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# Beating the market with small portfolios: Evidence from Brazil

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## Abstract

Optimal portfolios with a restriction on the number of assets, also referred to as cardinality-constrained portfolios, have been receiving attention in the literature due to its popularity among market practitioners and retail investors. In most cases, however, the interest is in proposing efficient optimization methods to solve the problem, with little or no attention to the characteristics of the resulting portfolio such as risk-adjusted performance and turnover. We address this question by implementing a tractable reformulation of the cardinality-constrained version of the minimum variance portfolio. We analyze the out-of-sample performance of cardinality-constrained portfolios according to alternative criteria and check the robustness of the results for portfolios with alternative number of assets and under alternative re-balancing frequencies. Our empirical application for the Brazilian equities market shows that cardinality-constrained minimum variance portfolios with very few assets, e.g. 3 stocks, can deliver statistically lower portfolio risk and higher Sharpe ratios in comparison to the market index. Similar results are obtained for constrained portfolios with 5 and 10 assets and under daily, weekly, and monthly re-balancing frequencies. Our evidence indicates that it is possible to obtain better risk-adjusted performance with *fewer* securities in the portfolio by using an improved allocation scheme.

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*Keywords:* Bootstrap; Cardinality constraint; Graduated non-convexity; Shrinkage estimator

## Resumo

Carteiras ótimas com restrições no número de ativos, também conhecidas como carteiras com restrições de cardinalidade, tem recebido grande atenção na literatura em função da sua popularidade entre praticantes de mercado e pequenos investidores. Na maioria dos casos, entretanto, o interesse está em propor métodos eficientes de otimização para resolver o problema, com pouca ou nenhuma atenção às características das carteiras como desempenho ajustado ao risco ou *turnover*. Abordamos esta questão através da implementação de uma reformulação tratável do problema de minimização da variância da carteira sujeita à restrição de cardinalidade. Analisamos o desempenho fora-da-amostra das carteiras com restrição de cardinalidade segundo diversos critérios e checamos a robustez dos resultados para carteiras com diversos números de ativos e também diversas frequências de rebalanceamento. Nossa aplicação empírica para o mercado acionário brasileiro mostra de carteiras de variância mínima com restrições de cardinalidade envolvendo um pequeno número de ativos, por exemplo, três ações podem obter níveis menores de risco e maiores índices de Sharpe em relação ao índice de mercado. Resultados semelhantes são obtidos para carteiras restringidas com cinco e 10 ativos e

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com frequências de rebalanceamento diária, semanal e mensal. Nossa evidência indica que é possível obter um melhor desempenho ajustado ao risco com menos ativos na carteira através do uso de uma melhor estratégia de alocação.

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*Palavras-chave:* Bootstrap; restrição de cardinalidade; não-convexidade graduada; estimador de encolhimento

## 1. Introduction

Both academics and market participants have been interested for a long time in knowing whether or not it is possible to build sensible portfolios with only few assets in the so-called cardinality-constrained approach. The literature dates back to [Evans and Archer \(1968\)](#) and [Jacob \(1974\)](#), who were among the first to study the characteristics of small portfolios.<sup>1</sup> This problem seems particularly relevant to investors with financial constraints as well as to financial institutions that customize portfolios for clientele with specific needs. [Blog et al. \(1983\)](#), for instance, point out that the small investor that wishes to come up with an optimal risk-return portfolio will be constrained as the efficient portfolios in the mean-variance setting of [Markowitz \(1952\)](#) sometimes contain too many securities to be attractive to the small investor. Therefore, the cardinality-constrained approach is mostly suitable for the small investor wishing to optimize the risk-return tradeoff for a limited number of assets.

Despite its practical appeal, obtaining a mean-variance portfolio with a constraint of the number of assets is a challenging optimization problem. The main difficulties are the non-differentiability and the discontinuity that arise due to the inclusion of a cardinality or counting function constraint. In this sense, a very large number of studies have been proposing alternative optimization methods to efficiently handle this NP-hard problem (see, for instance, [Jacob, 1974](#); [Faaland, 1974](#); [Blog et al., 1983](#); [Chang et al., 2000](#); [Jobst et al., 2001](#); [Jansen and Van Dijk, 2002](#); [Maringer and Kellerer, 2003](#); [Li et al., 2006](#); [Maringer and Oyewumi, 2007](#); [Cura, 2009](#); [Brodie et al., 2009](#); [Kopman et al., 2009](#); [Bertsimas and Shioda, 2009](#); [Canakgoz and Beasley, 2009](#); [Wang et al., 2012](#); [Chen and Kwon, 2012](#)). These references suggest that a fairly large number of approaches ranging from integer programming methods to heuristic-based algorithms are currently available in order to overcome the difficulties in solving the cardinality-constrained problem.

It is also worth noting that the current relevance of cardinality-constrained portfolios is controversial. The main reason is straightforward: with the advent of new financial instruments such as exchange traded funds (ETF), nowadays investors can easily buy or sell a benchmark portfolio by investing only in a *single* asset, thus overcoming potential restrictions that arise when many assets have to be traded.<sup>2</sup> Therefore, the practical relevance of the cardinality-constrained portfolio is questionable as the financial market provides specific instruments that replicates near-to-perfection alternative benchmark portfolios such as stock market indices.

We argue in this paper that the construction of cardinality-constrained portfolios only makes sense if these portfolios are able to provide an attractive risk-return profile. To be more precise, holding a small portfolio is interesting provided that this portfolio outperforms the benchmark portfolio in terms of risk-adjusted performance. Otherwise, the investor would be better off by investing in a single asset that replicates a large, diversified portfolio such as ETF-like securities. In this sense, the decision to invest in an optimized cardinality-constrained portfolio must be preceded by comparative analysis on the risk-adjusted performance of the small portfolio with respect to an adequate benchmark index. Therefore, this paper asks a fundamental question: How good (or bad) is an optimal cardinality-constrained portfolio with respect to the ETF-indexable market portfolio?

One potential concern that arises when addressing this question is the underdiversification of the resulting portfolios. That is, investors holding a cardinality-constrained portfolio might be subjected to a high level of unsystematic risk. In fact, [Statman \(1987\)](#) shows that for the US market a diversified portfolio would contain at least 30 stocks. However, the study of the relationship between portfolio size and risk in [Statman \(1987\)](#) is based on equally weighted randomly

<sup>1</sup> The work of [Evans and Archer \(1968\)](#) was followed by two discussion notes in [Whitmore \(1970\)](#) and [Johnson and Shannon \(1974\)](#). These notes provide additional analysis and formalization of the problem initially proposed in [Evans and Archer \(1968\)](#).

<sup>2</sup> One example of an ETF traded in the Brazilian equities market is the BlackRock's iShares BOVA11, which replicates the Ibovespa index.

selected assets. In this sense, [Jacob \(1974\)](#) and [Johnson and Shannon \(1974\)](#) argue that it is possible to obtain the same level of variation with far greater average portfolio returns and – more importantly – with *fewer* securities in the portfolio by using an alternative allocation scheme, thus enforcing the argument in favor of the optimal cardinality-constrained portfolios.

In this paper we obtain cardinality-constrained minimum variance portfolios by implementing the reformulation proposed by [Coleman et al. \(2006\)](#) of the original (cardinality-unconstrained) problem. The advantages of this approach are threefold. First, it approximates the discontinuous counting function with a sequence of continuously differentiable non-convex piecewise quadratic functions which approaches the original non-differentiable counting function in the limit. Second, the approach can be easily implemented in most commercial packages and standard optimization algorithms such as Matlab's *fmincon*. Third, simulation results reported in [Coleman et al. \(2006\)](#) show that this approach outperforms other approaches such as the one proposed in [Jansen and Van Dijk \(2002\)](#) in yielding cardinality-constrained portfolios with lower tracking error.

We provide empirical evidence involving a large data set consisting of daily returns of 45 stocks traded at the Brazilian equities markets from March/2009 to November/2011. We implement the cardinality-constrained minimum variance using the reformulation proposed in [Coleman et al. \(2006\)](#) along with a robust estimate of the covariance matrix of stock returns as proposed in [Ledoit et al. \(2004\)](#). We consider optimal portfolios with 3, 5, and 10 assets and a daily, weekly, and monthly re-balancing frequencies. Moreover, we formally test the differences in portfolio risk (standard deviation) and in risk-adjusted performance measured in terms of Sharpe ratios with respect to the Ibovespa index by employing a bootstrap approach proposed in [Politis and Romano \(1994\)](#). In this sense, the paper adds to the literature by providing a realistic out-of-sample implementation and evaluation of the cardinality-constrained portfolios and provide comparative analysis with respect to the main benchmark index under alternative re-balancing frequencies.

The results are favorable to the cardinality-constrained portfolios considered in the paper. We find that optimal portfolios containing only 3 assets outperform the market portfolio in terms of lower risk and higher Sharpe ratios. Moreover, this result is also robust to the choice of portfolio re-balancing frequency. For instance, in the case of monthly re-balancing the portfolio standard deviation of the cardinality-constrained minimum variance portfolio with only 3 assets is 1.14, whereas the same figure for the Ibovespa is 1.43. The Sharpe ratio of the minimum variance portfolio with 3 assets is 0.068 whereas the same figure for the Ibovespa is -0.038. The results with daily and weekly re-balancing frequencies are even more favorable to the cardinality-constrained portfolios, although this is also accompanied by an increase in portfolio turnover. Overall, our results corroborate the evidence in [Jacob \(1974\)](#) and [Johnson and Shannon \(1974\)](#) as we find that it is possible to obtain better risk-adjusted performance with *fewer* securities in the portfolio by using an alternative allocation scheme.

The paper is organized as follows. In [Section 2](#) we detail the formulation of the cardinality-constrained minimum variance portfolios and the methodology to evaluate portfolio performance. [Section 3](#) presents the data used in the paper. [Section 4](#) discusses the results of the empirical application. Finally, [Section 5](#) brings concluding remarks.

## 2. Cardinality-constrained minimum variance portfolios

In this section, we review the cardinality-constrained minimum variance portfolio and discuss the implementation strategy and methodology to assess out-of-sample performance. We start by defining a vector of portfolio returns  $R_t = R_{1t}, \dots, R_{Nt}$  where  $N$  is the number of assets in the portfolio and  $t = 1, \dots, T$  is the number of observations. Returns are computed as the first differences in log prices. The cardinality-constrained optimal portfolio problem is given by

$$\begin{aligned}
 & \min_{w \in \mathbb{R}^N} f(w) \\
 & \text{subject to} \\
 & \sum_{i=1}^N \Lambda(w_i) \leq K \\
 & \sum_{i=1}^N w_i = 1 \\
 & w \geq 0,
 \end{aligned} \tag{1}$$

where  $w$  is the vector of portfolio weights,  $f(w)$  is the portfolio objective function and  $\Lambda(w_i) = 1$  if  $w_i \neq 0$  and  $\Lambda(w_i) = 0$  otherwise.  $w \geq 0$  is a shortselling restriction. The cardinality constraint,  $\sum_{i=1}^N \Lambda(w_i) \leq K$ , is a counting function and can be interpreted as an upper bound on the number of assets allowed to enter the portfolio. Coleman et al. (2006) point out that this problem is NP-hard and the existing methods for solving (1) are heuristic-based.

The choice of the portfolio objective function  $f(w)$  in Eq. (1) will determine the nature of the problem. A common choice for  $f(w)$  is the mean-variance function according to Markowitz (1952), i.e.  $f(w) = w' \Sigma w - (1/\delta)w' R$  where  $\Sigma$  is a positive-definite covariance matrix,  $w' R$  is the portfolio return, and  $\delta$  is the risk-aversion coefficient. In this framework, individuals choose their allocations in risky assets based on the trade-off between expected return and risk. In order to implement the mean-variance optimization in practice, it is common to obtain estimates of the vector of expected returns and its covariance matrix and plug these estimators in an analytical or a numerical solution to the mean-variance problem. This problem is known to be very sensitive to estimation of the mean returns (e.g. Michaud, 1989; Jagannathan and Ma, 2003). Very often, the estimation error in the mean returns degrade the overall portfolio performance and introduces an undesirable level of portfolio turnover. In fact, existing evidence suggest that the performance of optimal portfolios that do not rely on estimated mean returns is better. Following Fan et al. (2012), to avoid the difficulty associated with estimation of the expected return vector, from now on, we consider the minimum variance portfolio objective function, i.e.  $f(w) = w' \Sigma w$ .

### 2.1. A reformulation based on graduated non-convexity

Standard optimization software does not apply to the minimum variance portfolio problem in Eq. (1) since the cardinality constraint function is discontinuous. Coleman et al. (2006) point out that one possible way of overcoming this difficulty is to consider a sequence of approximations which approach the variance minimization problem in the limit. We depart by considering an equivalent form of the minimum variance problem in Eq. (1) given by

$$\begin{aligned} & \min_{w \in \mathbb{R}^N} \left( w' \Sigma w + \mu \sum_{i=1}^N \Lambda(w_i) \right) \\ & \text{subject to} \\ & \sum_{i=1}^N w_i = 1 \\ & w \geq 0, \end{aligned} \tag{2}$$

where  $\mu > 0$  is a penalty parameters. By varying  $\mu > 0$ , solutions of Eq. (2) yield optimal minimum variance portfolios with different number of assets.

To handle the discontinuity introduced by the counting function  $\Lambda(w_i)$ , we follow Coleman et al. (2006) and solve a sequence of approximations  $\{\mathcal{P}_k\}_{k=1,2,\dots}$  to the minimum variance problem in Eq. (1) based on approximating the counting function  $\Lambda(w_i)$  with continuous differentiable piecewise quadratic functions with graduated non-convexity. The solution of the approximation problem  $\{\mathcal{P}_{k-1}\}$  is used as the starting point for the approximation problem  $\{\mathcal{P}_k\}$ .

We approximate the discontinuous function  $\Lambda(w_i)$  by the following continuous function  $h_\lambda(z)$

$$h_\lambda(z) = \begin{cases} \lambda z^2 & \text{if } |z| \leq \sqrt{1/\lambda} \\ 1 & \text{otherwise,} \end{cases}$$

where  $\lambda > 0$  is a large constant (which set to  $10^8$  in our implementation). The function  $h_\lambda(z)$  is continuous but not everywhere differentiable and it has many local minimizers. In this sense, we consider a continuous and differentiable approximation to the function  $h_\lambda(z)$  given by the function  $g_\lambda(z; \rho)$ ,

$$g_\lambda(z; \rho) = \begin{cases} \lambda z^2 & \text{if } |z| \leq q \\ 1 - \frac{\rho}{2}(|z| - r)^2 & q \leq |z| < r \\ 1 & \text{otherwise,} \end{cases}$$

where  $r^2 = ((2/\rho) + (1/\lambda))$ ,  $q = (1/\lambda r)$ , and  $\rho > 0$ . The function  $g_\lambda(z; \rho)$  is piecewise quadratic with a concave quadratic piece for  $z \in (q, r)$ .

To see how the functions  $g_\lambda(z; \rho)$  approach  $h_\lambda(z)$ , let  $\{\rho_k\}$  be a monotonically increasing sequence which converges to  $+\infty$ . As  $\rho_k$  increases, the curvature of the quadratic function of  $g_\lambda(z; \rho)$  for  $z \in [q_k, r_k]$  becomes more negative, introducing graduated non-convexity. In addition, as  $\rho_k \rightarrow +\infty$ ,  $r_k, q_k$  converge to  $\sqrt{1/\lambda}$  and the functions  $g_\lambda(z; \rho)$  approach  $h_\lambda(z)$ ; see [Coleman et al. \(2006\)](#) for a graphical visualization of the approximations. Finally, replacing  $g_\lambda(z; \rho)$  for  $\Lambda(w_i)$  in Eq. (2), the sequence of approximations  $\{\mathcal{P}_k\}_{k=1,2,\dots}$  to the variance minimization problem is given by

$$\begin{aligned} & \min_{w \in \mathcal{R}^N} \left( w' \Sigma w + \mu \sum_{i=1}^N g_\lambda(w_i; \rho_k) \right) \\ & \text{subject to} \\ & \sum_{i=1}^N w_i = 1 \\ & w \geq 0. \end{aligned} \tag{3}$$

The implementation of the sequence of approximation problems in Eq. (3) works as follows. First, we start solving the cardinality-unconstrained version of the minimum variance minimization problem in Eq. (1). Second, we use the resulting minimizer of the first step as starting values for solving the approximation problem  $\{\mathcal{P}_k\}$  in Eq. (3) for a given value of  $\rho_k$ . Third, we use the resulting minimizer in  $\{\mathcal{P}_k\}$  as starting values for solving the approximation problem  $\{\mathcal{P}_{k+1}\}$  using  $\rho_{k+1}$ . As  $\rho_k$  converges to  $+\infty$ , the approximate problems approach the variance minimization problem in Eq. (1).

So far we have considered the solution of the approximation problem in Eq. (3) in which the size of the optimal minimum variance portfolio is chosen by varying the parameter  $\mu$ . In this sense, it is necessary to experiment with different values of  $\mu$  in order to generate an optimal portfolio with a desired number of assets. However, we can be more explicit in obtaining a desired number of assets by considering the following exact penalty formulation

$$\begin{aligned} & \min_{w \in \mathcal{R}^N} \left( w' \Sigma w + \mu \max \left( \sum_{i=1}^N g_\lambda(w_i; \rho_k) - K, 0 \right) \right) \\ & \text{subject to} \\ & \sum_{i=1}^N w_i = 1 \\ & w \geq 0, \end{aligned} \tag{4}$$

where  $\mu$  is the exact parameter corresponding to the cardinality constraint. For a sufficient large  $\mu$ , the minimization of Eq. (4) yields a minimum variance portfolio of no more than  $K$  assets. The intuition behind this result is as follows. First, when  $\sum_{i=1}^N g_\lambda(w_i; \rho_k) > K$  decreasing the objective function in Eq. (4) leads to decrease in the objective function in Eq. (3). Second, for initial approximate problems  $\{\mathcal{P}_k\}$  with small  $\rho_k$ , then  $\sum_{i=1}^N g_\lambda(w_i; \rho_k) < K$ . As  $\rho_k$  converges to  $+\infty$ , more negative curvature is gradually introduced into the objective function via  $g_\lambda(w_i; \rho_k)$  and the objective function in Eq. (3) is gradually decreased. The graduated non-convexity process is terminated when, for all  $i$ , either  $(w_k)_i < q$  (i.e. the  $i$ th asset is not in the minimum variance portfolio) or  $(w_k)_i < r$  (i.e. the  $i$ th asset is in the minimum variance portfolio).

### 2.1.1. Choosing a good covariance matrix

One fundamental issue in our empirical implementation is the choice of the specification of the covariance matrix of returns,  $\Sigma$ , in Eq. (1). In this paper we consider the shrinkage estimator of [Ledoit et al. \(2004\)](#). Shrinkage estimators are becoming very popular in the portfolio construction literature due to their ability to reduce the estimation error in large covariance matrices. For instance, [Ledoit and Wolf \(2003\)](#), [Ledoit et al. \(2004\)](#) report improved results in terms of portfolio performance when the shrinkage estimator is used vis-à-vis traditional estimators such as the sample covariance matrix. The shrinkage estimator of [Ledoit et al. \(2004\)](#) is defined as an optimally weighted average of the sample covariance matrix and the covariance matrix based on the constant correlation model, which is simply the

average of all the sample correlations. The intuition behind this shrinkage estimator is to come up with an optimal convex combination between an unbiased covariance matrix estimator that may be subject to substantial estimation error (i.e. the sample covariance matrix) and another estimator that possibly is biased but has considerably less estimation error (i.e. the covariance matrix from the constant correlation model). The shrinkage estimator of Ledoit and Wolf (denoted by  $\Sigma_{LW}$ ) is then defined as

$$\Sigma_{LW} = \psi\Phi + (1 - \psi)\Sigma, \quad (5)$$

where  $\psi$  is the shrinkage intensity,  $\Sigma$  is the sample covariance matrix, and  $\Phi$  is the (target) covariance matrix based on the constant correlation model. A closed-form solution for the optimal shrinkage intensity (minimizing the distance between the true and estimated covariance matrices based on the Frobenius norm) is provided by Ledoit et al. (2004).

## 2.2. Methodology to evaluate portfolio performance and implementation details

The performance of the cardinality-constrained minimum variance portfolios is evaluated in terms of average return ( $R_p$ ), standard deviation (volatility) of returns ( $\sigma$ ), Sharpe ratio (SR), and turnover. These statistics are calculated as follows:

$$\begin{aligned} R_p &= \frac{1}{T-1} \sum_{t=1}^{T-1} w_t' R_{t+1} \\ \sigma &= \sqrt{\frac{1}{T-1} \sum_{t=1}^{T-1} (w_t' R_{t+1} - R_p)^2} \\ SR &= \frac{R_p}{\hat{\sigma}} \\ \text{Turnover} &= \frac{1}{T-1} \sum_{t=1}^{T-1} \sum_{j=1}^N (|w_{j,t+1} - w_{j,t}|), \end{aligned}$$

where  $w_{j,t}$  is the weight of the asset  $j$  in the portfolio in period  $t$  before the re-balancing, while  $w_{j,t+1}$  is the desired weight of the asset  $j$  in period  $t+1$ . As pointed out by DeMiguel et al. (2009), the turnover as defined above, can be interpreted as the average fraction of wealth traded in each period.

In order to assess the relative performance of the cardinality-constrained minimum variance portfolios, we consider as main benchmark the Ibovespa index, which is the main stock market index in the Brazilian market. The stationary bootstrap of Politis and Romano (1994) with  $B=1.000$  resamples and block size  $b=5$  was used to test the statistical significance of differences between portfolio standard deviation and Sharpe ratios of optimal portfolios relative to the benchmark portfolio. The methodology suggested in Ledoit and Wolf (2008, Note 3.2) was used to obtain  $p$ -values.

The implementation and out-of-sample of the cardinality-constrained portfolios are based on a rolling window scheme. First, a covariance estimate based on the Ledoit et al. (2004) shrinkage estimator is made using an estimation window of  $L=252$  observations, which for daily data corresponds to 1 years. Second, using this estimate we compute the optimal cardinality-constrained minimum variance portfolio. Three, we repeat this procedure for the next period, by including the data for the new date and dropping the data for the earliest period. We continue doing this until the end of the data set is reached. At the end of this process, we have generated  $T-L$  portfolio weight vectors, where  $T$  is the total number of observations in the data set. Holding the portfolio  $w_t$  yields the out-of-sample return in period  $t+1$ :  $\hat{R}_{p,t+1} = w_t' R_{t+1}$ . The performance evaluation according to the statistics discussed above are solely based on the  $T-L$  out-of-sample observations.

## 3. Data

The data used in this paper consists of daily returns of  $N=45$  stocks traded in the São Paulo's BMFBovespa stock exchange in Brazil from 02/03/2009 to 24/11/2011, yielding a total of  $T=677$  observations. All selected stocks belonged to the composition of the Ibovespa stock market index during the sample period. Table 1 reports the ticker of each

Table 1  
Descriptive statistics.

Ticker	Mean return (%)	Standard deviation (%)	Minimum (%)	Maximum (%)	Skewness	Kurtosis
PETR4	-0.006	1.821	-7.885	6.836	-0.100	4.284
VALE5	0.085	1.960	-9.621	9.249	0.293	6.094
BVMF3	0.092	2.482	-9.523	10.536	0.199	4.459
CSNA3	0.020	2.251	-12.401	8.753	0.063	5.559
VALE3	0.071	2.098	-10.007	10.314	0.325	5.890
PETR3	-0.026	1.917	-8.230	8.230	0.040	4.261
BBDC4	0.090	1.841	-9.221	7.131	0.063	4.464
USIM5	-0.012	2.525	-6.666	12.359	0.366	3.767
GGBR4	0.025	2.466	-11.434	9.849	0.166	4.174
BBAS3	0.110	2.018	-8.504	7.440	0.186	4.245
ITSA4	0.091	1.966	-11.591	8.884	0.064	6.129
CMIG4	0.066	1.610	-7.024	4.767	-0.108	3.712
BRAP4	0.086	2.266	-8.741	10.307	0.170	5.061
TNLP4	-0.051	2.006	-11.889	12.449	0.171	6.825
CESP6	0.131	1.994	-6.681	7.758	0.085	3.620
LAME4	0.145	2.404	-7.443	9.580	0.317	3.821
AMBV4	0.191	1.498	-4.914	5.743	0.034	3.834
ELET3	-0.001	1.962	-7.872	9.483	0.354	5.484
CYRE3	0.119	3.068	-7.543	13.076	0.321	3.864
ELET6	0.024	1.831	-6.280	10.571	0.611	5.724
RDGD3	0.060	2.248	-11.381	7.459	-0.130	4.378
GFS3	0.032	3.075	-8.880	14.980	0.425	4.383
BTOW3	-0.107	2.820	-8.256	11.449	0.366	4.273
LREN3	0.206	2.679	-8.492	11.034	0.263	4.142
GOAU4	0.021	2.474	-8.615	9.876	0.311	3.961
NATU3	0.090	1.876	-6.153	7.067	0.139	3.568
GOLL4	0.052	3.067	-24.360	16.351	-0.215	10.320
JBSS3	0.036	2.842	-13.045	9.973	0.157	4.332
CPL6	0.085	1.556	-6.476	6.072	0.107	3.888
TAMM4	0.126	2.930	-13.251	24.408	0.868	10.823
CCRO3	0.134	1.919	-8.688	7.568	0.039	4.115
USIM3	0.091	2.940	-9.168	13.103	0.297	3.605
CSAN3	0.153	2.570	-12.751	11.495	0.255	5.669
CPFE3	0.088	1.456	-5.844	5.160	-0.099	3.665
CRUZ3	0.163	1.780	-6.482	5.689	-0.126	3.820
BRKM5	0.146	2.564	-11.043	10.175	0.120	4.603
PCAR4	0.121	1.897	-8.059	11.903	0.387	6.429
RSID3	0.168	3.259	-8.708	14.553	0.214	3.480
EMBR3	0.098	2.409	-13.251	12.117	0.136	7.220
TNLP3	-0.068	2.308	-17.424	11.081	-0.270	8.787
TRPL4	0.060	1.406	-5.216	5.099	-0.003	3.688
SBSP3	0.126	2.071	-9.578	7.416	0.104	4.309
KLBN4	0.150	2.426	-8.536	9.510	0.315	4.205
TMAR5	0.004	2.327	-10.541	9.795	0.066	5.138
LIGT3	0.062	1.602	-6.079	6.645	-0.060	3.905

stock along with descriptive statistics. We observe that the data displays stylized facts in financial time series, such as mean returns close to zero and excess kurtosis with respect to that of the Gaussian distribution.

#### 4. Results

We report in [Table 2](#) the performance statistics of the cardinality-constrained minimum variance portfolio and of the benchmark portfolios. We obtain cardinality-constrained portfolios with a target number of  $K = \{3, 5, 10\}$  assets in the portfolio. Also, we consider as benchmark portfolios the Ibovespa index and also an equally weighted portfolio that



Table 2  
Performance of cardinality-constrained minimum variance portfolios

	Mean return (%)	Standard deviation (%)	Sharpe ratio	Turnover
<i>Daily re-balancing</i>				
$K = 1$	0.097	1.209*	0.080*	0.089
$K = 5$	0.076	1.120*	0.067*	0.097
$K = 10$	0.059	0.956*	0.062*	0.068
<i>Weekly re-balancing</i>				
$K = 1$	0.077	1.213*	0.063*	0.035
$K = 5$	0.074	1.127*	0.066*	0.044
$K = 10$	0.061	0.961*	0.063*	0.039
<i>Monthly re-balancing</i>				
$K = 1$	0.083	1.222*	0.068*	0.021
$K = 5$	0.069	1.143*	0.060*	0.029
$K = 10$	0.058	0.968*	0.060*	0.026
<i>Benchmarks</i>				
Ibovespa	−0.055	1.436	−0.038	
Equally weighted	−0.036	1.298	−0.028	

The Table reports performance statistics for the cardinality-constrained minimum variance portfolios using 45 Brazilian stocks and for the benchmarks (Ibovespa index and the equally weighted portfolio). The portfolios are obtained using a target number of  $K = \{3, 5, 10\}$  assets in the portfolio. The statistics of returns and standard deviation and Sharpe ratio are based on daily returns. Optimal portfolios are re-balanced on a daily, weekly and monthly basis. Asterisks indicate that the standard deviation and the Sharpe ratio are statistically different at the 10% level with respect to those obtained with the Ibovespa index.

assigns equal weights in all the 45 stocks in the portfolio.<sup>3</sup> Table reports the mean return, standard deviation of returns, Sharpe ratio, and portfolio turnover. All figures are based on daily returns. Portfolio compositions are re-balanced on a daily basis. However, the transaction costs involved in this re-balancing frequency might degrade the performance of the portfolios and hinder its implementation in practice. Thus, the performance of optimized portfolios is also evaluated in the case of weekly and monthly re-balancing frequencies. A potentially negative effect of adopting a lower re-balancing frequency is that the optimal compositions may become outdated.

The results in Table 2 reveal that the cardinality-constrained portfolios outperform the benchmark index in *all* cases. For instance, under daily re-balancing frequency the minimum variance portfolio with 3, 5 and 10 assets deliver statistically lower portfolio risk (1.209, 1.120, and 0.956, respectively) in comparison to the Ibovespa index (1.436) and also in comparison to the equally weighted portfolio (1.298). This result also holds when the optimal portfolios are re-balanced on a weekly and monthly basis. The most important message from this result is that it is possible to build a portfolio less riskier than the market portfolio with very few assets, therefore corroborating the evidence in Jacob (1974) and Johnson and Shannon (1974).

The risk-adjusted performance of the cardinality-constrained portfolios measured by the Sharpe ratio is also substantially better than that of the benchmark. Under daily re-balancing, the Sharpe ratio of the minimum variance portfolios with 3, 5 and 10 assets deliver Sharpe ratios of 0.08, 0.07, and 0.06, whereas the same figure the the Ibovespa index and the equally weighted portfolio are, respectively, −0.04 and −0.03. Similar as before, this result also holds when the optimal portfolios are re-balanced on a weekly and monthly basis.

As expected, we find that lowering the portfolio re-balancing frequency leads to a substantial decrease in portfolio turnover. For instance, the turnover of the minimum variance portfolio with 3 assets under daily, weekly and monthly re-balancing are, respectively, 0.089, 0.035, and 0.021. As for the minimum variance portfolio with 10 assets, the same figures are 0.068, 0.039, and 0.026. We also find that, on average, lowering the portfolio re-balancing frequency does not lead to decreases in the risk-adjusted performance measured by the Sharpe ratio. This result suggests that cardinality-constrained portfolios can “beat the market” even in low portfolio re-balancing frequencies.

<sup>3</sup> The equally weighted portfolio has been extensively studied in the empirical literature. For instance, DeMiguel et al. (2009) find that this portfolio composition outperforms (in terms of Sharpe ratio and turnover) 14 widely used portfolio strategies, such as mean-variance and minimum variance. Therefore, it seems natural to compare against this simple but powerful portfolio in which all the assets have the same weight.

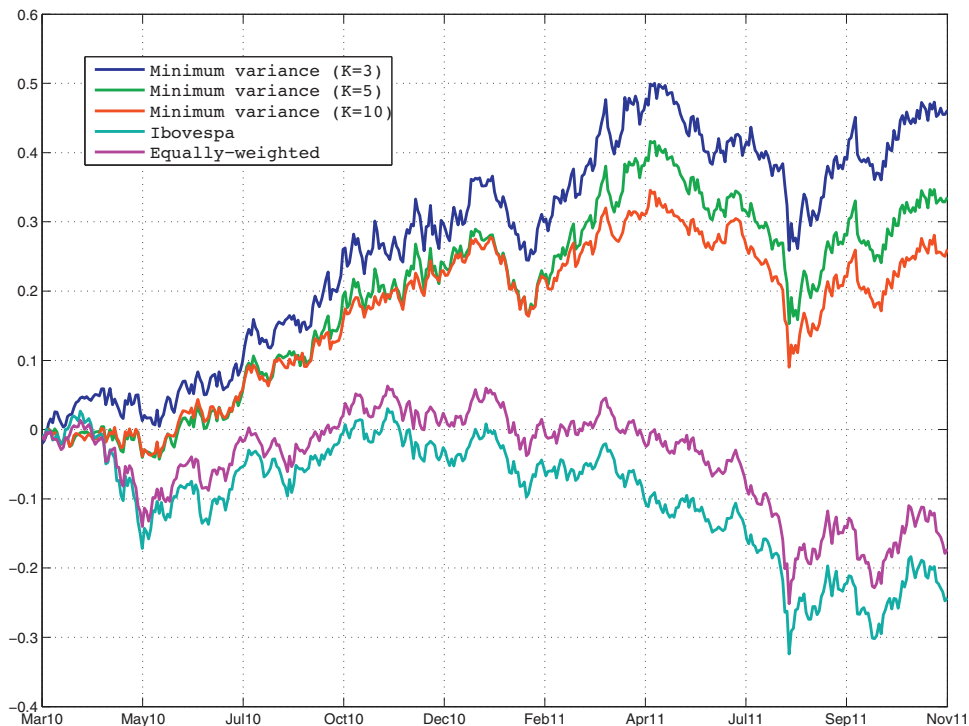


Fig. 1. Cumulative returns

To further illustrate the results, we plot in Fig. 1 the cumulative returns under daily re-balancing of the cardinality-constrained minimum variance portfolios with 3 (blue line), 5 (green line), and 10 (red line) assets along with the cumulative returns of the benchmark index and of the equally weighted portfolio. The figure shows a stellar difference in cumulative returns across the out-of-sample period: while the minimum variance portfolio achieves a cumulative portfolio return of approximately 45% throughout the period, the benchmark index and the equally weighted portfolio deliver cumulative returns of  $-25\%$  and  $-15\%$  approximately.

## 5. Concluding remarks

The implementation of the mean-variance portfolio policy of Markowitz (1952) for small investors can be problematic, as efficient portfolios sometimes contain too many securities. One alternative to overcome this difficulty is the cardinality-constrained version of the mean-variance portfolio optimization problem, in which the optimal portfolio is restricted to a limited number of assets. Despite its practical relevance, this approach has a fundamental difficulty as the resulting optimization problem becomes non-differential as discontinuous due to the inclusion of a cardinality constraint.

We implement a tractable reformulation of the cardinality-constrained minimum variance portfolio problem that has numerous advantages with respect to other existing approaches, and obtain cardinality-constrained portfolios for the Brazilian equities market. We consider optimal portfolios with 3, 5, and 10 assets and a daily, weekly, and monthly re-balancing frequencies. Moreover, we formally test the differences in portfolio risk (standard deviation) and in risk-adjusted performance measured in terms of Sharpe ratios with respect to the Ibovespa index by employing a bootstrap test.

The empirical evidence of this paper provides favorable evidence to the cardinality-constrained portfolios. We find that optimal portfolios containing only 3 assets outperform the market portfolio in terms of lower risk and higher Sharpe ratios, and that this result is robust to the choice of portfolio re-balancing frequency. Overall, our results corroborate the evidence in Jacob (1974) and Johnson and Shannon (1974) as we find that it is possible to obtain better risk-adjusted performance with *fewer* securities in the portfolio by using an alternative allocation scheme.

## References

- Bertsimas, D., Shioda, R., 2009. [Algorithm for cardinality-constrained quadratic optimization](#). *Comput. Optim. Appl.* 43 (1), 1–22.
- Blog, B., Van Der Hoek, G., Rinnooy Kan, A.H.G., Timmer, G.T., 1983. [The optimal selection of small portfolios](#). *Manage. Sci.* 29 (7), 792–798.
- Brodie, J., Daubechies, I., De Mol, C., Giannone, D., Loris, I., 2009. [Sparse and stable Markowitz portfolios](#). *Proc. Natl. Acad. Sci.* 106 (30), 12267–12272.
- Canakgoz, N.A., Beasley, John E., 2009. [Mixed-integer programming approaches for index tracking and enhanced indexation](#). *Eur. J. Oper. Res.* 196 (1), 384–399.
- Chang, T-J., Meade, Nigel, Beasley, John E., Sharaiha, Yazid M., 2000. [Heuristics for cardinality constrained portfolio optimisation](#). *Comput. Oper. Res.* 27 (13), 1271–1302.
- Chen, Chen, Kwon, Roy H., 2012. [Robust portfolio selection for index tracking](#). *Comput. Oper. Res.* 39 (4), 829–837.
- Coleman, Thomas F., Li, Yuying, Henniger, Jay, 2006. [Minimizing tracking error while restricting the number of assets](#). *J. Risk* 8 (4), 33.
- Cura, Tunchan, 2009. [Particle swarm optimization approach to portfolio optimization](#). *Nonlinear Anal. Real World Appl.* 10 (4), 2396–2406.
- DeMiguel, V., Garlappi, L., Uppal, R., 2009. [Optimal versus naive diversification: how inefficient is the 1/N portfolio strategy?](#) *Rev. Financ. Stud.* 22 (5), 1915–1953.
- Evans, John L., Archer, Stephen H., 1968. [Diversification and the reduction of dispersion: an empirical analysis](#). *J. Financ.* 23 (5), 761–767.
- Faaland, Bruce, 1974. [An integer programming algorithm for portfolio selection](#). *Manage. Sci.* 20 (10), 1376–1384.
- Fan, Jianqing, Zhang, Jingjin, Yu, Ke, 2012. [Vast portfolio selection with gross-exposure constraints](#). *J. Am. Stat. Assoc.* 107 (498), 592–606.
- Jacob, Nancy L., 1974. [A limited-diversification portfolio selection model for the small investor](#). *J. Financ.* 29 (3), 847–856.
- Jagannathan, R., Ma, T., 2003. [Risk reduction in large portfolios: Why imposing the wrong constraints helps](#). *J. Financ.* 58 (4), 1651–1684.
- Jansen, Roel, Van Dijk, Ronald, 2002. [Optimal benchmark tracking with small portfolios](#). *J. Portf. Manage.* 28 (2), 33–39.
- Jobst, Norbert J., Horniman, Michael D., Lucas, Cormac A., Mitra, Gautam, 2001. [Computational aspects of alternative portfolio selection models in the presence of discrete asset choice constraints](#). *Quant. Financ.* 1 (5), 489–501.
- Johnson, Keith H., Shannon, Donald S., 1974. [A note on diversification and the reduction of dispersion](#). *J. Financ. Econ.* 1 (4), 365–372.
- Kopman, Leonid, Scott Liu, Shucheng, Shaw, Dong, 2009. [Using lagrangian relaxation to obtain small portfolios](#). *J. Portf. Manage.* 35 (2), 75–79.
- Ledoit, O., Wolf, M., 2003. [Improved estimation of the covariance matrix of stock returns with an application to portfolio selection](#). *J. Empir. Financ.* 10 (5), 603–621.
- Ledoit, O., Wolf, M., 2008. [Robust performance hypothesis testing with the Sharpe ratio](#). *J. Empir. Financ.* 15 (5), 850–859.
- Ledoit, Olivier, Wolf, Michael, 2004. [Honey, I shrunk the sample covariance matrix](#). *J. Portf. Manage.* 30 (4), 110–119.
- Li, Duan, Sun, Xiaoling, Wang, Jun, 2006. [Optimal lot solution to cardinality constrained mean-variance formulation for portfolio selection](#). *Math. Financ.* 16 (1), 83–101.
- Maringer, Dietmar, Kellerer, Hans, 2003. [Optimization of cardinality constrained portfolios with a hybrid local search algorithm](#). *Or Spectr.* 25 (4), 481–495.
- Maringer, Dietmar, Oyewumi, Olufemi, 2007. [Index tracking with constrained portfolios](#). *Intel. Syst. Account. Financ. Manage.* 15 (1-2), 57–71.
- Markowitz, Harry, 1952. [Portfolio selection](#). *J. Financ.* 7 (1), 77–91.
- Michaud, Richard O., 1989. [The Markowitz optimization enigma: is 'optimized' optimal?](#) *Financ. Anal. J.*, 31–42.
- Politis, Dimitris N., Romano, Joseph P., 1994. [The stationary bootstrap](#). *J. Am. Stat. Assoc.* 89 (428), 1303–1313.
- Statman, Meir, 1987. [How many stocks make a diversified portfolio?](#) *J. Financ. Quant. Anal.* 22 (03), 353–363.
- Wang, Meihua, Xu, Chengxian, Xu, Fengmin, Xue, Hongang, 2012. [A mixed 0-1 LP for index tracking problem with CVaR risk constraints](#). *Ann. Oper. Res.* 196 (1), 591–609.
- Whitmore, G.A., 1970. [Diversification and the reduction of dispersion: a note](#). *J. Financ. Quant. Anal.* 5 (02), 263–264.