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## Article

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# The option to expand a project: its assessment with the binomial options pricing model<sup>☆</sup>



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## ABSTRACT

Traditional methods of investment appraisal, like the Net Present Value, are not able to include the value of the operational flexibility of the project. In this paper, real options, and more specifically the option to expand, are assumed to be included in the project information in addition to the expected cash flows. Thus, to calculate the total value of the project, we are going to apply the methodology of the Net Present Value to the different scenarios derived from the existence of the real option to expand. Taking into account the analogy between real and financial options, the value of including an option to expand is explored by using the binomial options pricing model. In this way, estimating the value of the option to expand is a tool which facilitates the control of the uncertainty element implicit in the project.

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## 1. Introduction

Real options have been widely studied from an academic point of view. A certain consensus has been reached among analysts, who recognize the fact that the use of real options can make a very interesting contribution to business practice. As a consequence of the absence of a single formula to calculate the option value, only the largest companies have studied the value of real options given that a high level of mathematical knowledge is required for its calculation [15]. Real options provide a tool capable of dealing with abstract concepts [16,24]. This enables decision-makers to have the flexibility necessary to respond to changing scenarios [19]. Real options improve the dynamic and proactive management since they allow for the incorporation of some aspects of a strategic nature in investment projects [18,32].

The option to expand a project offers the possibility (not the obligation) of increasing the productive scale of the project by making an additional investment. It is equivalent to an American *call* option, given that it allows for the consideration of making additional follow-on investments (e.g. in order to increase the production rate) if the project is profitable [7].

This type of option has a high strategic value since the decision-maker has the possibility to decide the size of the project, and

even organize the necessary investment in successive stages, thus dividing the project into a sequence of smaller units [5]. Indeed, in the case of a multi-stage project, the element of risk for the company is reduced, since before increasing the investment in the project, the original forecasts for the results of the project can be reconsidered in the light of actual market developments [22].

Examples of such options are the opportunity of introducing a new product, or entering a new market without barriers to competitive entry [29]. Moreover “expanding into new geographic areas and investing in research and development also involve staged investments of this kind” [6]. The option to expand is considered when the market conditions are better than initially expected. In response to positive market conditions, the company may decide to expand production, which in turn requires additional investment [1].

According to Kogut [17], the investment in a new market may be considered as the right to a future purchase. In those sectors where the investment facilitates access to future opportunities, the option to expand may represent a substantial element in assessing the project's viability. There are even some projects with a negative Net Present Value (NPV) which provide the company with the possibility to implement this option in the future. In these cases, the option to expand may represent a major part of the overall value of the project for potential investors [27].

In general terms, the consideration of real options should be included in the initial presentation of any project. Driouchi and Bennett [12] reveal that “the real option attention, knowledge and management can be more crucial for organizations than real op-

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**Table 1**  
Relationship between real options and project magnitudes. Source: Own elaboration.

Magnitude	Real option
Cash flow	To expand/reduce
Residual value	To abandon
Maturity	To defer

tions opportunities". In this paper our aim is to derive a formula to calculate the value of the real option to expand as a part (percentage) of the actual value of the project. All the steps of this procedure are explained in detail throughout this paper. The nomenclature used is that found in the NPV formula since it is familiar to all companies.

After defining the concept of a real option and specifically the option to expand, in Section 2 we explain the NPV method to assess investment projects, in order to introduce the nomenclature. In addition, Section 3 explains the multiplicative binomial model to calculate the value of an option. Later, we develop the expression of the option to expand in one, two and  $n$  periods, in the Sections 4, 5 and 6, respectively. Finally, Section 7 summarizes and concludes.

## 2. Some limitations of the Net Present Value methodology

Traditional methods to assess investments are characterized by the lack of flexibility when implementing the project. Some methods such as the NPV (one of the most common expressions employed in project appraisal) are not able to take into account the value of the operative flexibility when managing a project. The well-known expression of the NPV is given by:

$$NPV = -I_0 + \sum_{k=1}^n CF_k \prod_{j=1}^k (1 + i_j)^{-1} + RV_n \prod_{k=1}^n (1 + i_k)^{-1},$$

where:

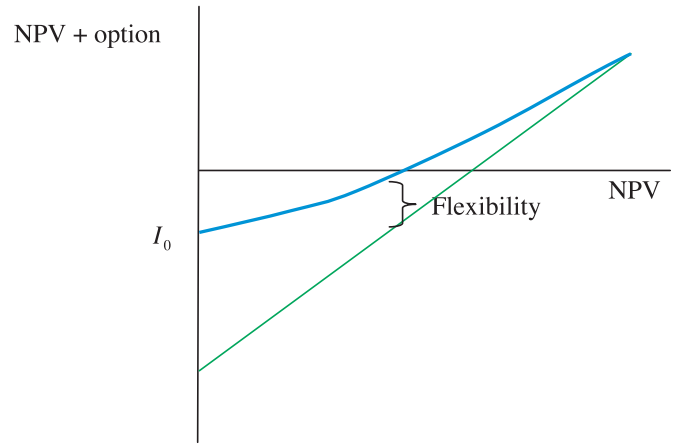
- $I_0$  is the initial investment at the present moment.
- $CF_k$  is the expected cash flow corresponding to period  $k$ , where  $k = 1, 2, \dots, n$ .
- $i_k$  is the interest rate corresponding to period  $k$ , where  $k = 1, 2, \dots, n$ .
- $i_j$  is the interest rate corresponding to period  $j$ , where  $j \leq k$ .
- $RV_n$  is the residual value of the project at moment  $n$ .

The method of the NPV has many limitations since its main parameters are of a random nature: cash flows, interest rate, residual value and project maturity. Following the evolution of any of the above magnitudes makes a compound model whose complexity increases as the project duration increases. In spite of this, NPV continues to be a universally used model.

As previously defined, the NPV is a static method which assumes that the basic conditions of the project cannot be modified [5]. The discount rate used will be greater only in investment projects with a high degree of uncertainty, making the project NPV lower. So, the use of traditional methodologies to assess projects associated with different alternatives and changing scenarios results in an undervalued project appraisal [25].

However, some investment projects incorporate several kinds of flexibility as a consequence of the options which they contemplate. Specifically, some project magnitudes may be related to a particular type of operational flexibility which may appear at any moment during the project: they are called real options [25]. Table 1 shows how each random magnitude is related to a real option:

In order to solve this problem, in this paper we are going to study the option to expand, which is most frequently used in business practice together with the option to defer and abandon [9].



**Fig. 1.** NPV + real option depending on NPV. Source: Own elaboration.

Real options are used as a complement to the static methodology of NPV in that both methods have to be used sequentially. Real options assessment offers a fundamental modification with regard to traditional methods: the incorporation of uncertainty as an element which adds value to the project [11], provided that this flexibility is identified and used to respond to any new information which may arise [2]. The increasing interest in the study of real options is justified by the current situation of investment opportunities which is characterized by a high level of risk.

## 3. The binomial options pricing model

The primary aim of this research is to derive an expression to calculate the value of the option to expand from a theoretical point of view. The procedure to be used is as follows:

1. First, to value the investment project, by including the real option.
2. Second, once it is mathematically proved that this value is higher than the project value without the real option, the value of the real option is obtained by the difference between the values including and not including the real option in the project.

Previous studies on real option assessment have been conducted using discrete and continuous approaches, depending on the degree of complexity and the nature of the option [3,8,23,26]. Specifically, given the greater manageability of discrete methods [4,30,31,34], in this paper we use the discrete model known as *binomial multiplicative*, first for one period, then for two periods, and finally for  $n$  periods.

The binomial options pricing model is one of the most frequently used methods in the field of financial options appraisal [28,33]. Given the conceptual analogy between financial and real options, in this work we will develop a new expression to assess the option to expand based on the multiplicative binomial method. Some authors, for example Damodaran [10] and Mascareñas [21], have already started to apply this simple and intuitive method to real options assessment.

In Fig. 1, the value represented by the curve in blue can be divided into two components: the static NPV (in green) and the value of the real option, graphically treated as the flexibility of the project [29].

Real options are ideal when the basic NPV of the project is close to zero, the risk is high and the options contained in the project are exclusive, that is to say, when flexibility brings an additional value to the whole investment project [14]. In this way, the project assessment using real options allows the integration of traditional

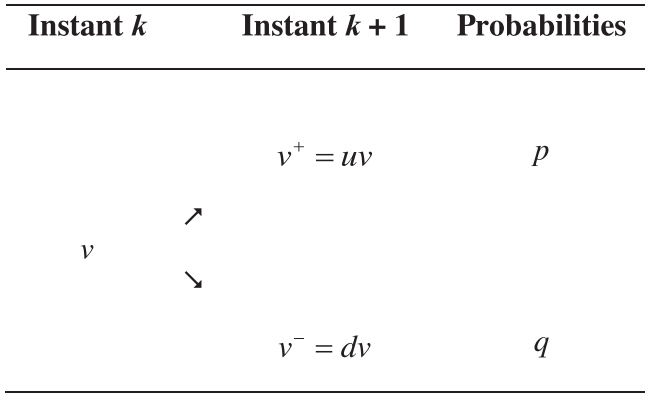


Fig. 2. Evolution of the project in one period. Source: Own elaboration.

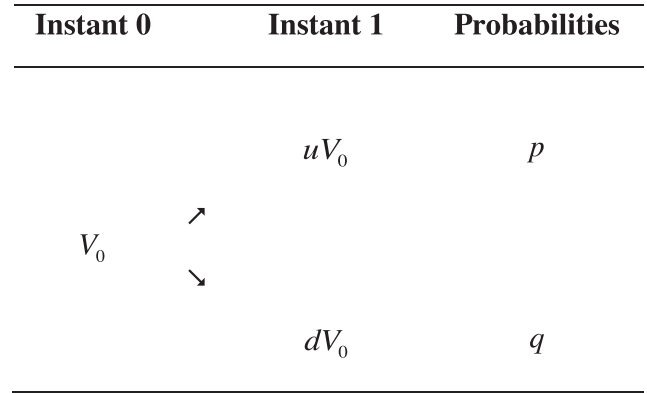


Fig. 4. Evolution of the project in the first period. Source: Own elaboration.

Instant $k$	Instant $k + 1$	Probability
$I_E^{(k)}$	$I_E^{(k+1)} = (1 + r_f)I_E^{(k)}$	1

Fig. 3. Evolution of the additional expenditure in one period. Source: Own elaboration.

financial theory and strategic planning, thus obtaining a more appropriate valuation of projects with a high level of uncertainty [13].

The value of a project with the option to expand depends on the evolution of its future value. Specifically, the exercise of the option to expand at any moment is only justified when the value of the project with the option is greater than the value of the project without it. Observe that we are referring to the “value” of the project and not to its “net value”. So, hereinafter and for the sake of simplicity, our reasoning will be made on the “present value” and not on the “Net Present Value”, i.e. without taking into account the initial investment.

In general terms, the present value of a project with the option to expand by a percentage  $x$ , by incurring an additional expenditure  $I_E^{(n)}$  at moment  $n$  (denoted by  $V(E)_0^{(n)}$ ), using continuous stochastic processes, is given by:

$$V(E)_0^{(n)} = \frac{1}{1 + r_f} \int_{-\infty}^{\infty} \max\{V_n(1 + x) - I_E^{(n)}, V_n\} f(V_n) dV_n,$$

where:

- $V_n$  is the random variable which describes the value of the project at moment  $n$ ,
- $f(V_n)$  is the probability density function of  $V_n$ , and
- $r_f$  is the risk-free interest rate.

In this paper, we will employ the multiplicative binomial process where the value,  $v$ , of the project at an instant  $k$  is assumed to fluctuate between an upper value  $v^+ := uv$  (where  $u > 1$ ) and a lower value  $v^- := dv$  (where  $d = 1/u$ ), with probabilities  $p = \frac{(1+r_f)-d}{u-d}$  and  $q = 1 - p$ , respectively. Fig. 2 displays the stochastic process for two arbitrary consecutive periods ( $k \geq 0$ ).

The accompanying evolution of the additional expenditure for two arbitrary consecutive periods ( $k \geq 1$ ) is shown in Fig. 3.

#### 4. Option to expand in the first period

In this case, the evolution of the present value,  $V_0$ , of the project is represented in Fig. 4.

Thus, the corresponding possible values of the project with the option to expand are:

$$v(E)^+ := \max\{uV_0(1 + x) - I_E^{(1)}, uV_0\} = uV_0 + \max\{uxV_0 - I_E^{(1)}, 0\}$$

and

$$v(E)^- := \max\{dV_0(1 + x) - I_E^{(1)}, dV_0\} = dV_0 + \max\{dxV_0 - I_E^{(1)}, 0\},$$

where their probabilities of occurrence are  $p$  and  $q = 1 - p$ , respectively. Therefore, the present value of the total project is given by the following expression:

$$V(E)_0^{(1)} = \frac{pv(E)^+ + qv(E)^-}{1 + r_f}. \quad (1)$$

In this way, the value of the option to expand will be positive when the additional expected benefits of the investment (at least in the best case, viz  $uxV_0$ ) exceed the additional payment necessary to finance its expansion. Thus, the present value of the project depends on whether this condition holds, resulting as follows:

$$V(E)_0^{(1)} = \begin{cases} V_0(1 + x) - \frac{I_E^{(1)}}{1 + r_f}, & \text{if } I_E^{(1)} < dxV_0 \\ \frac{p[uV_0(1+x) - I_E^{(1)}] + qdV_0}{1 + r_f}, & \text{if } dxV_0 \leq I_E^{(1)} < uxV_0 \\ V_0, & \text{if } uxV_0 \leq I_E^{(1)} \end{cases} \quad (2)$$

As a first step to study the value of the option to expand, we will prove that, as expected, the project value including the option is greater than the value of the project without it.

**Proposition 1.** The present value of an investment project with the option to expand in one period is always greater than or equal to  $V_0$ .

**Proof.** We will consider all cases displayed in expression (2):

1. If  $I_E^{(1)} < dxV_0$ , the present value of the investment project with the option to expand is  $V_0(1 + x) - \frac{I_E^{(1)}}{1 + r_f}$ . As  $\frac{I_E^{(1)}}{1 + r_f} < I_E^{(1)} < dxV_0 < xV_0$ , it is proved that:

$$V_0(1 + x) - \frac{I_E^{(1)}}{1 + r_f} = V_0 + xV_0 - \frac{I_E^{(1)}}{1 + r_f} > V_0.$$

2. If  $dxV_0 \leq I_E^{(1)} < uxV_0$ ,  $V(E)_0^{(1)}$  is equal to  $\frac{p[uV_0(1+x) - I_E^{(1)}] + qdV_0}{1 + r_f}$ . According to Copeland et al. [7], the absence of arbitrage condition ( $pu + qd = 1 + r_f$ ) leads to:

$$\frac{qd}{1 + r_f} = 1 - \frac{pu}{1 + r_f}. \quad (3)$$

Therefore,

$$V(E)_0^{(1)} = V_0 + \frac{p(uxV_0 - I_E^{(1)})}{1 + r_f}.$$

As  $I_E^{(1)} < uxV_0$ , the following inequality holds:

$$V_0 + \frac{p(uxV_0 - I_E^{(1)})}{1 + r_f} > V_0.$$

3. Finally, if  $uxV_0 \leq I_E^{(1)}$ , the project value with the option to expand in one period is:

$$V(E)_0^{(1)} = V_0,$$

which confirms the required equality/inequality.  $\square$

In this way, the three following scenarios may arise when valuing a project with the option to expand within the first period:

- The first scenario, when  $I_E^{(1)} < dxV_0$ , is favorable to the expansion as the increase in the value of the project exceeds the additional expenditure. Therefore, the option value will be positive.
- On the other hand, if  $dxV_0 \leq I_E^{(1)} < uxV_0$ , the investment could become profitable, given that the incremental benefits obtained by the expansion may be greater than the required additional investment costs. Therefore, in this case, the value of the option to expand will also be positive.
- Finally, if  $uxV_0 \leq I_E^{(1)}$ , expanding the project is not feasible. In this way, the additional investment would not be undertaken since the corresponding investment costs are greater than the increase in value of the project, even in the most favorable situation. Thus, in this case, the value of the option to expand is 0.

**Example 1.** We will now consider the present value of the project and the value of the option to expand for each period studied. To do this, we will discuss the example of an investment project of an oil company [20] which provides the following information (in millions of euros):

- Initial investment to carry out the project,  $I_0 = 104$ .
- Present value of the project,  $V_0 = 100$ .
- Risk-free interest rate,  $r_f = 5\%$ .
- Up and down factors affecting the project value,  $u = 1.8$  and  $d = 0.56$ , respectively. Thus, the risk-neutral probabilities [7] are  $p = 39.5\%$  and  $q = 1 - p = 60.5\%$ .
- Option to expand by 50% ( $x = 0.50$ ) its productive capacity in a given period.

Starting from this information, the project value with the option to expand in one period,  $V(E)_0^{(1)}$ , can be expressed according to the additional investment required to undertake such an expansion (see Eq. (2)):

$$V(E)_0^{(1)} = \begin{cases} 150 - 0.9524I_E^{(1)}, & \text{if } I_E^{(1)} \leq 28 \\ 133.83 - 0.3762I_E^{(1)}, & \text{if } 28 < I_E^{(1)} \leq 90 \\ 100, & \text{if } 90 < I_E^{(1)} \end{cases}$$

whose graphical representation is as follows:

In Fig. 5, we can confirm that the project value decreases as the value of the additional expenditure required to carry out the project expansion increases. Once the project value with the real option has been calculated, the value of the option to expand within the first period can be obtained.

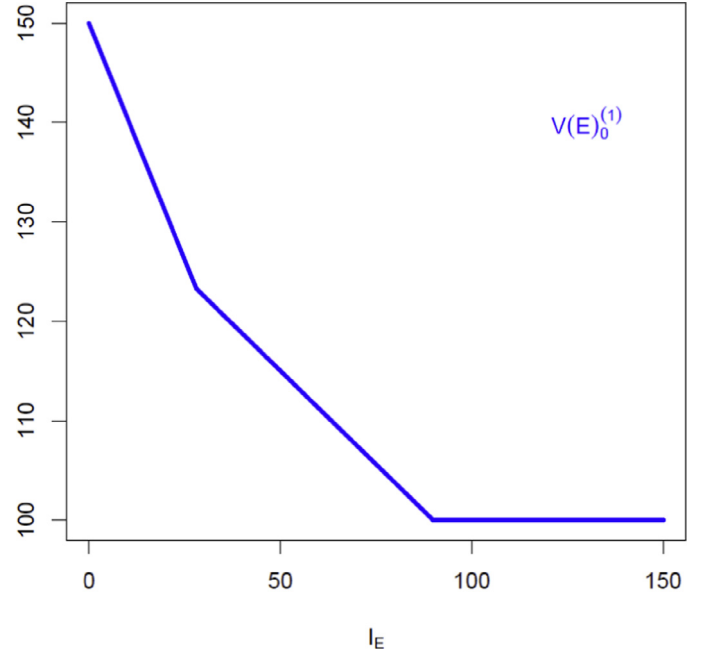


Fig. 5. Project value with the option to expand in the first period. Source: Own elaboration.

**Corollary 1.** The value of the option to expand in one period ( $O_E^{(1)}$ ) is equal to:

$$O_E^{(1)} = \begin{cases} xV_0 - \frac{I_E^{(1)}}{1 + r_f}, & \text{if } I_E^{(1)} < dxV_0 \\ \frac{p(uxV_0 - I_E^{(1)})}{1 + r_f}, & \text{if } dxV_0 \leq I_E^{(1)} < uxV_0 \\ 0, & \text{if } uxV_0 \leq I_E^{(1)} \end{cases}$$

**Proof.** To calculate the option value, we have to use the formula  $O_E^{(1)} = V(E)_0^{(1)} - V_0$ . The proof of the third case is obvious (see the proof of Proposition 1), from which we can deduce the situations corresponding to the first and second case:

- If  $I_E^{(1)} < dxV_0$ ,

$$O_E^{(1)} = V_0(1 + x) - \frac{I_E^{(1)}}{1 + r_f} - V_0 = xV_0 - \frac{I_E^{(1)}}{1 + r_f}.$$

- If  $dxV_0 \leq I_E^{(1)} < uxV_0$ , one obviously has (see the proof of Proposition 1):

$$O_E^{(1)} = V_0 + \frac{p(uxV_0 - I_E)}{1 + r_f} - V_0 = \frac{p(uxV_0 - I_E)}{1 + r_f}.$$

$\square$

**Example 2.** The value of the option to expand the productive capacity of the oil company by 50% in one period is:

$$O_E^{(1)} = \begin{cases} 50 - 0.9524I_E^{(1)}, & \text{if } I_E^{(1)} \leq 28 \\ 33.83 - 0.3762I_E^{(1)}, & \text{if } 28 < I_E^{(1)} \leq 90 \\ 0, & \text{if } 90 < I_E^{(1)} \end{cases}$$

Fig. 6 displays the option value.

Observe that the value of the option to expand is decreasing with respect to the investment necessary to expand the project scale.

## 5. Option to expand in the second period

The value of the option to expand the investment project within two periods from the present moment (denoted by  $O_E^{(2)}$ ) can be

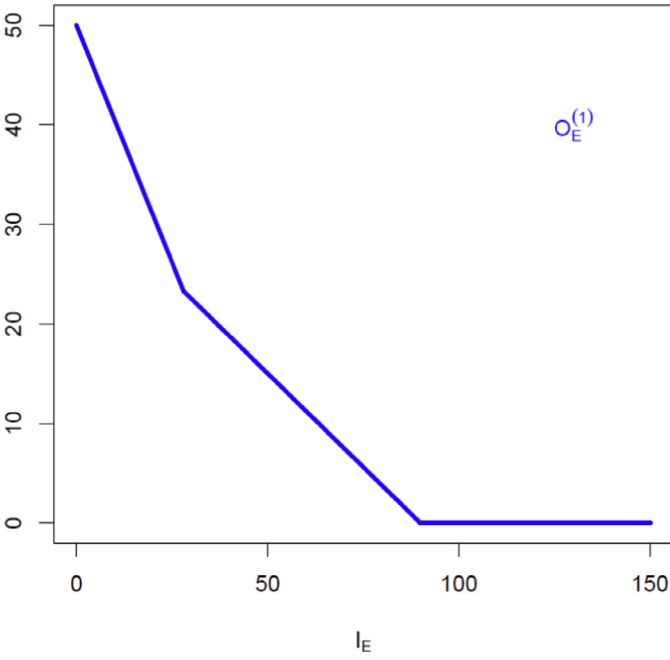


Fig. 6. Value of the option to expand in the first period. Source: Own elaboration.

calculated by the difference between the present value of the project with the option to expand (denoted by  $V(E)_0^{(2)}$ ) and the present value of the project without this option:

$$O_E^{(2)} = V(E)_0^{(2)} - V_0.$$

For this calculation it is necessary to consider the possible evolution of the project value in the first two periods. In this way, the possible values are given by:

$$\begin{aligned} v(E)^{++} &:= \max\{u^2V_0(1+x) - I_E^{(2)}, u^2V_0\} \\ &= u^2V_0 + \max\{u^2xV_0 - I_E^{(2)}, 0\}, \end{aligned}$$

$$\begin{aligned} v(E)^{+-} &:= \max\{udV_0(1+x) - I_E^{(2)}, udV_0\} \\ &= udV_0 + \max\{udxV_0 - I_E^{(2)}, 0\} \end{aligned}$$

and

$$\begin{aligned} v(E)^{-} &:= \max\{d^2V_0(1+x) - I_E^{(2)}, d^2V_0\} \\ &= d^2V_0 + \max\{d^2xV_0 - I_E^{(2)}, 0\}, \end{aligned}$$

where their probabilities of occurrence are  $p^2$ ,  $2pq$  and  $q^2$ , respectively, and  $I_E^{(2)}$  represents the additional investment to implement the project expansion at instant 2. The evolution of the project value in the two first periods can be represented as in Fig. 7.

On the other hand, the evolution of the additional investment required to implement the expansion in the second period can be seen in Fig. 8.

The equation to calculate the present value of the project with the option to expand within two periods is the following:

$$V(E)_0^{(2)} = \frac{p^2v(E)^{++} + 2pqv(E)^{+-} + q^2v(E)^{-}}{(1+r_f)^2}, \quad (4)$$

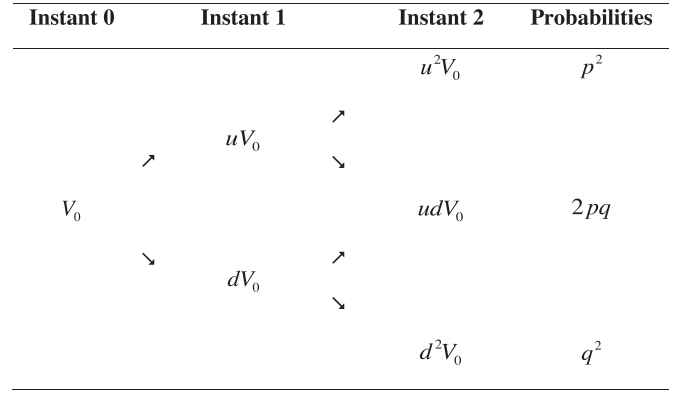


Fig. 7. Evolution of the project in two periods. Source: Own elaboration.

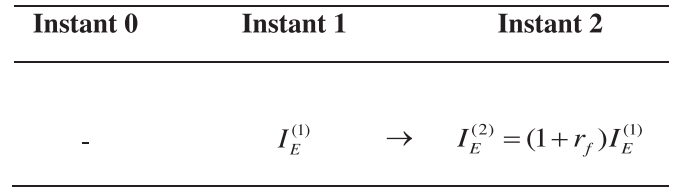


Fig. 8. Evolution of the additional expenditure in the second period. Source: Own elaboration.

Table 2  
Possible intervals for  $I_E^{(1)}$  and  $I_E^{(2)}$ . Source: Own elaboration.

$I_E^{(1)} < dxV_0$ ( $I_{11}$ )	$dxV_0 \leq I_E^{(1)} < uxV_0$ ( $I_{12}$ )	$uxV_0 \leq I_E^{(1)}$ ( $I_{13}$ )
$I_E^{(2)} < d^2xV_0$ ( $I_{21}$ )	$d^2xV_0 \leq I_E^{(2)} < xV_0$ ( $I_{22}$ )	$xV_0 \leq I_E^{(2)} < u^2xV_0$ ( $I_{23}$ )
		$u^2xV_0 \leq I_E^{(2)}$ ( $I_{24}$ )

which can be detailed by the following piece-wise function:

$$V(E)_0^{(2)} = \begin{cases} V_0(1+x) - \frac{I_E^{(2)}}{(1+r_f)^2}, & \text{if } I_E^{(2)} < d^2xV_0 \\ \frac{p^2[u^2V_0(1+x) - I_E^{(2)}] + 2pq[udV_0(1+x) - I_E^{(2)}] + q^2d^2V_0}{(1+r_f)^2}, & \text{if } d^2xV_0 \leq I_E^{(2)} < udxV_0 \\ \frac{p^2[u^2V_0(1+x) - I_E^{(2)}] + 2pqudV_0 + q^2d^2V_0}{(1+r_f)^2}, & \text{if } udxV_0 \leq I_E^{(2)} < u^2xV_0 \\ V_0, & \text{if } u^2xV_0 \leq I_E^{(2)} \end{cases}$$

We are going to prove that the present value of the project with the option to expand after two periods is greater than the present value of the project without this option and greater than the present value of the project with the option to expand within one period. Subsequently, we will be able to deduce the expression to obtain the value of the option to expand the project.

**Proposition 2.** The present value of the project with the option to expand within two periods is always greater than or equal to  $V_0$ , as well as greater than or equal to the corresponding value when there is the option to expand within one period.

**Proof.** In order to facilitate the understanding of the proof of this proposition, Table 2 shows the possible values of  $I_E^{(2)}$  in period 2 (in orange) and their correspondence with the possible values of  $I_E^{(1)}$  in period 1 (in green).

In effect,

1. If  $I_E^{(2)} < d^2xV_0$ , it is proved that  $I_E^{(1)} < \frac{d^2xV_0}{1+r_f} < dxV_0$ , given that  $d < 1 + r_f$ . So,  $I_{21}$  implies  $I_{11}$ .



2. If  $d^2xV_0 \leq I_E^{(2)} < xV_0$ , it is proved that  $\frac{d^2xV_0}{1+r_f} \leq I_E^{(1)} < \frac{xV_0}{1+r_f}$ , so  $I_E^{(1)} \leq xV_0$ , given that  $1+r_f > 1$ . So,  $I_{22}$  implies  $I_{11} \cup I_{12}$ .
3. If  $xV_0 \leq I_E^{(2)} < u^2xV_0$ , it is proved that  $\frac{udxV_0}{1+r_f} \leq I_E^{(1)} < \frac{u^2xV_0}{1+r_f}$ , so  $I_E^{(1)} \geq dxV_0$ , given that  $u > 1+r_f$ . So,  $I_{23}$  implies  $I_{12} \cup I_{13}$ .
4. Finally, if  $u^2xV_0 \leq I_E^{(2)}$ , it is proved that  $\frac{u^2xV_0}{1+r_f} \leq I_E^{(1)}$ , so  $I_E^{(1)} \geq uxV_0$ , given that  $u > 1+r_f$ . So,  $I_{24}$  implies  $I_{13}$ .

We will now consider the proof of the proposition for all possible cases.

1. If  $I_E^{(2)} < d^2xV_0$ , the project value is:

$$V_0 + xV_0 - \frac{I_E^{(2)}}{(1+r_f)^2}.$$

As  $\frac{I_E^{(2)}}{(1+r_f)^2} < I_E^{(2)} < d^2xV_0 < xV_0$ , then

$$V(E)_0^{(2)} = V_0 + xV_0 - \frac{I_E^{(2)}}{(1+r_f)^2} > V_0.$$

In this case, the incremental value of the project is higher than the additional investment needed for its implementation. In this way, we conclude that the expression which represents the present value of the total project satisfies the required inequality/equality.

2. If  $d^2xV_0 \leq I_E^{(2)} < udxV_0$ , the present value of the project with the option to expand within two periods is:

$$V(E)_0^{(2)} = V_0 + \frac{p^2(u^2xV_0 - I_E^{(2)}) + 2pq(udxV_0 - I_E^{(2)})}{(1+r_f)^2}.$$

As  $I_E^{(2)} < udxV_0 < u^2xV_0$ , we can confirm that, in this case, the present value of the project is greater than  $V_0$ .

In the following paragraphs, we are going to demonstrate that the present value of the project with the option to expand within two periods is greater than the value of one with the option to expand within one period:

- If  $I_E^{(1)} < dxV_0$ , the present value of the project with the option to expand at instant 1 is:

$$V(E)_0^{(1)} = V_0(1+x) - \frac{I_E^{(1)}}{1+r_f}.$$

Moreover, the present value of the project with the option to expand at instant 2 can be written as:

$$V(E)_0^{(2)} = V_0 + \frac{(pu+qd)^2xV_0}{(1+r_f)^2} - \frac{q^2d^2xV_0}{(1+r_f)^2} - \frac{(p+q)^2I_E^{(2)}}{(1+r_f)^2} + \frac{q^2I_E^{(2)}}{(1+r_f)^2}.$$

As  $d^2xV_0 \leq I_E^{(2)}$ ,  $V(E)_0^{(2)}$  is obviously greater than

$$V_0(1+x) - \frac{I_E^{(1)}}{1+r_f}.$$

- If  $dxV_0 \leq I_E^{(1)} < uxV_0$ , the present value of the project with the option to expand within one period is:

$$V_0 + \frac{p(uxV_0 - I_E)}{1+r_f}.$$

On the other hand, by using inequality (3), we can re-write the expression of the present value of the project with the option to expand within two periods:

$$V(E)_0^{(2)} = V_0 + \frac{puxV_0}{1+r_f} + \frac{pqudxV_0 - p^2I_E^{(2)} - 2pqI_E^{(2)}}{(1+r_f)^2},$$

which is greater than or equal to

$$V_0 + \frac{puxV_0}{1+r_f} + \frac{pqI_E^{(2)} - p^2I_E^{(2)} - 2pqI_E^{(2)}}{(1+r_f)^2} = V_0 + \frac{p(uxV_0 - I_E)}{1+r_f},$$

which is the present value of the project with the option to expand within one period.

3. If  $duxV_0 \leq I_E^{(2)} < u^2xV_0$ , the present value of the project with the option to expand within two periods is:

$$V(E)_0^{(2)} = \frac{p^2[u^2xV_0(1+x) - I_E^{(2)}] + 2pqudxV_0 + q^2d^2xV_0}{(1+r_f)^2}.$$

This expression can be written as:

$$V_0 + \frac{p^2(u^2xV_0 - I_E^{(2)})}{(1+r_f)^2}.$$

As  $I_E^{(2)} < u^2xV_0$ , one has:

$$V_0 + \frac{p^2u^2xV_0 - p^2I_E^{(2)}}{(1+r_f)^2} > V_0.$$

Observe that the previous paragraph can also be used to demonstrate that the value of the option to expand a project within two periods is greater than the value of the option to expand it within one period, if  $uxV_0 < I_E^{(1)}$ . Now we are going to see whether this inequality holds if  $dxV_0 \leq I_E^{(1)} < uxV_0$ , i.e. when the project value with the option to expand within two periods is:

$$V(E)_0^{(2)} = V_0 + \frac{p^2(u^2xV_0 - I_E^{(2)})}{(1+r_f)^2}.$$

Again, by inequality (3),  $V(E)_0^{(1)}$  can be re-written as follows:

$$V_0 + \frac{puxV_0}{1+r_f} - \frac{pqudxV_0 - p^2I_E^{(2)}}{(1+r_f)^2},$$

which is greater than

$$V_0 + \frac{puxV_0 - pqI_E - p^2I_E}{1+r_f} = V_0 + \frac{p(uxV_0 - I_E)}{1+r_f}.$$

Therefore, the inequality  $V(E)_0^{(2)} \geq V(E)_0^{(1)}$  is shown.

4. If  $u^2xV_0 \leq I_E^{(2)}$ , the present value of the project with the option to expand within one and two periods is:

$$V(E)_0^{(1)} = V(E)_0^{(2)} = V_0,$$

so the required inequality/equality is thereby demonstrated.  $\square$

**Example 3.** Assuming that the oil company of [Example 2](#) has the option to expand its productive capacity by 50% within two periods, the project value with this option is given by the following piece-wise function:

$$V(E)_0^{(2)} = \begin{cases} 150 - 0.9070I_E^{(2)}, & \text{if } I_E^{(2)} \leq 15.68 \\ 144.74 - 0.575I_E^{(2)}, & \text{if } 15.68 < I_E^{(2)} \leq 50.4 \\ 122.88 - 0.1415I_E^{(2)}, & \text{if } 50.4 < I_E^{(2)} \leq 162 \\ 100, & \text{if } 162 < I_E^{(2)} \end{cases}$$

whose graphical representation is (see [Fig. 9](#)):

[Fig. 9](#) shows that the value of the project with the option to expand within two periods (in green) is greater than, or at least equal to, the value of the project with the option to expand within one period (in blue), as shown in [Proposition 2](#).

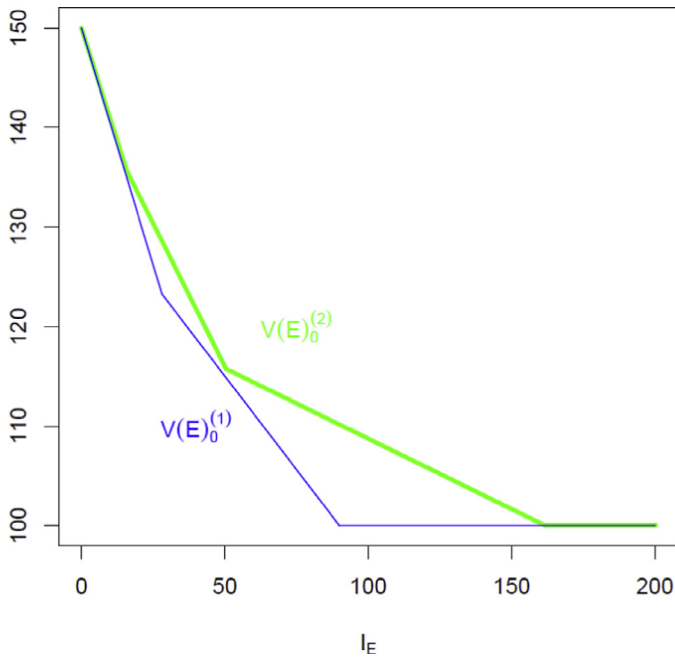


Fig. 9. Present value of the project with the option to expand within one and two periods. Source: Own elaboration.

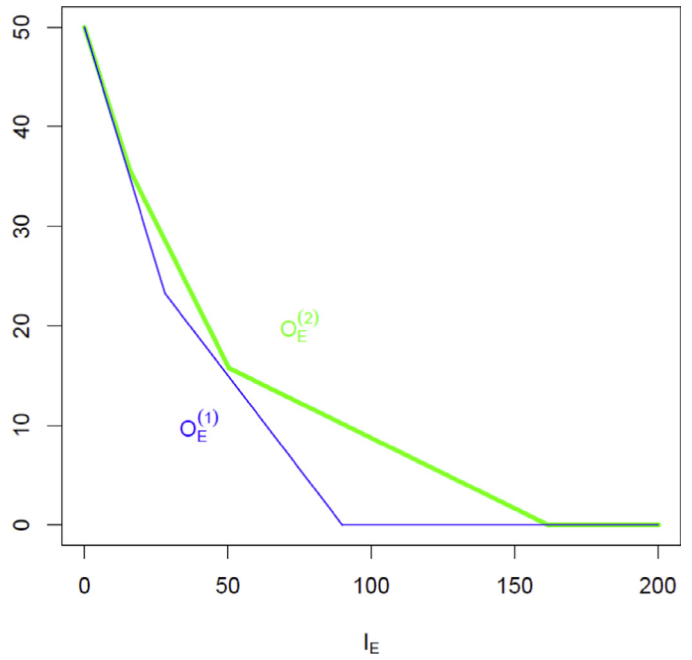


Fig. 10. Value of the option to expand in one and two periods. Source: Own elaboration.

**Corollary 2.** The value of the option to expand within two periods (denoted by  $O_E^{(2)}$ ) is:

$$O_E^{(2)} = \begin{cases} xV_0 - \frac{I_E^{(2)}}{(1+r_f)^2}, & \text{if } I_E^{(2)} < d^2xV_0 \\ \frac{xV_0(p^2u^2 + 2pqud) - I_E^{(2)}(p^2 + 2pq)}{(1+r_f)^2}, & \text{if } d^2xV_0 \leq I_E^{(2)} < udxV_0 \\ \frac{p^2u^2xV_0 - p^2I_E^{(2)}}{(1+r_f)^2}, & \text{if } udxV_0 \leq I_E^{(2)} < u^2xV_0 \\ 0, & \text{if } u^2xV_0 \leq I_E^{(2)} \end{cases}$$

**Proof.** The proof is obvious in every case since the option can be calculated as the difference between the present value of the project with the option ( $V(E)_0^{(2)}$ ) and the value of the project without this option ( $V_0$ ):

$$O_E^{(2)} = V(E)_0^{(2)} - V_0.$$

□

**Example 4.** The value of the option to expand within two periods of the oil company hypothesized in Example 3 is given by:

$$O_E^{(2)} = \begin{cases} 50 - 0.9070I_E^{(2)}, & \text{if } I_E^{(2)} \leq 15.68 \\ 44.74 - 0.575I_E^{(2)}, & \text{if } 15.68 < I_E^{(2)} \leq 50.4 \\ 22.88 - 0.1415I_E^{(2)}, & \text{if } 50.4 < I_E^{(2)} \leq 162 \\ 0, & \text{if } 162 < I_E^{(2)} \end{cases}$$

whose graphical representation is (see Fig. 10):

Fig. 10 shows that the value of the real option is increasing with respect to the option maturity.

## 6. Option to expand in the $n$ th period

First, the present value of the project with the option to expand within  $n$  periods (denoted by  $V(E)_0^{(n)}$ ) is ( $I_E^{(n)} := (1+r_f)^{n-1}I_E^{(1)}$ ):

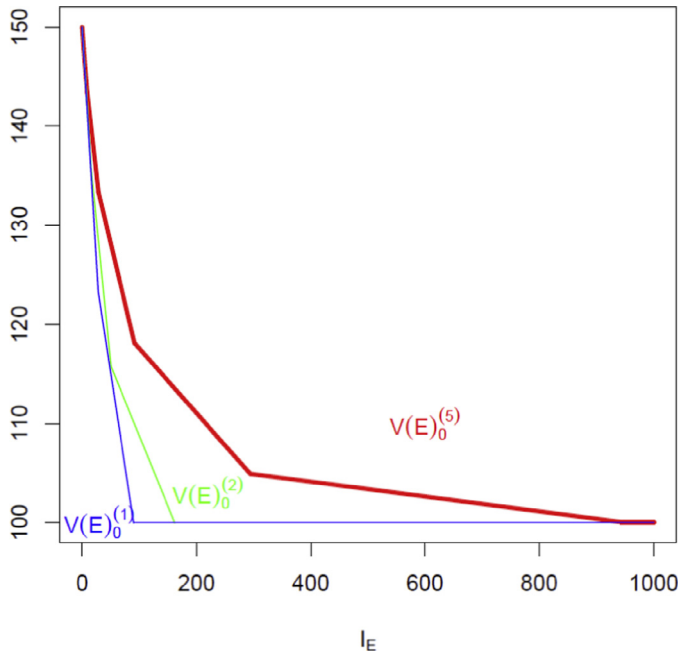
$$V(E)_0^{(n)} = \begin{cases} V_0(1+x) - \frac{I_E^{(n)}}{(1+r_f)^n}, & \text{if } I_E^{(n)} \leq d^nV_0 \\ \vdots & \vdots \\ \sum_{k=0}^s \binom{n}{k} \frac{p^k q^{n-k}}{(1+r_f)^n} [u^k d^{n-k} V_0(1+x) - I_E^{(n)}] + & \text{if } d^{n-s+1}u^{s-1}V_0 < I_E^{(n)} \leq d^{n-s}u^sV_0 \\ \sum_{k=s+1}^n \binom{n}{k} \frac{p^k q^{n-k}}{(1+r_f)^n} V_0, & \\ \vdots & \vdots \\ V_0, & \text{if } u^nV_0 < I_E^{(n)} \end{cases}$$

**Example 5.** The present value of the project with the option to expand by 50% the productive capacity of the oil company of Example 2 within five periods is given by:

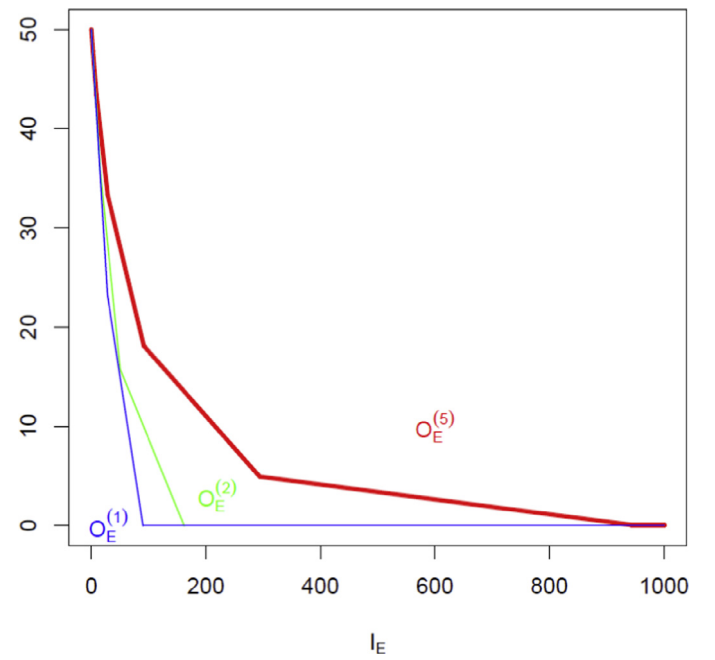
$$V(E)_0^{(5)} = \begin{cases} 150 - 0.7835I_E^{(5)}, & \text{if } I_E^{(5)} \leq 2.75 \\ 153.78 - 0.7200I_E^{(5)}, & \text{if } 2.75 < I_E^{(5)} \leq 8.85 \\ 151.94 - 0.5127I_E^{(5)}, & \text{if } 8.85 < I_E^{(5)} \leq 28.5 \\ 144.24 - 0.2420I_E^{(5)}, & \text{if } 28.5 < I_E^{(5)} \leq 91.45 \\ 128.08 - 0.0652I_E^{(5)}, & \text{if } 91.45 < I_E^{(5)} \leq 293.93 \\ 111.12 - 0.0075I_E^{(5)}, & \text{if } 293.93 < I_E^{(5)} \leq 944.78 \\ 100, & \text{if } 944.78 \leq I_E^{(5)} \end{cases}$$

whose graphical representation is (see Fig. 11):





**Fig. 11.** Value of the project with option to expand in one, two and five periods. Source: Own elaboration.



**Fig. 12.** Value of the option to expand in one, two and five periods. Source: Own elaboration.

Next, the value of the corresponding option to expand (denoted by  $O_E^{(n)}$ ) is:

$$O_E^{(n)} = \begin{cases} xV_0 - \frac{I_E^{(n)}}{(1+r_f)^n}, & \text{if } I_E^{(n)} \leq d^n V_0 \\ \vdots & \vdots \\ \left( \sum_{k=s}^n \binom{n}{k} \frac{p^k u^k q^{n-k} d^{n-k}}{(1+r_f)^n} \right) xV_0 - \frac{I_E^{(n)}}{(1+r_f)^n}, & \text{if } d^{n-s+1} u^{s-1} V_0 < I_E^{(n)} \leq d^{n-s} u^s V_0 \\ \sum_{k=s}^n \binom{n}{k} \frac{p^k q^{n-k}}{(1+r_f)^n} I_E^{(n)}, & \vdots \\ \vdots & \vdots \\ 0, & \text{if } u^n V_0 < I_E^{(n)} \end{cases}$$

**Example 6.** The expression of the option value to expand by 50% the productive capacity of the oil company within five periods is:

$$O_E^{(5)} = \begin{cases} 50 - 0.7835I_E^{(5)}, & \text{if } I_E^{(5)} \leq 2.75 \\ 49.78 - 0.7200I_E^{(5)}, & \text{if } 2.75 < I_E^{(5)} \leq 8.85 \\ 47.94 - 0.5127I_E^{(5)}, & \text{if } 8.85 < I_E^{(5)} \leq 28.5 \\ 40.24 - 0.2420I_E^{(5)}, & \text{if } 28.5 < I_E^{(5)} \leq 91.45 \\ 24.08 - 0.0652I_E^{(5)}, & \text{if } 91.45 < I_E^{(5)} \leq 293.93 \\ 7.12 - 0.0075I_E^{(5)}, & \text{if } 293.93 < I_E^{(5)} \leq 944.78 \\ 0, & \text{if } 944.78 \leq I_E^{(5)} \end{cases}$$

whose graphical representation is (see Fig. 12):

## 7. Conclusions

The primary objective of this paper has been to determine the present value of a project with the option to expand its productive capacity by a given percentage within one, two and, in general,  $n$  periods of time. It has been shown that this value is increasing with respect to the maturity option. The methodology employed has been the binomial options pricing model. As a result of this,

the value of the option to expand has been derived, and it has been demonstrated that this value is also increasing with respect to the option expiration.

More specifically, the procedure adopted has consisted in deriving a piece-wise function depending on the relative position of the additional investment needed to implement the expansion of the project. It has been based on a detailed reconstruction of all possible future scenarios and their respective probabilities of occurrence.

This analysis has been initially made on the basis of real options to expand whose maturity is one and two periods. This has allowed us to derive, by recurrence, the expression of an option expiring in  $n$  periods. In all cases, it has been shown that the value of the option to expand is greater than or equal to zero, and that, as the maturity of a real option increases, its value also increases, as is the case with *put* and *call* options. The mathematical analysis has been completed with the graphical representation of both the project value and the option value, by using a numerical example based on the information from a hypothetical oil company.

In summary, this paper provides an important quantitative analysis of the value of the option to expand, being one of the real options most commonly used in business practice. Defining the boundaries of the option value is a useful tool which allows an increased control over the uncertainty which surrounds any project, which in turn reduces the element of risk for companies.

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