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Article

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Priority degrees for hesitant fuzzy sets: Application to multiple attribute decision making



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ABSTRACT

In this paper, some drawbacks to existing ordering relations for hesitant fuzzy sets are examined using examples. To overcome these flaws, a priority degree formula for comparing two hesitant fuzzy sets is presented and the desirable priority degree properties studied. Then, based on an introduced priority degree formula, a new hesitant fuzzy multiple attribute decision making methodology is proposed. Finally, a numerical example together with a comparison analysis is given to illustrate the effectiveness and feasibility of the new approach to decision making applications.

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1. Introduction

The fuzzy sets theory introduced by Zadeh [1] has been very successful in dealing with problems involving uncertainty. With an increase in inaccurate and vague information in real life problems, several extensions of the fuzzy set have been developed, one of which is the intuitionistic fuzzy set (IFS) pioneered by Atanassov [2], which has a membership function, a non-membership function and a hesitancy function. Zadeh [3] presented a type-2 fuzzy set that allowed the membership of a given element to be a fuzzy set. The type- n fuzzy set [4] generalized type-2 fuzzy set, thereby permitting the membership to be a type- $n-1$ fuzzy set. The fuzzy multiset introduced by Yager [5] allowed elements to be repeated more than once.

In practical applications, because of a lack of knowledge, time pressure and other reasons, people do not often agree on specific elements in complex decisions, which means that it is often difficult to reach agreement. For example, two decision makers may discuss the membership degree of an element x to a set A , for which one decision maker wishes to assign 0.4 but the other wishes to assign 0.8. Accordingly, the difficulty in establishing a common membership degree is not because there is a margin of error or some possibility distribution values, but because there is a set of possible values [6]. To deal with such cases, Torra [7] and Torra and Narukawa [8] proposed the concept of the hesitant fuzzy set (HFS), which permitted membership to have a set of possible values.

Since the concept of the hesitant fuzzy set was established, it has gained increasing attention [9–17] and has been successfully applied to many uncertain decision making problems. Many studies have also been conducted on the application of HFS aggregation operators [18–20] and distance and similarity measures [21–23] to multi-criteria decision making problems.

Multi-criteria decision making has been widely applied in many scientific fields [24–26], such as medical care [27], engineering [28], social sciences [29] and economics [30]. In general, multiple attribute decision making problems have two phases; aggregation and exploitation. Of these, the aggregation phase is more important, so significant aggregation techniques have been developed for decision-making processes, in which the experts express their assessments using HFSs. Xia and Xu [31] presented a hesitant fuzzy weighted averaging operator and a hesitant fuzzy weighted geometric operator, to which they gave different extensions and generalizations; a generalized hesitant fuzzy weighted averaging operator, a generalized hesitant fuzzy weighted geometric operator, a hesitant fuzzy hybrid averaging operator, and a hesitant fuzzy hybrid geometric operator. Yu et al. [32] proposed a new hesitant fuzzy aggregation operator based on the Choquet integral which included the importance of the elements, their ordered positions and a fuzzy measure. Motivated by the idea of prioritized aggregation operators, Wei [33] proposed a hesitant fuzzy prioritized weighted average and hesitant fuzzy prioritized weighted geometric aggregation operators, which accounted for the different criteria priority levels in multi-criteria decision-making problems. Yu and Zhou [34] defined a generalized hesitant fuzzy Bonferroni mean which extended the Bonferroni mean to a hesitant fuzzy environment. Other extensions of the Bonferroni mean were

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proposed in [35,36]. Xia et al. [37] introduced a new HFS operator by extending the quasi-arithmetic means. Bedregal et al. [38] presented two methodologies to develop triangular hesitant aggregation functions over all THFS. Xu and Xia [39] proposed several distance and similarity measures and studied the properties and relationships between them. Zhou and Li [40] modified the axiom definitions for the distance and similarity measures developed by Xu and Xia [39], and proposed some new distance and similarity measures between HFSs based on Hamming and Euclidean distances. Peng et al. [41] presented a novel generalized hesitant fuzzy synergetic weighted distance measure which reflected both individual distances and their ordered positions. In addition to these, many approaches can be used to deal with multiple attribute decision making problems (see, [42–47]). An exploitation phase was developed to build the preference relations between the alternatives and a nondominant choice degree was applied to obtain a solution set of alternatives for multiple attribute decision making problems.

Ordering relations play an important role in decision making and some HFS ordering relations have been proposed. Rodriguez et al. [48] gave a definition for order relations between HFSs, and then used aggregation operators to determine the order relations between them. Xia and Xu [14] introduced a comparison law by defining a score function to determine the order relations between HFSs. Farhadinia [49] also developed two ordering methods for HFSs. Zhou [50] introduces the intuitionistic fuzzy ordered weighted cosine similarity (IFOWCS) measure by using the cosine similarity measure of intuitionistic fuzzy sets and the generalized ordered weighted averaging (GOWA) operator. Zhou [51] develops the continuous intuitionistic fuzzy ordered weighted distance (C-IFOWD) measure by using the continuous intuitionistic fuzzy ordered weighted averaging (C-IFOWA) operator in the interval distance. Wei et al. [52] define the Shapley value-based L_p -metric and extend VIKOR method with the L_p -metric to deal with the correlative multiple criteria decision making (MCDM) problem under hesitant fuzzy environment. Wei et al. [53] developed hesitant fuzzy choquet ordered averaging (HFCHOA) operator and hesitant fuzzy choquet ordered geometric (HFCHOG) operator, and apply the HFCHOA and HFCHOG operators to hesitant fuzzy multiple attribute decision making. Zhao et al. [54] utilize Einstein operations to develop hesitant fuzzy Einstein correlated averaging (HFCECA) operator and hesitant fuzzy Einstein correlated geometric (HFCECG) operator. They can not only consider the importance of the elements or their ordered positions, but also reflect the correlation among the elements or their ordered positions. In addition to these, many aggregation operators and methods can be used to rank HFSs (see, [18,55–58]). However, the existing order relations for HFSs are defective (see examples 2–4). For example, Jack and Tom play a game that has three turns. Jack has three cards; a 9 of spades, a 6 of spades and a 3 of spades; and Tom has three cards; an 8 of spades, a 5 of spades and a 2 of spades. The rules of the game are: (1) Each person can only select one of their own cards to play in each turn; (2) The card with the higher points wins in each turn; (3) The person who wins two turns is the final winner. Although Jack's cards have a points' advantage, it is not certain that Jack can win the game, as his winning probability is 0.6667. From the example above, we can construct two HFES; $H_1(x) = \{0.9, 0.6, 0.3\}$ and $H_2(x) = \{0.8, 0.5, 0.2\}$. From the existing HFS order relations, we have $H_1(x) \succeq H_2(x)$, which does not conform to the actual situation.

This paper seeks to overcome the flaws outlined above. The remainder of this paper is organized as follows: Section 2 reviews some basic concepts and order relations. Section 3 analyzes the order relations between HFSs, and in Section 4, a priority degree formula for comparing two hesitant fuzzy sets is presented and the desirable priority degree properties studied. In Section 5, based on the proposed formula for the priority degree, a new approach to hesitant fuzzy multiple attribute decision making is developed.

Section 6 gives an example to illustrate the rationality and applicability of the new method and in Section 7 conclusions are given.

2. Preliminaries

In this section, the HFS concept and order relations are briefly reviewed.

2.1. Hesitant fuzzy sets

As people are usually hesitant when making decisions, it is often difficult to reach a final agreement. With these difficulties in mind, Torra [59] developed the following hesitant fuzzy set definition:

Definition 1. [59] Given a fixed set X , then a hesitant fuzzy set (HFS) on X is a function that when applied to X returns a subset of values in $[0,1]$.

For convenience, Wei [33] completed the original HFS definition by including the HFS mathematical representation as follows:

$$E = (\langle x, h_E(x) \rangle | x \in X).$$

where $h_E(x)$ is a set of some values in $[0,1]$, and denotes the possible membership degree of the element $x \in X$ to the set E ; $h(x) = h_E(x)$ is called a hesitant fuzzy element (HFE).

Example 1. Suppose that $X = \{x_1, x_2, x_3\}$ is the discourse set, and $h_M(x_1) = \{0.8, 0.5, 0.3\}$, $h_M(x_2) = \{0.6, 0.4\}$ and $h_M(x_3) = \{0.6, 0.3, 0.2\}$ are the HFES for $x_i (i = 1, 2, 3)$ to a set E . Then E can be considered a HFS; i.e.

$$E = \{\langle x_1, \{0.8, 0.5, 0.3\} \rangle, \langle x_2, \{0.6, 0.4\} \rangle, \langle x_3, \{0.6, 0.3, 0.2\} \rangle\}.$$

Further operations on HFES can be seen in [7,14].

2.2. Order relations between HFSs

Order relations play an important role in decision making. We first review some order relations.

Definition 2. [60] Given two HFSs, H_1 and H_2 on X of the same cardinality, it is defined that $H_1 \succeq H_2$ if $H_1(x) \succeq H_2(x)$ for all x . Note that $H_1(x)$ and $H_2(x)$ are HFES. Here, $h_1 \succeq h_2$ for HFES h_1 and h_2 if $h_1^{\sigma(j)} \succeq h_2^{\sigma(j)}$ for all $j = \{1, \dots, |H_1|\}$, where $h^{\sigma(j)}$ is the j th element in h when they are ordered in a decreasing order.

Definition 3. [60] Let φ be a function on the HFSs such that the cardinality of φ is the same for all HFSs. We then say that φ is monotonic when $\varphi(E) \succeq \varphi(E')$ for all $E = \{H_1, \dots, H_n\}$ and $E' = \{H'_1, \dots, H'_n\}$ such that $H'_i \succeq H_i$ for all $i = \{1, \dots, n\}$.

Definition 4. [60] Let $E = \{H_1, \dots, H_n\}$ be a set of n HFSs and Θ a function, $\Theta: [0, 1]^n \rightarrow [0, 1]$, we then export the Θ on the fuzzy sets to HFSS, which is defined as:

$$\Theta_E = \cup_{\gamma \in H_1(x) \times \dots \times H_n(x)} \{\Theta(\gamma)\} \quad (1)$$

Proposition 1. [60] Let $E = \{H_1, \dots, H_n\}$ and $E' = \{H'_1, \dots, H'_n\}$ such that $H'_i \succeq H_i$ for all $i = \{1, \dots, n\}$. Then, if Θ is a monotonic function, Θ_E is monotonic.

In practical applications, situations arise in which the number of the elements in the different hesitant fuzzy elements may vary. For correct operations, Xu and Xia [6] proposed the following regulation: the shorter element is extended by adding the minimum value, the maximum value, or any value until it has the same length as the longer element. The selection of this value depends mainly on the decision makers' risk preferences. Optimists expecting desirable outcomes may add the maximum value, while pessimists expecting unfavorable outcomes may add the minimum value.

When the cardinality between two hesitant fuzzy elements is different, the method in Definition 2 leads to failure. To solve this problem, Xia and Xu [14] established a new order between HFEs by defining the following score function:

Definition 5. [14] Let h be an HFE, with a score function of h defined by

$$s(h) = \frac{1}{l(h)} \sum_{\gamma \in h} \gamma \quad (2)$$

where $l(h)$ is the number of elements in h .

Let h_1 and h_2 be two HFEs, then
 if $s(h_1) > s(h_2)$, then $h_1 > h_2$;
 if $s(h_1) = s(h_2)$, then $h_1 = h_2$.

Except for the score function, the order for the HFSs was defined by Farhadinia[49] as follows:

Definition 6. [49] Let E_1 and E_2 be two HFSs on $X = \{x_1, x_2, \dots, x_n\}$. $E_1 \geq E_2$ if and only if $Score(E_1) \geq Score(E_2)$, where $Score(E_1) = \frac{1}{n} \sum_{i=1}^n s(h_{E_1}(x_i))$ and $Score(E_2) = \frac{1}{n} \sum_{i=1}^n s(h_{E_2}(x_i))$.

In addition to the score function, many other aggregation operators(see,[54]) can be used for the same purpose.

3. Analysis of the HFS ordering relations

In this section, we analyze the HFS ordering relations using examples.

Example 2. Suppose that $X = \{x\}$, $H_1(x) = \{0.9, 0.6, 0.3\}$ and $H_2(x) = \{0.8, 0.5, 0.2\}$ are two HFSs on X .

From Definition 2, we have $H_1(x) \geq H_2(x)$. In general, aggregation operators are monotonic increasing functions. Using aggregation operators to deal with the ordering of $H_1(x)$ and $H_2(x)$, we also have $H_1(x) \geq H_2(x)$.

TOPSIS can be used to deal with the order relations between $H_1(x)$ and $H_2(x)$. Let $H^-(x) = \{0, 0, 0\}$ and $H^+(x) = \{1, 1, 1\}$. Using the distance formula proposed by Xu and Xia [39], we have

$$d_h(H_1(x), H^-(x)) = 0.6, \quad d_h(H_2(x), H^-(x)) = 0.5.$$

$$d_h(H_1(x), H^+(x)) = 0.4, \quad d_h(H_2(x), H^+(x)) = 0.5.$$

$$\frac{d_h(H_1(x), H^-(x))}{d_h(H_1(x), H^-(x)) + d_h(H_1(x), H^+(x))} = 0.6.$$

$$\frac{d_h(H_2(x), H^-(x))}{d_h(H_2(x), H^-(x)) + d_h(H_2(x), H^+(x))} = 0.5.$$

$$\frac{d_h(H_1(x), H^-(x))}{d_h(H_1(x), H^-(x)) + d_h(H_1(x), H^+(x))} > \frac{d_h(H_2(x), H^-(x))}{d_h(H_2(x), H^-(x)) + d_h(H_2(x), H^+(x))}.$$

from which $H_1(x) > H_2(x)$. The same result can be derived using other formulas.

However, $(u, v) \in H_1(x) \times H_2(x)$ are real elements that satisfy $u < v$; for example $(0.6, 0.8)$. As in the example in the first section of this paper, it is uncertain whether $H_1(x) \geq H_2(x)$. This indicates that using Definition 2 is inflexible when ranking HFSs as significant information may be lost.

Example 3. Suppose that $X = \{x\}$. $H_1(x) = \{0.9, 0.4\}$ and $H_2(x) = \{0.8, 0.5, 0.2\}$ are two HFSs on X .

Employing the score function, we have

$$s(H_1(x)) = 0.65, \quad s(H_2(x)) = 0.3.$$

From Definition 5, we have $H_1(x) > H_2(x)$.

In this example, the number of elements in the two hesitant fuzzy sets is different. For correct operations, based on the regulation proposed by Xu and Xia [6], the shorter set is extended by adding the minimum value; that is, $H_1(x) = \{0.9, 0.4, 0.4\}$. TOPSIS is then used to deal with the order relations between $H_1(x)$ and $H_2(x)$. Let $H^-(x) = \{0, 0, 0\}$ and $H^+(x) = \{1, 1, 1\}$. Using the distance formula proposed by Xu and Xia [39], we have

$$d_h(H_1(x), H^-(x)) = 0.57, \quad d_h(H_2(x), H^-(x)) = 0.5.$$

$$d_h(H_1(x), H^+(x)) = 0.43, \quad d_h(H_2(x), H^+(x)) = 0.5.$$

$$\frac{d_h(H_1(x), H^-(x))}{d_h(H_1(x), H^-(x)) + d_h(H_1(x), H^+(x))} = 0.57.$$

$$\frac{d_h(H_2(x), H^-(x))}{d_h(H_2(x), H^-(x)) + d_h(H_2(x), H^+(x))} = 0.5.$$

$$\frac{d_h(H_1(x), H^-(x))}{d_h(H_1(x), H^-(x)) + d_h(H_1(x), H^+(x))} > \frac{d_h(H_2(x), H^-(x))}{d_h(H_2(x), H^-(x)) + d_h(H_2(x), H^+(x))}.$$

so we also have $H_1(x) > H_2(x)$. The result is the same if other formulas are used to deal with this problem.

However, real elements $(u, v) \in H_1(x) \times H_2(x)$ exist that satisfy $u < v$; for example, $(0.4, 0.8)$. Therefore, it is uncertain whether $H_1(x) \geq H_2(x)$, which indicates that Xia and Xu's [14] comparison law to rank HFSs is also inflexible and significant information loss could occur.

Example 4. Suppose that $X = \{x_1, x_2\}$. $E_1 = (\langle x_1, \{0.5, 0.4, 0.3\} \rangle, \langle x_2, \{0.9, 0.8, 0.7, 0.1\} \rangle)$, $E_2 = (\langle x_1, \{0.5, 0.3\} \rangle, \langle x_2, \{0.6, 0.5, 0.3\} \rangle)$ and $E_3 = (\langle x_1, \{0.8, 0.7, 0.4, 0.3\} \rangle, \langle x_2, \{0.7, 0.4, 0.2\} \rangle)$ are three hesitant fuzzy sets on X .

By employing the score function, we have

$$s(h_{E_1}(x_1)) = 0.4, \quad s(h_{E_1}(x_2)) = 0.625,$$

$$s(h_{E_2}(x_1)) = 0.4, \quad s(h_{E_2}(x_2)) = 0.4667,$$

$$s(h_{E_3}(x_1)) = 0.55, \quad s(h_{E_3}(x_2)) = 0.4333.$$

$$Score(E_1) = 0.5125, \quad Score(E_2) = 0.4335, \quad Score(E_3) = 0.4917.$$

$Score(E_1) > Score(E_3) > Score(E_2)$. According to Definition 6, we have $E_1 > E_3 > E_2$.

Since $s(h_{E_1}(x_1)) < s(h_{E_3}(x_1))$ and $s(h_{E_1}(x_2)) > s(h_{E_3}(x_2))$, elements $(u_1, v_1) \in h_{E_1}(x_1) \times h_{E_3}(x_1)$ exist to satisfy $u_1 < v_1$ and $(u_2, v_2) \in h_{E_1}(x_2) \times h_{E_3}(x_2)$ satisfies $u_2 < v_2$; however, from the decision making viewpoint, it is not clear whether $E_1 > E_3$.

Many other aggregation operators (see [61]) have been used to rank HFSs. However, the same situation as shown in the above examples has been encountered, indicating that using aggregation operators to deal with HFS ordering is inflexible and could lead to information loss.

In the following, a priority degree definition for hesitant fuzzy sets is proposed and then a new method is introduced to solve the above problem.

4. Priority degrees for hesitant fuzzy sets

To overcome the current flaws, in this section, a priority degree formula is proposed to deal with the ordering relations between HFSSs.

Definition 7. Given a fixed set $X = \{x_1, x_2, \dots, x_n\}$, suppose that $E_1 = \{ \langle x_i, h_{E_1}(x_i) \rangle \mid x_i \in X \}$ and $E_2 = \{ \langle x_i, h_{E_2}(x_i) \rangle \mid x_i \in X \}$ are two hesitant fuzzy sets. The priority degree for $E_1 > E_2$ is defined by

$$P(E_1 > E_2) = \begin{cases} \frac{\sum_{i=1}^n S_1(x_i)}{\sum_{i=1}^n S_1(x_i) + \sum_{i=1}^n S_2(x_i)}, & \sum_{i=1}^n S_1(x_i) + \sum_{i=1}^n S_2(x_i) \neq 0, \\ 0.5, & \sum_{i=1}^n S_1(x_i) + \sum_{i=1}^n S_2(x_i) = 0. \end{cases} \quad (5)$$

where

$$h_{E_1}(x_i) = \{u_l(x_i) \mid u_l(x_i) \in [0, 1], l = 1, 2, \dots, r_i\},$$

$$h_{E_2}(x_i) = \{v_t(x_i) \mid v_t(x_i) \in [0, 1], t = 1, 2, \dots, s_i\},$$

$$A_1(x_i) = \{ (u_l(x_i), v_t(x_i)) \mid u_l(x_i) - v_t(x_i) > 0, (u_l(x_i), v_t(x_i)) \in h_{E_1}(x_i) \times h_{E_2}(x_i) \},$$

$$A_2(x_i) = \{ (u_l(x_i), v_t(x_i)) \mid u_l(x_i) - v_t(x_i) < 0, (u_l(x_i), v_t(x_i)) \in h_{E_1}(x_i) \times h_{E_2}(x_i) \},$$

$$A_3(x_i) = \{ (u_l(x_i), v_t(x_i)) \mid u_l(x_i) - v_t(x_i) = 0, (u_l(x_i), v_t(x_i)) \in h_{E_1}(x_i) \times h_{E_2}(x_i) \},$$

$$S_1(x_i) = \begin{cases} \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_1(x_i)} (u_l(x_i) - v_t(x_i)), & A_1(x_i) \neq \emptyset; \\ 0, & A_1(x_i) = \emptyset. \end{cases}$$

$$S_2(x_i) = \begin{cases} \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_2(x_i)} (v_t(x_i) - u_l(x_i)), & A_2(x_i) \neq \emptyset; \\ 0, & A_2(x_i) = \emptyset. \end{cases}$$

Remark 1. $S_1(x_i)$ represents the average residual amount for $h_{E_1}(x_i)$ over $h_{E_2}(x_i)$. Corresponding, $S_2(x_i)$ represents the average residual amount for $h_{E_2}(x_i)$ over $h_{E_1}(x_i)$, and $P(E_1 > E_2)$ represents the priority degree for $E_1 > E_2$. For example, $P(E_1 > E_2) = 0.5$ indicates that the priority degree for $E_1 > E_2$ is 0.5. Note that $E_1 >_{P(E_1 > E_2)} E_2$ does not mean that E_1 is absolutely superior to E_2 ; it just indicates that the priority degree for $E_1 > E_2$ is $P(E_1 > E_2)$. In fact, the priority degree for $E_2 > E_1$ is $1 - P(E_1 > E_2)$.

Let $X = \{x_1, x_2, \dots, x_n\}$. $E_1 = \{ \langle x_i, h_{E_1}(x_i) \rangle \mid x_i \in X \}$, $E_2 = \{ \langle x_i, h_{E_2}(x_i) \rangle \mid x_i \in X \}$ and $E_3 = \{ \langle x_i, h_{E_3}(x_i) \rangle \mid x_i \in X \}$ are three hesitant fuzzy sets on X , where

$$h_{E_1}(x_i) = \{u_l(x_i) \mid u_l(x_i) \in [0, 1], l = 1, 2, \dots, r_i\}$$

$$h_{E_2}(x_i) = \{v_t(x_i) \mid v_t(x_i) \in [0, 1], t = 1, 2, \dots, s_i\}$$

$$h_{E_3}(x_i) = \{w_k(x_i) \mid w_k(x_i) \in [0, 1], k = 1, 2, \dots, q_i\}$$

$$\bar{u}(x_i) = \frac{1}{r_i} \sum_{l=1}^{r_i} u_l(x_i), \quad \bar{v}(x_i) = \frac{1}{s_i} \sum_{t=1}^{s_i} v_t(x_i),$$

we can conclude some properties about priority degree as follows:

Property 2. (normalization). $0 \leq P(E_1 > E_2) \leq 1, 0 \leq P(E_2 > E_1) \leq 1$.

Property 3. (complementarity). $P(E_1 > E_2) + P(E_2 > E_1) = 1$.

Proof. It is obvious that properties 2–3 hold. \square

Property 4. (intuition). $P(E_1 > E_2) = 1$ if and only if $u_l(x_i) \geq v_t(x_i)$ for all x_i, l, t and $\exists l', t', u_{l'}(x_i) > v_{t'}(x_i)$.

Proof. $u_l(x_i) \geq v_t(x_i)$ for all x_i, l, t and $\exists l', t', u_{l'}(x_i) > v_{t'}(x_i) \Leftrightarrow \sum_{i=1}^n S_2(x_i) = 0 \Leftrightarrow P(E_1 > E_2) = 1$. \square

Property 5. $P(E_1 > E_2) \geq \frac{1}{2}$ if and only if $\sum_{i=1}^n \bar{u}(x_i) \geq \sum_{i=1}^n \bar{v}(x_i)$.

Proof.

$$\begin{aligned} P(E_1 > E_2) &\geq \frac{1}{2} \Leftrightarrow \sum_{i=1}^n S_1(x_i) \geq \sum_{i=1}^n S_2(x_i) \\ &\Leftrightarrow \sum_{i=1}^n \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_1(x_i)} (u_l(x_i) - v_t(x_i)) \\ &\geq \sum_{i=1}^n \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_2(x_i)} (v_t(x_i) - u_l(x_i)) \\ &\Leftrightarrow \sum_{i=1}^n \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_1(x_i)} u_l(x_i) + \sum_{i=1}^n \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_2(x_i)} u_l(x_i) \\ &\quad + \sum_{i=1}^n \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_3(x_i)} u_l(x_i) \\ &\geq \sum_{i=1}^n \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_1(x_i)} v_t(x_i) + \sum_{i=1}^n \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_2(x_i)} v_t(x_i) \\ &\quad + \sum_{i=1}^n \frac{1}{r_i s_i} \sum_{(u_l(x_i), v_t(x_i)) \in A_3(x_i)} v_t(x_i) \\ &\Leftrightarrow \sum_{i=1}^n \frac{1}{r_i} \sum_{l=1}^{r_i} u_l(x_i) \geq \sum_{i=1}^n \frac{1}{s_i} \sum_{t=1}^{s_i} v_t(x_i) \Leftrightarrow \sum_{i=1}^n \bar{u}(x_i) \geq \sum_{i=1}^n \bar{v}(x_i). \end{aligned}$$

\square

Property 6. (weak transitivity). If $P(E_1 > E_2) \geq \frac{1}{2}, P(E_2 > E_3) \geq \frac{1}{2}$, then $P(E_1 > E_3) \geq \frac{1}{2}$.

Proof. $P(E_1 > E_2) \geq \frac{1}{2} \Leftrightarrow \sum_{i=1}^n \bar{u}(x_i) \geq \sum_{i=1}^n \bar{v}(x_i)$ and $P(E_2 > E_3) \geq \frac{1}{2} \Leftrightarrow \sum_{i=1}^n \bar{v}(x_i) \geq \sum_{i=1}^n \bar{w}(x_i)$, we have $\sum_{i=1}^n \bar{u}(x_i) \geq \sum_{i=1}^n \bar{w}(x_i) \Rightarrow P(E_1 > E_3) \geq \frac{1}{2}$, so property 6 holds. \square

Example 5. The proposed priority degree is employed to deal with Example 2 above, so we have

$$S_1(x) = 0.2, \quad S_2(x) = 0.1,$$

$$P(H_1(x) > H_2(x)) = 0.6667, \quad P(H_2(x) > H_1(x)) = 0.3333,$$

$$H_1(x) >_{(0.6667)} H_2(x), \quad H_2(x) >_{(0.3333)} H_1(x).$$

The result only show that the priority degree for $H_1(x) > H_2(x)$ is 0.6667 and the priority degree for $H_2(x) > H_1(x)$ is 0.3333; however, this does not mean that $H_1(x)$ is absolutely superior to $H_2(x)$.

Example 6. The proposed priority degree is employed to deal with Example 4 above, so we have

$$S_1(x_1) = 0.05, \quad S_1(x_2) = 0.25,$$

$$S_2(x_1) = 0.05, \quad S_2(x_2) = 0.0917,$$

$$P(E_1 > E_2) = 0.6792, \quad P(E_2 > E_1) = 0.3208,$$

Similarly,

$$P(E_1 > E_3) = 0.5362, \quad P(E_3 > E_1) = 0.4638,$$

$$P(E_2 > E_3) = 0.3138, \quad P(E_3 > E_2) = 0.6862.$$

$$E_1 >_{(0.6792)} E_3 >_{(0.6862)} E_2, \quad E_3 >_{(0.6862)} E_2 >_{(0.3208)} E_1$$

$$E_3 >_{(0.4638)} E_1 >_{(0.6792)} E_2, \quad E_2 >_{(0.3138)} E_3 >_{(0.4638)} E_1$$

Table 1
Hesitant fuzzy decision matrix.

	P_1	P_2	P_3	P_4
A_1	{0.5, 0.4, 0.3}	{0.9, 0.8, 0.7, 0.1}	{0.5, 0.4, 0.2}	{0.9, 0.6, 0.5, 0.3}
A_2	{0.5, 0.3}	{0.9, 0.7, 0.6, 0.5, 0.2}	{0.8, 0.6, 0.5, 0.1}	{0.7, 0.4, 0.3}
A_3	{0.8, 0.7, 0.4, 0.3}	{0.7, 0.4, 0.2}	{0.8, 0.1}	{0.9, 0.8, 0.6}
A_4	{0.9, 0.7, 0.6, 0.3, 0.1}	{0.8, 0.7, 0.6, 0.4}	{0.9, 0.8, 0.7}	{0.9, 0.7, 0.6, 0.3}

Table 2
Converted decision matrix.

	P_1	P_2	P_3	P_4
A_1	{0.075, 0.06, 0.045}	{0.27, 0.24, 0.21, 0.03}	{0.1, 0.08, 0.04}	{0.315, 0.21, 0.5, 0.105}
A_2	{0.075, 0.045}	{0.27, 0.21, 0.18, 0.15, 0.06}	{0.16, 0.12, 0.1, 0.02}	{0.245, 0.14, 0.105}
A_3	{0.12, 0.105, 0.06, 0.045}	{0.21, 0.12, 0.06}	{0.16, 0.02}	{0.315, 0.28, 0.21}
A_4	{0.135, 0.105, 0.09, 0.045, 0.015}	{0.12, 0.105, 0.09, 0.06}	{0.135, 0.12, 0.105}	{0.135, 0.105, 0.09, 0.045}

These results only show the degree of priority but not a absolute order relations. Compared with existing HFS order relations, the proposed priority degree formula conforms to the actual situation and has the following advantages: it is flexible, it does not loss decision making information, the results meet the conditions of Property 2. For convenient computation, before a new multiple attribute decision making method is proposed, we first introduce a new HFE operator.

Definition 8. For any HFE h , a new HFE operator is defined by

$$\lambda \odot h = \cup_{r \in h} \{\lambda r\}, \quad 0 < \lambda \leq 1 \quad (6)$$

5. A new approach to multiple attribute decision making

Multiple attribute decision making has attracted significant attention [62,63] recently due to its practicability. In this section, we develop a new approach to hesitant fuzzy multiple attribute decision making based on the proposed priority degree.

Generally speaking, let $A = \{A_1, A_2, \dots, A_m\}$ be the discrete set of alternatives, and $C = \{C_1, C_2, \dots, C_n\}$ be the set of attributes. For each alternative $A_i \in A$, an expert gives a preference value a_{ij} with respect to attribute $C_j \in C$ [64]. $w = (w_1, w_2, \dots, w_n)^T$ are the attribute weight vectors, for which $w_j \in [0, 1]$, $j = 1, 2, \dots, n$ and $\sum_{j=1}^n w_j = 1$. Note that a_{ij} represents the assessment information given by an expert about the j th attribute for alternative A_i , where a_{ij} is a hesitant fuzzy element, and all preference values for the alternatives make up the decision-making matrix $A = (a_{ij})_{m \times n}$.

In the following, a new method for multiple attribute decision making problems is proposed, in which the attribute weight information is known and the attribute values take the form of hesitant fuzzy elements. This new approach has the following steps:

Step 1: Let $A = (a_{ij})_{m \times n}$ be a hesitant fuzzy decision-making matrix in which a_{ij} is each alternative $A_i \in A$ for each attribute $C_j \in C$, and let $w = (w_1, w_2, \dots, w_n)^T$ be the attribute weight vector in which $w_j \in [0, 1]$, $j = 1, 2, \dots, n$, and $\sum_{j=1}^n w_j = 1$.

Step 2: Convert $(a_{ij})_{m \times n}$ into $(h_{ij})_{m \times n} = (w_j \odot a_{ij})_{m \times n}$ using Definition 8.

Step 3: Let $C = \{C_1, C_2, \dots, C_n\}$. The hesitant fuzzy sets are constructed on C as follows,

$$E_{A_i} = \{ \langle C_j, h_{ij} \rangle \mid j = 1, 2, \dots, n \}, i = 1, 2, \dots, m.$$

Step 4: Calculate the priority degree between E_{A_i} and E_{A_j} ($i \neq j, i, j = 1, 2, \dots, m$).

Step 5: Rank the alternatives.

Step 6: End.

Table 3
Results obtained using priority degree.

	E_{A_1}	E_{A_2}	E_{A_3}	E_{A_4}
E_{A_1}	0.5	0.2734	0.3721	0.7672
E_{A_2}	0.7266	0.5	0.3777	0.7194
E_{A_3}	0.6279	0.6223	0.5	0.8019
E_{A_4}	0.2328	0.2806	0.1981	0.5

6. Illustrative example

In this section, a numerical example and a comparison analysis are given to show the feasibility and validity of the proposed priority degree.

Example(adapted from [22]). Consider a decision-making problem that requires the assessment of various engines. Suppose that there are four engine brands (alternatives) A_i ($i = 1, 2, 3, 4$) to be assessed, from which the best needs to be selected. Four attributes are considered: P_1 : responsiveness, P_2 : fuel economy, P_3 : vibration, and P_4 : start. Assume that the attribute weight vector is $\omega = (0.15, 0.3, 0.2, 0.35)$. A hesitant fuzzy decision matrix is used to display the assessment values given by the decision makers as shown in Table 1.

In the following, assuming $X = \{P_1, P_2, P_3, P_4\}$, the hesitant fuzzy sets are constructed as follows (Table 2):

$$\begin{aligned} E_{A_1} &= \{ \langle P_1, \{0.075, 0.06, 0.045\} \rangle, \langle P_2, \{0.27, 0.24, 0.21, 0.03\} \rangle, \\ &\quad \langle P_3, \{0.1, 0.08, 0.04\} \rangle, \langle P_4, \{0.315, 0.21, 0.5, 0.105\} \rangle \}, \\ E_{A_2} &= \{ \langle P_1, \{0.075, 0.045\} \rangle, \langle P_2, \{0.27, 0.21, 0.18, 0.15, 0.06\} \rangle, \\ &\quad \langle P_3, \{0.16, 0.12, 0.1, 0.02\} \rangle, \langle P_4, \{0.245, 0.14, 0.105\} \rangle \}, \\ E_{A_3} &= \{ \langle P_1, \{0.12, 0.105, 0.06, 0.045\} \rangle, \langle P_2, \{0.21, 0.12, 0.06\} \rangle, \\ &\quad \langle P_3, \{0.16, 0.02\} \rangle, \langle P_4, \{0.315, 0.28, 0.21\} \rangle \}, \\ E_{A_4} &= \{ \langle P_1, \{0.135, 0.105, 0.09, 0.045, 0.015\} \rangle, \\ &\quad \langle P_2, \{0.12, 0.105, 0.09, 0.06\} \rangle, \langle P_3, \{0.135, 0.12, 0.105\} \rangle, \\ &\quad \langle P_4, \{0.135, 0.105, 0.09, 0.045\} \rangle \} \end{aligned}$$

Compute the priority degree of each using Eq.(5), the results for which are shown in Table 3.

Analyzing the above priority degrees, the ranking of the alternatives is as follows:

$$A_3 \succ_{(0.6223)} A_2 \succ_{(0.7266)} A_1 \succ_{(0.7672)} A_4.$$

These results only show the degree of priority between the alternatives not the absolute order relations; that is, A_3 is not necessarily the optimal alternative. Similarly, A_4 is not necessarily the worst alternative.

Li et al. [22] used distance and a similarity measures to deal with this problem, the results for which were as follows:

Table 4

Results obtained using the distance measure d_{ophug} in [22] with $\alpha = 0.8$, $\beta = 0.2$.

	A_1	A_2	A_3	A_4	Rankings
$\lambda = 1$	0.6814	0.6639	0.6195	0.6675	$A_3 > A_2 > A_4 > A_1$
$\lambda = 2$	0.6967	0.6802	0.6380	0.6909	$A_3 > A_2 > A_4 > A_1$
$\lambda = 6$	0.7212	0.7159	0.6782	0.7251	$A_3 > A_2 > A_1 > A_4$
$\lambda = 10$	0.7346	0.7355	0.7059	0.7379	$A_3 > A_1 > A_2 > A_4$

Table 5

Results obtained using the distance measure d_{og} in [22].

	A_1	A_2	A_3	A_4	Rankings
$\lambda = 1$	0.4779	0.5027	0.4292	0.3558	$A_4 > A_3 > A_1 > A_2$
$\lambda = 2$	0.5378	0.5451	0.5052	0.4129	$A_4 > A_3 > A_1 > A_2$
$\lambda = 6$	0.6599	0.6476	0.6704	0.5699	$A_4 > A_2 > A_1 > A_3$
$\lambda = 10$	0.7213	0.7046	0.7373	0.6537	$A_4 > A_2 > A_1 > A_3$

Results obtained using the score function in Definition 6.

$$Score(A_1) = 0.1508 \quad Score(A_2) = 0.1193$$

$$Score(A_3) = 0.1427 \quad Score(A_4) = 0.1009$$

$$Score(A_1) > Score(A_3) > Score(A_2) > Score(A_4)$$

According to Definition 6, we conclude

$$A_1 > A_3 > A_2 > A_4.$$

The above result is still too absolutely, contrast to the proposed method, which is short of flexible and can led to the decision information loss.

α and β are the preference ratio between the hesitance degree and the membership values. When the ratio of the hesitance degree is greater than the membership values, the result is similar to that when using the proposed priority degree. Otherwise, the result is different from that obtained using the distance measure in [22].

The results shown in Tables 4 and 5 are absolutely order relations. Compared to the results obtained using the priority degree in Table 3, it can be concluded that the proposed priority degree formula better conforms to the actual situation, it is more flexible and retains the decision making information. In contrast, Tables 4 and 5 indicate that using distance and similar measures to determine the HFS order is inflexible and could lead to a loss of decision making information.

7. Conclusions

In this paper, we analyzed the existing order relations for HFSs and provided a priority degree definition. Then, some desirable properties for the proposed priority degree formula were studied. For computation convenience, a new operator between the hesitant fuzzy elements was proposed. Based on the presented priority degree, a new multiple attribute decision making method was obtained. Finally, an illustrative example was given to show the rationality of the proposed priority degree.

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