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Supply Function Equilibrium over a Constrained Transmission Line II: Multiple Plants and Nodal Price Derivatives

Keith Ruddell

Supply function equilibrium over a constrained transmission line II: multiple plants and nodal price derivatives

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Abstract

Market power in electricity wholesale markets arises when generators have incentives to mark up their offers above the cost of production.

I model a transmission network with a single line. I derive optimality conditions for supply functions for generators who supply energy at both ends of the line, and also for generators who hold financial derivatives on the locational prices. These financial derivatives include contracts for differences as well as financial transmission rights.

One way that generators can manipulate prices in their favor is by inducing congestion in the network. I find that dispersed ownership and financial transmission rights are both effective ways to reduce strategic congestion of the line. I also find that certain portfolios of contracts for differences can lead to multiple supply function equilibria.

JEL: C62, D43, D47, L13, L94

Keywords: supply function equilibrium; electricity markets; market power; financial transmission rights.

1 Introduction

In this paper are extended the methods developed in Ruddell (2018) to solve for supply function equilibrium (SFE) under joint ownership of plants or financial derivatives, in electricity markets with nodal pricing and transmission constraints. In that previous paper it was shown that generators competing in supply functions may have incentives to withhold energy from the market in order to cause congestion on the transmission line. Here I show how joint ownership and financial derivatives alter those incentives, and lead to a more efficient equilibrium.

Wilson (2008) derived optimality conditions for generators who offer power at more than one node in the network. We adapt these to the market distribution function approach of Anderson

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and Philpott (2002). We solve for SFE under conditions where the plants can be classified as two types, with all plants of each type having identical objective functions.

We shall consider SFE under both contracts for difference (CfDs) and financial transmission rights. Green (1999) observed that pool-market CfDs are equivalent to forward sales in an SFE market. He finds that selling forward contracts reduces the incentive for suppliers to mark up prices, as they only benefit from market power over the un-contracted part of their output. His results agree with the well-known Cournot model of Allaz and Vila (1993) that the spot price decreases as suppliers make more forward sales.

Under locational pricing, a contract for differences is indexed to the price at a particular location. When a generator sells CfDs to a customer, they may index them to the generator's grid injection point or to the grid exit point of the customer. For CfDs traded on exchanges in a nodal-priced market, contracts will only be listed at a small subset of the nodes — this is in order to guarantee depth in the market for each contract. With an exchange-traded CfD it can be that neither the holder nor the counterparty are located at the node where the contract is indexed.

A financial transmission right (FTR), in the sense of Hogan (1992), is a swap contract on nodal prices. The holder of an FTR for a given volume from source to sink gives up the nodal price at the source and receives instead the nodal price at the sink, for the stated volume.

FTRs are used to mitigate locational risk — whenever a party holds or writes a forward contract indexed to a different location from where they actually access the transmission grid, they bear a locational price risk. Congestion between the two locations can lead significant differences in price.

It is usual for the transmission operator to be a major supplier of FTRs. For them, selling FTRs is a way to swap the highly variable income from congestion rentals for a fixed payment. The purchaser uses the FTRs to convert a distant CfD into a CfD indexed at their local node. Joskow and Tirole (2000) analyzed a two-node model where firms hold financial transmission rights and compete with Cournot conjectures; they found that it was possible for generators to increase their market power by buying transmission rights.

Asset swaps are a regulatory intervention that can be been applied to improve competition in a concentrated and congested market. In dividing up a monopoly, creating generating companies with geographically diversified plants leads to increased local competition. On other occasions *virtual* asset swaps have been implemented. In a virtual asset swap the ownership and operation of power plants is not altered, but generators swap contracts for difference at each other's locations. In this way both generators obtain locational price hedges. The spot-market incentives created by an FTR and a virtual asset swap are identical, since holding either is equivalent to buying CfDs at one location and selling at another.

Outline

The paper is laid out as follows. Section 2 presents the network, pricing rules, stochastic demand, and some general optimality conditions for constrained supply functions. In Section 3 we consider generators with plants at multiple nodes.

The network SFE model is then extended to allow for financial derivatives held by generators. We consider supply functions with locational CfDs in Section 4. Two-way financial transmission rights (FTRs) and virtual asset swaps are also covered. Option FTRs require more detailed analysis of congestion probabilities, which we carry out in Section 5.

2 Optimal supply functions

Formally we suppose that each supplier submits to the system operator a piecewise differentiable curve

$$\mathcal{S} = \{(q(t), p(t)) : t \in [0, T]\},$$

whose quantity and price components q and p are non-decreasing with respect to the parameter t

$$\dot{q} = \frac{dq}{dt} \geq 0 \text{ and } \dot{p} = \frac{dp}{dt} \geq 0. \quad (1)$$

These are the *monotonicity constraints*. We will usually have $\dot{p} > 0$, so that the supply function can be reparametrized as a function of price $q(p)$. Each supplier chooses its supply curve \mathcal{S} to maximize an expected profit functional of the form

$$\Pi[\mathcal{S}] = \int_{\mathcal{S}} R(q(t), p(t)) d\psi(q(t), p(t)) = \int_0^T R(q(t), p(t)) (\psi_q \dot{q} + \psi_p \dot{p}) dt, \quad (2)$$

i.e. a line integral along the curve \mathcal{S} . We suppose that R , ψ_q and ψ_p are (weakly) differentiable.

Following Anderson and Philpott (2002) we define the field

$$Z(q, p) = \frac{\partial R}{\partial q} \psi_p - \frac{\partial R}{\partial p} \psi_q, \quad (3)$$

which we will call the *first-variation derivative*.

The *constrained offer optimization problem* is to maximize Π in (2), subject to the monotonicity constraints (1) and the state constraints on output and price.

$$\begin{aligned} q &\in [\underline{q}, \bar{q}], \text{ and} \\ p &\in [\underline{p}, \bar{p}]. \end{aligned} \quad (4)$$

These constraints arise from the physical production limits of the electricity generators, and the market-imposed bounds on offer prices.

An *adjoint function* to \mathcal{S} for the constrained offer optimization problem is a function $w : [0, T] \rightarrow \mathbb{R}$ with

$$\dot{w} = \frac{dw}{dt} = Z(q(t), p(t)). \quad (5)$$

Given a candidate curve \mathcal{S} , we can calculate an adjoint function w from

$$w(t) = w_0 + \int_0^t Z(q(\tau), p(\tau)) d\tau. \quad (6)$$

So the adjoint function is defined up to a constant of integration w_0 . The following theorem gives a necessary conditions for an optimal constrained supply function in terms of the adjoint.

Theorem 1 (Slope-constrained optimal supply curve). *Let \mathcal{S}^* be a supply curve with quantity and price components $q(t)$, $p(t)$. If the curve \mathcal{S}^* is a maximum for the constrained offer optimization problem — with the constraints (1) and (4) — then there exists an adjoint function w such that*

$$\dot{q}(t) \cdot w(t) \leq 0 \leq \dot{p}(t) \cdot w(t) \quad (7)$$

for all $t \in [0, T]$.

Proof. See Anderson and Philpott (2002). □

2.1 Constrained demand distribution

To aid in the calculation, we define a conditional probability distribution that captures the effect of the transmission constraint on distribution of a generator's residual demand.

First, some notation: at every node m the system operator will choose a nodal price p_m . Each power plant (identified by its owner i and node m) submits an offer curve $q_{i,m}(p_m)$ for the amount offered as a function of the local price. The market demand at m is the sum of a price-responsive component $D_m(p_m)$ and a stochastic shock ε_m . For the total offers net of price-responsive demand at node m , write

$$S_m(p_m) := \sum_{j \in m} q_{j,m}(p_m) - D_m(p_m),$$

where $q_{j,m}(p_m)$ is the quantity offered by generator j at node m at the local price and D_m is the nodal demand function representing price responsive but non-strategic demand at node m . For the total net offers of all generators, other than i , at node m write

$$S_{-i,m}(p) = \sum_{j \in m, j \neq i} q_{j,m}(p_m) - D_m(p_m),$$

where $j \in m$ means that agent j is located at node m . The vector S_{-m} consists of the net offers at all nodes other than m .

As shown by Wilson (2008) and Holmberg and Philpott (2017), in a lossless and loop-free network the market distribution function $\psi_{i,n}(q_{i,n}, p_n; S_{-(i,n)})$ of a generator i located at node n will depend, besides the generator's own offer $q_{i,n}$ and local price p_n , only on the net offers at the other nodes at price p_n . Let S be a vector of net injections at each node of the network, and define the *constrained demand distribution* $\phi_n(S)$ to be the probability that the quantity dispatched at node n is less than S_n , given that other nodes have net offers S_{-n} at the same local price.¹

So ϕ_n is the joint distribution on the net injection at each node induced by the demand shock distribution and the transmission constraint. We can calculate ϕ_n by integrating over the space of demand shocks. See the Appendix for an explanation of the calculation of ϕ .

3 Supply functions at multiple locations

We now consider SFE for generators who own several generating plants at different locations in a transmission network. When there is a transmission line between two plants owned by the same market agent, the offer curve of one affects the residual demand of the other. The strength of the effect depends on the likelihood of congestion in the transmission network. In the extreme case of a line that never congests, and without losses, the output of the two plants are perfect substitutes. In the other extreme, of a line that is always congested, neither plant can influence the residual demand of the other, and the supply functions can be optimized independently.

Generators with plants at multiple nodes submit separate supply functions for each node. We construct models in which there are two types of plant, so that the inward and outward algo-

¹In a network with loops, it can arise that there are congested and uncongested paths between a pair of nodes on a loop. The supply curve of a generator at one of these nodes contributes to the residual demand function of a generator in the other, even though their local prices differ. Thus a generator's market distribution function will depend on their competitors' offers *over a range of prices*. In such a network the system of first-order optimality conditions is a system of *integro-differential equations*, which is a rather different problem mathematically to the ordinary differential equations we consider in this work.

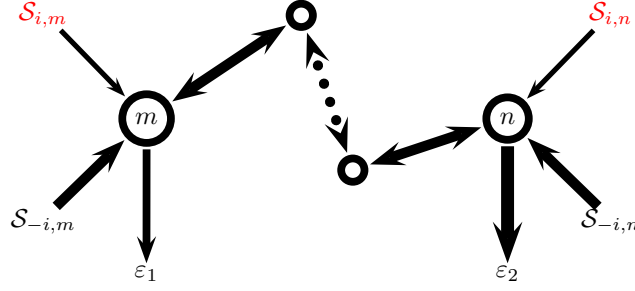


Figure 1: Generator i offers supply curves at nodes m and n in a loop-free network.

rithms may be applied. All plants of a given type are owned by identical firms and face identical market conditions, hence they offer identical supply functions.

Suppose a generator i has plant located at two nodes of a loop-free transmission network, nodes m and n . The situation is illustrated in Figure 1. She submits supply curves

$$\mathcal{S}_{i,m} = \{(q_{i,m}(t), p_m(t)) : t \in [0, T]\} \text{ and } \mathcal{S}_{i,n} = \{(q_{i,n}(t), p_n(t)) : t \in [0, T]\}$$

and her total profit will be a functional depending on both curves:

$$\Pi[\mathcal{S}_{i,m}, \mathcal{S}_{i,n}],$$

which we may think of as a function of two variables. A necessary condition for the maximization of a function of two variables is that the gradient — the two partial derivatives — be zero. Each partial derivative is simply the derivative of the function with respect to one of the variables while the other is held constant. It is exactly the same for this functional over two curves: at a local maximum the first-variation derivative of Π with respect to each of the curves, while holding the other constant, must be zero.

The generator's objective functional Π is the sum of expected revenue at each node

$$\Pi[\mathcal{S}_{i,m}, \mathcal{S}_{i,n}] = \Pi^m[\mathcal{S}_{i,m}, \mathcal{S}_{i,n}] + \Pi^n[\mathcal{S}_{i,m}, \mathcal{S}_{i,n}]. \quad (8)$$

Thus the first-variation derivative of Π with respect to, for instance, $\mathcal{S}_{i,m}$ is the sum

$$Z_m = Z_m^m + Z_m^n, \quad (9)$$

where Z_m^m and Z_m^n are the first-variation derivatives of Π^m and Π^n .

Let us suppose that the market clears by uniform nodal pricing and that the generator's plant at each node n has production cost function $C_{i,n}(q_n)$. Then the profit function at node n can be written

$$R_n(q_{i,n}, p_n) = p_n q_{i,n} - C_{i,n}(q_{i,n}). \quad (10)$$

The expected profit at node n can be written as a line integral along $\mathcal{S}_{i,n}$:

$$\Pi^n = \int_{\mathcal{S}_{i,n}} R_n(q_{i,n}(t), p_n(t)) d\phi_n(q_{i,n} + S_{-i,n}(p), S_{-n}(p)), \quad (11)$$

with $d\phi_n(S_n, S_{-n})$ is the constrained demand density, the total derivative of ϕ_n with respect to the parameter t . It gives the probability density for the price p_n , given the network constraints

and the aggregate nodal offers at $p = p_n$. The profit Π^m at node m is similarly defined as a line integral along $\mathcal{S}_{i,m}$. To apply (9), we must take ‘partial derivatives’ Z_m^m and Z_m^n .

To calculate Z_m^m , we can just apply (3) with $\psi_{i,n}(q, p) = \phi_n(q_{i,n} + S_{-i,n}(p), S_{-n}(p))$. The fact that the supply curve $\mathcal{S}_{i,n}$ at node n is offered by the same generator makes no difference to the calculation, since we treat it as a fixed function $q_{i,n}(p_n)$ anyway. We obtain

$$Z_m^m(q_{i,m}, p_m) = (p_m - C'_{i,m}(q_{i,m})) \left(\phi_{m,m} S'_{-i,m} + \sum_{\ell \neq m} \phi_{m,\ell} S'_\ell \right) - \phi_{m,m} q_{i,m}. \quad (12)$$

For the Z_m^n term there is a problem, which is that Π^n is a line integral over \mathcal{S}_n , a different curve to \mathcal{S}_m , the curve to be optimized. Fortunately, the complementarity between price differences and market interaction in a loop-free and lossless network allows us to calculate Z_m^n as a function of the local price at node m . The calculation gives the following result.

Proposition 2. *In a lossless loop-free network, the first-variation derivative of Π^n in (11) with respect to a supply curve $\mathcal{S}_{i,m} = \{(q_{i,m}(t), p_m(t)) : t \in [0, T]\}$ offered at node $m \neq n$ is*

$$Z_m^n(q_{i,m}, p) = - \left(\frac{\partial R_n}{\partial p_n} + q'_n(p) \frac{\partial R_n}{\partial q_{i,n}} \right) \phi_{n,m}. \quad (13)$$

If we think in terms of the expectation of the revenue R_n , conditional on the controls $q_{i,m}$ and p_m , as well as the interaction probabilities, we arrive at the following interpretation of (13). By manipulating the market from node m , an agent can only affect outcomes at node n through control of prices; the payoff function is independent of the quantity offered at m , i.e. $\frac{\partial R_n}{\partial q_{i,m}} = 0$. The term

$$\left(\frac{\partial R_n}{\partial p_n} + \frac{\partial R_n}{\partial q_n} q'_{i,n}(p_m) \right)$$

is the total derivative of R_n with respect to p_m , conditional on prices matching $p_m = p_n$. The condition that prices match is corrected by the other term $\phi_{n,m}$. Recalling the interaction probability from Section 2, we can rewrite $\phi_{n,m}$ as follows

$$\phi_{n,m} = \phi_{m,m} \Pr[p_m = p_n].$$

So (13) is the product of three terms: the change in profit at n , conditional on prices matching between m and n ; the probability that prices match; and the local residual demand density $\phi_{m,m}$.

Summing up Z_m^m and Z_m^n gives the first-variation derivative of $\Pi[\mathcal{S}_{i,m}, \mathcal{S}_{i,n}]$:

$$\begin{aligned} Z_{i,m}(q_{i,m}, p) &= (p - C'_{i,m}(q_{i,m})) \left(S'_{-i,m} \phi_{m,m} + S'_{-i,n} \phi_{m,n} + \sum_{\ell \neq m,n} S'_\ell \phi_{m,\ell} \right) \\ &\quad - q_{i,m} \phi_{m,m} - q_{i,n} \phi_{n,m} \\ &\quad + q'_{i,n} ((p - C'_{i,m}(q_{i,m})) \phi_{m,n} - (p - C'_{i,n}(q_{i,n})) \phi_{n,m}). \end{aligned} \quad (14)$$

Along un-constrained segments of the supply function, we have $Z_{i,m} = 0$. This condition implicitly defines the optimal supply function. We can derive several properties of the optimal curve by examination of (14). First, when we divide through by $\phi_{m,m}$, we see that local effects are more important than actions at other points in the network, insofar as $\frac{\phi_{n,m}}{\phi_{m,m}} < 1$.

Second, the derivative $q'_{i,n}$ of the generator's own offer at the node n appears twice, with slightly different coefficients,

$$(p - C'_{i,m}(q_{i,m})) \phi_{m,n}, \text{ and } -(p - C'_{i,n}(q_{i,n})) \phi_{n,m}. \quad (15)$$

The partial derivatives $\phi_{m,n}$ and $\phi_{n,m}$ are equal, so the only thing that would keep the two terms from exactly canceling out is a possible difference in production costs at the two nodes.

Third, the quantity terms

$$q_{i,m} \phi_{m,m} - q_{i,n} \phi_{n,m} \quad (16)$$

are more heavily weighted towards the local offer, since $\phi_{m,m} > \phi_{n,m}$. The generator must choose their offer quantities to balance the difference in cost in (15) with the difference in congestion probability relative to output in (16).

Supply function equilibrium

There is a separate optimization for every plant (not just every firm) offering to the market. Each generator i optimizes a supply curve $\mathcal{S}_{i,n}$ at each node n where they have production. Each pair (i, n) is a power plant; the plant is owned by agent i and located at node n . With this in mind, the first order condition for a supply function equilibrium is

$$Z_{i,n} = 0 \text{ for all plants } (i, n). \quad (17)$$

Again this is generally a system of non-linear ordinary differential equations of order equal to the number of plants. We modify the assumptions made in Ruddell (2018) to reduce this to two types.

Assumption 3 (Two-type linearizable market).

1. *Demand is perfectly inelastic in price.*
2. *The demand shock has a bivariate normal distribution.*
3. *Producers have identical and constant marginal costs of production c .*
4. *There are two types of generation plant.*
5. *All plants of the each type has identical*
 - *production capacity, and*
 - *residual demand distribution*
 - *number of other plants of each type with the same owner.*

Under Assumption 3, the equilibrium conditions (17) reduce to a linear ODE of order two. We then solve this ODE to find candidate trajectories for the SFE.

3.1 Plants across two nodes

There are three ownership arrangements of plants in the two-node network that satisfy Assumption 3. The first is as in the previous examples, where each type corresponds to a node and each generator owns one plant. This is shown in Figure 2.

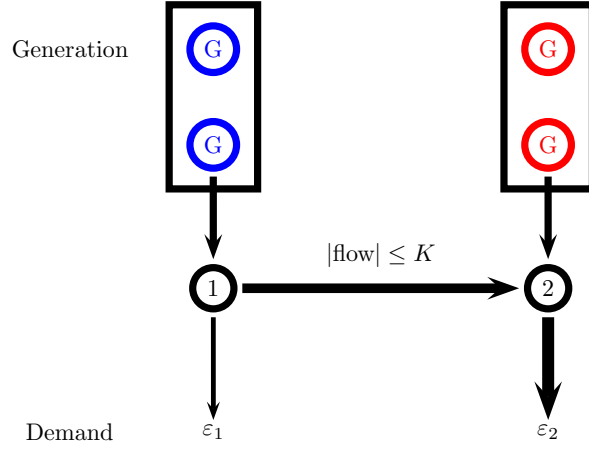


Figure 2: Two types of plant differentiated by node.

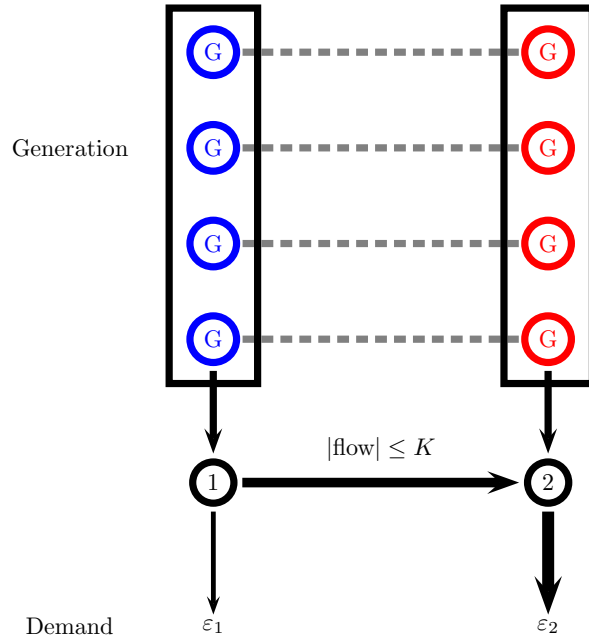


Figure 3: Two types of plant with cross-ownership

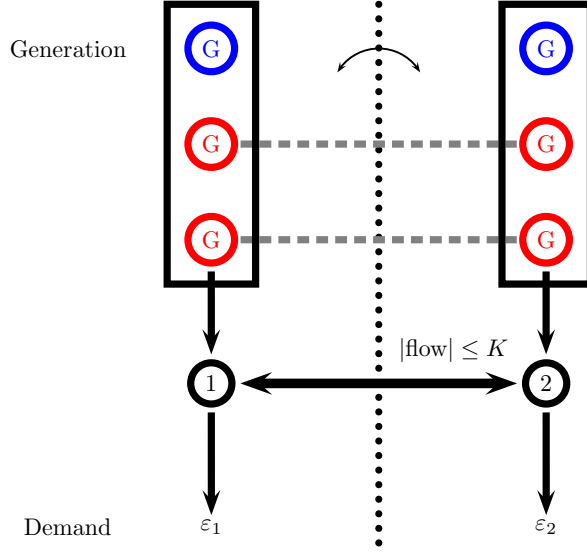


Figure 4: Type 1 plants (blue) have no co-ownership and the type 2 plants (red) are co-owned with plants on the other end of the line.

In the second arrangement, we match every plant with a co-owned plant in the other node — symmetry within the nodes, as shown in Figure 3. And in the third arrangement, the demand distribution is symmetric and some plants are co-owned with identical plants in the other node, while other plants remain independent — symmetry between the nodes, as shown in Figure 4.

The second and third models have first-order equilibrium conditions of the form

$$Z = U\dot{Q} - VQ = 0$$

where U and V are 2×2 matrices and Q is a 2-component curve representing the offer trajectory. We shall use the inward and outward algorithms, for which we must transform the ODE to an explicit form, and also define the ironing function Y . We define

$$A := V^{-1}U \tag{18}$$

so that $A\dot{Q} = Q$.

We derive an ironing function Y like that in Ruddell (2018). The ironing function gives the value of Z_m when the constraint \dot{Q}_m binds. Suppose that the generators at node 1 iron, so $\dot{Q}_1 = 0$. The generators at node 2 offer supply curves that satisfy

$$Z_2 = u_{22}\dot{Q}_2 - v_{21}Q_1 - v_{22}Q_2 = 0,$$

which gives

$$\dot{Q}_2 = \frac{1}{u_{22}} (v_{21}Q_1 + v_{22}Q_2).$$

The ironing function for node 1, Y_1 , is equal to Z_1 with this \dot{Q}_2 and $\dot{Q}_1 = 0$, i.e.

$$\begin{aligned} Y_1(Q) &= u_{12}\dot{Q}_2 - v_{11}Q_1 - v_{12}Q_2 \\ &= \frac{u_{12}}{u_{22}} (v_{21}Q_1 + v_{22}Q_2) - v_{11}Q_1 - v_{12}Q_2. \end{aligned}$$

By a similar argument, we obtain the ironing function for node 2

$$Y_2(Q) = \frac{u_{21}}{u_{11}} (v_{11}Q_1 + v_{12}Q_2) - v_{21}Q_1 - v_{22}Q_2.$$

Observing that

$$\mathbf{U}^{-1} = \frac{1}{|\mathbf{U}|} \begin{bmatrix} u_{22} & -u_{12} \\ -u_{21} & u_{11} \end{bmatrix},$$

we find that

$$Y(Q) = \begin{bmatrix} Y_1(Q) \\ Y_2(Q) \end{bmatrix} = -|\mathbf{U}| \begin{bmatrix} \frac{1}{u_{22}} & 0 \\ 0 & \frac{1}{u_{11}} \end{bmatrix} \mathbf{U}^{-1} \mathbf{V} Q. \quad (19)$$

To apply the algorithms of Ruddell (2018), we verify the relation between the signs of Y and the unconstrained ODE $\mathbf{A}\dot{Q} = Q$.

Lemma 4. *Let \mathbf{U} and \mathbf{V} be 2×2 matrices which may depend on Q . Suppose that \mathbf{U} has positive diagonal elements and $|\mathbf{V}| > 0$. Let \mathbf{A} and Y be as in (18) and (19). Then the function Y has the property that if $Y(Q) = 0$, then either $Q = 0$ or $|\mathbf{A}| = 0$.*

Furthermore, the derivative $\dot{Q} = \mathbf{A}^{-1}Q$ of the candidate trajectory has both components positive if and only if either

- $|\mathbf{A}| < 0$ and $Y_1, Y_2 > 0$, or
- $|\mathbf{A}| > 0$ and $Y_1, Y_2 < 0$.

Lemma 4 controls where candidate trajectories can have corners. The algorithms of Ruddell (2018) can be applied to the two-type model. These tools will suffice to solve for SFE with co-owned plants in the setting we consider here.

Symmetry within the nodes

One ownership arrangement that satisfies Assumption 3 is to have each of the N firms owns one plant in each node, and thus one of each type.

Figure 3 shows this. The equilibrium condition (14) simplifies to the 2×2 system of ODEs

$$(p - c)\mathbf{U}Q' - \mathbf{V}Q = 0$$

with

$$\mathbf{U} = (N - 1) \begin{bmatrix} \phi_{1,1} & \phi_{1,2} \\ \phi_{2,1} & \phi_{2,2} \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \phi_{1,1} & \phi_{1,2} \\ \phi_{2,1} & \phi_{2,2} \end{bmatrix}.$$

Under the change of variable $p - c = e^t$, this becomes the autonomous system

$$\mathbf{U}\dot{Q} - \mathbf{V}Q = 0.$$

The matrix \mathbf{A} and the ironing function Y are

$$\begin{aligned} \mathbf{A} &= \mathbf{V}^{-1}\mathbf{U} \\ &= (N - 1)\mathbf{I} \end{aligned}$$

and

$$Y(Q) = -(\phi_{1,1}\phi_{2,2} - \phi_{1,2}\phi_{2,1}) \begin{bmatrix} \frac{1}{\phi_{2,2}} & 0 \\ 0 & \frac{1}{\phi_{1,1}} \end{bmatrix} Q.$$

It is clear that the determinant $|\mathbf{A}|$ is always positive. Since \mathbf{A} is constant, we can use the constant-coefficient algorithm from Ruddell (2018) to calculate solutions.

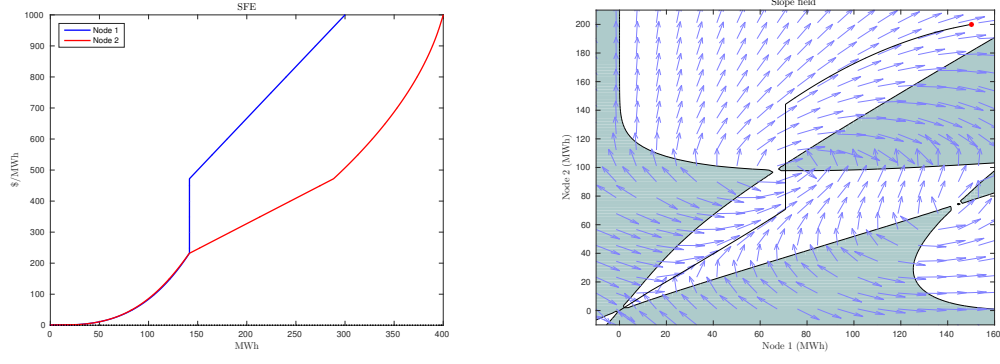


Figure 5: Concentrated ownership: four generators, two at each node

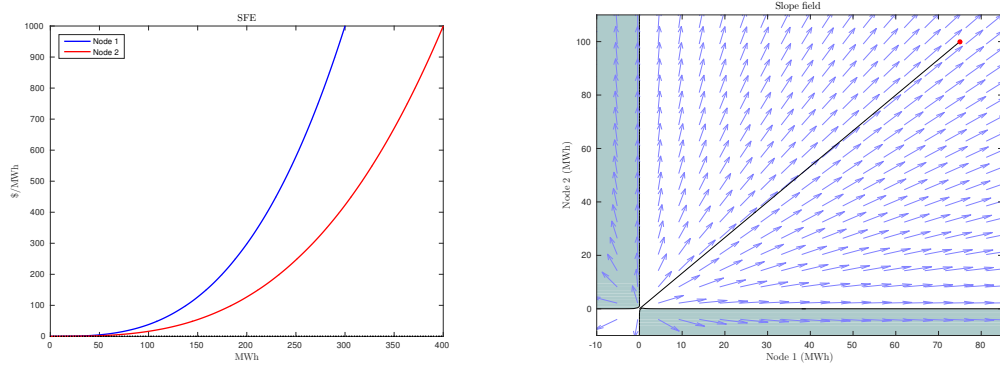


Figure 6: Dispersed generators: four generators, each with a plant at each node

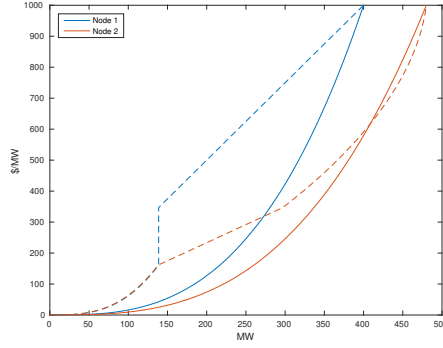


Figure 7: SFE for dispersed (solid) and concentrated (dashed) plant ownership

Example 5. Make Assumption 3. Let there be four firms and eight identical generation plants, four on either side of a transmission link as shown in Figure 3. We can solve SFE to compare two different ownership structures.

1. Each firm owns two plants in the same node. So there are two firms operating entirely at node 1 and two entirely at node 2, as in Figure 2.
2. Each firm owns one plant in each node, as in Figure 3.

In the first structure, it is as though there are four firms with one plant each, but those plants have twice the production capacity.

The total number of firms competing strategically in both structures is four. The type 1 plants, in node 1, can each produce up to 75MW, at zero marginal cost, while the type 2 plants can each produce up to 100MW, also at zero marginal cost. The transmission line has a capacity of 80MW and the demand shocks are independently and identically normally distributed with means (80, 80)MW and standard deviations (60, 40)MW.

Figure 5 shows the SFE and trajectory for the first structure, without co-ownership across nodes. The node 1 generators withhold in order to congest the line. To achieve this they iron across the price range \$300-550.

Figure 6 shows the SFE and trajectory for the second structure, where each firm has a plant in each node. There is no ironing and the SFE curves are all cubics.

Figure 7 compares the nodal offer curves for these two ownership structures. Prices are clearly lower in the case where ownership is dispersed throughout the network and no-one can create a local monopoly or even duopoly.

Symmetry between the nodes

We now consider a third ownership arrangement. Instead of all plants in a given node being identical, we make the two nodes identical to each other. Figure 4 shows this in a diagram. The two types of plant are the independent plants (type 1, in blue), which are not co-owned with any other plant, and the co-owned plants (type 2, in red). Each firm owns either a single type 1 plant or one type 2 plant in each node.

In order for the market distribution functions to be identical between plants of the same type in different nodes, we require that the distribution of the demand shock have a symmetrical

distribution in the following sense. Let f be the density function of the shock distribution. The demand shock has a symmetrical distribution if

$$f(\varepsilon_1, \varepsilon_2) = f(\varepsilon_2, \varepsilon_1),$$

for all $\varepsilon_1, \varepsilon_2$.

A consequence of the symmetry in the demand shock distribution is that the Jacobian of the constrained demand distribution is both symmetric and *persymmetric*,² and so has only two distinct elements. The Jacobian matrix is

$$\begin{bmatrix} \frac{\partial \phi_m}{\partial S_n} \end{bmatrix} = \begin{bmatrix} \phi_o & \phi_x \\ \phi_x & \phi_o \end{bmatrix},$$

with on-diagonal entries ϕ_o and cross-terms ϕ_x . We have $0 < \phi_x \leq \phi_o$, so the matrix is both positive and positive-definite. The system of ODEs becomes

$$\mathbf{U}\dot{\mathbf{Q}} - \mathbf{V}\mathbf{Q} = 0$$

with

$$\mathbf{U} = \begin{bmatrix} (N_1 - 1)\phi_o + N_1\phi_x & N_2(\phi_o + \phi_x) \\ N_1(\phi_o + \phi_x) & (N_2 - 1)(\phi_o + \phi_x) \end{bmatrix} \quad \text{and} \quad \mathbf{V} = \begin{bmatrix} \phi_o & 0 \\ 0 & \phi_o + \phi_x \end{bmatrix}.$$

The matrix \mathbf{A} and the ironing function Y are

$$\begin{aligned} \mathbf{A} &= \mathbf{V}^{-1}\mathbf{U} \\ &= \begin{bmatrix} (N_1 - 1) + N_1 \frac{\phi_x}{\phi_o} & N_2 \left(1 + \frac{\phi_x}{\phi_o}\right) \\ N_1 & (N_2 - 1) \end{bmatrix} \end{aligned}$$

and

$$Y(Q) = \begin{bmatrix} \phi_o & \frac{-N_2(\phi_o + \phi_x)}{N_2 - 1} \\ \frac{-N_1(\phi_o + \phi_x)\phi_o}{(N_1 - 1)\phi_o + N_1\phi_x} & \phi_o + \phi_x \end{bmatrix} Q.$$

The ratio $\frac{\phi_x}{\phi_o}$ is the probability of uncongested flow between the two nodes, conditional on the local price. It can take values between zero and one. When we take the determinant of \mathbf{A}

$$\begin{aligned} |\mathbf{A}| &= \left(\left(1 + \frac{\phi_x}{\phi_o}\right) N_1 - 1 \right) (N_2 - 1) - \left(1 + \frac{\phi_x}{\phi_o}\right) N_1 N_2 \\ &= 1 - \left(1 + \frac{\phi_x}{\phi_o}\right) N_1 - N_2 < 0 \end{aligned}$$

we find that it is always negative. The matrix \mathbf{A} is not constant because $\frac{\phi_x}{\phi_o}$ varies. We use the outward algorithm from Ruddell (2018) to find solutions.

Example 6. There are $N_1 = 2$ independent (type 1) plants and $N_2 = 4$ plants in co-ownership (type 2), as in Figure 4. Thus the total number of firms competing strategically is four. The type 1 plants can each produce up to 50MW, at zero marginal cost, while the type 2 plants can each produce up to 30MW, also at zero marginal cost. The transmission line has a capacity

²Symmetric in the SW-NE diagonal.

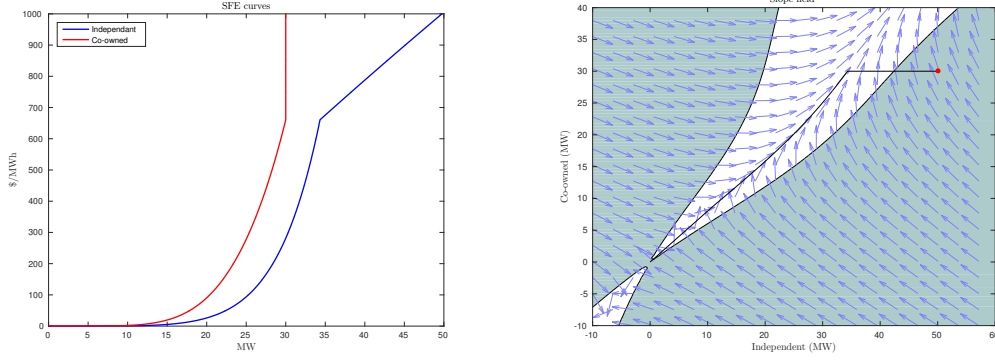


Figure 8: SFE and slope field for independent and co-owned plant types

of 20MW and the demand shocks are independently and identically normally distributed with means (80, 80)MW and standard deviations (60, 60)MW.

Figure 8 shows the equilibrium supply functions for the two types of plant and the slope field. The independent plants offer to the market more competitively than the co-owned plants, even though all the plants have the same marginal costs. The different production capacities mean that the co-owned plants reach their maximum output at a lower price than the independent plants.

Hence, given a fixed number of firms, it is pro-competitive overall to have firms own plants throughout the network, in cases where there are both independently owned and co-owned plants, it is the independent plants that offer more competitively.

4 Supply functions with nodal price derivatives

We now extend the network SFE model to allow for financial hedging instruments held or written by generators. These hedging instruments include CfDs and options, as well as financial transmission rights and virtual asset swaps.

The generator writes derivatives contracts and must pay out to their counterparty an amount which depends on the clearing price at the index node n . Define $B_{i,n}(p_n)$ the *nodal contract obligation function* of generator i at node n to be the net obligation for derivatives written by that generator at that nodal price. For instance, a generator i holding a contract for difference of volume $b_{i,n}$ indexed to the node n price p_n and with strike price $\pi_{i,n}$ has obligation function

$$B_{i,n}(p_n) = b_{i,n}(p_n - \pi_{i,n}).$$

Options and portfolios of options can give non-linear obligation functions, as in Oum and Oren (2010).

4.1 Optimal supply curves with contracts for differences

Suppose we have a uniform nodal-price auction over a loop-free (radial) transmission network, and that contracts for differences are available at several nodes throughout the network. We consider a generator with plant at a single node. A portfolio consisting of CfDs and options at different nodes will have a side-payment function that is the sum of the contract obligation

functions of all the nodes:

$$B_i(p) = \sum_{n \in \mathcal{N}} B_{i,n}(p_n). \quad (20)$$

Here p is the vector of prices at all nodes with components p_n .

Consider the generator's optimal offer problem at node m . The generator i offers a supply curve $\mathcal{S}_{i,m} = \{(q(t), p(t)) : t \in [0, T]\}$ at node m , the only node where they have a plant. The total expected profit is of the form (8), with payoff terms for the contracts indexed at each node. The generator is not offering at nodes $n \neq m$, and this is the same as offering a supply curve at n whose quantity component is identically zero. So we can apply Theorem 2 with

$$R_{i,n}(q_{i,n}, p_n) = -B_{i,n}(p_n),$$

and $q_{i,n} \equiv 0$ for $n \neq m$, to obtain the first-variation derivative for the expected outlay for contracts indexed at node n

$$Z_m^n(q_{i,m}, p) = B'_{i,n}(p) \phi_{n,m}.$$

To obtain the total first-variation derivative for generator i who offers electricity at node m and has additively separable contract obligation function $B_i(p)$, we sum up the Z_m^n over all $n \in \mathcal{N}$ and add it to the first-variation derivative of the expected profit from sales of electricity. This gives

$$\begin{aligned} Z_{i,m} = (p - C'_{i,m}(q)) & \left(S'_{-i,m}(p) \phi_{m,m} + \sum_{n \neq m} S'_n(p) \phi_{m,n} \right) \\ & - q \phi_{m,m} + \sum_{n \in \mathcal{N}} B'_{i,n}(p) \phi_{n,m}. \end{aligned} \quad (21)$$

With this Z , we can find SFE in the two-node network under Assumption 3, in which generators write financial derivative contracts indexed to nodal prices. We will start by looking at locational CfDs.

For a portfolio of nodal CfDs, the marginal nodal obligation functions

$$B'_{i,n}(p_n) = b_{i,n}$$

are constant. The generator's portfolio of CfDs across all nodes is a vector

$$b_i = \{b_{i,n}\}_{n \in \mathcal{N}}, \quad (22)$$

whose components $b_{i,n}$ give the net volume of CfDs indexed at node n that the generator has written. With such a portfolio, the generator's first-order optimality condition is

$$\begin{aligned} (p - C'_{i,m}(q)) & \left(S'_{-i,m}(p) \phi_{m,m} + \sum_{n \neq m} S'_n(p) \phi_{m,n} \right) \\ & - (q - b_{i,m}) \phi_{m,m} + \sum_{n \neq m} b_{i,n} \phi_{n,m} = 0. \end{aligned} \quad (23)$$

In (23), we observe that the CfDs sold in a generator's own node $b_{i,m}$, have the same co-efficient $\phi_{m,m}$ and opposite sign to their output q . Thus CfDs sold at a generator's own node are exactly equivalent to forward sales. However, CfDs sold at a different node do not directly cancel with output; their equivalence to forward sales is mitigated by the probability of congestion $\left(1 - \frac{\phi_{n,m}}{\phi_{m,m}}\right)$.

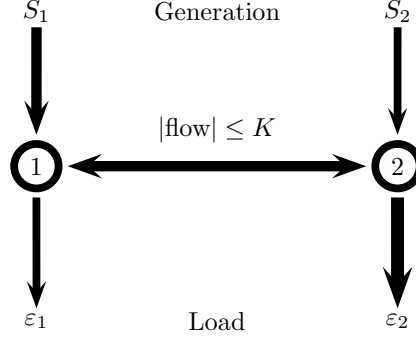


Figure 9: The two-node network.

4.2 Locational CfDs in the two-node network

For what follows, we work in the two-node network shown in Figure 9, and again make Assumption 3. Under this assumption and with the change of variable $p - c = ae^t$, the first-order equilibrium condition is

$$Z = \mathbf{U}\dot{Q} - \mathbf{V}Q + B = 0 \quad (24)$$

with

$$\mathbf{U} = \begin{bmatrix} (N_1 - 1)\phi_{1,1} & N_2\phi_{1,2} \\ N_1\phi_{2,1} & (N_2 - 1)\phi_{2,2} \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \phi_{1,1} & 0 \\ 0 & \phi_{2,2} \end{bmatrix} \quad (25)$$

and

$$B = \begin{bmatrix} \phi_{1,1}b_{1,1} + \phi_{1,2}b_{1,2} \\ \phi_{2,1}b_{2,1} + \phi_{2,2}b_{2,2} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

The matrices \mathbf{U} and \mathbf{V} and the vector B are all functions of the offer quantities Q through the constrained demand distribution functions $\phi_m(N_1Q_1, N_2Q_2)$, $m = 1, 2$.

Define

$$\begin{aligned} \mathbf{A}(Q) &:= \mathbf{V}^{-1}\mathbf{U} \\ &= \begin{bmatrix} (N_1 - 1) & N_2 \frac{\phi_{1,2}}{\phi_{1,1}} \\ N_1 \frac{\phi_{2,1}}{\phi_{2,2}} & (N_2 - 1) \end{bmatrix} \end{aligned} \quad (26)$$

and

$$Y(Q) = \begin{bmatrix} \frac{N_2}{N_2-1} \frac{\phi_{1,2}}{\phi_{2,2}} (\phi_{2,2}Q_2 - B_2) - \phi_{1,1}Q_1 + B_1 \\ \frac{N_1}{N_1-1} \frac{\phi_{2,1}}{\phi_{1,1}} (\phi_{1,1}Q_1 - B_1) - \phi_{2,2}Q_2 + B_2 \end{bmatrix}. \quad (27)$$

With this Y , and keeping in mind that $|\mathbf{A}|$ and $|\mathbf{U}|$ have the same sign, we can check the sign relationships hold so that our integration algorithms can be applied.

Lemma 7. *Let $\mathbf{U}(Q)$, $\mathbf{V}(Q)$ and $Y(Q)$ be as in (25) and (27). The slope $\dot{Q} = \mathbf{U}^{-1}(\mathbf{V}Q - B)$ has both components positive if and only if either*

$|\mathbf{U}| < 0$ and $Y_1, Y_2 > 0$, or

$|\mathbf{U}| > 0$ and $Y_1, Y_2 < 0$.

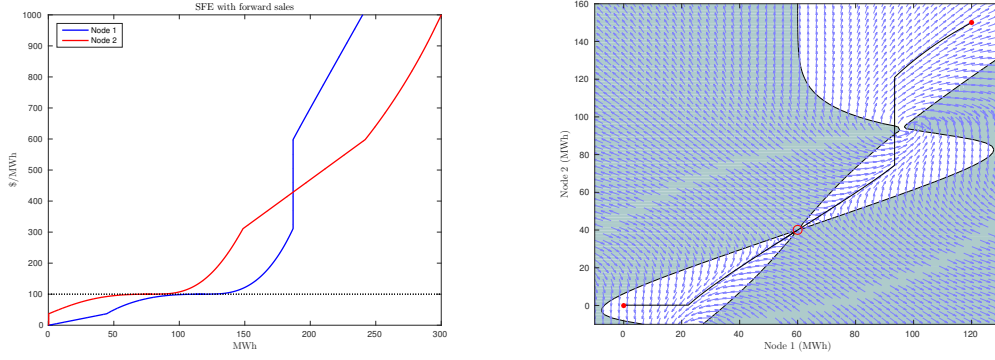


Figure 10: Four generators sell CfDs in their own nodes

With these modified \mathbf{A} and \mathbf{Y} , we can use the inward and outward algorithms to solve (24) for candidate trajectories.

4.3 CfDs at grid injection points

We consider a simple case of CfDs in a network where each generator writes CfDs only at the node where they sell energy. Suppose generator i sells volume $b_{i,m}$ CfDs at node m and none elsewhere, i.e. $b_{i,n} = 0$ if i is not at node n . The generator's unconstrained first-order optimality condition is

$$Z_{i,m} = (p - c) \left(\phi_{m,m} S'_{-i,m} + \sum_{n \neq m} \phi_{m,n} S'_n \right) - \phi_{m,m} (q_{i,m} - b_{i,m}) = 0.$$

Under the change of variable

$$\tilde{q}_{i,m} = q_{i,m} - b_{i,m},$$

this reduces to (12), so we can apply the inward and outward algorithms to solve for SFE.

Example 8 (Own-node CfDs). Generators have plant in one node each and sell CfDs at the node where they have plant. There are two generators in each node. Each generator in node 1 sells $b_1 = 60$ MW and each in node 2 sells $b_2 = 40$ MW of CfDs. The node 1 plants each have a production capacity of 120 MW and those in node 2 have 150 MW. All plants have constant marginal cost of production of \$100/MWh.

The demand shock distribution is an independent bivariate normal with means (120, 80) MW and standard deviations (60, 40) MW. The line has capacity of 70 MW.

Figure 10 shows the SFE and the supply curve trajectories. In the slope field, we see that the 'origin' of the system has been translated to the contract point (60, 40). This corresponds to the crossing of the aggregate supply functions and the marginal cost curve at the price \$100/MWh and quantities $S_1 = 120$ MW and $S_2 = 80$ MW. Node 1 generators iron over the price range \$320-600/MWh.

Below the marginal cost curve at $p = c = 100$, we apply the change of variables

$$\begin{aligned}\hat{q} &= b - q \text{ and} \\ \hat{p} &= c - p.\end{aligned}$$

and then solve using the outward algorithm. The node 2 generators reach zero output at a higher price than the node 1 ones do, so they iron at zero output until the node 1 generators also hit zero output.

4.4 CfDs at a trading hub

In an electricity market with nodal pricing, there may be hundreds of nodes. When generators trade CfDs, they must trade with reference to the price at a specific node. For an over-the-counter trade this can be chosen at the generator's grid injection point or the customer's grid offtake point. For exchange-traded CfDs it is usual to restrict the number of trading hubs to major nodes of the network. In New Zealand there are two principal trading nodes for CfDs (Benmore and Otāhuhu), out of the over 200 nodes for which distinct prices are calculated. Thus it will often be the case that a generator sells CfDs at a different node from where they are supplying power to the network. Transmission congestion between their grid injection point and the trading hub will affect their offering strategy.

Suppose we are in a two-node network in which CfDs are traded only at node 1. For a generator i located at node 1, the first-order optimality condition for a supply function (23) is

$$Z_1 = (p - C'_i(q_i)) (\phi_{1,1} S'_{-i,1} + \phi_{1,2} S'_2) - \phi_{1,1} (q_i - b_i) = 0,$$

where b_i is her volume of forward sales at node 1. The generator optimizes for sales net of her contract position.

On the other hand, a generator j in node 2 has first-order condition

$$Z_2 = (p - C'_j(q_j)) (\phi_{2,1} S'_1 + \phi_{2,2} S'_{-j,2}) - \phi_{2,2} q_j + \phi_{2,1} b_j = 0,$$

where b_j is her volume of forward sales at node 1. Here the generator's offer quantity and contract quantity have different coefficients. If we write

$$\phi_{2,2} q_j - \phi_{2,1} b_j = \phi_{2,2} \left(q_j - \frac{\phi_{2,1}}{\phi_{2,2}} b_j \right),$$

then we can see that a generator selling CfDs at a distant hub will optimize for sales net of *congestion-adjusted contracts*.

For any generator j at node 2, there is an *effective contract point* (q_j, p) where $p - C'_j(q_j) = 0$ and $q_j - \frac{\phi_{2,1}}{\phi_{2,2}} b_j = 0$. Under certain conditions, this contract point is unique, as we show in Lemma 9 below. Note that if there is a non-zero probability of congestion, then the factor $\frac{\phi_{2,1}}{\phi_{2,2}} \leq 1$. So selling CfDs at node 1 gives an effective contract point for node 2 generators with a quantity less than the volume of contracts sold: i.e. where $q_j < b_j$.

Lemma 9 (Unique contract point). *Under Assumption 3, if node 1 is the trading hub, and the constrained distribution function $\phi_2(S_1, S_2)$ is such that*

$$\frac{\partial}{\partial S_2} \left[\frac{\phi_{2,1}(S_1, S_2)}{\phi_{2,2}(S_1, S_2)} \right] < \frac{1}{b_j} \quad (28)$$

for all nodal outputs (S_1, S_2) and all generating plants j at node 2, then there is a unique effective contract point $q_{j,2} = \frac{\phi_{2,1}}{\phi_{2,2}} b_j$ for every generating plant.

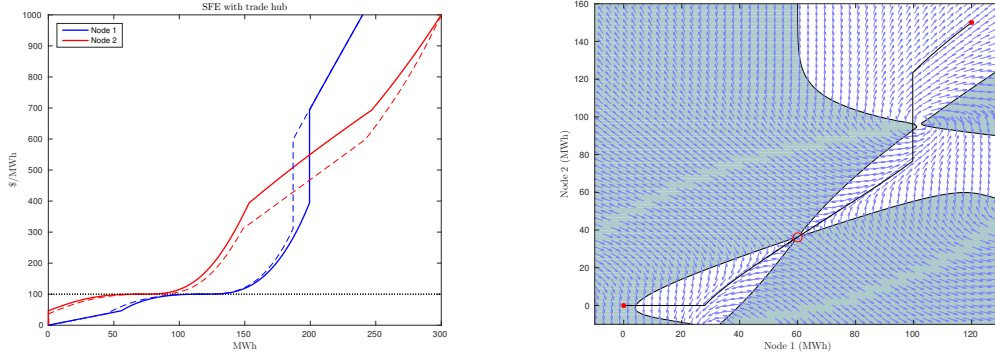


Figure 11: Four generators sell CfDs at the trading hub, node 1 (solid curves). SFE when selling at their own nodes as dashed curves.

Proof. With constant marginal costs, the contract points are the $q_{j,2}$ which solve

$$q_{j,2} - \frac{\phi_{2,1}(S_1, S_{-j,2} + q_{j,2})}{\phi_{2,2}(S_1, S_{-j,2} + q_{j,2})} b_j = 0.$$

This equation has a unique solution if the left-hand side is strictly monotone in $q_{j,2}$. The derivative of the left-hand side with respect to $q_{j,2}$ is

$$1 - b_j \frac{\partial}{\partial S_2} \left[\frac{\phi_{2,1}}{\phi_{2,2}} \right].$$

By (28), this is positive, so the contract point is unique. \square

Lemma 9 states that if the probability of uncongested flow does not increase too quickly, and if not too many contracts are sold, then there will be a unique contract point. See Example 12 below for an example of a CfD portfolio and a normally distributed shock in the two-node network that gives rise to multiple contract points.

In a general loop-free network, where contracts are written at many nodes, the effective contract point for generator i offering at node m and holding a CfD portfolio $\{b_{i,n}\}_{n \in \mathcal{N}}$ is a price and quantity pair (p, q) where

$$p - C'_{i,m}(q) = 0 \text{ and } \phi_{m,m}q - \sum_n \phi_{m,n}b_{i,n} = 0. \quad (29)$$

In the two-node network, the contract points are solutions to $Y(Q_1, Q_2) = 0$, with Y ironing function defined in (27). This is a necessary but not sufficient condition, as we see in the example below.

Example 10 (Separation from forward trading node). Forward trading is only carried out at node 1. Node 2 generators sell forward using CfDs indexed at node 1 but the transmission constraint makes the contract an imperfect proxy for forward sales.

The plants, capacities and demand distribution are identical to the previous Example 8. Even the contracted amounts are the same, the only difference being that the node 2 generators now sell their CfDs at node 1.

The effective contract point is at $(60, 36.3)$, so the node 2 generators do not quite offset the full face value of their forward commitments in constructing their offer

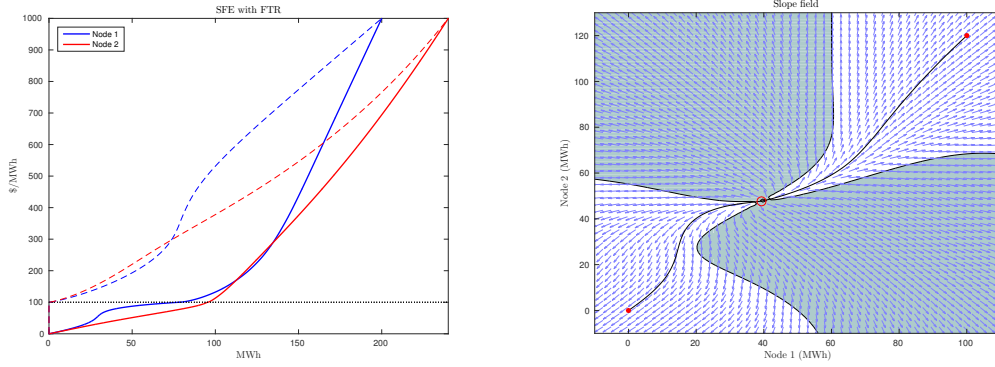


Figure 12: SFE with a small line; with FTRs (solid) and without (dashed).

Figure 11 shows the aggregate supply curves for each node on the left and the slope field with the candidate trajectory on the right. The effective contract point is circled. It is unique; the other crossing of $Y_1 = 0$ and $Y_2 = 0$ is on the line where \mathbf{A} is singular. The node 1 generators iron across the singularity in \mathbf{A} as in the previous example. The dashed supply curves are the supply curves from the SFE in the previous example, where node 2 generators sell CfDs at their own node. When they sell CfDs at their own node, the node 2 generators offer slightly more energy to the market across all prices, while the node 1 generators offer less competitively.

4.5 Two-way financial transmission rights

Financial transmission rights, like contracts for differences, divide into obligation (two-way) and option (one-way) contracts. In this section, we look at two-way FTRs, which can have positive or negative payoffs according to the direction of price difference.

A two-way FTR is a swap contract on nodal prices that can have positive or negative payoff. The payoff of an FTR or virtual asset swap of volume x from node m to node n is

$$x(p_n - p_m).$$

A two-way FTR reduces to a portfolio of CfDs, since an FTR from node m to node n can be constructed by buying a CfD at m and selling one at n , both for volume x . This portfolio of CfDs is additively separable in the prices, so we can apply Theorem 2. The first-variation derivative of an offer at node ℓ with respect to a single unit FTR from m to n is thus

$$Z_\ell^{FTR(m,n)} = x(\phi_{m,\ell} - \phi_{n,\ell}).$$

Option FTRs have obligation functions, which are not additively separable, and so cannot be written in the form (20). We derive the first-variation derivative for an option FTR in the following Section 5.

Example 11 (SFE with FTRs). Here we give two examples of SFE in which generators trade FTRs amongst themselves. They differ in the size of the transmission line.

There are two generators in each node. The node 1 plants each have a production capacity of 100MW and those in node 2 have 120MW. All plants have constant marginal cost of production of \$100/MWh. The demand shock distribution is an independent bivariate normal with means (120, 80)MW and standard deviations (60, 40)MW. Each generator holds an FTR for 60MW from their own local node to the other. The rights are complementary, so the generators can serve as counterparties to each other.

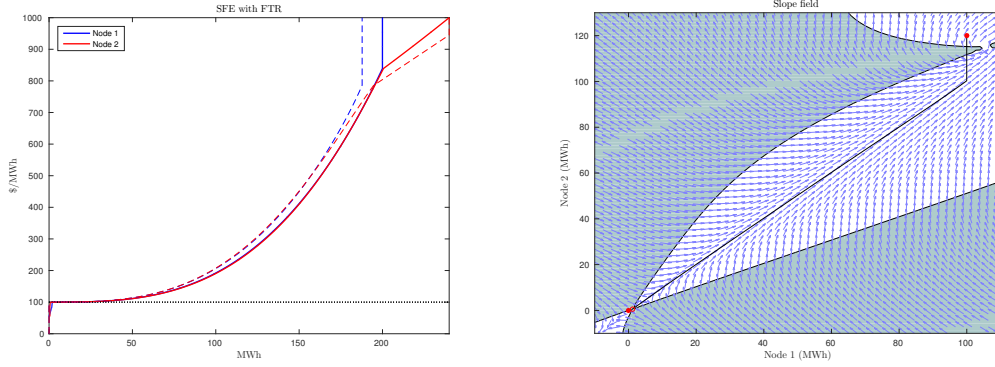


Figure 13: SFE with a large line; with FTRs (solid) and without (dashed).

In the small-line example, the line capacity is only 20MW. FTRs are like CfDs because the line is very often congested. Figure 12 shows the SFE and trajectory. The contract point is at (39.3, 47.8). Both components of this are less than 60MW because the generators consider the payoff from the other-node CfD part of the FTR as outside their control.

In the large-line example, the line capacity is 100MW, and the strategic effects of the FTRs are negligible. Figure 13 shows the SFE and trajectory. Because the line is so seldom congested, the expected payoff from the FTRs is almost zero, and so they make little difference to the equilibrium offers. In fact the SFE is almost identical to a pool market SFE where transmission constraints are ignored entirely.

Non-uniqueness of SFE under FTRs

As seen in Ruddell (2018), there can be multiple equilibrium trajectories in the two node model. In that example, the multiple trajectories were the result of the two generator types ironing in one order or another near the price cap. We will now examine a different sort of non-uniqueness that arises from multiple contract points when generators hold FTR portfolios.

There will be a unique contract point when only small amounts are traded or constrained distribution functions are flat, as in Lemma 9. However, with large volumes traded there can be multiple such points at the same price, even with constant $B'_{i,m}$, as we shall see in Example 12 below. An extra step we need to carry out in verifying uniqueness of SFE is to check that the contract point is unique in our particular market. Such contract points will appear as nodes in the slope-field diagram.

Example 12 (Multiple contract points). The demand shock distribution is as in Example 11. There are two generators in each node. Node 1 generators have production capacity of 20MW and node 2 generators have 40MW. The line has a transmission capacity of 74MW. Each generator holds a 60MW FTR from their own node to the other.

Figure 14 shows the zero contours of Y_1 , Y_2 and $|\mathbf{A}|$. The contract points are all the intersections of $Y_1 = 0$ and $Y_2 = 0$ where $|\mathbf{A}| \neq 0$, which are marked with red circles. There are three of them, located at

$$\begin{aligned} Q^A &= (8.7860, 8.3386), \\ Q^B &= (4.1772, 25.4681), \text{ and} \\ Q^C &= (8.4463, 37.9700). \end{aligned}$$

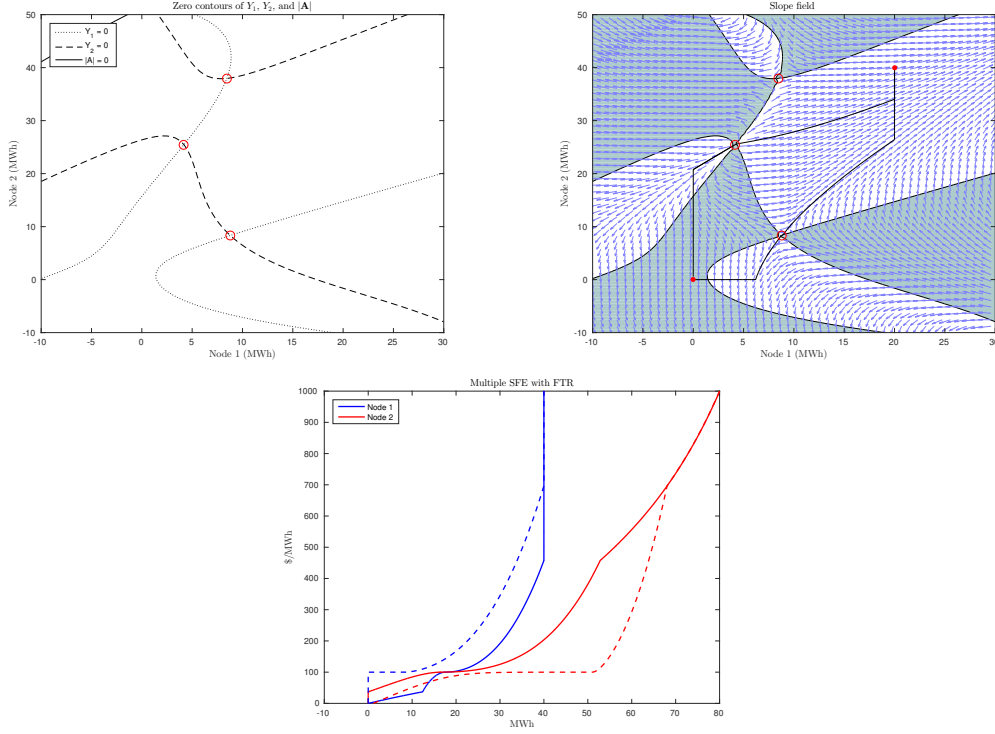


Figure 14: Multiple contract points, multiple SFE. Nodal aggregate supply curves are plotted for effective contract points Q^A (solid) and Q^B (dashed).

On the slope field diagram are traced equilibrium trajectories emanating from two of the contract points, Q^A and Q^B . The second-order optimality conditions³ are satisfied for both trajectories, so they give SFE. The nodal supply curves for the two SFE are plotted in the bottom of Figure 14. The dashed SFE is very flat near the contract point; for instance, the entire non-ironed segment of the trajectory between zero output at node 1 and the contract point is traversed over a price interval of $[99.96, 100]$.

The candidate trajectory from the third contract point Q^C is not able to be found using the Outward algorithm. This is because the set $Y_2 > 0$ is not simply connected. It has one component on whose boundary Q^C lies and another containing $(Q_1, Q_2) = (0, 0)$.

4.6 Case study: virtual asset swaps in New Zealand

In this section we illustrate the models by applying them to a regulatory action carried out in New Zealand a few years ago. New Zealand has two main islands, North and South, connected by an underwater HVDC cable. Meridian Energy is a generator, all of whose plants are hydro-electric plants in the mountains of the South Island. They have a retail arm, and some of their retail customers are located in the North Island. At certain times the HVDC cable runs at maximum capacity and the price of electricity in the North Island can greatly exceed the price in the South. Thus Meridian Energy runs the risk of having to sell electric power from their hydro-electric plants at a low price and buy electric power to serve their northern customers at a high price. This situation puts Meridian Energy at a competitive disadvantage in the northern retail market.

³Theorems 1 and 2 in Ruddell (2018)

Another generator-retailer is at a complementary disadvantage. Mighty River Power have all their hydro-electric plants on the Waikato River in the North Island, and also some customers in the South. They are at a disadvantage in the southern retail market and are wary of pursuing more business there. Due to the cascading river chains of the hydro systems — particularly the Waikato system which only has one reservoir of significance — redistributing ownership of the dams and stations would threaten the efficient management of those catchment basins. The New Zealand Electricity Authority decided instead, in 2010, to impose a virtual asset swap by forcing an exchange of contracts for differences between Meridian Energy and Mighty River Power. The contracts amount to swapping 80MW of virtual base-load generation (equal to roughly 2% of national generation) as a CfD with the same volume at every hour of the day. In effect, Meridian Energy acquires a financial transmission right for 80MW from south to north and Mighty River Power acquires a financial transmission right for 80MW from north to south.

Real asset dispersal has been supported by the regulators of the New Zealand electricity market as it has de-regulated. Contact Energy was the first part of the old ECNZ to be privatized, and was created with both South Island hydroelectric plants (on the Clutha) and thermal generation in the North Island. Also, at the same time as the 2010 virtual asset swap, Genesis energy purchased the Tekapo A and B hydro plants from Meridian Energy. Genesis previously only held North Island thermal generation. The hydro plants are at the head of the Waitaki system, which was previously entirely owned by Meridian. The sale is an exception to the rule of keeping river chains under unified management, justified by the large Benmore reservoir downstream of Pukaki, which remains under the control of Meridian.

Example 13 (Meridian and Mighty River virtual asset swap). Before the virtual asset swap, Meridian Energy offers a supply curve $q_{M,1}(p_1)$ at node 1, to maximize a profit functional Π^I . This profit functional has first-variation derivative Z^I . The virtual asset swap is a financial transmission right with volume $x = 80\text{MW}$, and its expected payoff Π^{swap} has first-variation derivative

$$Z^{swap} = (\phi_{S,S} - \phi_{N,S})x.$$

Consider the first-variation Z of the total profit $\Pi^I + \Pi^{swap}$, and the partial derivative of Z with respect to the swap volume:

$$\begin{aligned} Z &= Z^I + Z^{swap} = Z^I + (\phi_{S,S} - \phi_{N,S})x, \text{ and} \\ \frac{\partial Z}{\partial x} &= (\phi_{S,S} - \phi_{N,S}) > 0. \end{aligned}$$

Since $\frac{\partial Z}{\partial p} > 0$ and $\frac{\partial Z}{\partial q} < 0$, it is clear (by the implicit function theorem) that increasing x will make the optimal offer lower in price and higher in volume.

By a similar argument Mighty River Power also has an incentive to bid more competitively. After the swap Meridian has greater incentive (or lesser disincentive) to congest the line in the south-to-north direction and they do this by offering energy more competitively in the south. Mighty River Power, on the other island, has greater incentive to prevent the line from congesting in this direction, which they do by offering electric power more competitively in the north.

The net result of these shifted incentives should be that the wholesale market becomes more competitive in both islands. The retail market is also more competitive because the price-risks of selling retail in the other island are now better-hedged.

The above analysis is by comparison generators' optimality conditions. We can, in our two-node model, calculate SFE for generators under virtual and (from Section 3.1) real asset swaps. In the following Example 14, we compare dispersal of generation plants with virtual asset swaps in how they affect competition on the wholesale spot market.

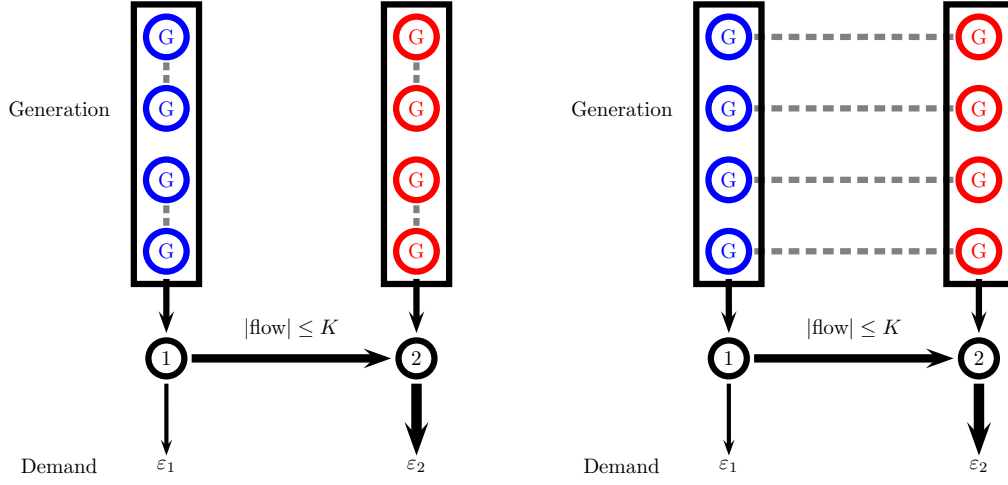


Figure 15: Plant ownership in Example 14. The two plant types are colored red and blue. They correspond to the two nodes. Grey dashed lines indicate co-ownership. In Scenarios 1 and 3, ownership is as on the left. In Scenario 2 it is as on the right.

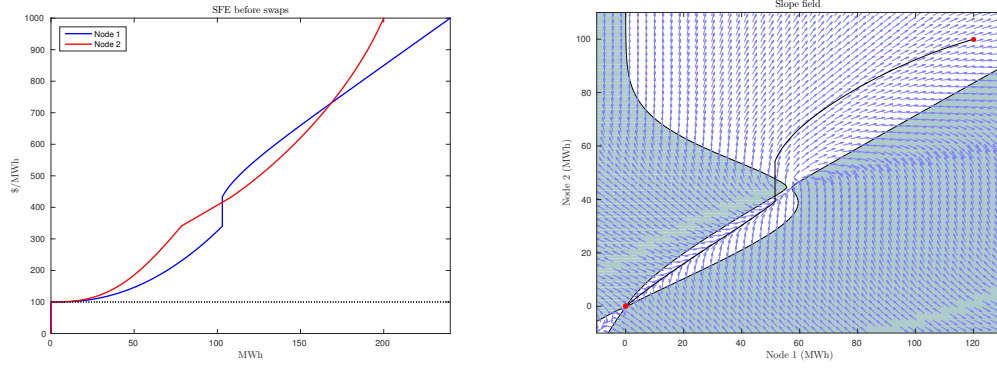


Figure 16: Two generators in each island

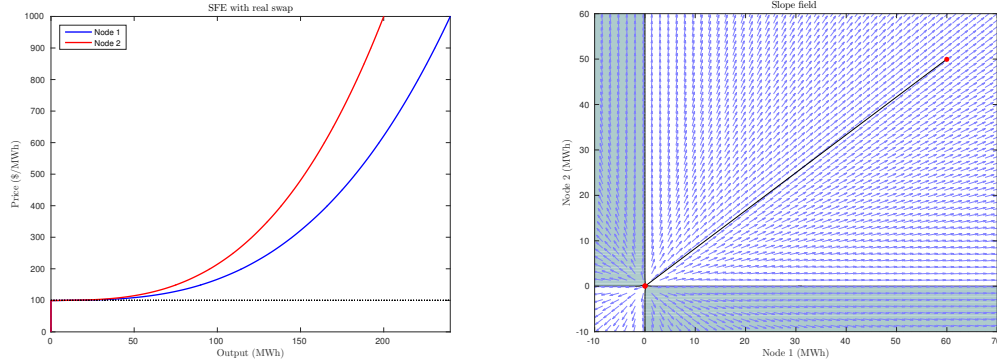


Figure 17: Real asset swap

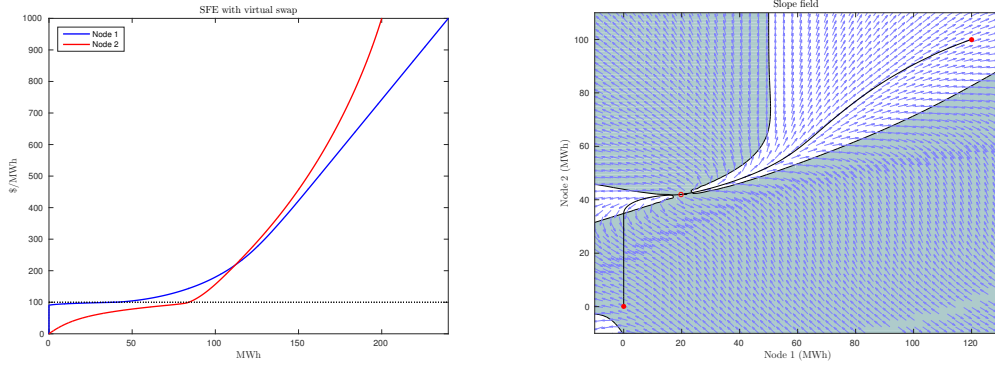


Figure 18: Virtual asset swap

Example 14 (Comparing real and virtual asset swaps). In this example we construct a simple model of the New Zealand market and examine the effects of either a real or virtual asset swap on the supply function equilibrium. There are four generators in the market and eight plants. All plants have identical and constant marginal cost of \$100/MWh. We model three scenarios:

1. Two generators in each island, each with two plants, no contracts.
2. Real asset swap: each of the four generators owns one plant in each island.
3. Virtual asset swap: two generators in each island as in Scenario 1. generators obtain FTRs from their island to the other, of total volume equal to expected nodal demand.

The two sorts of ownership arrangement are illustrated in Figure 15. The demand shock has an independent bivariate normal distribution with means (160, 40) and standard deviations (70, 30). There are eight plants, four in node 1 and four at node 2. The node 1 plants each have a production capacity of 60MW and the node 2 ones each have 50MW. All plants have constant marginal cost of production of \$100/MWh.

In Scenario 1, each generator has two plants in the same island. These are combined when they are offered to the spot market, so effectively there are two node 1 generators with 120MW capacity and two node 2 generators with 100MW.

In Scenario 2, each generator has an identical set of generating plants, one in node 1 with 60MW capacity and one in node 2 with 50MW.

In Scenario 3, the plant ownership is as in scenario 1, but the node 1 generators each have a 50MW virtual asset swap with a node 2 generator. The equivalent CfD portfolio is given by the following matrix of $b_{i,m}$:

$$\begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix} = \begin{bmatrix} 50 & -50 \\ -50 & 50 \end{bmatrix}.$$

The SFE are plotted in Figures 16–18.

Figure 16 shows the SFE without swap arrangements. We see some ironing and some withholding of energy to increase congestion.

Figure 17 shows the SFE after a real asset swap. The ownership arrangements are like in Figure 3, but with four plants in each node. All plants are offered with smooth supply functions. Each plant is always in direct competition with at least three other plants in this ownership arrangement.

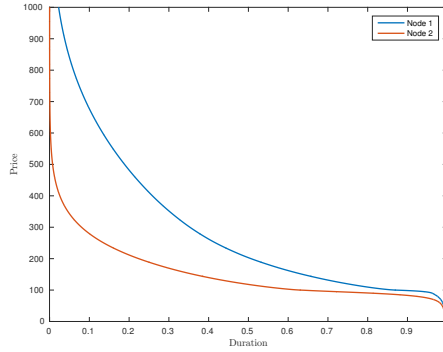


Figure 19: Nodal price duration curves with virtual asset swap.

Figure 18 shows the SFE after a virtual asset swap. There is almost as much improvement in competitiveness as with the real swap. The node 2 generators treat the virtual asset swap more like a CfD than the node 1 generators do. The contract point is at (19.8167, 41.9805). Above prices of \$400/MWh, the transmission constraint binds most of the time for node 1 generators, so they offer an almost straight supply function, like a local duopoly. Figure 19 shows the nodal price-duration curves under this SFE. We see that node 2 almost never experiences prices over \$400/MWh, even though node 1 does. This is a symptom of the congestion from node 2 to node 1.

We cannot say from these SFE solutions how the generators will rearrange other parts of their contracting strategy to compensate for the swaps. Nevertheless, this example suggests that virtual asset swaps are almost as effective as changing the ownership of plant at mitigating the exercise of market power arising from the ability to congest transmission lines.

5 Option FTRs

An option FTR from node m to node n with volume x will pay x times the difference in nodal prices $p_n - p_m$ if $p_n > p_m$, otherwise it pays nought. The payoff function (negative of the obligation function) for the holder of the contract is thus

$$-B(p) = x(p_n - p_m)^+ = x \cdot \max(p_n - p_m, 0).$$

This function is *not* additively separable in the nodal prices.

Unlike a two-way FTR, an option FTR has no downside beyond the price paid to purchase the option. If prices differ in the in-the-money direction, then the option payout is positive, otherwise it is zero. An obligation or two-way FTR, on the other hand, has the potential to become a large liability if price differences run in an unfavorable direction. For this reason, option FTRs are considerably more popular in real world trading than obligation FTRs. This is the reverse of the situation with CfDs, where options are considerably less popular than the two-sided contracts.

An option FTR from node m to node n is a sort of call option, while the option in the opposite direction is a put option, on the price difference $p_m - p_n$. An option FTR is in the money only in certain congestion states. We write $\Omega = \{\omega\}$ for the set of congestion states in the network. In the two-node network of Section 2, there are only three congestion states,

$$\Omega = \{\omega_1, \omega_2, \omega_3\};$$

the uncongested state is ω_1 , while ω_2 has the line congested in the $1 \rightarrow 2$ direction, and ω_3 has the line congested in the $2 \rightarrow 1$ direction. In general, in a loop-free network, with $n = |\mathcal{N}|$ nodes there will be $(n - 1)$ lines and

$$3^{n-1}$$

congestion states. Thus the number of congestion states grows geometrically with the number of nodes. An option FTR can be characterized by the two nodal prices it references and subset of congestion states in which it pays out.

5.1 Option FTRs in the two-node network

The key to optimal offer conditions for option FTRs is that they only pay out under certain congestion states of the network. We shall calculate first-order conditions for an option FTR between the two nodes in our two-node model from Section 2, with its lossless transmission line. The general principle of taking conditional expectations over congestion states will apply to any loop-free network, though in a larger network the set of congestion states in which an FTR pays out may contain more than one element.

Optimal offers with option FTRs

Recall also the three possible congestion states of the two-node network. In state ω_1 the line is uncongested. In state ω_2 it is congested from node 1 to node 2, while in state ω_3 it is congested from node 2 to node 1.

The payout of the option FTR contract is conditional on the congestion state of the network. An option FTR from node 1 to node 2 will only be exercised if the line is congested from node 1 to node 2, which is congestion state ω_2 . An option FTR from node 2 to node 1 is only exercisable in congestion state ω_3 . We shall calculate the first-variation derivative term for the holder of a call FTR from node 1 to node 2.

The expected payoff to a holder of such an FTR with volume x is

$$\Pi^{FTR(\omega_2)} = x \mathbb{E} \left[p_2 - p_1 \mid \omega = \omega_2 \right] \Pr(\omega_2). \quad (30)$$

By similar reasoning, we find that a generator holding an option FTR of volume x that pays out in congestion state ω_3 will have expected payoff

$$\Pi^{FTR(\omega_3)} = x \mathbb{E} \left[p_1 - p_2 \mid \omega = \omega_3 \right] \Pr(\omega_3). \quad (31)$$

The first-variation derivative of an option FTR's expected payout with respect to a supply function involves the joint density function $P_m(S, \omega)$ of the offers at node m 's local price and the congestion state.

Theorem 15. *The first-variation derivative of the expected payoff (31) with respect to a supply curve $\mathcal{S}_{i,m} = \{(q_{i,m}(t), p_m(t)) : t \in [0, T]\}$ offered at node $m \in \{1, 2\}$ is*

$$Z^{FTR(m,\omega)}(q_{i,m}, p) = \iota_{m,\omega} x P_m(S, \omega) \quad (32)$$

for $p \in (\underline{p}, \bar{p})$, where

$$\iota_{m,\omega} = \begin{cases} +1 & \text{if line congested away from node } m \text{ in state } \omega \\ -1 & \text{if line congested towards node } m \text{ in state } \omega. \end{cases}$$

The term (32) is supported by the following intuitions. The option FTR contract only pays out in the particular congestion state ω , but in this congestion state our generator at node m has no control over the price at the other node. The payout of the option FTR has two parts, one at the local price, and the other at the price of the other node. The holder of the FTR never has control over the other node price when the contract pays out, so they can only control the local, obligation, part of the contract. The ratio

$$\frac{P_m(S, \omega)}{\phi_{m,m}(S)}$$

is the conditional probability of congestion state ω given offers S at node m 's local price, so holding the option FTR is like holding a local CfD except that it only pays out in a particular congestion state. The sign function $\iota_{m,\omega}$ indicates whether the generator holds an option CfD directed towards $(-)$ or away from $(+)$ the node where they supply energy. This determines whether their payoff is improved by increasing or decreasing the local price.

Let $x_{i,m}^{\omega_2}$ and $x_{i,m}^{\omega_3}$ be the volume of option CfDs held by agent i at node m that pay out in congestion states ω_2 and ω_3 respectively. Then the first-order optimality condition for that agent's offer at node m is

$$\begin{aligned} Z_{i,m}(q_{i,m}, p) = & (p - C'_{i,m}) \left(\phi_{m,m} S'_{-i,m} + \sum_{n \neq m} \phi_{m,n} S'_n \right) - q_{i,m} \phi_{m,m} \\ & + \iota_{m,\omega_2} x_{i,m}^{\omega_2} P_m((q_{i,m} + S_{-i,m}, S_{-m}), \omega_2) \\ & + \iota_{m,\omega_3} x_{i,m}^{\omega_3} P_m((q_{i,m} + S_{-i,m}, S_{-m}), \omega_3). \end{aligned}$$

Under Assumption 3, together with the symmetry assumption that all agents at a given node m hold identical option FTRs, i.e.

$$x_{i,m}^\omega = x_m^\omega \text{ for all } i \text{ at } m,$$

we obtain the following system of ODEs for the equilibrium supply functions at each node:

$$Z = U\dot{Q} - VQ + B,$$

where

$$U = \begin{bmatrix} (N_1 - 1)\phi_{1,1} & N_2\phi_{1,2} \\ N_1\phi_{2,1} & (N_2 - 1)\phi_{2,2} \end{bmatrix}, \quad V = \begin{bmatrix} \phi_{1,1} & 0 \\ 0 & \phi_{2,2} \end{bmatrix}$$

and

$$B = \begin{bmatrix} x_1^{\omega_2} P_1(\cdot, \omega_2) - x_1^{\omega_3} P_1(\cdot, \omega_3) \\ -x_2^{\omega_2} P_2(\cdot, \omega_2) + x_2^{\omega_3} P_2(\cdot, \omega_3) \end{bmatrix}.$$

Define \mathbf{A} and \mathbf{Y} as in (26) and (27). Lemma 7 holds, so the inward and outward Algorithms from Ruddell (2018) can be applied.

Example 16 (SFE with option FTRs). Node 1 generators each buy some option FTRs from the transmission operator. There are two of them and they each have a production capacity of 40MW. The two generators in node 2 have a production capacity of 50MW. The line has a capacity of 20MW. The demand shock distributions are again independent and normal, with means (30, 30)MW and standard deviations (50, 30)MW.

Each generator in node 1 holds a 10MW option FTR from node 1 to node 2 so

$$x_1^{\omega_2} = 10,$$

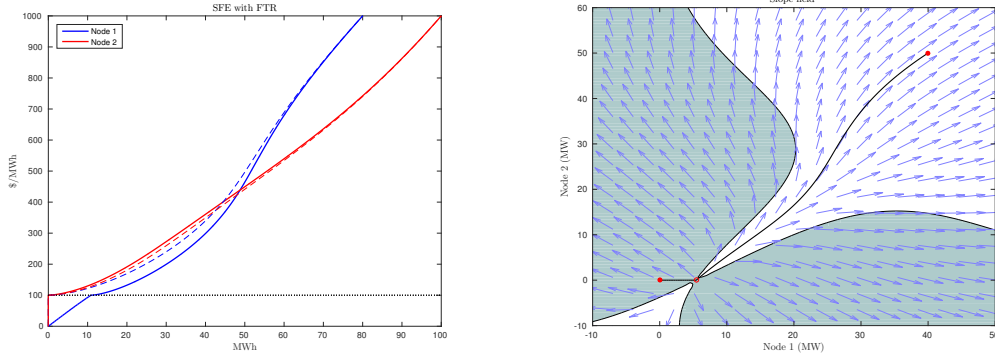


Figure 20: SFE with (solid) and without (dashed) FTRs. Slope field with FTRs

and no other contracts are held, so $x_1^{\omega_3} = x_2^{\omega_2} = x_2^{\omega_3} = 0$.

In Figure 20 we plot the SFE with (solid) and without (dashed) the option FTR. The effective contract point is at $(Q_1, Q_2) = (5.3, 0)$, which reflects that the line congests from node 1 to node 2 (congestion state ω_2) with probability 0.53. The node 1 generators (blue curve) mark down prices when they hold the FTR because their effective contracted volume increases. Because the slope of the node 1 supply functions Q'_1 is less with the option FTR, the offers of the node 2 generators (red curve) move slightly to the left.

In this example, the node 1 generators hold FTRs which are directed away from them, and the resulting incentive is to mark down offers. If they held FTRs directed towards them, we would obtain a result similar to those of Joskow and Tirole (2000) or Joung, Baldick, and Son (2008): that buying FTRs is equivalent to buying CfDs, which increases the un-contracted output of the generator, which increases the incentive to mark up offers.

6 Conclusion

In this paper, we derive optimality conditions for offering supply functions for dispersed generation plants. The supply curve of each plant accounts for the revenues earned by the other plants that a generator owns. We apply these optimality conditions to find SFE for three different plant-ownership arrangements in the two-node network.

The first-order conditions we derive for the two-node network are generalizable to larger loop-free (radial) networks. Even when we are unable to solve for a full SFE, a best-response analysis like in Example 13 can give insight into the strategic effect of a financial derivative or other contract.

We also calculate SFE for generators in the two-node network holding a portfolio of nodal-price derivatives. Nodal contracts for differences, when they are indexed at the same node as a generator supplies energy to the network, are equivalent to forward sales. For CfDs traded at a node distant from a generator's point of injection, the effective equivalent forward sales volume is mitigated by the probability of congestion between the forward sale and spot sale locations.

We found that certain portfolios of nodal CfDs can lead to multiple effective contract points, and non-unique SFE. We examined virtual asset swaps, which have been enforced by the New Zealand regulator in cases where changing the real ownership of generation assets was ruled out. We find that the virtual asset swaps are effective at reducing incentives to create local market power, and that they are almost as effective at this as real asset swaps.

Hogan’s (1992) financial transmission rights have obligation functions equal to a particular portfolio of nodal CfDs. Holding an FTR from a source to a sink is equivalent to writing a CfD at the source and buying a CfD at the sink node.

Option financial transmission rights only pay out when congestion occurs in a particular direction. Hence in the two-node network they only pay out in a particular congestion state of the network. We derived optimality conditions for generators holding option FTRs, in terms of the joint distribution of price and congestion state. We also solved SFE for generators holding option FTRs in the two-node network.

An important contribution of the present work is to show non-uniqueness of SFE in a network setting. Uniqueness of equilibrium is a desirable property in market models. Lemma 9 tells us that when the network model is a small perturbation of the pool market, uniqueness results will hold, but for large perturbations network effects can destroy the uniqueness of solutions. We find two sorts of non-uniqueness in the two-node model. In Example 12, the nodal offers net of contracts weighted by the probability of uncongested flow can have multiple zeros when generators hold FTRs or CfDs at nodes other than where they supply the grid. Each zero gives rise to a distinct candidate trajectory. In both these examples, all the candidate trajectories can be equilibria.

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Appendix

Proof of Theorem 2. To take the partial derivative of Π^n with respect to the curve $\mathcal{S}_{i,m}$, we first write (11) in the form

$$\Pi^n[S_{i,m}, \cdot] = \int_0^T (u(q_{i,m}, p_m) \dot{q}_{i,m} + v(q_{i,m}, p_m) \dot{p}_m) dt. \quad (33)$$

In a lossless loop-free network, the market distribution functions depend on the offers at other nodes only when the prices match. Without loss of generality we can set the price parts of all generators' supply curves at all nodes to be equal

$$p_n(t) = p(t) \text{ for all nodes } n.$$

Also notice that, since R_n is a function of $q_{i,n}$ and p_n , we have

$$\frac{\partial R_n}{\partial q_{j,\ell}} = 0$$

for every $(j, \ell) \neq (i, n)$.

Recall the notation for the total net nodal supply

$$\begin{aligned} S_m(p) &:= \sum_{j \in m} q_{j,m}(p) - D_m(p), \\ S_{-m} &:= \{S_n\}_{n \neq m}, \text{ and} \\ S_{-i,m}(p) &:= \sum_{j \in m, j \neq i} q_{j,m}(p) - D_m(p). \end{aligned}$$

We can rewrite $\psi_{i,n}$, with its dependence on all other offers as a functional, in terms of ϕ_n , which depends on the other offers as a function of the amount offered at the common price p

$$\psi_{i,n}(q_{i,n}, p; \mathcal{S}_{(-n)}) = \phi_n(q_{i,n} + S_{-i,n}(p), S_{-n}(p)).$$

Note that ϕ_n is a function of the offer quantity at every node, in particular the node m . We may rewrite the arguments of ϕ_n to emphasize this:

$$\psi_{i,n}(q_{i,n}, p) = \phi_n(q_{i,n} + S_{-i,n}(p), S_{-n}(p)) = \phi_n(q_{i,m} + S_{-i,m}(p), S_{-m}(p)).$$

We expand out $\frac{d\psi_{i,n}}{dt}$ in terms of the control variables $q_{i,m}$ and p

$$\frac{d\psi_{i,n}}{dt} = \phi_{n,m} \dot{q}_{i,m} + \left(\phi_{n,m} S'_{-i,m}(p) + \sum_{\ell \neq m} \phi_{n,\ell} S'_\ell(p) \right) \dot{p},$$

giving

$$\begin{aligned} u(q_{i,m}, p) &= R_n(q_{i,n}(p), p) \phi_{n,m} \\ v(q_{i,m}, p) &= R_n(q_{i,n}(p), p) \left(\phi_{n,m} S'_{-i,m}(p) + \sum_{\ell \neq m} \phi_{n,\ell} S'_\ell(p) \right) \end{aligned}$$

in the profit functional (33). Note that, because the supply curve $\mathcal{S}_{i,n}$ is fixed, we treat $q_{i,n}$ as a function of p here, like all the other summands that go into $S_n(p)$.

We calculate the necessary partial derivatives, remembering the arguments $(q_{i,m} + S_{-i,m}(p), S_{-m}(p))$ of $\phi_{n,m}$ and $\phi_{n,\ell}$,

$$\frac{\partial u}{\partial p} = \phi_{n,m} \left(\frac{\partial R_n}{\partial p} + q'_{i,n}(p) \frac{\partial R_n}{\partial q_{i,n}} \right) + R_n(q_{i,n}(p), p) \left(\phi_{n,m,m} S'_{-i,m}(p) + \sum_{\ell \neq m} \phi_{n,m,\ell} S'_\ell(p) \right)$$

and

$$\frac{\partial v}{\partial q_{i,m}} = R_n(q_{i,n}(p), p) \left(\phi_{n,m,m} S'_{-i,m}(p) + \sum_{\ell \neq m} \phi_{n,\ell,m} S'_\ell(p) \right)$$

Hence, applying

$$Z_m^n = \frac{\partial v}{\partial q_{i,m}} - \frac{\partial u}{\partial p},$$

we obtain (13) □

Proof of Lemma 4. Suppose $Y(Q) = 0$. Then

$$\begin{aligned} (u_{22}v_{11} - u_{12}v_{21})Q_1 + (u_{22}v_{12} - u_{12}v_{22})Q_2 &= 0 \text{ and} \\ (u_{11}v_{21} - u_{21}v_{11})Q_1 + (u_{11}v_{22} - u_{21}v_{12})Q_2 &= 0. \end{aligned}$$

This implies that if one of Q_1, Q_2 is zero, the other is too. Suppose both are non-zero. Then

$$-|\mathbf{U}| \begin{bmatrix} \frac{1}{u_{22}} & 0 \\ 0 & \frac{1}{u_{11}} \end{bmatrix} \mathbf{U}^{-1} \mathbf{V}$$

is a singular matrix. Since $|\mathbf{V}| > 0$, it must be that $|\mathbf{U}| = 0$, but then $|\mathbf{A}| = 0$ by (18).

For the “furthermore” part, substitute $\dot{Q} = \mathbf{A}^{-1}Q$ into (19) to obtain

$$Y(Q) = -|\mathbf{U}| \begin{bmatrix} \frac{1}{u_{22}} & 0 \\ 0 & \frac{1}{u_{11}} \end{bmatrix} \dot{Q},$$

from which it is clear that if both components of \dot{Q} are positive, then $|\mathbf{U}|$ and the Y_1, Y_2 must have opposite signs. Since $|\mathbf{V}| > 0$, the sign of $|\mathbf{A}|$ is the same as that of $|\mathbf{U}|$. □

Proof of Lemma 7. Write out $\dot{Q} = \mathbf{U}^{-1}(\mathbf{V}Q - B)$ in full:

$$\begin{aligned} \dot{Q} &= \frac{1}{|\mathbf{U}|} \begin{bmatrix} (N_2 - 1)\phi_{2,2} & -N_2\phi_{1,2} \\ -N_1\phi_{2,1} & (N_1 - 1)\phi_{1,1} \end{bmatrix} \begin{bmatrix} \phi_{1,1}Q_1 + \phi_{1,1}b_{1,1} + \phi_{1,2}b_{1,2} \\ \phi_{2,2}Q_2 + \phi_{2,1}b_{2,1} + \phi_{2,2}b_{2,2} \end{bmatrix} \\ &= \frac{1}{|\mathbf{U}|} \begin{bmatrix} (N_2 - 1)\phi_{2,2}(\phi_{1,1}Q_1 + \phi_{1,1}b_{1,1} + \phi_{1,2}b_{1,2}) - N_2\phi_{1,2}(\phi_{2,2}Q_2 - \phi_{2,1}b_{2,1} - \phi_{2,2}b_{2,2}) \\ -N_1\phi_{2,1}(\phi_{1,1}Q_1 - \phi_{1,1}b_{1,1} - \phi_{1,2}b_{1,2}) + (N_1 - 1)\phi_{1,1}(\phi_{2,2}Q_2 + \phi_{2,1}b_{2,1} + \phi_{2,2}b_{2,2}) \end{bmatrix} \\ &= \frac{1}{|\mathbf{U}|} \begin{bmatrix} -(N_2 - 1)\phi_{2,2} & 0 \\ 0 & -(N_1 - 1)\phi_{1,1} \end{bmatrix} Y(Q). \end{aligned}$$

Since $(N_1 - 1)$, $(N_2 - 1)$, $\phi_{1,1}$, and $\phi_{2,2}$ are all positive, it follows that when both components of \dot{Q} are positive, Y_1 and Y_2 have opposite sign to $|\mathbf{U}|$. □

In order to prove Theorem 15 below, we require the following lemma on the P_n functions from Ruddell (2018).

Recall that, for a node n , $\Gamma_{n,\omega}(S)$ are the sets of demand shocks ε for which the market clears in congestion state ω and the total offers at all nodes at the price $p = p_n$ is the vector S . The conditional density $P_n(S, \omega)$ is the integral density of the shock density over $\Gamma_{n,\omega}$:

$$P_n(S, \omega) = \int_{\Gamma_{n,\omega}} df(\varepsilon). \quad (34)$$

Lemma 17. For the functions P_1 and P_2 , defined in (34), the following identity holds

$$\frac{\partial P_1}{\partial S_2}(S, \omega_2) = -\frac{\partial P_2}{\partial S_1}(S, \omega_2).$$

Proof. Calculate the left-hand side

$$\begin{aligned} P_1(S, \omega_2) &= \int_{S_2+K}^{\infty} f(S_1 - K, t) dt \\ \frac{\partial P_1}{\partial S_2}(S, \omega_2) &= -f(S_1 - K, S_2 + K) \end{aligned}$$

and the right-hand side

$$\begin{aligned} P_2(S, \omega_2) &= \int_{-\infty}^{S_1-K} f(t, S_2 + K) dt \\ \frac{\partial P_2}{\partial S_1}(S, \omega_2) &= f(S_1 - K, S_2 + K), \end{aligned}$$

and the result is clear. \square

Proof of Theorem 15. We prove the case $m = 1$, $\omega = \omega_2$; the other cases only differ by a change of sign. We first write (30) as an integral over prices, plus some extra terms for the price floor and price cap. Expectation is linear, so

$$\begin{aligned} \Pi^{FTR(1, \omega_2)} &= x \mathbb{E}[p_2 | \omega_2] \Pr(\omega_2) - x \mathbb{E}[p_1 | \omega_2] \Pr(\omega_2) \\ &= \int_{\underline{p}}^{\bar{p}} (x p h_2(p | \omega_2) \Pr(\omega_2) - x p h_1(p | \omega_2) \Pr(\omega_2)) dp \\ &\quad + x \underline{p} \Pr[p = \underline{p} \wedge \omega_2] + x \bar{p} \Pr[p = \bar{p} \wedge \omega_2], \end{aligned} \tag{35}$$

where $h_m(p | \omega_2)$ is the conditional density of the price at node m , given the congestion state ω_2 . The *scrap-value terms* $x \underline{p} \Pr[p = \underline{p} \wedge \omega_2]$ and $x \bar{p} \Pr[p = \bar{p} \wedge \omega_2]$ do not appear in the first-variation derivative at prices in the interior of $[\underline{p}, \bar{p}]$ (see, e.g. Seierstad and Sydsaeter 1987).

To derive the joint distribution of price and congestion state, we start by manipulating the marginal dispatch probabilities $P_m(S, \omega)$, defined in (34) as integrals of the shock density over the sets $\Gamma_{m, \omega}(S)$. When the price at node m increases infinitesimally, the set $\Gamma_{m, \omega}(S)$ moves infinitesimally in the direction of the increase in offer quantities at nodes connected to node m by an uncongested path in congestion state ω . By summing over all the nodes connected to node m , we obtain the joint density function of node m prices and congestion state

$$h_m(p, \omega) = P_m(S(p), \omega) \sum_{n \sim_{\omega} m} S'_n(p), \tag{36}$$

where $n \sim_{\omega} m$ denotes that node n is connected to node m by an uncongested path in congestion state ω .

By definition of conditional probability and (36),

$$\begin{aligned} h_m(p | \omega_2) \Pr(\omega_2) &= h_m(p, \omega_2) \\ &= P_m(S(p), \omega_2) \sum_{n \sim_{\omega_2} m} S'_n(p). \end{aligned}$$

In congestion state ω_2 in the two-node network, the one transmission line is congested, so each node is connected by an uncongested path only to itself. Hence

$$\begin{aligned} h_1(p|\omega_2) \Pr(\omega_2) &= P_1(S(p), \omega_2) S'_1(p) \text{ and} \\ h_2(p|\omega_2) \Pr(\omega_2) &= P_2(S(p), \omega_2) S'_2(p). \end{aligned} \quad (37)$$

We wish to take the first-variation derivative of $\Pi^{FTR(1, \omega_2)}$ with respect to the supply curve $\mathcal{S}_{i,1}$. To do this we must transform the integral term in (35) into one with an integrand of the form

$$\Pi^{FTR(1, \omega_2)}[\mathcal{S}_{i,1}] = \int_0^T \left(u(q_{i,m}, p) \dot{q}_{i,m} + v(q_{i,m}, p) \dot{p} \right) dt, \quad (38)$$

for some functions u and v . The first variation is then given by

$$Z^{FTR(1, \omega_2)}(q_{i,m}, p) = \frac{\partial v}{\partial q_{i,m}} - \frac{\partial u}{\partial p}. \quad (39)$$

By (37), we can write the integral term in (35) as

$$\begin{aligned} I &= \int_{\underline{p}}^{\bar{p}} x p h_2(p|\omega_2) \Pr(\omega_2) - x p h_1(p|\omega_2) \Pr(\omega_2) dp \\ &= \int_{\underline{p}}^{\bar{p}} x p \left(P_2(S, \omega_2) S'_2(p) - P_1(S, \omega_2) S'_1(p) \right) dp. \end{aligned}$$

Without loss of generality, we can suppose that all generators' supply curves have identical price components. We make the change of variable $p = p(t)$, to obtain

$$I = \int_0^T x p \dot{p} \left(P_2(S, \omega_2) S'_2(p) - P_1(S, \omega_2) S'_1(p) \right) dt.$$

The total offer at node 1 is

$$S_1 = q_{i,1} + S_{-i,1}(p).$$

So

$$\dot{p} S'_1(p) = \dot{q}_{i,1} + \dot{p} S'_{-i,1}(p),$$

which gives

$$I = \int_0^T x \dot{p} p \left(P_2(S, \omega_2) S'_2(p) - P_1(S, \omega_2) S'_{-i,1}(p) \right) - x \dot{q}_{i,1} p P_1(S, \omega_2) dt.$$

Thus, besides the scrap-value terms, the expected FTR payoff functional for a generator i offering a supply curve at node 1 (30) has an integrand of the form (38), with

$$\begin{aligned} u &= -x p P_1(S, \omega_2) \text{ and} \\ v &= x p \left(P_2(S, \omega_2) S'_2(p) - P_1(S, \omega_2) S'_{-i,1}(p) \right) \end{aligned}$$

(a generator at node 2 would have a positive u at this point in the calculation). We take partial derivatives with respect to the controls

$$\frac{\partial u}{\partial p} = -x P_1(S, \omega_2) - x p \left(S'_{-i,1}(p) \frac{\partial P_1}{\partial S_1}(S, \omega_2) + S'_2(p) \frac{\partial P_1}{\partial S_2}(S, \omega_2) \right)$$

and

$$\frac{\partial v}{\partial q_{i,1}} = x p \left(\frac{\partial P_2}{\partial S_1} (S, \omega_2) S'_2(p) - \frac{\partial P_1}{\partial S_1} (S, \omega_2) S'_{-i,1}(p) \right).$$

Now apply (39) to obtain the first-variation derivative

$$\begin{aligned} Z_1^{FTR(1, \omega_2)}(q_{i,1}, p) &= \frac{\partial v}{\partial q_{i,1}} - \frac{\partial u}{\partial p} \\ &= x P_1(S, \omega_2), \end{aligned} \tag{40}$$

using the fact that $\frac{\partial P_1}{\partial S_2} = -\frac{\partial P_2}{\partial S_1}$ from Lemma 17. \square