

Cunha, George H.M.; Oliveira, Michel A.C.; Oliveira, Michel A.C.; Sosa, Wilfredo

Article

Remarks on Fenchel-Moreau conjugate in the setting of consumer theory

EconomiA

Provided in Cooperation with:

The Brazilian Association of Postgraduate Programs in Economics (ANPEC), Rio de Janeiro

Suggested Citation: Cunha, George H.M.; Oliveira, Michel A.C.; Oliveira, Michel A.C.; Sosa, Wilfredo (2016) : Remarks on Fenchel-Moreau conjugate in the setting of consumer theory, EconomiA, ISSN 1517-7580, Elsevier, Amsterdam, Vol. 17, Iss. 2, pp. 176-184, <https://doi.org/10.1016/j.econ.2016.04.001>

This Version is available at:

<https://hdl.handle.net/10419/179619>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.



<https://creativecommons.org/licenses/by-nc-nd/4.0/>

Remarks on Fenchel–Moreau conjugate in the setting of consumer theory[☆]

George H.M. Cunha, Michel A.C. Oliveira, Wilfredo Sosa*

Graduate School in Economics, Catholic University of Brasilia, SGAN 916 – Modulo B, 70790-000 Brasilia, DF, Brazil

Received 3 December 2014; received in revised form 10 September 2015; accepted 1 April 2016

Available online 14 April 2016

Abstract

In this manuscript we consider the conjugate notion focused from consumer theory as an interesting tool. According to us, conjugate notion remained undeveloped in economic theory because Fenchel's conjugate notion was introduced exclusively for proper convex lower semi continuous functions and convexity assumption is not natural in economic theory. Nevertheless, we introduce necessary and sufficient optimality conditions for consumer problem. Also, we consider a particular version of Fenchel–Moreau conjugate notion, for lower semi continuous functions recently introduced in the literature as a generalization of Fenchel's conjugate. Finally, we adapt it to closed functions in order to define a convex dual problem for consumer problem.

© 2016 National Association of Postgraduate Centers in Economics, ANPEC. Production and hosting by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

Keywords: Convex analysis; Fenchel's conjugate; Economic theory

JEL classification: C02; C61; D11

Resumo

Neste artigo consideramos a noção de conjugada focalizada na teoria do consumidor como uma ferramenta interessante. Segundo nosso entendimento, a noção de conjugada ainda não foi desenvolvida na teoria econômica, porque a Conjugada de Fenchel foi introduzida exclusivamente para funções próprias, convexas semi contínuas inferiormente e a convexidade não é uma hipótese natural na teoria econômica. Mesmo assim, introduzimos condições necessárias e suficientes de otimalidade para o problema do consumidor. Finalmente, considerando uma versão particular da Conjugada de Fenchel-Moreau para funções semi contínuas inferiores, recentemente introduzida na literatura como uma generalização da conjugada de Fenchel, e à adaptamos para as funções fechadas com o objetivo de definir um problema dual convexo para o problema do consumidor.

© 2016 National Association of Postgraduate Centers in Economics, ANPEC. Production and hosting by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

Palavras-chave: Análise convexa; Conjugação de Fenchel; Teoria Econômica

* Corresponding author.

E-mail addresses: george@ucb.br (G.H.M. Cunha), michel@ucb.br (M.A.C. Oliveira), sosa@ucb.br (W. Sosa).

[☆] Peer review under responsibility of National Association of Postgraduate Centers in Economics, ANPEC.

1. Preliminaries

Respect to notions coming from convex analysis, used here, were adopted those from [Rockafellar \(1974\)](#) (convexity, concavity, inner product, lower (upper) semi continuity, proper functions, etc.).

It is well known that Fenchel's conjugate plays an important role for instance in Functional Analysis, Convex Analysis and Optimization theory. From mathematical point is view, there are a lot of works in the literature. For example in [Rockafellar \(1974\)](#) make a systematic study for Convex Analysis, the series of works ([Singer, 1986, 1989, 1991](#)) treat the Duality Theory for Optimization Theory, in [Martinez Legaz \(2005\)](#) treat generalized convex duality and its economical applications, etc.

In Production Theory, “revenue minus production costs generate profit firm”, so Fenchel's conjugate ([Fenchel, 1949](#)) of a proper convex lower semi continuous function, which represent production cost of a firm is nothing else that maximum profile (see Section 2). As this natural interpretation of Fenchel's conjugate notion, there are many properties of Fenchel's conjugate, for considering it as an interesting tool in Economic Theory. For example, the involution property for proper convex lower semi continuous functions. This involution property say that the biconjugate of a proper convex lower semi continuous function is exactly the original function, because biconjugate is nothing else that closed convexification of original function. This property is very important in Convex Duality Theory, because the optimal value of dual problem of a convex problem, when it is generated by a proper convex lower semi continuous perturbed function, is exactly the optimal values of the original convex problem. When it occur, we say that there is no duality gap. Unfortunately, Fenchel's conjugate notion was introduced exclusively for proper convex lower semi continuous functions and convexity (or concavity) is no a natural assumption in Economic Theory.

Twenty one years after to Fenchel contribution, Moreau generalized Fenchel's conjugate ([Moreau, 1970](#)), but this involution property no hold in general and dual problem may be no convex. Recently (2011), was introduced a Fenchel–Moreau conjugate for lower semi continuous functions, where this involution holds and so we can again maintain the economical interpretation of Fenchel's conjugate. In particular, our work try to applied Fenchel–Moreau conjugate to consumer problem.

Firstly, in Section 2, we make a briefly introduction to Fenchel and Fenchel–Moreau conjugate and its importance in convex duality. we finished Section 2 introducing upper, lower closed functions. Here, the family of lower (upper) semi continuous functions are included strictly in the family of lower (upper) closed function. Moreover, Representation Theorem (Theorem I in [Debreu et al., 1983](#), p. 108) establish that when \mathbb{R}^n is completely ordered by the order \leq we have that: If for any $x' \in \mathbb{R}^n$ the sets $\{x \in \mathbb{R}^n : x \leq x'\}$ and $\{x \in \mathbb{R}^n : x' \leq x\}$ are closed, there exists on \mathbb{R}^n a continuous, real, order preserving function. We point out that there exists a family of closed real order preserving functions (closed function is such that it is lower and upper closed simultaneously). For this reason we work with upper (lower) closed functions.

Then, in Section 3 we establish son results in order to characterize the solution set of consumer problem using Fenchel–Moreau conjugate.

Finally, in Section 3.1 we build a dual problem for the consumer problem, adapting the conjugate for lower semi continuous functions introduced in [Cotrina et al. \(2011\)](#) to lower closed functions.

2. Fenchel and Fenchel–Moreau conjugate

In 1949, Fenchel introduced the conjugate notion for convex and lower semi continuous functions based on the well known fact that many inequalities used in functional analysis (such as Minkowski, Jensen, and Young) may be considered as a consequence of the convexity of a pair of functions, which Fenchel called “conjugate functions.” In a more precise formulation, Fenchel's result is the following: To each proper convex and lower semi continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$, there corresponds a function $f^* : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ with the same properties of f , such that

$$\langle x, y \rangle \leq f(x) + f^*(y),$$

for all x and y in \mathbb{R}^n . Here, functions f and f^* are called conjugate functions, and f^* is defined as follows:

$$f^*(y) := \sup\{\langle y, x \rangle - f(x) : x \in \mathbb{R}^n\}.$$

From economical viewpoint, taking f as a production cost, \bar{y} as a production supply and \bar{p} as a vector price. Here, conjugate function f^* represent maximum profile in the production. Unfortunately, these economical

interpretations hold when f is convex and lower semi continuous, because $f^{**} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+, -\infty\}$ defined by $f^{**}(x) := \sup\{\langle x, y \rangle - f^*(y) : y \in \mathbb{R}^n\}$ is the closed convexication of f ($f^{**} = f$). When f is not convex we lose all these economical interpretations because $f^{**} \neq f$.

The good news is that there exists in the literature extensions of Fenchel’s conjugate, for instance we have Fenchel–Moreau conjugate for lower semi continuous functions introduced in [Cotrina et al. \(2011\)](#), defined by

$$f^*(p) := \sup_{x \in \mathbb{R}^n} \{\langle p(x), x \rangle - f(x)\},$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semi continuous function and $p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous operator. So, $f^* : C(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R} \cup \{+\infty\}$ when f is proper and $C(\mathbb{R}^n, \mathbb{R}^n)$ is the continuous operator space. Following [Cotrina et al. \(2011\)](#), two functions $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : C(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R} \cup \{+\infty\}$ are conjugate functions if and only if $g = f^*$ and $f = g^*$. Moreover a linear subspace $S \subset C(\mathbb{R}^n, \mathbb{R}^n)$ which contains all constant operators of $C(\mathbb{R}^n, \mathbb{R}^n)$, is a dual conjugate space for f if and only if

$$f(y) = f^{**}(y) := \sup_{p \in S} \{\langle p(y), y \rangle - f^*(p)\}.$$

For example in Bertrand oligopoly, firms compete for price and we can consider the price as a continuous operator which depend of demand, so when cost functions are not convex, Fenchel–Moreau conjugate is welcome.

2.1. Convex duality

Without loss of generality, we can consider the consumer problem and its general model as follows:

$$(CP) \quad \begin{cases} \text{maximize} & u(x) \\ x \in & BS \end{cases} \tag{1}$$

where $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is an utility function and $\emptyset \neq BS \subset \mathbb{R}^n$ is a budget set.

Consumer problem is extremely difficult to solve as it has no structure. For example, utility functions are neither differentiable nor convex, budget set is not convex.

If we define $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ by

$$f(x) := \begin{cases} -u(x) & x \in BS \\ +\infty & x \notin BS \end{cases}$$

, the general model for consumer problem can be reduced to:

$$(PC) \quad \begin{cases} \text{minimize} & f(x) \\ x \in & \mathbb{R}^n \end{cases} \tag{2}$$

When f is convex and lower semi continuous (u is concave and upper semi continuous on BS), Fenchel’s conjugate is the key in order to build a dual problem for PC . The scheme (for built a dual problem) consider a convex lower semi continuous function $\varphi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ satisfying $\varphi(x, 0) = f(x) \forall x \in \mathbb{R}^n$ (φ is called as perturbation function), which generates the following dual problem:

$$(PCD) \quad \begin{cases} \text{maximize} & -\varphi^*(0, y^*) \\ y^* \in & \mathbb{R}^m \end{cases} \tag{3}$$

If for each $y \in \mathbb{R}^m$ we consider the following optimization problem

$$(PC_y) \quad \begin{cases} \text{minimize} & \varphi(x, y) \\ x \in & \mathbb{R}^n \end{cases} \tag{4}$$

The function value associated to the family of optimization problems is $v : \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+, -\infty\}$, defined by

$$v(y) = \inf\{\varphi(x, y) : x \in \mathbb{R}^n\} \tag{5}$$

We have that $\varphi^*(0, y^*) = v^*(y^*) \forall y^* \in \mathbb{R}^m$, and hence the optimal value is $v^*(0)$. Note that $v^*(0) = v(0)$ and $v(0)$ is the optimal value of primal problem PC .

Since $\varphi^*(0, y^*) = v^*(y^*) \forall y^* \in \mathbb{R}^m$, then dual problem *PCD* is equivalent to

$$(PCD) \quad \begin{cases} \text{minimize} & v^*(y^*) \\ & y^* \in \mathbb{R}^m \end{cases} \quad (6)$$

and φ^* is a perturbation function of v^* , following again this dual scheme we have that the dual problem of dual problem *PCD* is exactly the primal problem *PC*, so this duality theory is fully symmetric, when applied to problem with lower semi continuous, proper, convex perturbed function φ .

In [Martinez Legaz \(2005\)](#), the author point out that “The above duality specializes very nicely in the case of inequality constrained minimization problem of the form

$$(MP) \quad \begin{cases} \text{minimize} & f(x) \\ & g(x) \leq 0 \end{cases} \quad (7)$$

with $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ and $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$; the inequality \leq in \mathbb{R}^m is to be understood in the component wise sense. The classical way to embed this problem into a family of perturbed ones is by introducing so-called vertical perturbations, namely, one considers the perturbed function $\varphi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ given by

$$\varphi(x, y) = \begin{cases} f(x) & \text{if } g(x) + y \leq 0 \\ +\infty & \text{otherwise} \end{cases} \quad (8)$$

This function is convex whenever f and the components functions of g are convex. Clearly, minimizing $\varphi(x, 0)$ shows that, in this case, the dual problem reduce to

$$(MPD) \quad \begin{cases} \text{maximize} & \inf\{L(x, y^*) : x \in \mathbb{R}^n\} \\ & y^* \geq 0 \end{cases} \quad (9)$$

$L : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ being the Lagrangian function defined by $L(x, y^*) := f(x) + \langle g(x), y^* \rangle$. In this way, the classical Lagrangian duality theory becomes a particular case of the perturbational duality theory we have briefly described.

It is worth mentioning that, historically, the first dual problems discovered in optimization theory were defined without using any perturbation; the perturbational approach to duality proposed by [Rockafellar \(1974\)](#) later on provided us with a unifying scheme for all those duals. In a series of papers (see [Singer, 1986, 1989, 1991](#)), Singer has shown that the converse way also works, that is, some unperturbational dual problems induce the perturbational dual problems”.

Unfortunately, many texts, for instance, those related to microeconomics, firm theory, and production theory, leave Fenchel’s conjugate tools undeveloped, because Fenchel’s conjugate was introduced exclusively for convex lower semi continuous functions, but in general convex functions are not natural in economic theory.

2.2. Lower and upper closed functions

The following definition (which is equivalent to other one appears in [Aliprantis and Border, 1999](#), p. 43) will be used in the sequel.

Definition 1. The function $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ ($u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$) is regarded as upper (lower) closed if the upper level set $U(u, u(x)) := \{y \in \mathbb{R}^n : u(y) \geq u(x)\}$ (lower level set $L(u, u(x)) := \{y \in \mathbb{R}^n : u(x) \geq u(y)\}$) is closed for all $x \in \mathbb{R}^n$.

If the function $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty, -\infty\}$ is simultaneously upper closed and lower closed, we say that u is a closed function.

It is well known that if X is a completely ordered subset, with order denoted by \preceq , of a finite Euclidean space and both sets $\{y \in X : y \preceq x\}$ and $\{y \in X : x \preceq y\}$ are closed in X , then there exists a continuous function $u : X \rightarrow \mathbb{R}$ such that $U(u, u(x)) = \{y \in X : x \preceq y\}$ and $L(u, u(x)) = \{y \in X : y \preceq x\}$. But this representation is not unique, because there exist infinite closed functions which represent the preference ordering. Note, that upper closed functions would be not semi continuous functions. So, in this sense it is important to consider upper closed functions, and we use it instead of upper semi continuous functions.

3. Existence result for the consumer problem

Here we introduce a characterization of the solution set for the consumer problem (PC), using Fenchel–Moreau conjugate.

$$(PC) \quad \begin{cases} \text{maximize} & u(x) \\ x & \in BS \end{cases} \quad (10)$$

If we define $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ by $f(x) := \begin{cases} -u(x) & x \in BS \\ +\infty & x \notin BS \end{cases}$ (see previous section for more details), the following results establish a necessary and sufficient optimality condition for this general model of consumer problem (without assumptions for both u and BS).

Theorem 1. *Problem PC has solutions if and only if the following set*

$$\{x \in BS : f^*(0) = u(x)\} \neq \emptyset \quad (11)$$

Moreover, $\operatorname{argmax}\{u(x) : x \in BS\} = \{x \in BS : f^*(0) = u(x)\}$.

Proof. We know that

$$\begin{aligned} f^*(0) &= \sup\{\langle 0, x \rangle - f(x) : x \in \mathbb{R}^n\} = \sup\{-f(x) : x \in \mathbb{R}^n\} \\ &= \sup\{-f(x) : x \in BS\} = \sup\{u(x) : x \in BS\} \end{aligned}$$

Problem PC has solutions if and only if $\operatorname{argmax}\{u(x) : x \in BS\} \neq \emptyset$ if and only if $\{x \in BS : f^*(0) = u(x)\} \neq \emptyset$. \square

The following results reveal that PC has at least one solution if BS is a compact and u is upper closed.

Theorem 2. *If $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is an upper closed function and BS is a compact set, then $\{x \in BS : f^*(0) = u(x)\} \neq \emptyset$. Moreover, the solution set of PC is nonempty and compact.*

Proof. Taking a maximizer sequence $\{x^k\} \subset BS$ of u without loss of generality, because BS is compact, consider that x^k converges to $\bar{x} \in BS$, $\|\bar{x} - x^k\| \geq \|\bar{x} - x^{k+1}\| \forall k$, $u(x^k) \leq u(x^{k+1}) \leq \lambda \forall k$, where $\lambda = \sup\{u(x) : x \in BS\}$ and $\{u(x^k)\} \rightarrow \lambda$. The sequence of sets defined by $F_k = BS \cap U(u, u(x^k)) \cap \bar{B}(\bar{x}, \|\bar{x} - x^k\|)$ are embedded closed sets, and $\operatorname{diam}(F_k) \rightarrow 0$. By applying Cantor's intersection theorem, we have that $\bigcap_{k \in \mathbb{N}} F_k \neq \emptyset$, taking $\hat{x} \in \bigcap_{k \in \mathbb{N}} F_k$, then $u(\hat{x}) \geq \lambda$. Since $f^*(0) = \sup\{u(x) : x \in BS\} \geq u(\hat{x})$, the statement follows. \square

For each $p \in C(\mathbb{R}^n, \mathbb{R}^n)$ and $\omega \in \mathbb{R}^n$ consider $B(p, \omega) := \{x \in \mathbb{R}^n : \langle p(x), x \rangle \leq \langle p(\omega), \omega \rangle\}$. So, given $\omega \in \mathbb{R}^n$ consider $\wp(\omega) := \{p \in C(\mathbb{R}^n, \mathbb{R}^n) : B(p, \omega) \text{ is compact}\}$.

Corollary 1. *Given $\omega \in \mathbb{R}^n$. For each $p \in \wp(\omega)$, u upper closed and $BS = BS(p, \omega)$, the respective PC problem always has solution.*

Proof. Follows from [Theorem 2](#). \square

For example, consider $p : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined by $p(x) := Ax$, where A is a nonsingular, symmetric and positive definite matrix (hence, a linear monotone operator) and $\omega \in \mathbb{R}^n$ such that $\langle p(\omega), \omega \rangle > 0$, we have that for any upper closed utility function and $BS = BS(p, \omega)$, the respective Consumer problem has at least one solution, since [Theorem 2](#).

As a consequence of [Theorem 2](#), we obtain the following result, which improve Weierstrass Theorem (remember that the family of closed upper functions contains the family of upper semi-continuous functions, but both families are different). Similar result you can find in [Aliprantis and Border \(1999\)](#), Theorem 2.43.

Corollary 2. *Every upper closed function $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ obtains its maximum value in any compact set $C \subset \mathbb{R}^n$.*

When BS is unbounded, we consider the following assumption:

WS For each $\{x^k\} \subset BS$ such that $\|x^k\| \rightarrow +\infty$ when $k \rightarrow +\infty$, there exists m such that $BS \cap U(u, u(x^m)) \cap B(0, \|x^m\|) \neq \emptyset$.

From optimization viewpoint, assumption WS, can be consider as a coercive condition. For more detailed see [Sosa \(2013\)](#) and references there in.

From consumer problem, assumption WS establish that for any sequence of choices $\{x^k\} \subset BS$ going to $+\infty$ in measure, always there exist an element x^m of the sequence and another one choice \hat{x} such that in measure \hat{x} is strictly less than x^m and $u(x^m) \leq u(\hat{x})$.

Theorem 3. *Given problem PC, where u is upper closed and BS is a nonempty closed set, the set $\{x \in BS : f^*(0) = u(x)\}$ is nonempty if and only if assumption WS holds.*

Proof. If $\{x \in BS : f^*(0) = u(x)\} \neq \emptyset$, then by [Theorem 1](#), there exists \bar{x} solution of PC. We consider any sequence (fixed) $\{x^k\} \subset BS$ such that $\|x^k\| \rightarrow +\infty$ when $k \rightarrow +\infty$. Taking m such that $\|x^m\| > \|\bar{x}\|$, assumption WS holds. Conversely, by [Theorem 2](#), $\operatorname{argmax}\{u(x) : x \in BS \cap \bar{B}(0, m)\} \neq \emptyset$, without loss of generality consider that $BS \cap \bar{B}(0, k) \neq \emptyset \forall k$ is nonempty. Taking $x^k \in \operatorname{argmax}\{u(x) : x \in BS \cap \bar{B}(0, k)\}$ such that $\|x^k\| \leq \|x\| \forall x \in \operatorname{argmax}\{u(x) : x \in BS \cap \bar{B}(0, k)\}$. Suppose that $\{x^k\}$ is unbounded, then without loss of generality consider $\|x^k\| \rightarrow +\infty$ when $k \rightarrow +\infty$. By assumption WS, there exists m such that $BS \cap U(u, u(x^m)) \cap B(0, \|x^m\|) \neq \emptyset$. Taking $\hat{x} \in BS \cap U(u, u(x^m)) \cap B(0, \|x^m\|)$, then $\hat{x} \in BS$, $\|\hat{x}\| < \|x^m\|$ and $u(\hat{x}) \geq u(x^m)$. Since $x^m \in \operatorname{argmax}\{u(x) : x \in BS \cap \bar{B}(0, m)\}$, then \hat{x} also belongs to $\operatorname{argmax}\{u(x) : x \in BS \cap \bar{B}(0, m)\}$. However, $\|x^m\| \leq \|x\| \forall x \in \operatorname{argmax}\{u(x) : x \in BS \cap \bar{B}(0, n)\}$, so $\|x^m\| \leq \|\hat{x}\|$. This contradiction implies that $\{x^k\}$ is bounded. Now, take any cluster point of $\{x^k\}$, and without loss of generality, consider that $\{x^k\}$ converges to \bar{x} . Now, take $x \in B$ as arbitrarily fixed. For any $k > \|x\|$, we have that $u(x^k) \geq u(x) \forall k$ such that $\|x\| < k$. It implies that $\{x^k\}_{k > \|x\|} \subset U(u, u(x))$, so $\bar{x} \in U(u, u(x))$. Since $x \in BS$ was arbitrary, we have that $u(\bar{x}) \geq u(x) \forall x \in BS$. The statement follows because $u(\bar{x}) = \sup\{u(x) : x \in BS\} = f^*(0)$. \square

As a consequence of [Theorem 3](#) we obtain the following result, adopting assumption WS for any function $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ and any set $C \subset \mathbb{R}^n$. We denote this assumption by WS(u,C).

Corollary 3. *Let $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ be an upper closed function and $C \subset \mathbb{R}^n$ be a nonempty closed set. The function u obtains its maximum value over C if and only if WS(u,C) holds.*

We finished this section with an academical example in order to understand condition WS. Consider PC where u is a Cobb–Douglas function defined by $u(x, y) = \begin{cases} x^\alpha y^{1-\alpha}, & (x, y) \in]0, +\infty[^2 \\ -\infty, & (x, y) \notin]0, +\infty[^2 \end{cases}$, and $BS(p, \omega) = \{(x, y) \in \mathbb{R}^2 : \langle p, (x, y) \rangle \leq \langle p, \omega \rangle\}$. Of course $BS(p, \omega)$ is unbounded. By applying [Theorem 3](#), PC has solution if and only if WS holds. The question is: What does mean WS in this academical example?. The answer is: If $BS(p, \omega) \cap \operatorname{dom}(u) \neq \emptyset$, then WS is equivalent to $p \in \operatorname{dom}(u)$. Of course, the reader can verify it without difficulties.

3.1. Duality scheme for the consumer problem

In this section, we adapt the conjugate for lower semi continuous functions introduced in [Cotrina et al. \(2011\)](#) to lower closed functions.

Given $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ a lower closed function, we define $\bar{f}(x) = \sup_{\epsilon > 0} \inf_{y \in B(x, \epsilon)} f(y)$ for each $x \in \mathbb{R}^n$.

Lemma 1. *If $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower closed function, then $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a lower semi continuous function and $\bar{f} \leq f$.*

Proof. For each $x \in \mathbb{R}^n$, $\bar{f}(x) > -\infty$ follows from [Theorem 2](#), because f is lower closed and the closed ball $B(x, \epsilon)$ is compact for every $\epsilon > 0$. It implies that $\bar{f} : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is well defined, lower semi continuous and $\bar{f} \leq f$.

Given $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ a lower closed function, we define $f^*(p) = \sup_{x \in \mathbb{R}^n} \{\langle p(x), x \rangle - f(x)\}$, for each $p \in C(\mathbb{R}^n, \mathbb{R}^n)$.

Lemma 2. *If $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is a proper lower closed function, then $f^* : C(\mathbb{R}^n, \mathbb{R}^n) \rightarrow \mathbb{R} \cup \{+\infty\}$ is a convex function.*

Proof. For each $x \in \mathbb{R}^n$ and each $\lambda \in \mathbb{R}$ with $\lambda < \bar{f}(x)$, from Lemma 4.5 in Cotrina et al. (2011), there exists $\bar{p} \in C(\mathbb{R}^n, \mathbb{R}^n)$ such that $\bar{f}(y) - \langle \bar{p}(y), y \rangle > \lambda - \langle \bar{p}(x), x \rangle$ for all $y \in \mathbb{R}^n$. It implies that $-\infty < f^*(\bar{p}) < +\infty$. The convexity follows because f^* is the supremum of affine linear functions.

Given the consumer problem P_C and its associate function f , we indicate that $\phi : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R} \cup \{+\infty\}$ is a perturbation function for f , if ϕ is proper and lower closed and $\phi(x, 0) = f(x) \forall x \in \mathbb{R}^n$. We define the conjugate function $\phi^* : C(\mathbb{R}^{n+m}, \mathbb{R}^{n+m}) \rightarrow \mathbb{R} \cup \{+\infty\}$ of ϕ by $\phi^*(r) = \sup\{\langle r(z), z \rangle - \phi(z) : z = (x, y) \in \mathbb{R}^{n+m}\}$ and $C(\mathbb{R}^n, \mathbb{R}^n) \times C(\mathbb{R}^m, \mathbb{R}^m) \subset C(\mathbb{R}^{n+m})$. So, for all $x \in BS(p, w)$ and $r \in C(\mathbb{R}^m)$, we have $\phi^*(0, r) \geq u(x)$.

Definition 2. The dual problem associated to P_C is formulated as follows:

$$\begin{aligned} (DSCP) \quad & \min \quad \phi^*(0, r) \\ & \text{subject to } r \in C(\mathbb{R}^m). \end{aligned} \tag{12}$$

Note that the dual problem is an unrestricted convex problem. Unfortunately, it holds in infinite dimensional space as a consequence of transforming a restricted problem in finite dimensional space without structure into an unrestricted convex problem. We highlight that, in general, the gap is not zero. It is

$$\sup\{u(x) : x \in B(p, w)\} < \inf\{\phi^*(r) : r \in C(\mathbb{R}^m)\}$$

The next definition will establish the involution of duality operation (it is, dual of dual is the original problem).

Definition 3. Let $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be a proper lower closed function. We consider the subspace $S \subset C(\mathbb{R}^n)$ as the conjugate dual space of f if $f_S^{**} = f$, where

$$f_S^{**}(x) := \sup\{\langle s(x), x \rangle - f(x) : s \in S\}$$

The following result reveals when the involution holds for the dual operation.

Theorem 4. *If $C(\mathbb{R}^n) \times C(\mathbb{R}^m)$ is the dual conjugate space of perturbation function ϕ of P_C , then the involution holds for dual operation.*

Proof. Given the perturbation function ϕ of P_C , we define the marginal function $h : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ as

$$h(y) = \inf\{\phi(x, y) : x \in \mathbb{R}^n\}. \tag{13}$$

Thus, conjugate of the marginal function h is given by $h^* : C(\mathbb{R}^m) \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$, where

$$\begin{aligned} h^*(r) &= \sup\{\langle r(y), y \rangle - h(y) : y \in \mathbb{R}^m\} \\ &= \sup\{\langle r(y), y \rangle - \inf_{x \in \mathbb{R}^n} \phi(x, y) : y \in \mathbb{R}^m\} \\ &= \sup\{\langle r(y), y \rangle - \phi(x, y) : (x, y) \in \mathbb{R}^n \times \mathbb{R}^m\} \\ &= \sup\{\langle (0, r(y)), (x, y) \rangle - \phi(x, y) : (x, y) \in \mathbb{R}^n \times \mathbb{R}^m\} \\ &= \phi^*(0, r). \end{aligned}$$

Consider $g : C(\mathbb{R}^n) \rightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ as a marginal function of ϕ^* . We elucidate the primary points of duality scheme between P_C and DP_C as follows:

$$\begin{array}{ll}
 P_C & DP_C \\
 f(x) = \phi(x, 0) \quad \forall x \in \mathbb{R}^n & h^*(r) = \phi^*(0, r) \quad \forall r \in C(\mathbb{R}^m) \\
 h(y) = \inf\{\phi(x, y) : x \in \mathbb{R}^n\} & g(q) = \inf\{\phi^*(q, r) : r \in C(\mathbb{R}^m)\} \\
 \alpha := \inf\{f(x) : x \in \mathbb{R}^n\} & \beta := \inf\{h^*(r) : r \in C(\mathbb{R}^m)\}
 \end{array}$$

As ϕ^* restricted to $C(\mathbb{R}^n) \times C(\mathbb{R}^m)$ is the perturbation function of h^* , we claim that ϕ_S^{**} will be a perturbation function of the objective function of the dual of DP_C , where $S = C(\mathbb{R}^n) \times C(\mathbb{R}^m)$. Similar to previously outlined scheme, primary points of duality scheme between DP_C and its dual are as follows:

$$\begin{array}{ll}
 DP_C & DDP_C \\
 h^*(r) = \phi^*(0, r) \quad \forall r \in C(\mathbb{R}^m) & k(x) := \phi_S^{**}(x, 0) \quad \forall x \in \mathbb{R}^n \\
 g(q) = \inf\{\phi^*(q, r) : r \in C(\mathbb{R}^m)\} & j(y) := \inf\{\phi_S^{**}(x, y) : x \in \mathbb{R}^n\} \\
 \beta = \inf\{h^*(r) : r \in C(\mathbb{R}^m)\} & \delta := \inf\{k(x) : x \in \mathbb{R}^n\}
 \end{array}$$

Since $S = C(\mathbb{R}^n) \times C(\mathbb{R}^m)$ is the conjugate dual space of ϕ , we have $\phi_S^{**} = \phi$. Hence, it follows that $k=f, j=h$, and $\delta=\alpha$. Therefore, involution holds. \square

As an example, consider an allocation time problem in the sense of Becker with a usual budget set and a linear transformation technology, which can be formulated as follows:

$$(P_C) \quad \begin{cases} \text{maximize } u(x) \\ Ax \leq b \\ x \geq 0 \end{cases} \tag{14}$$

where $u : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{-\infty\}$ is an upper closed utility function, A is a matrix with size $m \times n$ and $b \in \mathbb{R}^m$. Following our scheme, the function $f : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ is defined by $f(x) = \begin{cases} -u(x) & Ax \leq b, x \geq 0 \\ +\infty & \text{other case} \end{cases}$. Now we define the per-

turbation function $\phi : \mathbb{R}^n \times C(\mathbb{R}^m, \mathbb{R}^m) \rightarrow \mathbb{R} \cup \{+\infty\}$ by $\phi(x, p) = \begin{cases} f(x) & (b - Ax - s) \in \text{Im}(p), x \geq 0, s \geq 0 \\ +\infty & \text{other case} \end{cases}$,

where the set $\text{Im}(p) = \{p(x) : x \in \mathbb{R}^n\}$. Here, $\phi^*(0, y) = \begin{cases} \langle b, y \rangle + g^*(-A^T y) & y \geq 0 \\ +\infty & y \neq 0 \end{cases}$, where $g : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$

defined by $g(x) = \begin{cases} f(x) & x \geq 0 \\ +\infty & \text{othercase} \end{cases}$. It is, dual problem is a convex problem in \mathbb{R}^m .

Acknowledgements

Wilfredo Sosa was partially supported by CNPq Grants 302074/2012-0 and 471168/2013-0. Part of this research was carried out during Wilfredo’s visit to IMPA of Rio de Janeiro.

References

Aliprantis, C.D., Border, K.C., 1999. *Infinite Dimensional Analysis: A Hitchhiker’s Guide*. Springer Verlag, New York.
 Cotrina, J., Karas, E., Ribeiro, A., Sosa, W., Yuan, J., 2011. Fenchel–Moreau conjugation for lower semi continuous functions. *Optimization* 60, 1045–1057.
 Debreu, G., 1983. *Mathematical Economics Twenty papers of Gerard Debreu*. Econometric Society Monographs.
 Fenchel, W., 1949. On conjugate convex functions. *Can. J. Math.* 1, 73–77.
 Martinez Legaz, J.E., 2005. Generalized convex duality and its economic applications. In: *Handbook of Generalized Convexity and Generalized Monotonicity, Nonconvex Optim. Appl.*, vol. 76. Springer, New York, pp. 237–292.

- Moreau, J.-J., 1970. Inf-convolution, sous-additivité, convexité des fonctions numériques. *J. Math. Pures Appl.* 49, 109–154.
- Rockafellar, R.T., 1974. *Conjugate duality and optimization*. SIAM.
- Singer, I., 1986. A general theory of dual optimization problems. *J. Math. Anal.* 116, 77–130.
- Singer, I., 1989. A general theory of dual optimization problems II. On the perturbational dual problem corresponding to an unperturbational dual problem. *Z. Oper. Res.* 33, 241–258.
- Singer, I., 1991. Some further relations between unperturbational and perturbational dual optimization problems. *Optimization* 22, 317–339.
- Sosa, W., 2013. From Weierstrass to Ky Fan Theorems and existence results on optimization and equilibrium problems. *Pesqui. Oper.* 33 (2), 199–215.