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## Steady states, stability and bifurcations in multi-asset market models

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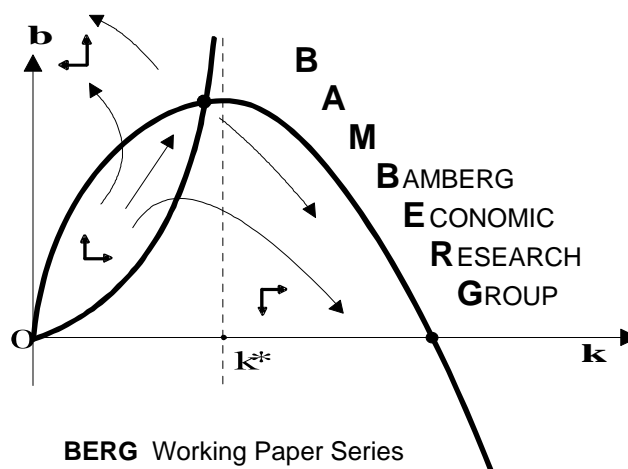
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# Steady states, stability and bifurcations in multi-asset market models

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# Steady states, stability and bifurcations in multi-asset market models<sup>☆</sup>

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## Abstract

We provide a full analytical treatment of a multi-asset market model in which speculators have the choice between two risky and one safe asset. As it turns out, the dynamics of our model is driven by a four-dimensional nonlinear map and may undergo a transcritical, flip or Neimark-Sacker bifurcation. While the first bifurcation is associated with an undervaluation of the risky assets, the latter two may trigger (complex) endogenous dynamics. To facilitate our analysis, we first study a simpler two-dimensional setup of our model in which speculators can only switch between one risky and one safe asset.

*Keywords:* Multi-asset markets, replicator dynamics, nonlinear maps, stability and bifurcation analysis

*JEL classification:* D84; G12; G41

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## 1. Introduction

We first explore a multi-asset market model in which speculators can either invest their money in a risky or safe asset. Speculators repeat their investment decisions every period depending on the attractiveness of the risky asset relative to the safe asset. In particular, the attractiveness of the risky asset depends on its momentum and mispricing. Another important feature of our model is that

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the price of the risky asset increases with the number of speculators who invest in it. The model, represented by a two-dimensional nonlinear map, admits two steady states: a fundamental steady state (FSS) in which the price of the risky asset mirrors its fundamental value and a nonfundamental steady state (NFSS) in which the price of the risky asset is either overvalued or undervalued. We show analytically that a transcritical bifurcation may cause a stability exchange between the FSS and the NFSS. Such a bifurcation may occur if the total number of speculators falls below a critical threshold. Since the trading strength of speculators is then limited, the price of the risky asset remains below its fundamental value – a scenario reminiscent of the famous limits of arbitrage argument by Shleifer and Vishny (1997). Moreover, the FSS may also become unstable due to a flip or Neimark-Sacker bifurcation. These bifurcation scenarios may occur if speculators’ participation in the risky asset market depends strongly on the mispricing or price trend of the risky asset. Numerical evidence indicates that the risky asset market is then subject to (complex) endogenous dynamics.

After having established these results for a rather simple setup, we add an additional risky asset to our framework. Despite the increased dimension of our new model – its dynamics is now driven by a four-dimensional nonlinear map – we are still able to show that the model’s FSS may exchange its stability with a NFSS via a transcritical bifurcation. However, our analysis also reveals that the NFSS may lose its stability via a flip or Neimark-Sacker bifurcation (which is not possible in the case of one risky asset). In particular, simulations reveal that the Neimark-Sacker bifurcation of the NFSS is associated with the emergence of endogenous dynamics occurring below the risky assets’ fundamental values. Put differently, limits of arbitrage prevent prices of risky assets from reaching their fundamental values, either in a steady-state environment or a dynamic context. Moreover, the FSS may also lose its stability via a flip or Neimark-Sacker bifurcation. For some parameter combinations, we observe quite complex asset price dynamics.

Note that our paper extends our previous work. Our two-dimensional multi-asset market model may be regarded as a generalization of the asset-pricing

model by Schmitt and Westerhoff (2016). In particular, Schmitt and Westerhoff (2016) use simple linear functions to describe the attractiveness of the risky asset relative to the safe asset while we use a fairly general nonlinear specification. In addition, they explicitly focus on the implications of flip and Neimark-Sacker bifurcations and disregard the transcritical bifurcation and the relationship between the FSS and the NFSS. Our four-dimensional multi-asset market model represents a generalization of the asset-pricing model by Dieci et al. (2018). Again, our setup is more general since it explicitly recognizes the effects of total market participation. Moreover, we are now able to provide a complete analytical treatment of the transcritical bifurcation for the underlying four-dimensional map. As we will see, this novel proof has a number of interesting economic implications. Finally, the current paper also provides a natural bridge between the one-risky-asset model and the two-risky-asset model, thereby fostering our understanding of multi-asset market dynamics.

Our paper is also part of a larger stream of literature seeking to explain the dynamics of financial markets via the interplay of heterogeneous and boundedly rational speculators. In most of these models, speculators only have access to one risky asset. Interesting dynamics may arise in them nevertheless since speculators switch between competing trading rules. Pioneering contributions in this direction include Day and Huang (1990), Chiarella (1992), Lux (1995) and Brock and Hommes (1998). So far, relatively few models have considered the fact that speculators are usually active in more than one risky asset market. For examples in this direction, see Chiarella et al. (2005, 2007), Westerhoff and Dieci (2006) and Schmitt and Westerhoff (2014). Our multi-asset market models differ quite substantially from these contributions. While almost all other models assume that speculators have sufficient funds to push asset prices towards their fundamental values, we assume that their financial means are restricted. Limits of arbitrage may thus result in a permanent undervaluation of risky assets.

The remainder of our paper is organized as follows. In Section 2, we explore some properties of our two-dimensional multi-asset market model. Equipped

with these insights, we then proceed to investigate our four-dimensional multi-asset market model in Section 3. Finally, we conclude our paper and point out some avenues for future research.

## 2. A financial market model with one risky asset

### 2.1. Model setup

Our analysis in this section is based on a generalized version of the asset-pricing model by Schmitt and Westerhoff (2016) in which speculators can choose whether to enter a market for a risky asset. Their market entry decisions are repeated at the beginning of each period and depend on observed price trends and fundamental conditions. Alternatively, speculators can invest their money in a safe asset. Since the price of the risky asset adjusts such that the market clears in every period, we have  $Q_t = S_t$ , where  $Q_t$  and  $S_t$  denote total demand and supply of the risky asset in the market at time step  $t$ , respectively. For simplicity, the total supply of the risky asset is assumed to be fixed over time, i.e.  $S_t = X$  with  $X > 0$ . Moreover, we assume that speculators are willing to invest a fixed amount of money in the risky asset market. Let  $I > 0$  represent their monetary engagements and  $P_t$  the unit price of the risky asset at time step  $t$ . Speculators' individual demands can then be expressed by the isoelastic demand function  $q_t = I/P_t$ . Thus, the total demand for the risky asset amounts to  $Q_t = q_t n_t$ , where  $n_t$  represents the number of speculators who are active in the risky asset market. Note that combining the above assumptions yields

$$P_t = \alpha n_t. \tag{1}$$

Since  $\alpha = \frac{I}{X} > 0$ , the price of the risky asset depends positively on market participation and investors' financial commitment.

Following Shiller (2015), speculators' decisions whether to enter the risky asset market depends on its momentum and mispricing. Accordingly, we model the relative attractiveness of the risky asset market by

$$A_t = f(\rho_t) + g(\delta_t). \tag{2}$$

Note that  $f$  and  $g$  are strictly increasing functions with  $f(0) = g(0) = 0$  and  $f' > 0, g' > 0$ . While  $\rho_t = \frac{P_t - P_{t-1}}{P_{t-1}}$  represents the relative price change of the risky asset,  $\delta_t = \frac{D}{P_t} - r$  captures the dividend-price ratio of the risky asset relative to an investment in a safe asset, where  $D$  and  $r$  denote constant dividend payments and the return of a safe asset, respectively. Hence, the first term of (2) indicates that the stronger the current price of the risky asset increases (decreases), the more (less) attractive the risky asset market. However, the second term suggests that increasing (decreasing) risky asset prices decrease (increase) the relative fundamental gain potential of the risky asset market, which makes it less (more) attractive. Speculators who are not active in the risky asset market invest their money at the constant rate  $r$ , and we assume that the attractiveness of this alternative is 0.

To describe the number of active speculators in the risky asset market, we follow Hofbauer and Sigmund (1988) and use exponential replicator dynamics, i.e.

$$n_{t+1} = N \frac{n_t}{n_t + (N - n_t) \exp[-\lambda A_t]}, \quad (3)$$

where  $N > 0$  stands for the total number of speculators and  $\lambda > 0$  represents speculators' intensity of choice. Accordingly, an increase in the relative attractiveness of the market leads to an increase in market participation; and the increase in market participation is stronger as speculators' intensity of choice increases. In particular, if  $\lambda$  goes to plus infinity, either none or all investors will enter the risky asset market, depending on whether its attractiveness is below or above the attractiveness of the safe asset. In contrast, if  $\lambda$  approaches zero, half of the investors will enter the risky asset market and the other half will choose the safe asset market, independently of their relative fitness.<sup>1</sup>

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<sup>1</sup>As we will see in the sequel, one attractive feature of exponential replicator dynamics is that it ensures a steady state in which the price of the risky asset mirrors its fundamental value, provided that investors have sufficient funds. See Dindo and Tuinstra (2011) for an insightful discussion of exponential replicator dynamics and Bischi et al. (2015) and Schmitt et al. (2017) for recent applications. However, Agliari et al. (2018) study a related stock market



## 2.2. Dynamical system and steady states

Due to (1), (2) and the above definitions of  $\rho_t$  and  $\delta_t$ , it turns out that attractiveness  $A_t$  depends on  $n_t$  and  $n_{t-1}$ . By setting  $x_t := n_t/N$ ,  $z_t := n_{t-1}/N$ , the recurrence relation (3) can be rewritten as a two-dimensional (2D) dynamical system in investors' proportions  $x_t$  and  $z_t$ :

$$x_{t+1} = F(x_t, z_t) := \frac{x_t}{x_t + (1 - x_t) \exp(-\lambda A(x_t, z_t))}, \quad z_{t+1} = x_t, \quad (4)$$

where

$$A(x_t, z_t) := f\left(\frac{x_t}{z_t} - 1\right) + g\left(\frac{D}{\alpha N x_t} - r\right).$$

As shown in the Appendix, the model admits two steady states:

(i) A fundamental steady state (FSS), where the attractiveness of the risky asset market is zero, while asset market participation and the price of the risky asset are given as

$$x^* = z^* = \frac{D}{\alpha N r}, \quad n^* = N x^* = \frac{D}{\alpha r}, \quad P^* = \alpha n^* = \frac{D}{r}, \quad (5)$$

provided that  $x^* < 1$ , i.e.

$$r > \frac{D}{\alpha N} := r_l. \quad (6)$$

At the FSS, the price of the risky asset reflects the present value of the dividend stream or, put differently, its dividend-price ratio is equal to  $r$ . However, the FSS is (economically) feasible only for a sufficiently high interest rate (i.e.  $r > r_l$ ) or - given  $r$  and  $D$  - if total market participation  $N$  and trading strength  $\alpha$  are sufficiently large.

(ii) A nonfundamental steady state (NFSS), where  $\hat{x} = \hat{z} = 1$ ,  $\hat{n} = N$  and  $\hat{P} = \alpha N$ , implying that the dividend-price ratio at the NFSS is equal to  $r_l$ . Unlike the FSS, this boundary steady state always exists. If the FSS is feasible, then  $\hat{P} > D/r := P^*$ , that is, the NFSS entails a kind of price bubble in this

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participation model on the basis of the discrete choice approach by Brock and Hommes (1998). To improve our understanding of multi-asset market dynamics, future work may also consider the transition probability approach by Lux (1995).

case. However, the attractiveness at the NFSS is given by  $\hat{A} = g(r_l - r)$  (where  $\hat{A} \neq 0$ , unless  $r = r_l$ ). Therefore, while  $\hat{A}$  is positive if the FSS is unfeasible, it is negative whenever an interior FSS exists.

### 2.3. Steady-state stability and bifurcations with one risky asset

The Appendix presents the analytical derivation of the stability conditions and local bifurcations of the FSS and NFSS. The main results are summarized by the following propositions, each followed by a brief discussion. Note that parameters  $\beta := f'(0)$  and  $\gamma := g'(0)$  denote sensitivity to observed price trends and mispricing at the FSS, respectively.

**Proposition 1** The FSS is locally asymptotically stable (LAS) in the region of the parameter space given as:

$$\beta_l^{(I)} := \frac{r}{2}\gamma - \frac{1}{\lambda} \frac{1}{1-x^*} < \beta < \frac{1}{\lambda} \frac{1}{1-x^*} := \beta_u^{(I)}, \quad (7)$$

where  $x^*$  is defined by (5). In addition, if  $\beta = \beta_l^{(I)}$  ( $\beta = \beta_u^{(I)}$ ) a flip bifurcation (Neimark-Sacker bifurcation) takes place.

Generally speaking, the flip bifurcation can be observed whenever  $\gamma$  becomes sufficiently large, which entails a strong reaction to fundamental mispricing. Numerical evidence indicates that the flip bifurcation is supercritical and that a stable orbit of period 2 replaces the destabilized steady state, possibly followed by a route to chaos. The Neimark-Sacker bifurcation takes place whenever (for not too large  $\gamma$ ) aggregate parameter  $\lambda\beta$  crosses  $1/(1-x^*) = \alpha Nr/(\alpha Nr - D)$ . Note that, in the case of an interior FSS,  $\alpha N > D/r := P^*$  and that quantity  $1/(1-x^*)$  decreases from  $+\infty$  to 1 when aggregate parameter  $\alpha N$  ranges over  $(D/r, +\infty)$ . Therefore,  $\lambda\beta < 1$  is a *sufficient* condition for stability (although *not necessary*). For given  $r$  and  $D$ , the system can be destabilized via a Neimark-Sacker bifurcation if  $\lambda$  (switching intensity) or  $\beta$  (sensitivity to observed price trends) becomes sufficiently large. The same bifurcation can also occur for increasing  $\alpha$  (trading strength) or  $N$  (total market participation), provided that  $\lambda\beta > 1$ .

**Proposition 2** The NFSS is LAS in the region defined by  $r < \frac{D}{\alpha N} = r_l$ . In addition, a transcritical bifurcation occurs for  $r = r_l$ .

The transcritical bifurcation of the NFSS is characterized by a ‘stability exchange’ between the NFSS and the FSS. The NFSS is stable for  $r < r_l$ , or  $\alpha N < D/r$  (weak total market participation or low individual investment). It becomes unstable as soon as  $r = r_l$  when, at the same time, it collides with the (unstable) FSS, previously located in the ‘economically unfeasible’ portion of the phase space. Note that, immediately after the bifurcation, quantity  $(1 - x^*)$  is strictly positive, yet very close to zero. Therefore, based on Proposition 1, the newborn FSS is necessarily stable, no matter how large the values of behavioral parameters  $\lambda$ ,  $\beta$  and  $\gamma$  are.

#### 2.4. Numerical illustration

In this section, we briefly illustrate our main analytical results and explore some further global properties of our 2D model. To be able to simulate our 2D model, we specify functions  $f$  and  $g$  by  $f(\rho_t) = \mu \arctan(\frac{\beta}{\mu} \rho_t)$  with  $\mu := \frac{2\kappa}{\pi}$  and  $g(\delta_t) = \gamma \delta_t$ , where  $\beta, \gamma, \kappa > 0$ . Note that function  $f$  is S-shaped and bounded between  $-\kappa$  and  $\kappa$  while function  $g$  is linear. Their derivatives (at any steady state) are given by  $\beta$  and  $\gamma$ , respectively. Figure 1 presents bifurcation diagrams for parameters  $N$  and  $\beta$ . On the left-hand side, we show the effect on the market share of speculators active in the risky asset market, while the right-hand side depicts the effect on the price of the risky asset. All simulations are based on  $D = 1$ ,  $r = 0.01$ ,  $\alpha = 1$  and  $\lambda = 1$ , implying that  $\hat{x} = 1$ ,  $\hat{P} = N$ ,  $x^* = 100/N$ ,  $1/(1 - x^*) = N/(N - 100)$  and  $P^* = 100$ . In the first line of Figure 1, we vary the total market participation between 1 and 300 units. The remaining parameters are set to  $\beta = 2$ ,  $\gamma = 5$  and  $\kappa = 0.5$ . The bifurcation diagrams confirm our analytical results. For  $1 < N < 100$ , the dynamics converges towards the NFSS. At  $N = 100$ , a transcritical bifurcation takes place, i.e. the NFSS exchanges its stability with the FSS. Between  $100 < N < 200$ , the FSS is locally stable. At  $N = 200$ , a Neimark-Sacker bifurcation occurs, triggering quasiperiodic motion. Interestingly, there seems to be a desirable range for

parameter  $N$ . If the total number of speculators is too low, the price of the risky asset remains below its fundamental value (investors' lack of financial means cause a limits of arbitrage problem). If the total number of speculators is too high, the price of the risky asset displays endogenous boom-bust dynamics (investors' excessive use of financial means cause endogenous dynamics).

The bifurcation diagrams depicted in the second line of Figure 1 are based on  $\beta = 1$ ,  $\gamma = 600$  and  $\kappa = 1.5$ . Again, the FSS is locally stable for intermediate values of  $N$ , namely for  $100 < N < 200$ . At  $N = 100$ , the FSS loses its stability due to a transcritical bifurcation while at  $N = 200$ , the FSS's stability loss is caused by a flip bifurcation. Note that a further increase in the total number of speculators is associated with a cascade of period-two cycles and the onset of complex dynamics. The bifurcation diagrams depicted in the third line of Figure 1 only differ from those depicted in the second line with respect to  $\kappa$ , which is now reduced to  $\kappa = 0.05$ . Since the stability condition of the FSS is independent of  $\kappa$ , the FSS remains locally stable between  $100 < N < 200$ . However, it is clear from the panels in the third line of Figure 1 that long-run fluctuations exist already for  $N < 200$ , which implies that the  $2D$  model possesses coexisting attractors. Depending on the initial conditions, the dynamics may converge towards the FSS or be subject to endogenous fluctuations.

The bottom line of Figure 1 shows bifurcation diagrams for increasing values of parameter  $\beta$ , assuming that  $N = 150$ ,  $\gamma = 1000$  and  $\kappa = 1.54$ . In line with our analytical results, the FSS is locally stable for  $2 < \beta < 3$ . If  $\beta$  falls below 2, a flip bifurcation occurs. As  $\beta$  decreases further, the period-two cycle turns into a period-four cycle, and increasingly more complex dynamics results. In fact, numerical tests indicate that the dynamics is chaotic for  $\beta = 1$ . Note that quasi-periodic motion emerges via a Neimark-Sacker bifurcation as  $\beta$  exceeds 3. Finally, a further global bifurcation occurs at  $\beta \approx 3.39$ , at which relatively modest and regular fluctuations turn quite abruptly into much more volatile and irregular dynamics.

Coexisting attractors constitute one of the most intriguing features of nonlinear dynamical systems and give rise to a number of puzzling economic phenom-

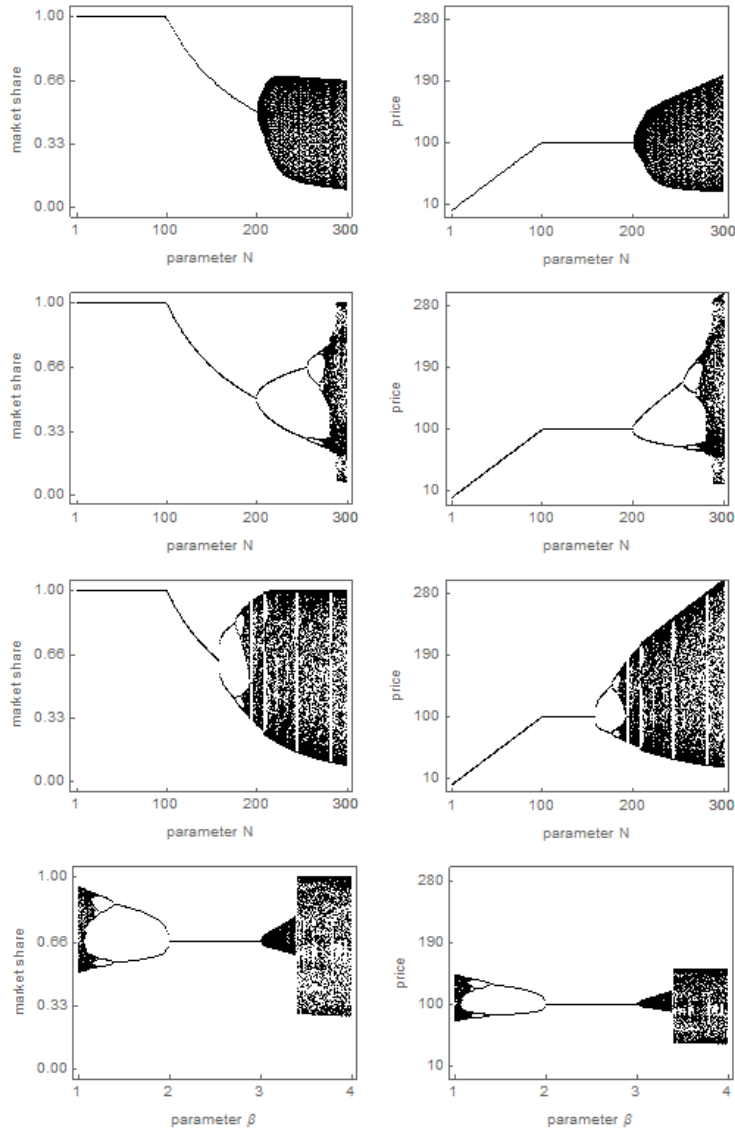


Figure 1: The dynamics of the 2D model. The first three lines of panels present bifurcation diagrams for the market share of speculators active in the risky asset market (left) and the price of the risky asset (right) with respect to parameter  $N$ . The bottom line of panels shows the same, but for parameter  $\beta$ . Parameter settings are provided in Section 2.4.

ena. In Figure 2, we thus briefly sketch two examples for coexisting attractors produced by our 2D model along with their intricate basins of attraction. In the

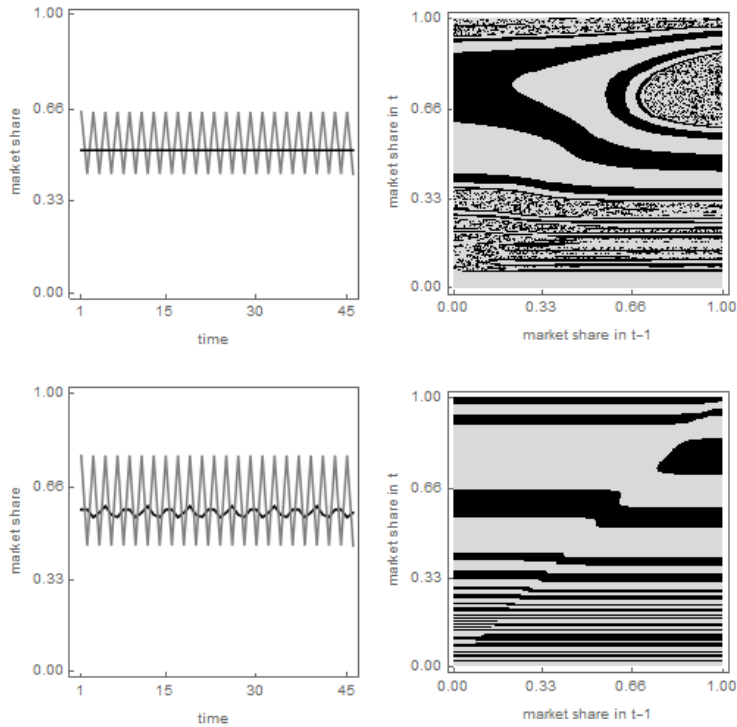


Figure 2: Coexisting attractors and basins of attraction of the 2D model. In the top left panel, the FSS (black line) coexists with a period 2 cycle (gray line). In the bottom left panel, a period 7 cycle (black line) coexists with a period 2 cycle (gray line). The panels on the right display the corresponding basins of attraction, using the same color coding. Parameter settings are provided in Section 2.4.

top line, we face a scenario in which the FSS (black color) coexists with a period 2 cycle (gray color). The parameter setting is as in the third line of Figure 1, except that  $\kappa = 0.6$  and  $N = 195$ . In the bottom line, we encounter a constellation in which a period 7 cycle (black color) coexists with a period 2 cycle (gray color). Simulations are based on the same parameter setting as before, except that  $\kappa = 0.15$ ,  $N = 175$  and  $\beta = 5$ , implying that a Neimark-Sacker bifurcation has destabilized the FSS. Note that the corresponding basins of attraction of these two examples suggest, amongst others, that sporadic exogenous shocks may entail a complex attractor switching process. In particular, the model dy-

namics may then be characterized by a rather low volatility regime that may, out of the blue, turn into a high volatility regime (and vice versa). For more background on the economic implications of coexisting attractors, how to manage the underlying dynamics and their basins of attraction and tools to characterize them, we refer the interested reader to Schmitt et al. (2017), Schmitt and Westerhoff (2015), Agliari et al. (2006) and Agliari and Dieci (2006), respectively.

### 3. A financial market model with two risky assets

#### 3.1. Model setup

In this section, we extend our model by considering two risky asset markets, indexed by  $i = 1, 2$ . Therefore, speculators can choose between entering one of the two risky asset markets and investing in a risk-free asset. The market clearing conditions are now expressed as  $Q_{i,t} = S_{i,t}$ , where  $S_{i,t} = X_i$  and  $Q_{i,t} = q_{i,t}n_{i,t}$  describe the total supply and demand in the risky asset market  $i$  in period  $t$ , respectively. While supplies  $X_i > 0$  are fixed, the total demand results from speculators' individual demands multiplied by the number of speculators active in asset market  $i$ . Speculators' individual demands are determined by the isoelastic demand functions  $q_{i,t} = I_i/P_{i,t}$ , where  $I_i > 0$  represents a fixed amount of money that speculators are willing to invest in market  $i$  and  $P_{i,t}$  denotes the price of the risky asset  $i$  in period  $t$ . By defining  $\alpha_i = \frac{I_i}{X_i}$ , we obtain

$$P_{i,t} = \alpha_i n_{i,t}. \quad (8)$$

Accordingly, the price of the risky assets increases with the number of speculators active in the respective market and their financial means.

Speculators' investment decisions are repeated at the beginning of each period and depend on the attractiveness of the risky assets relative to the safe asset. The attractiveness of the risky asset market  $i$  in period  $t$  is defined by

$$A_{i,t} = f(\rho_{i,t}) + g(\delta_{i,t}), \quad (9)$$

where  $f', g' > 0$ ,  $f(0) = g(0) = 0$ . Moreover,  $\rho_{i,t} = \frac{P_{i,t} - P_{i,t-1}}{P_{i,t-1}}$  is the observed price trend and  $\delta_{i,t} = \frac{D_i}{P_{i,t}} - r$  is the current deviation of the dividend-price ratio from the interest rate. Note that  $D_i$  represents constant dividends of asset  $i$ . Hence, speculators tend to enter asset market  $i$  when asset price  $i$  increases, but also tend to exit asset market  $i$  in periods of overvaluation.

The number of speculators active in the two risky asset markets evolve again via the exponential replicator dynamics. For  $i = 1, 2$ , we thus obtain

$$n_{i,t+1} = N \frac{n_{i,t} \exp(\lambda A_{i,t})}{n_{1,t} \exp(\lambda A_{1,t}) + n_{2,t} \exp(\lambda A_{2,t}) + (N - n_{1,t} - n_{2,t})}, \quad i = 1, 2, \quad (10)$$

which is interpreted similarly to equation (3). Of course, the number of speculators who are not active in one of the two risky asset markets is given by  $N - n_{1,t} - n_{2,t}$ .

### 3.2. Dynamical system and steady states

Since attractiveness  $A_{i,t}$  in equation (9) depends on  $n_{i,t}$  and  $n_{i,t-1}$ , the recurrence relations (10) can be rewritten as a four-dimensional (4D) dynamical system in variables  $x_{i,t} := n_{i,t}/N$ ,  $z_{i,t} := n_{i,t-1}/N$ ,  $i = 1, 2$ . More precisely, for  $i = 1, 2$ , we have

$$x_{i,t+1} = F_i(x_{1,t}, x_{2,t}, z_{1,t}, z_{2,t}) := \frac{1}{U_t} x_{i,t} \exp(\lambda A_i(x_{i,t}, z_{i,t})), \quad z_{i,t+1} = x_{i,t}, \quad (11)$$

where

$$A_i(x_{i,t}, z_{i,t}) := f\left(\frac{x_{i,t}}{z_{i,t}} - 1\right) + g\left(\frac{D_i}{\alpha_i N x_{i,t}} - r\right) \quad (12)$$

and

$$U_t = x_{1,t} \exp(\lambda A_1(x_{1,t}, z_{1,t})) + x_{2,t} \exp(\lambda A_2(x_{2,t}, z_{2,t})) + (1 - x_{1,t} - x_{2,t}).$$

Similar to the baseline case, the 4D model admits two steady states (see Appendix for details):

(i) A FSS, characterized by the condition  $A_i = 0$ ,  $i = 1, 2$ , which yields

$$x_i^* = z_i^* = \frac{D_i}{\alpha_i N r}, \quad n_i^* = N x_i^* = \frac{D_i}{\alpha_i r}, \quad P_i^* = \alpha_i n_i^* = \frac{D_i}{r}, \quad (13)$$



provided that  $x_1^* + x_2^* < 1$ , that is

$$r > \frac{1}{N} \left( \frac{D_1}{\alpha_1} + \frac{D_2}{\alpha_2} \right) := r_l. \quad (14)$$

Similar comments as for the baseline case apply. In particular, the FSS is only feasible for sufficiently large  $r$ ,  $N$  and  $\alpha_i$ .<sup>2</sup>

(ii) A NFSS, implying

$$\hat{x}_1 = \frac{\alpha_2 D_1}{\alpha_1 D_2 + \alpha_2 D_1}, \quad \hat{x}_2 = \frac{\alpha_1 D_2}{\alpha_1 D_2 + \alpha_2 D_1}, \quad (15)$$

and where  $\hat{n}_i$  and  $\hat{P}_i$  are defined accordingly, based on  $\hat{P}_i = \alpha_i \hat{n}_i = \alpha_i N \hat{x}_i$ ,  $i = 1, 2$ . One can check that quantity  $r_l$  defined in equation (14) represents the dividend-price ratio at the NFSS (identical for both risky assets), i.e.  $r_l = D_i / \hat{P}_i$ ,  $i = 1, 2$ . Other steady-state properties are similar to the baseline case. In particular, if the FSS is feasible ( $r > r_l$ ), then  $\hat{P}_i > D_i / r := P_i^*$ ,  $i = 1, 2$ . Moreover, the (common) attractiveness of both risky assets at the NFSS,  $\hat{A}_i = g(r_l - r)$ ,  $i = 1, 2$ , is negative (positive) if and only if the FSS is feasible (unfeasible).

### 3.3. Steady-state stability and bifurcations with two risky assets

The main results about the stability and local bifurcations of the FSS and NFSS are summarized by the following propositions. The related discussions highlight how this situation resembles or differs from the case with one risky asset. The analytical proofs are provided in the Appendix.

**Proposition 3** The parameter domain in which the FSS is LAS is identified by the following double inequality:

$$\beta_l^{(II)} := \frac{r}{2} \gamma - \frac{1}{\lambda} < \beta < \frac{1}{\lambda} := \beta_u^{(II)}. \quad (16)$$

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<sup>2</sup> With an abuse of notation,  $r_l$  denotes the ‘feasibility threshold’ for both the 2D and 4D model. A comparison of  $r_l = \frac{D}{\alpha N}$  (2D model) and  $r_l = \frac{D_1}{\alpha_1 N} + \frac{D_2}{\alpha_2 N}$  (4D model) reveals, amongst others, that for  $D = D_1 = D_2$  and  $\alpha = \alpha_1 = \alpha_2$  (symmetric markets) twice as many investors are needed to make the FSS feasible.

In addition, if  $\beta = \beta_l^{(II)}$  ( $\beta = \beta_u^{(II)}$ ) a flip bifurcation (Neimark-Sacker bifurcation) takes place.

Again, a strong reaction to fundamental conditions (large  $\gamma$ ) may bring about a flip bifurcation of the FSS. The Neimark-Sacker bifurcation of the FSS occurs whenever (for not too large  $\gamma$ )  $\lambda\beta$  increases above 1, that is, if sensitivity to observed trends or the switching intensity become sufficiently large. Note that the stability condition (16) of the 4D model is tighter than the corresponding condition (7) of the 2D model.<sup>3</sup> In particular, total market participation  $N$  has no effect on the stability of the FSS. However,  $N$  continues to play a role as a key bifurcation parameter in the stability exchange between the FSS and the NFSS, as shown below.

**Proposition 4** The NFSS is LAS in the region defined by  $r < r_l$  and

$$\frac{r_l}{2}\hat{\gamma} - \frac{1}{\lambda} < \beta < \frac{1}{\lambda}, \quad (17)$$

where  $\hat{\gamma} := g'(r_l - r)$ . If  $r < r_l$ , violation of the inequality on the right (on the left) in (17) leads to a Neimark-Sacker bifurcation (flip bifurcation). In addition, if (17) holds but  $r$  crosses  $r_l$ , a transcritical bifurcation occurs.

Note that  $\hat{\gamma}$  is generally different from  $\gamma$  (unless  $g$  is linear). Of course, at the transcritical bifurcation,  $\hat{\gamma} = \gamma$ . Moreover, the stability of the NFSS requires  $r < r_l$ , or  $N < \frac{1}{r} \left( \frac{D_1}{\alpha_1} + \frac{D_2}{\alpha_2} \right)$ , i.e. sufficiently weak market activity and participation (as well as the absence of a feasible FSS). However, unlike the one-risky-asset case, the loss of stability of the NFSS is not necessarily associated with the appearance of the FSS. As a matter of fact, even with  $r < r_l$ , condition (17) may cease to hold if one of the behavioral parameters  $\lambda$ ,  $\beta$  or  $\gamma$  becomes sufficiently large, similar to condition (16) for the FSS. Interestingly, numerical investigations show that the fluctuations of investor shares generated by this loss of stability remain confined to the subset defined by  $x_{1,t} + x_{2,t} = 1$ . While investors start to switch across risky asset markets, no one invests in the safe

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<sup>3</sup>We have also observed numerically the same Neimark-Sacker bifurcation value  $\beta_u^{(II)}$  with a three-risky-asset extension of the present model.

asset. In addition, and similar to the case with one risky asset, the NFSS may become unstable as soon as the FSS appears in the feasible region, i.e. for  $r > r_l$ . Again, the transcritical bifurcation of the NFSS leads to a stability exchange between the two steady states. Note that a transcritical bifurcation actually occurs (at  $r = r_l$ ) only if (17) holds, i.e. if behavioral parameters  $\lambda$ ,  $\beta$  and  $\gamma$  are not too large.<sup>4</sup> Accordingly, immediately after the bifurcation, condition (16) is also satisfied and therefore the FSS is LAS.

### 3.4. Numerical illustration

Let us finally illustrate the main stability and bifurcation properties of our 4D model. The left panels of Figure 3 show a bifurcation diagram for  $\beta$  versus market shares  $x_1$ ,  $x_2$  and  $1 - x_1 - x_2$ , a bifurcation diagram for  $\beta$  versus risky asset prices  $P_1$  and  $P_2$  and the evolution of market shares  $x_1$ ,  $x_2$  and  $1 - x_1 - x_2$  in the time domain (blue: risky asset 1, red: risky asset 2, black: safe asset). Functions  $f$  and  $g$  are specified as in Section 2.4, while the underlying parameter setting is given by  $D_1 = 1.25$ ,  $D_2 = 0.75$ ,  $r = 0.01$ ,  $N = 400$ ,  $\alpha_1 = \alpha_2 = 1$ ,  $\beta = 1.25$ ,  $\gamma = 220$ ,  $\kappa = 0.75$  and  $\lambda = 1$ . Straightforward computations reveal that the FSS, represented by  $x_1^* = 0.3125$ ,  $x_2^* = 0.1875$ ,  $1 - x_1^* - x_2^* = 0.5$ ,  $P_1^* = 125$  and  $P_2^* = 75$ , is feasible. Moreover, the FSS is locally stable for  $0.1 < \beta < 1$ . At  $\beta = 0.1$ , the stability loss of the FSS is caused by a (supercritical) flip bifurcation while at  $\beta = 1$ , its stability loss is due to a Neimark-Sacker bifurcation, as confirmed by the bifurcation diagrams. In fact, the market shares oscillate for  $\beta = 1.25$  in a countercyclical manner around their steady-state values. The same is true for the prices of the risky assets (not depicted). Of course, the NFSS, i.e.  $\hat{x}_1 = 0.625$ ,  $\hat{x}_2 = 0.375$ ,  $1 - \hat{x}_1 - \hat{x}_2 = 0$ ,  $\hat{P}_1 = 250$  and  $\hat{P}_2 = 150$ , is unstable.

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<sup>4</sup>Simulation results reveal that interesting associated phenomena occur also when threshold  $r_l$  is crossed with parameters that do not satisfy condition (17). In this case, the motion of investor shares, previously confined to subset  $x_{1,t} + x_{2,t} = 1$ , takes place in the inner part of the phase space, with investors switching across all assets (including the safe asset) and prices fluctuating below their NFSS values  $\hat{P}_i = D_i/r_l$ ,  $i = 1, 2$ .

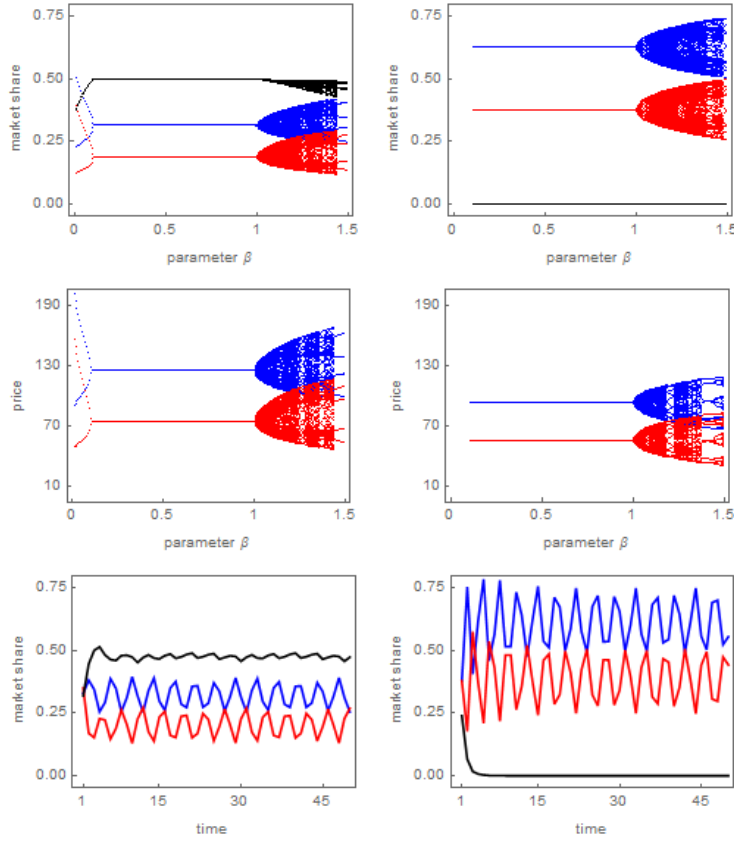


Figure 3: The dynamics of the 4D model. The left-hand panels visualize the flip and Neimark-Sacker bifurcation scenario of the FSS (blue: risky asset 1, red: risky asset 2, black: safe asset). The right-hand panels show the same but for the NFSS. Parameter settings are provided in Section 3.4.

Note that a stable FSS signals market efficiency, at least in the sense that the prices of the risky assets correspond to their fundamental values. Whether risky asset markets can achieve these values depends crucially on the total number of speculators. Given the above parameter setting, our analytical results indicate that a transcritical bifurcation appears at  $N = 200$ , triggering a stability exchange between the FSS and the NFSS. The effects of such a parameter change are depicted in the right-hand panels of Figure 3. In particular, setting the total number of speculators to  $N = 150$  implies that the new coordinates of the

FSS, i.e.  $x_1^* = 0.833$ ,  $x_2^* = 0.5$ ,  $1 - x_1^* - x_2^* = -0.333$ ,  $P_1^* = 125$  and  $P_2^* = 75$ , are unfeasible. Instead, the NFSS is now given by  $\hat{x}_1 = 0.625$ ,  $\hat{x}_2 = 0.375$ ,  $1 - \hat{x}_1 - \hat{x}_2 = 0$ ,  $\hat{P}_1 = 93.75$  and  $\hat{P}_2 = 56.25$ . Apparently, the NFSS implies that all speculators enter the risky asset markets but also that their buying pressure is not strong enough to ensure that the prices of the risky assets reach their fundamental values. In fact,  $\hat{P}_1 < P_1^*$  and  $\hat{P}_2 < P_2^*$  reveal that limits of arbitrage destroy market efficiency. In the right-hand panels of Figure 3, we have additionally adjusted parameter  $\gamma$ , whereas all other parameters remain as before. Setting  $\gamma = 165$  implies that the flip bifurcation occurs again at  $\beta = 0.1$  (since this bifurcation is now subcritical, no attractor is visible for  $\beta < 0.1$ ). Note also that the market shares of speculators active in the risky asset markets evolve again countercyclically to each other, while the market shares of speculators investing in the safe asset rapidly approach zero. After that has occurred, the dynamics of the  $4D$  model is effectively driven by a  $2D$  subsystem (in the invariant set  $x_{1,t} + x_{2,t} = 1$ ). However, exogenous noise (not considered here) can revive the full  $4D$  model. Moreover, the prices of the risky assets fluctuate after the Neimark-Sacker bifurcation below the FSS – another interesting limits of arbitrage effect.

Since the transcritical bifurcation represents a highlight of our paper and total market participation plays a crucial role in it, we finally plot in Figure 4 the positions of the FSS and NFSS for  $1 < N < 600$ , assuming the same color coding and parameter setting as in Figure 3. However,  $\beta$  and  $\gamma$  are set such that neither the flip nor the Neimark-Sacker bifurcation boundary is violated. Solid lines represent stable steady states whereas dotted lines denote unstable steady states. The left-hand panel of Figure 4 reveals the distribution of speculators across the three asset markets while the right-hand panel of Figure 4 shows the corresponding reaction of risky asset prices. As mentioned above, the stability exchange between the FSS and the NFSS takes place at  $N = 200$ .

To be able to appreciate the above results, let us put them into perspective. Recall that the famous noise trader approach, summarized by Shleifer and Summers (1990), rests on two core assumptions. First, some investors are

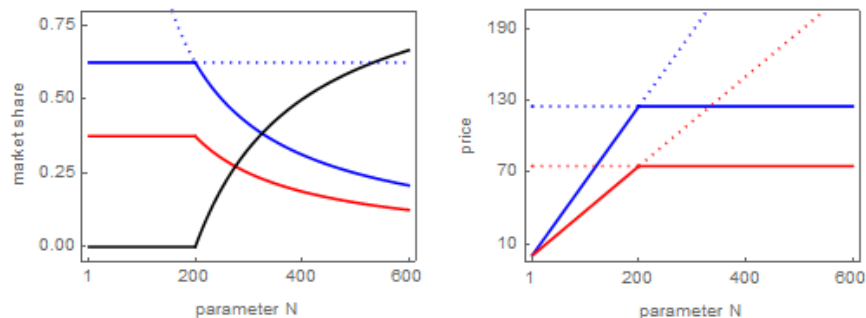


Figure 4: Some steady state implications of the transcritical bifurcation. The panels depict how the FSS and NFSS change with respect to parameter  $N$ . Solid (dotted) lines indicate stable (unstable) steady states (left: market shares, right: risky asset prices). Color coding and base parameter setting as in Figure 3, except that  $\beta$  and  $\gamma$  are such that neither the flip nor the Neimark-Sacker bifurcation boundary is violated.

not fully rational. In particular, their demand for risky assets is affected by sentiments that are not fully consistent with economic fundamentals (Barberis et al. 1998). Shleifer and Summers (1990) identify two kinds of sources for shifts in investor sentiment: the belief in pseudo signals and the use of popular models. An example for a pseudo signal (Black 1986), also called noise, may be the advice of a financial guru. The expression popular models (Shiller 1990) refers to the models that are used by the broad masses of economic actors to form their decisions, such as the widespread use of technical analysis. Second, arbitrage – defined as trading by fully rational investors not subject to such sentiments – is limited in reality because it requires capital (Shleifer and Vishny 1997). Based on this approach, De Long et al. (1990a) demonstrate that if a market is dominated by investors who follow positive feedback strategies it may even pay for arbitrageurs to jump on the bandwagon. Moreover, De Long et al. (1990b) show that noise traders may be willing to take higher risks and, consequently, earn higher profits than rational arbitrageurs. Overall, the noise trader approach suggests that shifts in investor sentiment are not fully countered by arbitrageurs and, in particular, that limits of arbitrage may hinder prices to reach their fundamental values. In contrast to the collection of the above pa-

pers, our point of reference is that all investors are boundedly rational and have a limited amount of capital at their disposal. In fact, our understanding of the empirical literature (Simon 1955, Kahneman and Tversky 1974) on investor behavior is that a clear distinction between rational and irrational investors is not appropriate. Behind this background, we hope that our results provide valuable additional insights that may help us to better understand the inefficiencies of financial markets.

#### 4. Conclusions

We explore a number of properties of two multi-asset market models in which speculators either have the choice between one or two risky assets and one safe asset. Both model versions admit two steady states, a fundamental steady state and a nonfundamental steady state. The fundamental steady state may lose its stability via a transcritical bifurcation, implying that the fundamental steady state exchanges its stability with the nonfundamental steady state, or via a flip or Neimark-Sacker bifurcation, giving birth to (complex) endogenous asset price dynamics. In the framework with two risky assets, however, the NFSS may also lose stability due to a flip or Neimark-Sacker bifurcation. Our paper makes clear that we can obtain valuable insights into the functioning of financial markets from a thorough stability and bifurcation analysis.

Of course, our multi-asset market framework may be extended in various directions. First of all, further risky assets may be added to the model. Preliminary investigations reveal that some markets may then be characterized by synchronous price movements while others display asynchronous price movements. Moreover, it could realistically be assumed that both total market participation and speculators' financial commitments are not constant, but endogenously affected by credit conditions and their borrowing capacity (particularly for very low interest rates  $r$ ). In such a context, boom-and-bust periods may become amplified. Moreover, such new endogenous forces could partly relax the limits of arbitrage arising in the present model. Finally, attempts may be made to

calibrate our model such that it replicates some statistical properties of actual financial markets in finer detail. To conclude, we hope that our paper results in further research that enables us to develop a better understanding of the intricate behavior of interacting financial markets.

### Appendix. Steady-states, stability and bifurcations

This appendix provides key details about the steady states, their local stability and possible bifurcations. We mainly focus on the  $4D$  model, since parallel results for the  $2D$  model can be derived along the same lines in a much easier manner. In any case, the  $2D$  model is briefly illustrated at the end of each of the two subsections of this Appendix.

*A1. Steady states.* Let us start with the case of two risky assets. Denote generic steady state quantities with an overbar and note first that  $\bar{x}_i = \bar{z}_i$ ,  $i = 1, 2$ , at any steady state. We neglect cases  $\bar{x}_i = \bar{z}_i = 0$ ,  $i = 1, 2$ , in which the attractiveness (12) is not defined. From (11), we obtain the following steady-state conditions:

$$\bar{x}_i = \frac{\bar{x}_i \exp(\lambda \bar{A}_i)}{\bar{x}_1 \exp(\lambda \bar{A}_1) + \bar{x}_2 \exp(\lambda \bar{A}_2) + 1 - \bar{x}_1 - \bar{x}_2}, \quad i = 1, 2 \quad (18)$$

where  $\bar{A}_i := A(\bar{x}_i, \bar{x}_i) = g\left(\frac{D_i}{\alpha_i N \bar{x}_i} - r\right)$ . This yields

$$\exp(\lambda \bar{A}_1) = \exp(\lambda \bar{A}_2) = \bar{x}_1 \exp(\lambda \bar{A}_1) + \bar{x}_2 \exp(\lambda \bar{A}_2) + 1 - \bar{x}_1 - \bar{x}_2,$$

implying that the two risky asset markets must have the same attractiveness, or equivalently the same *dividend-price ratio*, in steady-state conditions,  $\bar{A}_1 = \bar{A}_2 := \bar{A}$ ,  $\frac{D_1}{\alpha_1 N \bar{x}_1} = \frac{D_2}{\alpha_2 N \bar{x}_2}$ . Moreover, by summing equation (18) over the two risky assets, one obtains

$$\bar{y} = \frac{\bar{y}}{\bar{x}_1 \exp(\lambda \bar{A}_1) + \bar{x}_2 \exp(\lambda \bar{A}_2) + \bar{y}}, \quad (19)$$

where  $\bar{y} := 1 - \bar{x}_1 - \bar{x}_2$ . Clearly, there exists one steady state characterized by  $\bar{y} = 0$ , i.e.  $\bar{x}_1 + \bar{x}_2 = 1$ , which is considered below. Assuming  $\bar{y} > 0$  instead, condition (19) can be rewritten as

$$\bar{x}_1 \exp(\lambda \bar{A}_1) + \bar{x}_2 \exp(\lambda \bar{A}_2) = \bar{x}_1 + \bar{x}_2,$$



which, along with condition  $\bar{A}_1 = \bar{A}_2$ , leads to  $\bar{A}_i = 0$ ,  $i = 1, 2$ . The latter implies

$$\frac{D_i}{\alpha_i N \bar{x}_i} = r, \quad i = 1, 2,$$

and defines the *fundamental* steady state (FSS), where, for  $i = 1, 2$ :

$$x_i^* = z_i^* = \frac{D_i}{\alpha_i N r}.$$

Accordingly,  $n_i^* = N x_i^* = \frac{D_i}{\alpha_i r}$  and  $P_i^* = \alpha_i n_i^* = \frac{D_i}{r}$ , implying that the dividend-price ratios at the FSS are equal to  $r$ .

As regards the boundary or *nonfundamental* steady state (NFSS), from condition  $\frac{D_1}{\alpha_1 N \bar{x}_1} = \frac{D_2}{\alpha_2 N \bar{x}_2}$ , along with  $\bar{x}_1 + \bar{x}_2 = 1$ , it follows that

$$\hat{x}_1 = \frac{\alpha_2 D_1}{\alpha_1 D_2 + \alpha_2 D_1}, \quad \hat{x}_2 = \frac{\alpha_1 D_2}{\alpha_1 D_2 + \alpha_2 D_1},$$

$$\hat{P}_1 = \alpha_1 N \hat{x}_1 = N D_1 \frac{\alpha_1 \alpha_2}{\alpha_1 D_2 + \alpha_2 D_1}, \quad \hat{P}_2 = \alpha_2 N \hat{x}_2 = N D_2 \frac{\alpha_1 \alpha_2}{\alpha_1 D_2 + \alpha_2 D_1},$$

where the common value of the dividend-price ratio is given by

$$\frac{D_1}{\hat{P}_1} = \frac{D_2}{\hat{P}_2} = \frac{1}{N} \left( \frac{D_1}{\alpha_1} + \frac{D_2}{\alpha_2} \right) := r_l.$$

The case of one risky asset can be easily worked out along similar lines, starting from equation (4). In steady-state conditions, share  $\bar{x}$  of investors active in the asset market needs to satisfy the equation  $\bar{x} + (1 - \bar{x}) \exp(-\lambda \bar{A}) = 1$ , where  $\bar{A} := A(\bar{x}, \bar{x}) = g\left(\frac{D}{\alpha N \bar{x}} - r\right)$ . This condition is satisfied for  $\bar{x} = 1$  (NFSS) whereas, for  $\bar{x} \neq 1$ , when  $\bar{A} = 0$  (FSS), as summarized in Section 2.2.

*A2. Stability analysis and bifurcations.* For the case of *two risky assets*, it is convenient to regard functions  $F_i, i = 1, 2$ , governing the exponential replicator dynamics in (11) as functions of  $x_1, x_2, w_1, w_2$ , where  $w_i = w_i(x_i, z_i) := \exp(\lambda A_i(x_i, z_i))$  and  $A_i(x_i, z_i)$  is the attractiveness defined by (12). More precisely, for  $i = 1, 2$ :

$$F_i(x_1, x_2, z_1, z_2) = \tilde{F}_i(x_1, x_2, w_1, w_2) = \frac{x_i w_i}{U(x_1, x_2, w_1, w_2)},$$

where  $U = U(x_1, x_2, w_1, w_2) := x_1 w_1 + x_2 w_2 + 1 - x_1 - x_2$ . By the chain rule, the derivatives of  $F_i$  have the following structure for  $i, j = 1, 2, i \neq j$ :

$$\frac{\partial F_i}{\partial x_i} = \frac{\partial \tilde{F}_i}{\partial x_i} + \frac{\partial \tilde{F}_i}{\partial w_i} \frac{\partial w_i}{\partial x_i}, \quad \frac{\partial F_i}{\partial z_i} = \frac{\partial \tilde{F}_i}{\partial w_i} \frac{\partial w_i}{\partial z_i}, \quad (20)$$

$$\frac{\partial F_i}{\partial x_j} = \frac{\partial \tilde{F}_i}{\partial x_j} + \frac{\partial \tilde{F}_i}{\partial w_j} \frac{\partial w_j}{\partial x_j}, \quad \frac{\partial F_i}{\partial z_j} = \frac{\partial \tilde{F}_i}{\partial w_j} \frac{\partial w_j}{\partial z_j}, \quad (21)$$

where

$$\begin{aligned} \frac{\partial \tilde{F}_i}{\partial x_i} &= \frac{w_i U - x_i w_i (w_i - 1)}{U^2}, & \frac{\partial \tilde{F}_i}{\partial w_i} &= \frac{x_i U - x_i^2 w_i}{U^2}, \\ \frac{\partial \tilde{F}_i}{\partial x_j} &= -\frac{x_i w_i (w_j - 1)}{U^2}, & \frac{\partial \tilde{F}_i}{\partial w_j} &= -\frac{x_i x_j w_i}{U^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial A_i}{\partial x_i} &= \frac{1}{z_i} f' \left( \frac{x_i}{z_i} - 1 \right) - \frac{D_i}{\alpha_i N x_i^2} g' \left( \frac{D_i}{\alpha_i N x_i} - r \right), & \frac{\partial A_i}{\partial z_i} &= -\frac{x_i}{z_i^2} f' \left( \frac{x_i}{z_i} - 1 \right), \\ \frac{\partial w_i}{\partial x_i} &= \lambda w_i \frac{\partial A_i}{\partial x_i}, & \frac{\partial w_i}{\partial z_i} &= \lambda w_i \frac{\partial A_i}{\partial z_i}. \end{aligned}$$

At the FSS  $\mathbf{s}^* := (x_1^*, x_2^*, x_1^*, x_2^*)$ , where  $x_i^* = \frac{D_i}{\alpha_i N r}$ , we also have  $\frac{D_i}{\alpha_i N x_i^*} = \frac{D_i}{P_i^*} = r$ ,  $A_i^* := A_i(x_i^*, x_i^*) = 0$ ,  $w_i^* := w_i(x_i^*, x_i^*) = 1$ ,  $U^* := U(x_1^*, x_2^*, w_1^*, w_2^*) = 1$ ,  $i = 1, 2$ . By setting  $\beta := f'(0)$ ,  $\gamma := g'(0)$  and by defining the aggregate parameters  $\eta := \lambda\beta$  and  $\sigma := \lambda(\beta - r\gamma)$ , the above derivatives result in the following Jacobian matrix at the FSS (the order of rows and columns corresponds to variables  $x_1, x_2, z_1$  and  $z_2$ , respectively):

$$\mathbf{J}(\mathbf{s}^*) = \begin{pmatrix} 1 + \sigma(1 - x_1^*) & -\sigma x_1^* & -\eta(1 - x_1^*) & \eta x_1^* \\ -\sigma x_2^* & 1 + \sigma(1 - x_2^*) & \eta x_2^* & -\eta(1 - x_2^*) \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad (22)$$

Tedious computations<sup>5</sup> show that the characteristic polynomial of  $\mathbf{J}(\mathbf{s}^*)$ , which we denote by  $\mathcal{P}(v) := \text{Det}(\mathbf{J}(\mathbf{s}^*) - v\mathbf{I}_4)$ , can be factorized as the product of two

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<sup>5</sup>Elementary row and column operations performed on matrix  $\mathbf{J}(\mathbf{s}^*) - v\mathbf{I}_4$  lead to a simplified matrix  $\mathbf{Q} = \mathbf{Q}(v)$  with additional zero entries. Then,  $\text{Det}(\mathbf{Q}) = \text{Det}(\mathbf{J}(\mathbf{s}^*) - v\mathbf{I}_4)$  can easily be computed by co-factor expansion, yielding a tractable expression for the characteristic polynomial (23).

second-degree polynomials, namely

$$\mathcal{P}(v) = \mathcal{P}_a(v)\mathcal{P}_b(v) = [v^2 - (1 + \sigma)v + \eta] [v^2 - (1 + \sigma y^*)v + \eta y^*], \quad (23)$$

where  $y^* := 1 - x_1^* - x_2^*$ . This property enables us to investigate the local stability conditions and possible bifurcations of the FSS by relying on standard techniques for two-dimensional systems. In particular (see, e.g. Medio and Lines 2001), the four characteristic roots of (22) are jointly less than one in modulus if and only if:

$$\mathcal{P}_a(1) > 0, \quad \mathcal{P}_a(-1) > 0, \quad 1 - \mathcal{P}_a(0) > 0, \quad (24)$$

$$\mathcal{P}_b(1) > 0, \quad \mathcal{P}_b(-1) > 0, \quad 1 - \mathcal{P}_b(0) > 0, \quad (25)$$

where conditions (24) (or (25)) are necessary and sufficient for the roots  $v_1$  and  $v_2$  of  $\mathcal{P}_a(v)$  (or  $v_3$  and  $v_4$  of  $\mathcal{P}_b(v)$ ) to have modulus smaller than unity. The above conditions jointly result in

$$\left\{ \begin{array}{l} \eta - \sigma > 0 \\ 2 + \eta + \sigma > 0 \\ \eta < 1 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} (\eta - \sigma)y^* > 0 \\ 2 + (\eta + \sigma)y^* > 0 \\ \eta y^* < 1 \end{array} \right. . \quad (26)$$

The set of conditions on the left (right) correspond to polynomial  $\mathcal{P}_a$  ( $\mathcal{P}_b$ ). Note that, for an interior FSS,  $0 < y^* < 1$ . Since  $\eta - \sigma = \lambda r \gamma > 0$ , the first condition is always true for both sets. Let us now turn to the second condition. If  $\eta + \sigma = \lambda(2\beta - r\gamma) \geq 0$ , the second condition is satisfied for both sets. If, on the contrary,  $\eta + \sigma < 0$ , i.e.  $\gamma > 2\beta/r$ , the second condition on the left turns out to be more restrictive than its counterpart on the right. Finally, the third condition on the left is more restrictive than the corresponding condition on the right. To summarize, the stability domain of the FSS is determined, in terms of the original parameters, by the following double inequality:

$$\frac{r}{2}\lambda\gamma - 1 < \lambda\beta < 1, \quad (27)$$

where violation of inequality on the left leads to a period-doubling bifurcation (since it entails  $\mathcal{P}_a(-1) \leq 0$ ), whereas violation of inequality on the right results in a Neimark-Sacker bifurcation (since it leads to  $\mathcal{P}_a(0) \geq 1$ ).

At the NFSS  $\hat{\mathbf{s}} := (\hat{x}_1, \hat{x}_2, \hat{x}_1, \hat{x}_2)$ , steady-state proportions  $\hat{x}_1$  and  $\hat{x}_2$  are determined according to (15), with  $\hat{x}_1 + \hat{x}_2 = 1$ . Moreover, for  $i = 1, 2$ :

$$\hat{A}_i := A_i(\hat{x}_i, \hat{z}_i) = f(0) + g\left(\frac{D_i}{\alpha_i N \hat{x}_i} - r\right) = g(r_l - r),$$

where  $r_l = \frac{1}{N} \left( \frac{D_1}{\alpha_1} + \frac{D_2}{\alpha_2} \right)$ , as defined by (14). Since  $\hat{A}_1 = \hat{A}_2$ , we also have, for  $i = 1, 2$ ,  $\hat{w}_i := w_i(\hat{x}_i, \hat{x}_i) = \hat{w} := \exp(\lambda g(r_l - r))$  and

$$\hat{U} := U(\hat{x}_1, \hat{x}_2, \hat{w}_1, \hat{w}_2) = \hat{x}_1 \hat{w}_1 + \hat{x}_2 \hat{w}_2 + 1 - \hat{x}_1 - \hat{x}_2 = (\hat{x}_1 + \hat{x}_2) \hat{w} = \hat{w} = \exp(\lambda g(r_l - r)).$$

By defining  $\hat{\gamma} := g'(r_l - r)$ ,  $\hat{\sigma} := \lambda(\beta - r_l \hat{\gamma})$  and, again,  $\eta := \lambda\beta$ , and by re-evaluating the general derivatives (20)-(21), the Jacobian matrix at the NFSS is determined as follows:

$$\mathbf{J}(\hat{\mathbf{s}}) = \begin{pmatrix} 1 + \tau \hat{x}_1 + \hat{\sigma} \hat{x}_2 & (\tau - \hat{\sigma}) \hat{x}_1 & -\eta \hat{x}_2 & \eta \hat{x}_1 \\ (\tau - \hat{\sigma}) \hat{x}_2 & 1 + \tau \hat{x}_2 + \hat{\sigma} \hat{x}_1 & \eta \hat{x}_2 & -\eta \hat{x}_1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad (28)$$

where  $\tau := (1 - \hat{w})/\hat{w} = \exp(-\lambda g(r_l - r)) - 1$ . The characteristic polynomial of  $\mathbf{J}(\hat{\mathbf{s}})$ , which we denote by  $\mathcal{Q}(m) := \text{Det}(\mathbf{J}(\hat{\mathbf{s}}) - m\mathbf{I}_4)$ , can be factorized as

$$\mathcal{Q}(m) = -m(1 + \tau - m)\mathcal{P}_c(m) = -m(1 + \tau - m) [m^2 - (1 + \hat{\sigma})m + \eta].$$

Therefore, two eigenvalues of (28) are equal to  $m_1 = 0$  and  $m_2 = 1 + \tau = 1/\hat{w} = \exp(-\lambda g(r_l - r))$ . The remaining eigenvalues,  $m_3$  and  $m_4$ , are the roots of the second-degree polynomial  $\mathcal{P}_c(m) = m^2 - (1 + \hat{\sigma})m + \eta$ . These are jointly less than one in modulus if and only if  $\mathcal{P}_c(1) > 0$ ,  $\mathcal{P}_c(-1) > 0$ ,  $1 - \mathcal{P}_c(0) > 0$ , which results in

$$\lambda r_l \hat{\gamma} > 0, \quad 2 + 2\lambda\beta - \lambda r_l \hat{\gamma} > 0, \quad \lambda\beta < 1, \quad (29)$$

formally similar to the stability conditions (27) of the FSS. As regards the eigenvalue  $m_2$ , since  $g$  is strictly increasing with  $g(0) = 0$ , one has that  $0 < m_2 < 1$  if and only if:

$$\exp(-\lambda g(r_l - r)) < 1 \iff r < r_l = \frac{1}{N} \left( \frac{D_1}{\alpha_1} + \frac{D_2}{\alpha_2} \right). \quad (30)$$

Assume first that  $r$  and  $r_l$  are fixed, with  $r < r_l$ , which implies that the NFSS  $\hat{\mathbf{s}}$  is the unique feasible steady state. It follows from (29) that  $\hat{\mathbf{s}}$  is locally stable provided that

$$\frac{r_l}{2}\lambda\hat{\gamma} - 1 < \lambda\beta < 1, \quad (31)$$

whereas violation of the left (right) inequality results in a flip (Neimark-Sacker bifurcation). However, numerical simulations reveal that the bifurcated orbits do not leave the subset of the state space of equation  $x_1 + x_2 = 1$ .<sup>6</sup> Now assume that (31) is satisfied and that one of the parameters appearing in (30) - in particular, parameter  $N$  - changes such that  $r$  becomes larger than  $r_l$ , which determines the birth of the FSS  $\mathbf{s}^*$  and the coexistence of two steady states. Based on our local stability results, this phenomenon corresponds to a *transcritical bifurcation* of the NFSS, namely, the NFSS collides with the (unstable) FSS previously existing in the ‘economically unfeasible’ portion of the phase space (characterized by  $x_1 + x_2 > 1$ ). While the NFSS loses its stability, the newborn FSS is LAS.<sup>7</sup> Of course, a collision between the two steady states when  $r$  crosses threshold  $r_l$  may also occur in a situation where condition (27) does not hold. In this case, while the number of steady states changes, this change is not characterized - strictly speaking - by a loss of stability of the NFSS. However, numerical simulations reveal that the orbits bifurcated from the NFSS, previously confined to the subset of equation  $x_1 + x_2 = 1$ , now tend to spill over to the full  $4D$  state space.

As regards the *one-risky-asset* model, the derivatives of function  $F$  in equa-

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<sup>6</sup>Note that the lower-dimensional set defined by  $x_1 + x_2 = 1$  is *invariant* for the map that governs the dynamical system. In fact, from  $x_{2,t} = 1 - x_{1,t}$  we obtain  $x_{2,t+1} = 1 - x_{1,t+1}$ , as can be checked directly from equations (11).

<sup>7</sup>At the transcritical bifurcation value, where  $r = r_l$ , stability condition (31) becomes identical to the stability condition (27) of the FSS.

tion (4) can be worked out along similar lines, yielding the Jacobian matrices

$$\mathbf{J}(\mathbf{s}^*) = \begin{pmatrix} 1 + \sigma(1 - x^*) & -\eta(1 - x^*) \\ 1 & 0 \end{pmatrix}, \quad \mathbf{J}(\hat{\mathbf{s}}) = \begin{pmatrix} 1/\hat{w} & 0 \\ 1 & 0 \end{pmatrix}, \quad (32)$$

at the FSS  $\mathbf{s}^* := (x^*, x^*)$  and the NFSS  $\hat{\mathbf{s}} := (\hat{x}, \hat{x})$ , respectively, where  $x^* = \frac{D}{\alpha N r}$  and  $\hat{x} = 1$ . In (32), aggregate parameters  $\sigma$  and  $\eta$  are defined as in matrix (22), whereas  $\hat{w} := \exp(\lambda g(r_l - r))$ , where  $r_l$  is given by (6).

The characteristic roots of  $\mathbf{J}(\mathbf{s}^*)$  are jointly smaller than one in modulus if and only if

$$\lambda r \gamma (1 - x^*) > 0, \quad 2 + \lambda(2\beta - r\gamma)(1 - x^*) > 0, \quad \lambda\beta(1 - x^*) < 1,$$

resulting in condition (7).

As for matrix  $\mathbf{J}(\hat{\mathbf{s}})$ , from its lower triangular structure it can be immediately concluded that one eigenvalue is equal to zero, while the second eigenvalue is equal to  $1/\hat{w} = \exp[-\lambda g(\frac{D}{\alpha N} - r)] > 0$ . It follows that the only way the NFSS can lose its stability is when quantity  $\exp[-\lambda g(\frac{D}{\alpha N} - r)]$  becomes larger than 1, as stated in Proposition 2. Since  $g$  is strictly increasing with  $g(0) = 0$ , this occurs for  $r > r_l$ , which is the same condition (6) for the existence of a (feasible and) interior FSS.

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