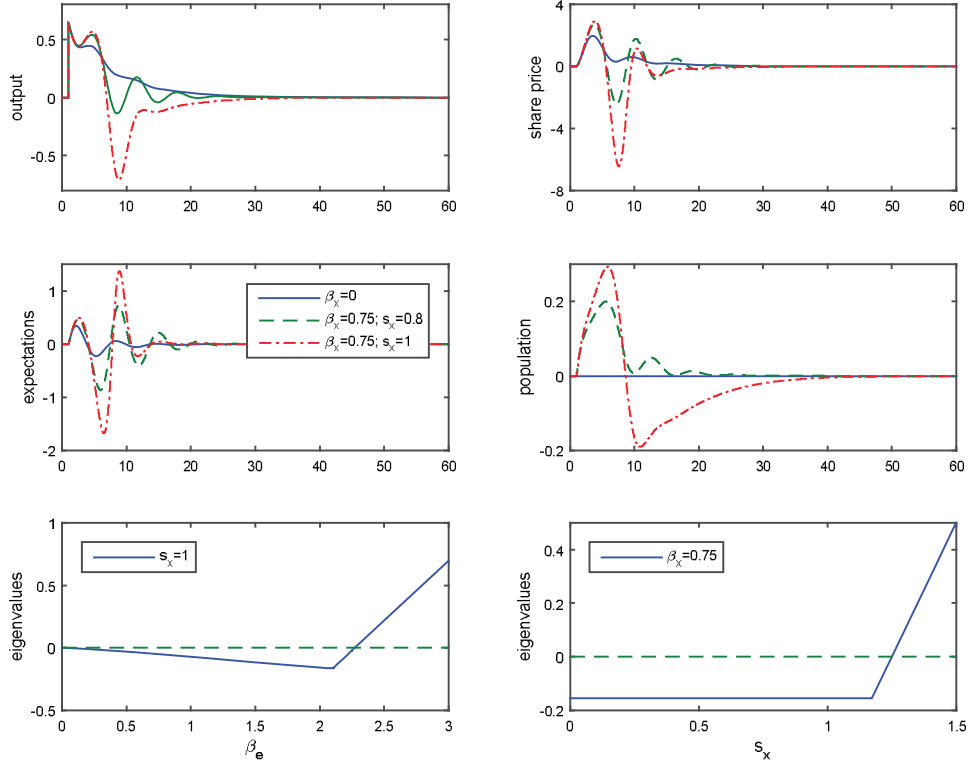


Figure 4: Dynamic adjustments to a one percent output shock in the 3D model (Y, p_e, π_e^e) and the 4D model (Y, p_e, π_e^e, x) (first two rows) and maximum eigenvalue diagrams (last row)

413 stated that the 2D model formed by equations (24) and (26) has a unique steady state if $s_x \in (0, 1/a_x)$
414 and is stable if $\beta_e < 2$. Similarly, as shown in section 3.2 above, the value of β_e affects the stability
415 of the 3D dynamical system formed by equations (16)-(18). This suggests that the parameter β_e may
416 play a key role in determining the stability properties of the whole system. The left figure of the third
417 panel in Figure 4 confirms this intuition: it plots the maximum real part of the eigenvalues of the
418 system around the steady state with $x_o = 0$ with respect to different values of β_e . The maximum
419 real part of the eigenvalues turns positive for β_e larger than 2.3, indicating that the 4D model loses
420 stability for large values of β_e . Comparably, the right panel of the third row displays the maximum
421 real part of the eigenvalues of the system around the steady state with $x_o = 0$ for s_x varying between
422 0 and 1.5. In line with the previous simulation, the system is stable when s_x is smaller than 1.25.
423 The system of equations has a unique steady state towards which the economy converges.

424 Next we analyze the dynamics of the 4D model assuming $s_{p_e} = s_{\pi_e^e} = 0$ with $s_x = 1.5$. Given
425 $a_x = 0.8$, these parameter values lead to the existence of three steady states, as discussed in Proposition
426 4. In this case, a negative shock on output steers the population dynamics towards a steady state
427 dominated by fundamentalists at $x_o = -0.65$ as illustrated in Figure 5. Given the parametrization

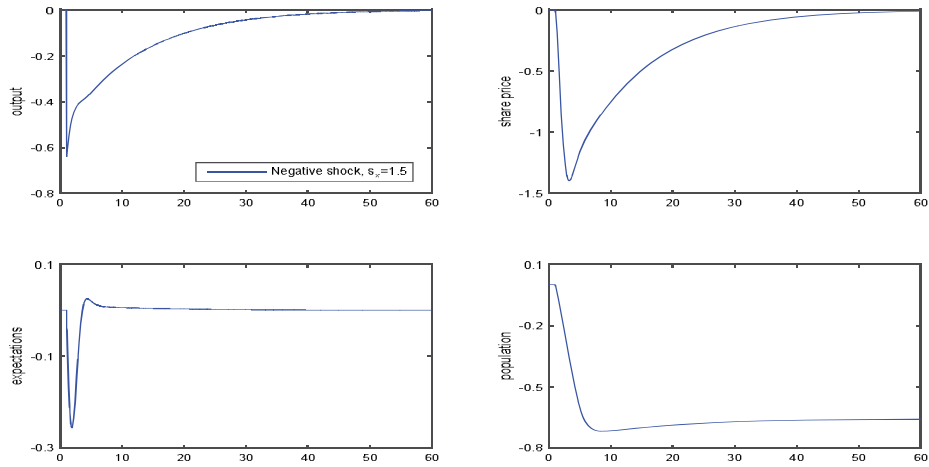


Figure 5: Dynamic adjustments to a negative one percent output shock in the 4D model.

428 of this simulation, output and share prices converge back to their corresponding steady states in a
 429 monotonic manner.

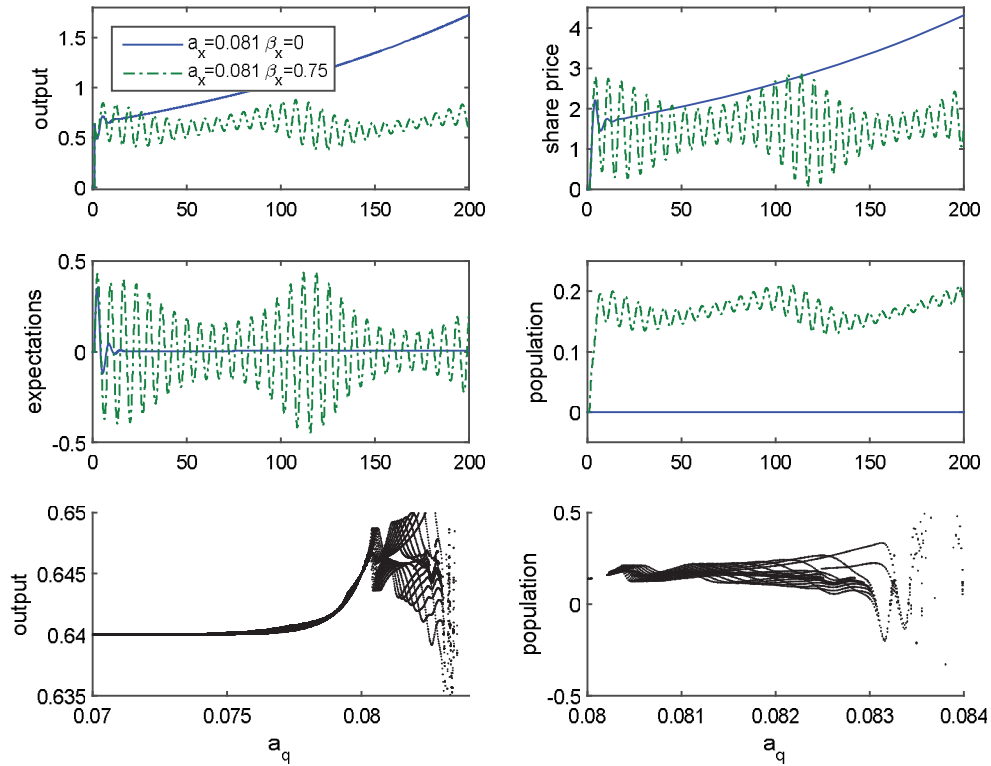


Figure 6: Explosive dynamics in the 3D model (Y, p_e, π_e^e) versus bounded dynamics in the 4D model (Y, p_e, π_e^e, x) .

430 While the aggregate sentiment dynamics tends to amplify financial instability in the proximity of
431 the steady state, the non-linearity embedded in the population dynamics generates forces that keep the
432 aggregate fluctuations within viable boundaries. Figure 6 illustrates how global stability is generated
433 by the sentiment dynamics. The solid blue line corresponds to the 3D model presented in Figure 2
434 with the parameter a_q (which represents the sensitivity of output to Tobin's q) increased from 0.05 to
435 0.081. For a value of $a_q = 0.081$, the 3D model is unstable as illustrated by the monotonically explosive
436 trajectory of output and of the price of equities in the top row, and of the capital gain expectations in
437 the left panel in the second row.¹⁶ The instability is located in the financial sector and arises because
438 of a positive feedback between the rate of return on equity, the price of equity, and its accelerator effect
439 on the real economy. The dashed line corresponds to the case where the 3D model is augmented by
440 aggregate sentiment dynamics with $\beta_x = 0.75$, $s_x = 0.8$, $s_y = 12.5$ and $s_{p_e} = s_{\pi_e} = 0$. The economy
441 does not display an explosive behavior now, being characterized instead by bounded cycles with high
442 frequency oscillations taking place around lower frequency fluctuations. The non-linearity embedded
443 in the sentiment dynamics sets an upper and a lower bound to the amplitude of the cycles. The lower
444 two panels plot the bifurcation diagrams for output and the relative size of the two populations for
445 $a_q \in [0.07; 0.084]$. The diagram shows the Hopf bifurcation for $a_q = 0.08$, beyond which the model
446 displays oscillations.

447 As already mentioned, the simulations of the 4D model shown in Figures 4 through 6 have all
448 considered a linear version of the sentiment switching index with s_{p_e} and s_{π_e} equal to zero in equation
449 (15). In Figure 7, we consider the case where the opinion switching index depends negatively on
450 the volatility of capital gain expectations and of the share price. As the graphs in Figure 7 show,
451 the activation of these nonlinear terms does modify the dynamics of the model. When the sentiment
452 switching index also depends on these two volatility terms, there is a coordination in the expectations of
453 financial market agents towards fundamentalism. We illustrate this emergent feature by the following
454 two examples.

455 The first example corresponds to the case where $\beta_e = 0.75$ and $s_x = 1$ and is illustrated in the
456 upper panels of Figure 7. Therein the blue line corresponds to the 4D model of Figure 4 with a linear
457 switching index specification ($s_{p_e} = s_{\pi_e} = 0$), while the green line corresponds to the case where the
458 switching index contains also nonlinear terms ($s_{p_e} = s_{\pi_e} = 20$), both with $\beta_e = 0.75$ and $s_x = 1$. As
459 it can be clearly observed, the extent of the dynamic reaction of the full nonlinear 4D model following
460 a positive output shock is smaller than the reaction of the 4D model with a linear switching index, as
461 the volatility in share price and capital gain expectations reduces the fluctuations in the population
462 dynamics.

463 The second example corresponds to the dynamically explosive case discussed for the 3D model
464 in Figure 6 and is illustrated in the lower panels of Figure 7. Therein, the blue line corresponds

¹⁶The scale of the graph gives the impression that π_e returns to its initial steady state value, but in fact it diverges, too, albeit very slowly.

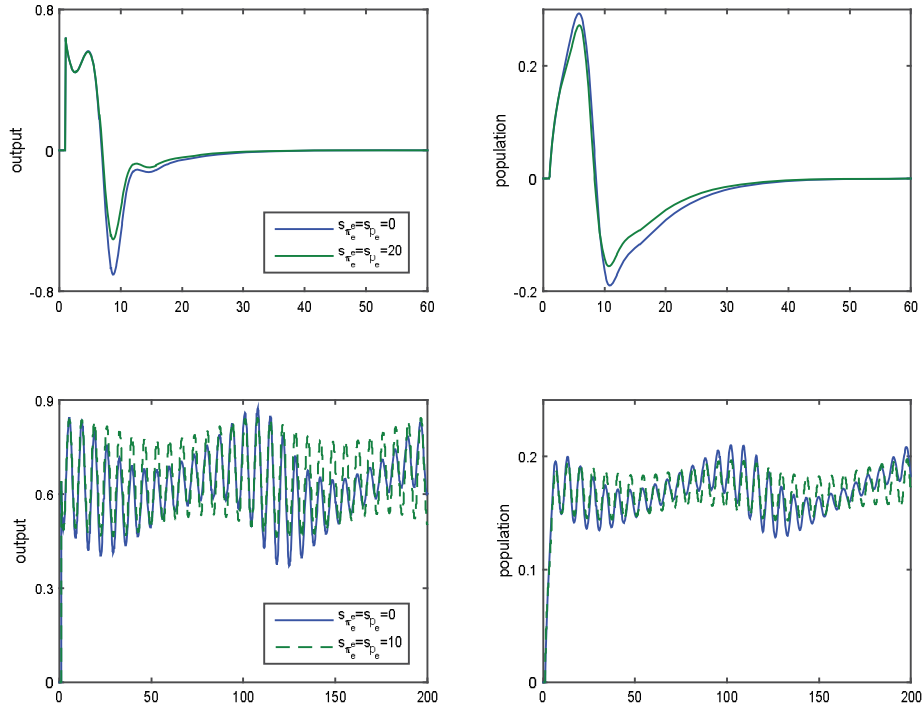


Figure 7: Dynamic adjustments of the full 4D model (Y, p_e, π_e^e, x) for different values of s_{p_e} and s_{π_e} for the dynamically stable case (upper panels) and the explosive case (lower panels).

465 to Figure 6 where the nonlinearity in the population dynamic stabilizes an otherwise explosive 3D
 466 model. More precisely, what characterized the dynamics of the 4D model shown in Figure 6 was that
 467 fluctuations took place along both high and low frequencies. Adding a second type of nonlinearity in
 468 the 4D model via the volatility terms in the sentiment switching index seems to reduce in particular
 469 the amplitude of the low frequency population fluctuations.¹⁷

470 5 Dynamics under Unconventional Monetary Policies

471 The previous numerical analysis showed the ambivalent effects of the interaction between capital gains
 472 expectations and the composition of the population of financial agents on the stability of our model
 473 economy. In this section, we briefly outline some policies that could stabilize both real *and* financial
 474 markets. Two policy proposals immediately come to mind, in the light of the current financial crisis
 475 and the measures adopted to tackle it.

476 Given the economic debate of the last years about a renewed regulation of international financial
 477 markets, it is natural to consider the impact of a tax on capital gains. Taxing finance either via a

¹⁷Appendix B contains additional simulations illustrating the properties of the full model highlighting in particular the possibility of complex dynamics and performing various robustness checks by means of bifurcation diagrams.

478 “Tobin Tax” or by increasing the marginal tax rate on capital is often suggested by policy makers as
 479 a way of curbing financial market instability, see e.g. [Admati and Hellwig \(2013\)](#). A second policy
 480 focuses on the ability of the Central Bank to reduce the pro-cyclicality of the sentiment switching
 481 index by convincing agents that it will act vigorously to prevent bubbles in financial markets. Indeed,
 482 as central banks greatly influence financial markets sentiments beyond the conventional interest rate
 483 policy via their communication policies, the ability of a central banker to coordinate financial traders’
 484 expectations on a stable equilibrium may be crucial in times of financial distress, see e.g. [Siklos and
 485 Sturm \(2013\)](#).

486 In Figure 8, the first two policies are assessed with respect to the dashed-dotted red line which
 487 corresponds to the green line in the top row of Figure 7 generated with $\beta_x = 0.75$ and $s_x = 1$. Further,
 488 we assume $s_{p_e} = s_{\pi_e} = 20$ as in Figure 7 of the previous section. In the following we thus simulate
 489 the impact of various policies in the full 4D model. Taxing capital gains is taken into account by
 490 introducing the tax rate τ_{p_e} in the equation for capital gain expectations (equation (18)).

$$\dot{\pi}_e^e = \beta_{\pi_e^e} \left[(1 - \tau_{p_e}) \left(\frac{1+x}{2} \right) \hat{p}_e - \pi_e^e \right]. \quad (27)$$

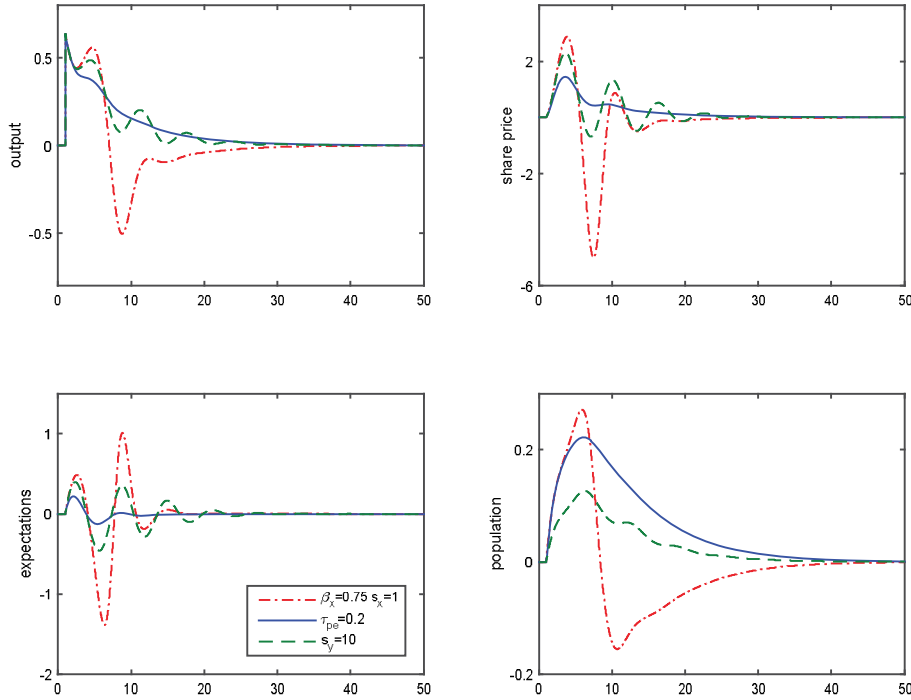


Figure 8: Dynamics under capital gains taxation and central bank communication policy in the full 4D model (Y, p_e, π_e^e, x) .

491 The dynamics illustrated by the continuous blue line was generated assuming a tax rate of 20%.
492 As it can be clearly observed, taxing capital gains has a strong impact on the output dynamics as it
493 almost entirely smooths out output fluctuations, and it also reduces the amplitude of the fluctuations
494 in expectations. A side effect is that the sentiment dynamics now follows a humped-shaped trajectory,
495 rather than an oscillating pattern. As a result, the fluctuations in share prices are much more limited
496 than in the case illustrated in the top row of Figure 7.¹⁸

497 The dashed green lines describe the dynamics of the 4D model under a successful central bank
498 communication policy which modifies the perceptions of financial market participants. We specify
499 this scenario in our stylized framework by a reduction of the sentiment index parameter s_y from 20 to
500 10. This type of policy has a direct impact on the volatility of financial markets and the real sector,
501 and the reduction in s_y translates into a sharp reduction in output fluctuations.

502 6 Conclusions

503 We have studied in this paper a stylized dynamic macroeconomic model of real-financial market
504 interactions with endogenous aggregate sentiment dynamics and heterogenous expectations in the
505 tradition of the Weidlich-Haag-Lux approach as recently reformulated by Franke (2012). Following
506 Blanchard (1981), we focused on the impact of equity prices on macroeconomic activity through the
507 Brainard-Tobin q , leaving the nominal interest rate fixed for the sake of simplicity, and also because
508 goods prices were assumed to be constant.

509 Using this extremely stylized but – due to the intrinsic nonlinear nature of the Weidlich-Haag-Lux
510 approach – complex theoretical framework, we showed that the interaction between real and financial
511 markets need not be necessarily stable, and might well be characterized by multiple equilibria (and even
512 complex dynamics, see Appendix B below). The crucial theoretical, empirical, and policy question,
513 then, is whether unregulated market economies contain some mechanisms ensuring the stability or
514 global boundedness of the economy, or whether centrifugal forces may prevail, making some equilibria
515 locally unstable and, potentially, the whole system globally unstable.

516 Our numerical simulations show that global stability can obtain if, far off the steady state, aggregate
517 sentiment dynamics favor fundamentalist behavior during booms and busts which ensures that there
518 are upper and lower turning points. Yet, both the local analysis and the simulations suggest that
519 market economies can be plagued by severe business fluctuations and recurrent crises. We showed
520 that two policy measures often advocated in the Keynesian literature, namely Tobin-type taxes (here
521 on capital gains), and Central Bank intervention, can mitigate these problems.

¹⁸Actually, the tax τ_{p_e} is not restricted to apply to actual transactions and is imposed on *both* actual *and* notional capital gains. Therefore, rather than a Tobin tax, it may be more appropriately interpreted as a wealth tax of the kind advocated by Piketty (2014). It is therefore quite interesting to note that, in addition to any redistributive effects, such a wealth tax may also help to mitigate business cycles and financial turbulence. We are grateful to Bruce Greenwald for pointing this out to us.

522 Our theoretical framework can be extended in a variety of directions. First, through the incorpo-
523 ration of a varying goods price level and an active conventional interest rate policy, the interaction
524 between macroprudential and conventional policies could be investigated. Also, given the central role
525 of aggregate sentiments and bounded rationality, we may use the model to investigate the efficiency of
526 these policies near or at the zero-lower bound of interest rates. Finally, we could analyze the dynamics
527 of the model under alternative heuristics than the traditional chartist and fundamentalist rules. We
528 intend to pursue some of these alternatives in future research.

References

- Admati, A.R. and M. Hellwig (2013), *The Bankers' New Clothes: What's Wrong with Banking and What to Do about It*, Princeton University Press, Princeton.
- Akerlof, G.A. (2002), 'Behavioral macroeconomics and macroeconomic behavior', *American Economic Review* **92**(3), 411–433.
- Akerlof, G.A. (2007), 'The missing motivation in macroeconomics', *American Economic Review* **97**(1), 3–36.
- Allen, H. and M.P. Taylor (1990), 'Charts, noise and fundamentals in the london foreign exchange market', *Economic Journal* **100**, 49–59.
- Beja, A. and B. Goldman (1980), 'On the dynamic behavior of prices in disequilibrium', *Journal of Finance* **35**, 235–48.
- Blanchard, O. (1981), 'Output, the stock market, and interest rates', *American Economic Review* **71**, 132–143.
- Blume, L. and S. Durlauf (2003), 'Equilibrium concepts for social interaction models', *International Game Theory Review* **5**, 193–209.
- Brainard, W.C. and J. Tobin (1968), 'Pitfalls in financial model building', *American Economic Review* **58**, 99–122.
- Branch, W. A. and B. McGough (2010), 'Dynamic predictor selection in a new keynesian model with heterogeneous expectations', *Journal of Economic Dynamics & Control* **34**, 1492–1508.
- Branch, W. A. and G. W. Evans (2011), 'Monetary policy and heterogeneous expectations', *Economic Theory* **47**, 365–393.
- Brock, W. and C. Hommes (1997), 'A rational route to randomness', *Econometrica* **65**, 1059–1095.
- Brock, W. and C. Hommes (1998), 'Heterogeneous beliefs and routes to chaos in a simple asset pricing model', *Journal of Economic Dynamics and Control* **22**, 1235–1274.
- Brock, W. and S. Durlauf (2001), 'Discrete choice with social interactions', *Review of Economic Studies* **68**, 235–260.
- Brock, W.A., J. Lakonishok and B. LeBaron (1992), 'Simple technical trading rules and the stochastic properties of stock returns', *Journal of Finance* **47**, 1731–1764.
- Brooks, R. and K. Ueda (2011), *User Manual for the Corporate Vulnerability Utility*, 4th edn, International Monetary Fund.

- 559 Brunnermeier, M.K. (2008), Bubbles, *in* S.Durlauf and L.Blume, eds, ‘The New Palgrave Dictionary
560 of Economics’, 2nd edn, Palgrave Macmillan, London.
- 561 Charpe, M., C. Chiarella, P. Flaschel and W. Semmler (2011), *Financial Assets, Debt and Liquidity*
562 *Crises: A Keynesian Approach*, Cambridge University Press, Cambridge.
- 563 Chiarella, C. (1992), ‘The dynamics of speculative behaviour’, *Annals of Operations Research* **37**, 101–
564 123.
- 565 Chiarella, C., C. Di Guilmi and T. Zhi (2015), Modelling the “animal spirits” of bank’s lending
566 behaviour, Working Paper 183, Finance Discipline Group, UTS Business School, University of
567 Technology, Sydney.
- 568 Chiarella, C. and P. Flaschel (2000), *The Dynamics of Keynesian Monetary Growth: Macrofounda-*
569 *tions*, Cambridge University Press, Cambridge.
- 570 Chiarella, C., P. Flaschel, Khomin A. and P. Zhu (2002), *The SND package: Applications to Keynesian*
571 *Monetary Growth Dynamics*, Peter Lang, Frankfurt am Main.
- 572 Chiarella, C., P. Flaschel and R. Franke (2005), *Foundations of a Disequilibrium Theory of the Business*
573 *Cycle*, Cambridge University Press, Cambridge.
- 574 Chiarella, C., P. Flaschel, R. Franke and W. Semmler (2009), *Financial Markets and the Macroecon-*
575 *omy. A Keynesian Perspective*, Routledge, London.
- 576 Chiarella, C., R. Dieci and L. Gardini (2006), ‘Asset price and wealth dynamics in a financial market
577 with heterogeneous agents’, *Journal of Economic Dynamics and Control* **30**, 1755–1786.
- 578 Chiarella, C. and X. He (2001), ‘Asset price and wealth dynamics under heterogeneous expectations’,
579 *Quantitative Finance* **1**, 509–526.
- 580 Chiarella, C. and X. He (2003), ‘Heterogeneous beliefs, risk and learning in a simple asset pricing
581 model with a market maker’, *Macroeconomic Dynamics* **7**, 503–536.
- 582 Day, R.H. and W. Huang (1990), ‘Bulls, bears and market sheep’, *Journal of Economic Behavior &*
583 *Organization* **14**, 299–329.
- 584 De Grauwe, P. (2011), ‘Animal spirits and monetary policy’, *Economic Theory* **47**, 423–457.
- 585 De Grauwe, P. (2012), ‘Booms and busts in economic activity: A behavioral explanation’, *Journal of*
586 *Economic Behavior & Organization* **83**(3), 484–501.
- 587 De Grauwe, P. and M. Grimaldi (2005), ‘Heterogeneity of agents, transaction costs and the exchange
588 rate’, *Journal of Economic Dynamics and Control* **29**, 691–719.

- 589 Dieci, R. and F. Westerhoff (2010), ‘Heterogeneous speculators, endogenous fluctuations and inter-
590 acting markets: A model of stock prices and exchange rates’, *Journal of Economic Dynamics and*
591 *Control* **24**, 743–764.
- 592 Flaschel, P., F. Hartmann, C. Malikane and C.R. Proaño (2015), ‘A behavioral macroeconomic model
593 of exchange rate fluctuations with complex market expectations formation’, *Computational Eco-*
594 *nomics* **45**, 669–691.
- 595 Franke, R. (2012), ‘Microfounded animal spirits in the new macroeconomic consensus’, *Studies in*
596 *Nonlinear Dynamics & Econometrics* **16**.
- 597 Franke, R. (2014), ‘Aggregate sentiment dynamics: A canonical modelling approach and its pleasant
598 nonlinearities’, *Structural Change and Economic Dynamics* **31**, 64–72.
- 599 Franke, R. and F. Westerhoff (2014), ‘Why a simple herding model may generate the stylized facts
600 of daily returns: Explanation and estimation’, *Journal of Economic Interaction and Coordination*
601 **11**, 1–34.
- 602 Franke, R. and J. Ghonghadze (2014), Integrating real sector growth and inflation into an agent-based
603 stock market dynamics, Working Paper 4, FinMaP.
- 604 Frankel, J.A. and K.A. Froot (1987), ‘Using survey data to test standard propositions regarding
605 exchange rate expectations’, *American Economic Review* **77**, 133–153.
- 606 Frankel, J.A. and K.A. Froot (1990), ‘Chartists, fundamentalists, and trading in the foreign exchange
607 market’, *American Economic Review* **80**(2), 181–185.
- 608 He, X., K. Li and L. Shi (2016), Social interaction, stochastic volatility,
609 and momentum. University of Technology Sydney (UTS), mimeo. URL:
610 https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2916490.
- 611 Hommes, C. (2006), Heterogeneous agent models in economics and finance, in L.Tesfatsion and
612 K.Judd, eds, ‘Handbook of Computational Economics, Vol. 2: Agent-Based Computational Eco-
613 nomics’, North-Holland, Amsterdam, pp. 1109–1186.
- 614 Kahneman, D. (2003), ‘Maps of bounded rationality: Psychology for behavioral economics’, *American*
615 *Economic Review* **93**, 1449–1475.
- 616 Kahneman, D. and A. Tversky (1973), ‘On the psychology of prediction’, *Psychological Review*
617 **80**, 237–251.
- 618 Karabarbounis, L. and B. Neiman (2014), ‘The global decline of the labor share’, *The Quarterly*
619 *Journal of Economics* **129**, 61–103.
- 620 Keynes, J.M. (1936), *The General Theory of Unemployment, Interest and Money*, Macmillan, London.

- 621 Keynes, J.M. (1937), ‘The general theory of employment’, *Quarterly Journal of Economics* **51**, 109–
622 123.
- 623 Kirman, A. (1993), ‘Ants, rationality, and recruitment’, *Quarterly Journal of Economics* **108**, 137–56.
- 624 LeRoy, S.F. and R.D. Porter (1981), ‘The present-value relation: Tests based on implied variance
625 bounds’, *Econometrica* **49**, 555–574.
- 626 Lojak, B. (2016), Sentiment-driven investment decisions and the emergence of co-existing business
627 regimes, BERG Working Paper 112, University of Bamberg.
- 628 Lux, T. (1995), ‘Herd behaviour, bubbles and crashes’, *Economic Journal* **105**, 881–889.
- 629 Mukherjee, S. and R. Bhattacharya (2010), Private sector consumption and government consump-
630 tion and debt in advanced economies: An empirical study, Working Paper 10/264, International
631 Monetary Fund.
- 632 Piketty, T. (2014), *Capital in the 21st Century*, Harvard University Press, Cambridge, MA.
- 633 Proaño, C.R. (2011), ‘Exchange rate determination, macroeconomic dynamics and stability under het-
634 erogeneous behavioral FX expectations’, *Journal of Economic Behavior & Organization* **77**(2), 177–
635 188.
- 636 Proaño, C.R. (2013), ‘Monetary policy rules and macroeconomic stabilization in small open economies
637 under behavioral FX trading: Insights from numerical simulations’, *Manchester School* **81**(6), 992–
638 1011.
- 639 Shiller, R.J. (1981), ‘Do stock prices move too much to be justified by subsequent changes in divi-
640 dends?’, *American Economic Review* **71**, 421–436.
- 641 Shiller, R.J. (1989), *Market Volatility*, MIT Press, Cambridge, MA.
- 642 Shiller, R.J. (2003), ‘From efficient market hypothesis to behavioral finance’, *Journal of Economic
643 Perspectives* **17**, 83–104.
- 644 Siklos, P.L. and J.E. Sturm (2013), *Central Bank Communication, Decision Making, and Governance*,
645 MIT Press, Cambridge, MA.
- 646 Simon, H.A. (1957), *Models of Man*, Wiley, New York, NY.
- 647 Turnovsky, S. (1995), *Methods of Macroeconomic Dynamics*, MIT Press, Cambridge, MA.
- 648 Weidlich, W. and G. Haag (1983), *Concepts and Models of a Quantitative Sociology. The Dynamics
649 of Interacting Populations*, Springer, Berlin.
- 650 Woodford, M. (2003), *Interest and Prices*, Princeton University Press, Princeton.

651 **Appendix A**

652 For any matrix J , let $\text{tr}(J)$ be the trace of J and let $|J|$ be its determinant.

653 **Proof of Proposition 1**

At a steady state, the Jacobian matrix J of equations (16) and (17) is:

$$J = \begin{pmatrix} -\beta_y(1 - a_y) & \beta_y a_q E \\ \frac{\beta_e b}{E} & -\beta_e \rho_{eo}^e \end{pmatrix}.$$

It is easy to see that $\text{tr}(J) < 0$. Furthermore, the determinant of J is

$$|J| = \beta_y(1 - a_y)\beta_e \rho_{eo}^e - \frac{\beta_y a_q E \beta_e b}{E}.$$

Therefore $|J| > 0$ if and only if

$$(1 - a_y)\rho_{eo}^e > a_q b.$$

654 Thus, $|J| > 0$ if and only if

$$\rho_{eo}^e > \frac{a_q b}{1 - a_y}. \quad (\text{Q.E.D.})$$

655 **Proof of Proposition 2**

656 For any $\nu_c \in [0, 1]$, at the steady state given by equations (20)-(22), the Jacobian of the 3D system
657 formed of equations (16), (17) and (18) is

$$J = \begin{pmatrix} -\beta_y(1 - a_y) & \beta_y a_q E & 0 \\ \frac{\beta_e b}{E} & -\beta_e \rho_{eo}^e & \beta_e p_{eo} \\ \frac{\beta_{\pi_e} \beta_e \nu_c b}{p_{eo} E} & -\frac{\beta_{\pi_e} \beta_e \nu_c \rho_{eo}^e}{p_{eo}} & \beta_{\pi_e} (\nu_c \beta_e - 1) \end{pmatrix}. \quad (28)$$

658 According to the Routh-Hurwitz theorem, the necessary and sufficient conditions for stability of
659 the system are:

660 (C1) $\text{tr}(J) < 0$;

661 (C2) $J_1 + J_2 + J_3 > 0$, where J_i represents the principal minor of order i of the matrix J ;

662 (C3) $|J| < 0$; and

663 (C4) $B = -\text{tr}(J)(J_1 + J_2 + J_3) + |J| > 0$.

664 Condition (C1) clearly holds. If $a_q < (1 - a_y)\rho_{eo}^e$, then (C2) and, since it can be proved that
 665 $|J| = -\beta_{\pi_e} J_3$, (C3) also hold. As for (C4):

$$\begin{aligned} -\text{tr}(J) (J_1 + J_2 + J_3) &= (\beta_y(1 - a_y) + \beta_e \rho_{eo}^e + \beta_{\pi_e}(\nu_c \beta_e - 1)) \\ &\cdot (\beta_e \rho_{eo}^e \beta_{\pi_e} - \beta_y(1 - a_y) \beta_{\pi_e}(\nu_c \beta_e - 1) + \beta_y(1 - a_y) \beta_e \rho_{eo}^e - \beta_y a_q \beta_e b), \end{aligned}$$

and

$$|J| = -\beta_{\pi_e} \left(\beta_y(1 - a_y) \beta_e \rho_{eo}^e - \frac{\beta_y a_q E_o \beta_e b}{E_o} \right).$$

666 Therefore, simplifying terms, $B > 0$ if and only if

$$\begin{aligned} &[\beta_y(1 - a_y) + \beta_e \rho_{eo}^e - \beta_{\pi_e}(\nu_c \beta_e - 1)] \{ \beta_e \beta_{\pi_e} \rho_{eo}^e - \beta_y(1 - a_y)(\nu_c \beta_e - 1) + \beta_y \beta_e [(1 - a_y) \rho_{eo}^e - a_q b] \} \\ &+ \beta_e \beta_{\pi_e} \beta_y [a_q b - (1 - a_y) \rho_{eo}^e] > 0 \end{aligned}$$

667 or, equivalently, after some straightforward algebra,

$$\begin{aligned} &[\beta_y(1 - a_y) + \beta_e \rho_{eo}^e] \{ \beta_e \beta_{\pi_e} \rho_{eo}^e + \beta_y(1 - a_y)(1 - \nu_c \beta_e) + \beta_y \beta_e [(1 - a_y) \rho_{eo}^e - a_q b] \} + \beta_{\pi_e}(1 - \nu_c \beta_e) \\ &\cdot [\beta_e \beta_{\pi_e} \rho_{eo}^e + \beta_y(1 - a_y)(1 - \nu_c \beta_e)] + \nu_c \beta_e \beta_e \beta_{\pi_e} \beta_y a_q b - \nu_c \beta_e \beta_e \beta_{\pi_e} \beta_y(1 - a_y) \rho_{eo}^e > 0 \end{aligned}$$

668

669 Note that if $1 > \beta_e$ and $(1 - a_y) \rho_{eo}^e > a_q b$ then all terms in the previous expression except for the
 670 last one are strictly positive. Then in order to prove that the desired inequality holds it suffices to
 671 note that

$$\beta_y(1 - a_y) \beta_e \beta_{\pi_e} \rho_{eo}^e - \nu_c \beta_e \beta_e \beta_{\pi_e} \beta_y(1 - a_y) \rho_{eo}^e = \beta_y(1 - a_y) \beta_e \beta_{\pi_e} \rho_{eo}^e (1 - \nu_c \beta_e) > 0. \quad (\text{Q.E.D.})$$

672 **Proof of Proposition 3**

673 Since condition (C1) does not hold for $\nu_c > \frac{\beta_y(1 - a_y) + \beta_e \rho_{eo}^e + \beta_{\pi_e}}{\beta_{\pi_e} \beta_e}$, the steady state of the 3D system is
 674 locally unstable. (Q.E.D.)

675 **Proof of Proposition 4**

676 Note that the steady state value of Y , p_e and π_e are uniquely determined independently of x by
 677 conditions (20)-(22) in Lemma 1. Given this, we focus on equation (19) where the probabilities and
 678 switching index are given by equations (13), (14) and (15), respectively. Let Y , p_e and π_e be equal to
 679 their steady state values so that $s = s_x x$.

680 Define then the following real valued function $g : (-1, +1) \rightarrow \Re$

$$g(x) := s_x x - \frac{1}{2a_x} [\ln(1+x) - \ln(1-x)] \quad (29)$$

681 This function has the property that $g(x) = 0$ if and only if $\dot{x} = 0$ as can be seen from (19) setting
 682 $\dot{x} = 0$ and taking the logs. The equation $g(x) = 0$ always has a solution for $x = 0$ and thus there is
 683 always a steady state with $x_o = 0$.

684 (i) Observe that

$$\lim_{x \rightarrow 1} g(x) = -\infty, \quad (30)$$

685

$$\lim_{x \rightarrow -1} g(x) = +\infty, \quad (31)$$

686 and the derivative of $g(x)$ is

$$g'(x) = s_x - \frac{1}{a_x(1-x^2)}. \quad (32)$$

687 Then if $s_x \leq \frac{1}{a_x}$, $g'(x) < 0$ and $g(x)$ is strictly decreasing for all $x \in (-1, 1)$. So, if $s_x \in (0, 1/a_x]$,
 688 $x_o = 0$ is the only value of x such that $g(x) = 0$ and so $\dot{x} = 0$.

689 (ii) By equation (32), $g(x)$ is increasing if and only if

$$g'(x) = s_x - \frac{1}{a_x(1-x^2)} \geq 0 \Leftrightarrow x^2 \leq \frac{s_x a_x - 1}{s_x a_x}.$$

690 Because $s_x a_x > 1$, it follows that $g(x)$ is strictly increasing for $x \in \left(-\sqrt{\frac{s_x a_x - 1}{s_x a_x}}, \sqrt{\frac{s_x a_x - 1}{s_x a_x}}\right)$ and
 691 strictly decreasing for $x \in \left(-1, -\sqrt{\frac{s_x a_x - 1}{s_x a_x}}\right) \cup \left(\sqrt{\frac{s_x a_x - 1}{s_x a_x}}, 1\right)$. Then, noting that $g(0) = 0$ and
 692 $g'(0) > 0$, by equations (30) and (31), and the continuity of $g(x)$, there exist three steady states:
 693 one with equal populations ($x_o = 0$), one where fundamentalists dominate ($x_o < 0$) and one
 694 where chartists dominate ($x_o > 0$). (Q.E.D.)

695 **Proof of Proposition 4**

696 The proof of Proposition 4 is a trivial modification of the proof of Proposition 3. (Q.E.D.)

697 **Proof of Proposition 5**

698 At any steady state $(x_o, \pi_{e_o}^e)$ with $\pi_{e_o}^e = 0$, the Jacobian of the system formed by equations (24)-(26)
 699 is:

$$J = \begin{pmatrix} \beta \pi_e^e \left[\frac{1+x_o}{2} \beta_e - 1 \right] & 0 \\ 0 & 2\beta_x \exp(a_x s_x x_o) \left[(1-x_o) a_x s_x - \frac{1}{1+x_o} \right] \end{pmatrix}. \quad (33)$$

700 (i) At the steady state with $x_o = 0$ and $\pi_{e_o}^e = 0$, the Jacobian becomes

$$J = \begin{pmatrix} \beta_{\pi_e^e} \left(\frac{\beta_e}{2} - 1 \right) & 0 \\ 0 & 2\beta_x(a_x s_x - 1) \end{pmatrix}. \quad (34)$$

701 Because $s_x \in (0, 1/a_x)$, if $\beta_e > 2$ then $|J| < 0$, and the steady state is an unstable saddle point.
 702 Conversely, if $\beta_e < 2$ then $\text{tr}J < 0$ and $|J| > 0$, and the steady state is stable.

703 (ii) The stability properties of the steady state with $x_o = 0$ and $\pi_{e_o}^e = 0$ can be derived with a
 704 straightforward modification of the argument in part (i) noting that $s_x > 1/a_x$.

705 In order to derive the stability properties of $e_f = (0, x_o^f)$ and $e_c = (0, x_o^c)$, note that $J_{22} \lesseqgtr 0$ if
 706 and only if $(1 - x_o)a_x s_x \lesseqgtr \frac{1}{1+x_o}$ or equivalently

$$x_o^2 \gtrless \frac{a_x s_x - 1}{a_x s_x}. \quad (35)$$

707 By the argument in part (ii) of Proposition 3, it follows that both at e_c and at e_f , $x_o^2 > \frac{a_x s_x - 1}{a_x s_x}$
 708 and therefore $J_{22} < 0$. (Q.E.D.)

709 Appendix B

710 In this appendix we present some additional simulations of the full model as well as bifurcation
 711 diagrams. Figure 9 illustrates the case where the relative population variable displays irregular yet
 712 persistent fluctuations. In this simulation, the adjustment speed of share price β_e is increased from
 713 2 to 2.5, while the sensitivity of the sentiment switching index to the output gap, s_y , is reduced to
 714 0.1. The fast adjustment of share price is a source of instability, which is counter-balanced by the
 715 nonlinearity in the opinion switching index ($s_{p_e} = 0.06$ and $s_{\pi_e^e} = 0.5$). The self-reflection parameter
 716 in the opinion switching index, s_x , is kept at 1.

717 The fluctuations in the population of traders are translated to capital gains expectations and the
 718 real economy. The relative size of the two groups (fundamentalists and chartists) fluctuates between
 719 -0.25 and 0 with oscillations differing in both amplitude and frequency. The stability in the fluctuation
 720 of the sentiment dynamics is related to the two volatility parameters in the switching equation – s_{p_e}
 721 and $s_{\pi_e^e}$ – which capture the idea that higher volatility leads agents to become fundamentalists.

722 We now turn to bifurcation diagrams based on the same calibration as in the lower panels of Figure
 723 9 in order to further illustrate the properties of the full model. The top panel of Figure 10 show the
 724 bifurcation diagrams of population dynamics and output with respect to the sensitivity of the opinion
 725 switching index to the self-reference element, with s_x varying between 0.4 and 1.5. For values of s_x
 726 between 0 and 0.5 there are four local minima and maxima for x . This number doubles between 0.5
 727 and 0.9. The number of local minima and maxima then goes back to four between 0.9 and 1 and

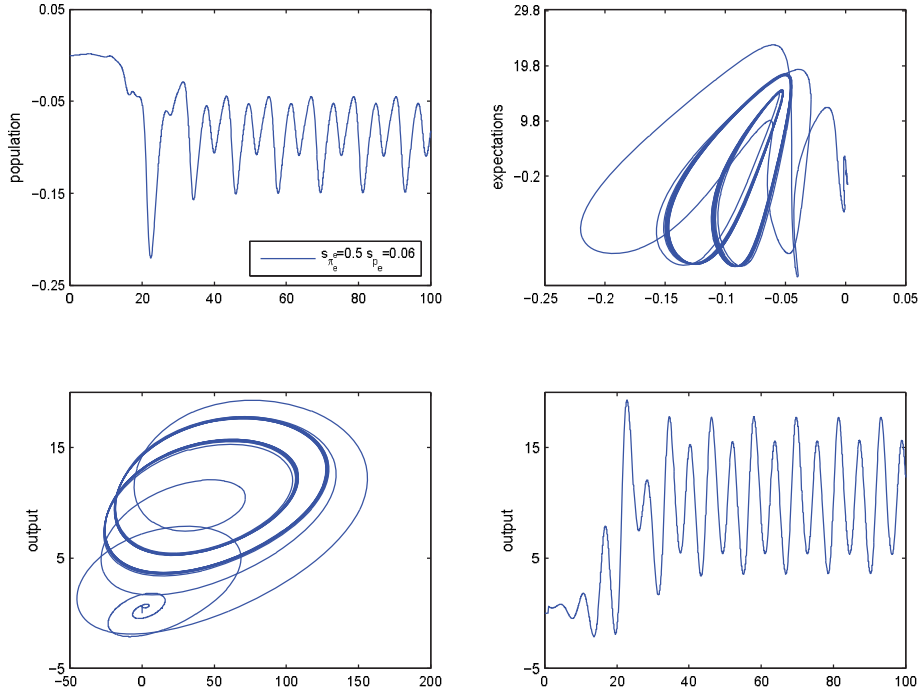


Figure 9: Complex dynamics in the 4D model (Y, p_e, π_e^e, x) .

728 further reduces to two between 1 and 1.25. Beyond 1.25 there is a unique steady state. A similar
 729 pattern describes the oscillation of output.

730 As shown in the next two panels, the number of local minima and maxima decreases with a_x from
 731 four over the range 0.7-0.8 to two over the range 0.8-1 and one when $a_x > 1$. This result is also
 732 consistent with the analysis in section 3.3.

733 The third row of Figure 10 shows bifurcation diagrams of the population dynamics with respect to
 734 the sensitivity of the opinion switching index to the output gap, s_y , and to capital gains expectations
 735 $s_{\pi_e^e}$. Values of s_y in the range $[0.15; 0.2]$ and $[0.27; 0.32]$ produce large fluctuations in the opinion
 736 dynamic. The population variable x goes either to -1 or to positive values when $s_y > 0.34$. For values
 737 of $s_{\pi_e^e} < 0.3$, the opinion dynamics displays large fluctuations over the range $[-0.6; 0]$ in line with the
 738 result that excess volatility favors fundamentalist expectations.

739 The fourth and fifth rows of Figure 10 summarize additional sensitivity analysis. The population
 740 dynamics is stable for either low or high values of the speed of adjustment of expectations, $\beta_{\pi_e^e}$, and
 741 the speed of adjustment of the price of capital, β_e . Interestingly, only a high speed of adjustment of
 742 population dynamics ($\beta_x > 0.8$) produces stability. Finally, the system produces oscillations when the
 743 sensitivity of aggregate demand to Tobin's q , a_q , is either small or larger than 0.8.

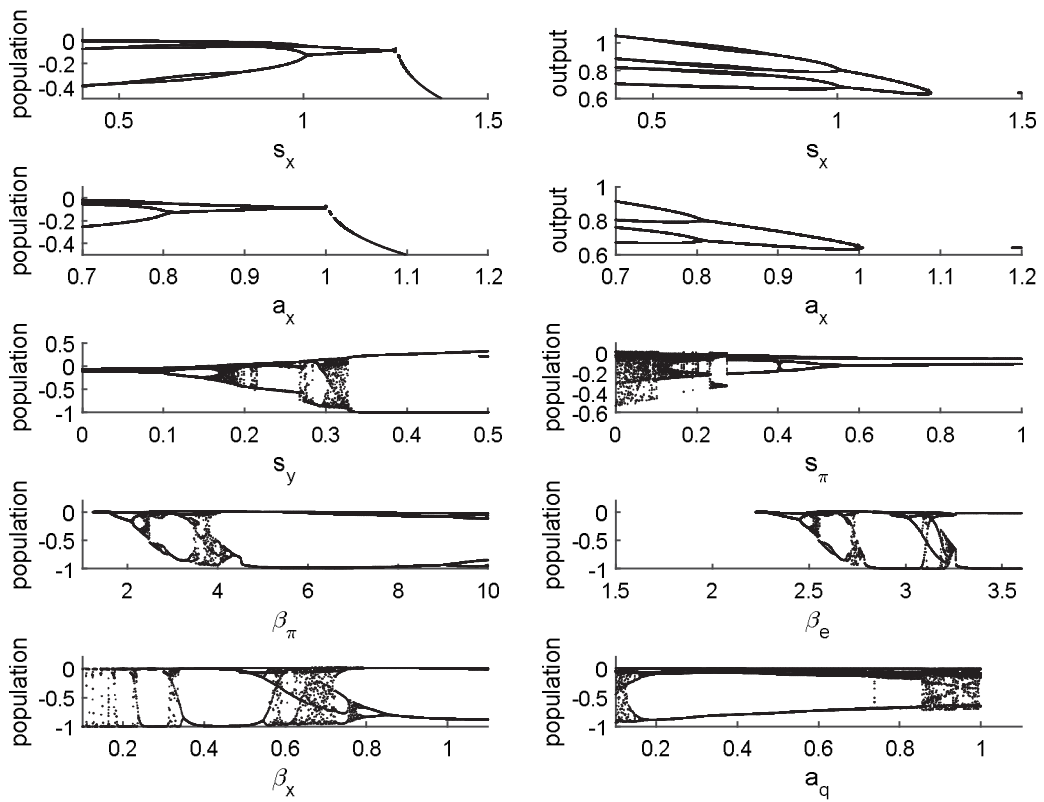


Figure 10: Bifurcation diagrams

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