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Analysis of an k -out-of- n :G system with repairman's single vacation and shut off rule



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ABSTRACT

This paper considers an k -out-of- n :G repairable system with repairman's single vacation and shut off rule where the working times and repair times of components follow exponential distributions, and the duration of the repairman's vacation is governed by a phase type distribution. Further, continuous operation as shut off rule in this paper means that all the non-failed components are still subject to failure when the system is down. We derive the transient system availability, the rate of occurrence of failures at time t , the stationary availability, the stationary rate of occurrence of failures along with other system performance measures by using the Markov process theory and the matrix analytical method. Moreover, employing the probabilistic properties of the pH distribution, we deduce the system reliability, the mean time to the first failure of the system and the waiting time distribution of an arbitrary failed component. Based on this, we discuss the time-dependent behavior of the system performance measures under different initial system states. Finally, the Monte Carlo simulation and special cases of the system are provided to show the correctness of our results.

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1. Introduction

In reliability theory, k -out-of- n :G system as a popular type of redundancy is firstly proposed by Birnbaum et al. [1]. For such a system, all the n components are working initially even though only k of them are required for the system to operate. Classical examples of its applications include the hydraulic control system with multiple pumps, the redundant inertial navigation system in an airborne integrated avionics system, the radar systems and the production systems. For illustration, we consider an active phased array radar (APAR) system with four faces, in which each face consists of hundreds of transmit and receive components. A certain percentage of the number of components is allowed to fail without losing the function of the radar system. If the total number of components is 400, and the allowed percentage is 10%, then the APAR system behaves as a 360-out-of-400:G system. Over the past decades, a few of many researchers have carried out the performance measures of k -out-of- n :G system owing to the importance in industry and system design. Comprehensive discussion of the

system are directed to the monographs by Kuo and Zuo [2], Cao and Cheng [3].

When k -out-of- n :G repairable system is considered, the exponential distributions of random times involved are usually supposed, for example, Smidt–Destombes et al. [4], Zhang and Wu [5], Jain and Gupta [6], Wu et al. [7]. They investigated such a redundant system with different assumptions. Wu et al. [8] considered a k -out-of- n :G system with N policy, repairman's vacations and one replaceable repair equipment where the working times and repair times of components, the repairman's vacation times, the operating times and replacement times of repair equipment follow exponential distributions. Applying the quasi-birth-and-death process theory, various steady-state system performance measures are derived. Although the memoryless property of the exponential distribution largely simplifies analysis, it may not precisely model real-world scenarios. To overcome this limitations, Neuts [9] introduced and analyzed the phase-type (PH) distribution firstly. The class of pH distribution is dense in the continuous non-negative class of distributions, and any such distribution can be approached by a pH distribution. Chakravarty et al. [10] studied a repairable k -out-of- n :G system with (N, T) policy and one unreliable repair equipment, in which the repair times, the vacation times and the fixing times are governed by pH distributions. A variety of steady-state system

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performance measures are evaluated by using the matrix analytical method. Later, Ruiz-Castro and Li [11] discussed a discrete k -out-of- n : G system with R repairmen where all random variables involved follow discrete phase type distributions. We refer the reader to Chakravarthy and Gómez-Corral [12], Pérez-Ocón et al. [13] and Wu et al. [14] for more details.

Traditionally, the repairman is assumed remaining idle until the failed components are presented or some repair control policies are met, see Krishnamoorthy et al. [15,16]. However, in vast majority of small-size and medium-size enterprises, repairman might have two roles: one for looking after the components and one for other duties. For example, Wang and Xu [17], Yu et al. [18], and Wu and Ke [19]. Yuan [20] studied a repairable k -out-of- n : G system with redundant dependency and repairmen's vacations, in which all random variables follow exponential distributions. In his/her model, when the system is down, none of the other functioning components are subject to failure. Later, Wu et al. [8,14,21] introduced the "repairman's vacation policy" into k -out-of- n : G system and deduced a variety of system performance measures in transient regime or in stationary regime. But, the vacation policy being considered in k -out-of- n : G repairable system is little more than a beginning. With this knowledge, the case of repairman's vacation policy is taken into account in this paper.

Another important factor in a k -out-of- n : G system is the shut off rules which describes the status of the non-failed components when the system is down. The most commonly used shut off rules are suspended animation (SA) and continuous operation (CO). The SA rule eliminates the possibility of additional failures of non-failed components when the system is down, while the CO rule assumes that all non-failed components are still subject to failure when the system is down. Many publication are reported for availability and reliability analysis of k -out-of- n : G systems assuming the SA rule. 2006, Li et al. [22] analyzed a repairable k -out-of- n : G system with CO rule. When the system is down, some of $(k-1)$ working components are suspended while others are still subject to failure. By utilizing the Markov analysis method, formulas for various system reliability measures including mean time between failures, mean working time in a failure repair cycle, and mean down time in a failure repair cycle are derived. Later, Moghaddas et al. [23,24] studied the k -out-of- n : G system with CO rule where random variables are all exponential distributions. However, the CO as the shut off rule for k -out-of- n : G system has so far been limited.

This study is motivated by some practical applications. For example, we consider a communication system with seven transmitters. The average message load may require at least five transmitters be operational, or some critical message may be lost. That as many as three transmitters fail is insufficient for smoothly communication. When one transmitter fails, it will be repaired by a technician. The technician may perform other job whenever there is no failed transmitter and then returns to the system after a random length of time. Upon returning from vacation, if there are failed transmitters in the system, he/she starts to repair them immediately until no failed transmitter presents, and then takes another assigned job. Otherwise, he/she remains idle till a breakdown to occur. When the system is down, the remaining non-failed transmitters are still in work state and subject to failure. This practical example can be viewed as a 5-out-of-7: G system with repairman's single vacation and CO rule.

In this paper, we consider a k -out-of- n : G repairable system with repairman's single vacation and CO rule. Applying the Markov process theory and matrix analytical method, the system availability and the rate of occurrence of failures are obtained in transient regime and in stationary regime. Further, the system reliability, the mean time to the first failure and the waiting time distribution of failed components are discussed by using the properties of pH distribution. In addition, we numerically show the influence of var-

ious parameters on the evolution of the system reliability measures under different initial states. The new contribution of this work is that the CO rule is considered and the repairman takes single vacation, the duration of which follows a phase type distribution. Moreover, we deduce some performance measures by using the probabilistic properties of pH distribution and discuss the time-dependent behavior of the system performance measures.

The rest of this paper is arranged as follows. Section 2 gives some definitions and assumptions of the model. The infinitesimal generator of the vector-valued Markov process that governs the system is constructed in Section 3. Moreover, system reliability measures are derived in Sections 4, 5 and 6. In Section 7, the waiting time of failed components is discussed. Section 8 reports some numerical results. Finally, conclusions are drawn in Section 9.

2. Some definitions and model description

2.1. Some definitions

Definition 1. [9] A distribution $H(x)$ on $[0, +\infty)$ is of phase type with representation (σ, S) , if it is the distribution of the time until absorption in a Markov process on the states $\{1, 2, \dots, r, r+1\}$ with infinitesimal generator

$$\begin{pmatrix} S_{r \times r} & S^0_{r \times 1} \\ \mathbf{0}_{1 \times r} & \mathbf{0}_{1 \times 1} \end{pmatrix},$$

and initial probability vector $(\sigma_{1 \times r}, \mathbf{0}_{1 \times 1})$. Assuming that the states $\{1, 2, \dots, r\}$ are all transient, and the state $r+1$ is absorbing. Hence the matrix S can be interpreted as the rate transient matrix among the transient states, while S^0 represents the column vector of absorption rates. The matrix S is non-singular with negative diagonal entries and non-negative off-diagonal entries and satisfies $-\mathbf{S}e_{r \times 1} = S^0$. The probability distribution $H(x) = 1 - \sigma_{1 \times r} \exp(Sx) e_{r \times 1}$, $x \geq 0$, and its mean $E(\chi) = -\sigma_{1 \times r} S^{-1} e_{r \times 1}$.

Definition 2. [25] Assume that χ has a pH distribution with irreducible pH representation (σ, S) and $\sigma e = 1$. Its probability distribution and mean value are given by $H(x)$ and $E[\chi]$, respectively. Then, the equilibrium distribution of χ is defined as

$$H_e(t) = \frac{1}{E[\chi]} \int_0^t (1 - H(x)) dx, \quad t \geq 0,$$

which is also a pH distribution with representation (π, S) . Here $\pi = \frac{1}{E[\chi]} (-\sigma S^{-1})$ satisfies $\pi(S + S^0 \sigma) = 0$, $\pi e = 1$.

2.2. Model description

The detailed assumptions of the system are described as follows.

Assumption 1. The k -out-of- n : G system is composed of n identical and independent components. It functions as long as there are at least k of them operate. The system is down until the number of working components goes down to $k-1$. When the system is down, the remaining $(k-1)$ working components are still subject to failure.

Assumption 2. The working time X of each component has an exponential distribution $F(t) = 1 - \exp(-\lambda t)$, $\lambda > 0$, $t \geq 0$. Failed components in the system form a single waiting line and receive repair provided by repairman in the order of their failures, i.e., FCFS discipline. The repair time Y of each failed component follows an exponential distribution $G(t) = 1 - \exp(-\mu t)$, $\mu \geq 0$, $t \geq 0$.

Assumption 3. The repairman leaves for a vacation whenever there is no failed component in the system. Upon returning from

his/her vacation, the repairman will start to work finding there exist failed components waiting for repair. Otherwise, the repairman remains idle in the system for the first failed component appears. The vacation time V is governed by a pH distribution with representation (θ, T) of order m . Its probability distribution and mean value are

$$V(t) = 1 - \theta \exp(Tt) \mathbf{e}_m, \quad t \geq 0, \quad E[V] = -\theta T^{-1} \mathbf{e}_m,$$

where

$$\theta = (\theta_1, \theta_2, \dots, \theta_m), \quad T = \begin{pmatrix} T_{1,1} & T_{1,2} & \cdots & T_{1,m} \\ T_{2,1} & T_{2,2} & \cdots & T_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ T_{m,1} & T_{m,2} & \cdots & T_{m,m} \end{pmatrix},$$

$$T^0 = \begin{pmatrix} T_1^0 \\ T_2^0 \\ \vdots \\ T_m^0 \end{pmatrix}.$$

Assumption 4. The random variables X , Y , and V are assumed to be independent of each other.

Remark 1. In order to make this work more practical and reasonable, we consider a data processing system with one repairman. A minimum of k active video displays can meet full data display. Once a working video display fails, it will be repaired by repairman. Further, the repairman may perform additional tasks (e.g., preventive maintenance or repair for other equipment) whenever there is no failed video display. Finding failed video displays in the system at the end of the additional task, he/she starts to repair them immediately. Otherwise, he/she waits until the first failed video display appears. Therefore, this research is helpful in some practical applications.

Furthermore, we define the following notations for use in the sequel. Let I_l be identity matrix of order l , \mathbf{e}_l be a column vector of order l of 1's, and $\mathbf{e}_l(r)$ be a column vector of order l with 1 in the r th position and 0 others. We denote by $\mathbf{0}_l$ a zero matrix of order l , by $\mathbf{0}_l \times_j$ a zero matrix of dimension $l \times j$.

3. Infinitesimal generator

The k -out-of- n :G repairable system as described above can be studied as a block-structured continuous-time Markov chain (CTCM). To see this, define $L(t)$ to be the number of failed components (either waiting or being repaired) at time t , it follows that $L(t) = i$ ($i = 0, 1, \dots, n$). Let $J(t)$ be the state of the repairman at

time t , and

$$J(t) = \begin{cases} -1 & \text{the repairman is idle in the system at time } t, \\ 0, & \text{the repairman is repairing failed component at time } t, \\ j, & \text{the repairman is on vacation at phase } j, \\ & j = 1, 2, \dots, m. \end{cases}$$

According to the model assumptions, one checks easily that the stochastic process $\mathcal{X} = \{L(t), J(t) : t \geq 0\}$ is a CTMC with state space

$$\Omega = \{(0, -1)\} \cup \{(0, j), j = 1, 2, \dots, m\} \cup \{(i, j), i = 1, 2, \dots, n, j = 0, 1, \dots, m\}.$$

Here, in order to give the reader a good intuitive understanding, we provide a state transition diagram of the system with the assumption that $m = 2$ (Fig. 1).

By partitioning the state space into levels with regard to the number of failed components and using lexicographical sequence for the state, we know that the corresponding infinitesimal generator matrix \mathbf{Q} of \mathcal{X} is of dimension $(n+1)(m+1)$, exhibiting the following block-structured form

$$\mathbf{Q} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{C}_0 & & & \\ \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{C}_1 & & \\ & \mathbf{B}_2 & \mathbf{A}_2 & \mathbf{C}_2 & \\ & & \ddots & \ddots & \ddots \\ & & & \mathbf{B}_{n-1} & \mathbf{A}_{n-1} & \mathbf{C}_{n-1} \\ & & & & \mathbf{B}_n & \mathbf{A}_n \end{pmatrix}. \quad (1)$$

Each block of the matrix \mathbf{Q} are defined in the following

$$\mathbf{B}_1 = \begin{pmatrix} \mathbf{0}_{1 \times 1} & \mu \mathbf{0}_{1 \times m} \\ \mathbf{0}_{m \times 1} & \mathbf{0}_{m \times m} \end{pmatrix},$$

$$\mathbf{B}_r = \begin{pmatrix} \mu & \mathbf{0}_{1 \times m} \\ \mathbf{0}_{m \times 1} & \mathbf{0}_{m \times m} \end{pmatrix}, \quad r = 2, 3, \dots, n,$$

$$\mathbf{A}_0 = \begin{pmatrix} -n\lambda & \mathbf{0}_{1 \times m} \\ \mathbf{T}^0 & \mathbf{T} - n\lambda \mathbf{I}_m \end{pmatrix},$$

$$\mathbf{A}_r = \begin{pmatrix} -\mu - (n-r)\lambda & \mathbf{0}_{1 \times m} \\ \mathbf{T}^0 & \mathbf{T} - (n-r)\lambda \mathbf{I}_m \end{pmatrix}, \quad r = 1, 2, \dots, n-1,$$

$$\mathbf{A}_n = \begin{pmatrix} -\mu & \mathbf{0}_{1 \times m} \\ \mathbf{T}^0 & \mathbf{T} \end{pmatrix},$$

$$\mathbf{C}_r = \begin{pmatrix} (n-r)\lambda & \mathbf{0}_{1 \times m} \\ \mathbf{0}_{m \times 1} & (n-r)\lambda \mathbf{I}_m \end{pmatrix}, \quad r = 0, 1, \dots, n-1.$$

4. Transient analysis

This section will discuss the transient behavior of system reliability measures including the availability $A(t)$ and the rate of occurrence of failures $m_f(t)$. We first define the following notations

$$P_{i,j}(t) = P\{L(t) = i, J(t) = j\}, \quad (i, j) \in \Omega,$$

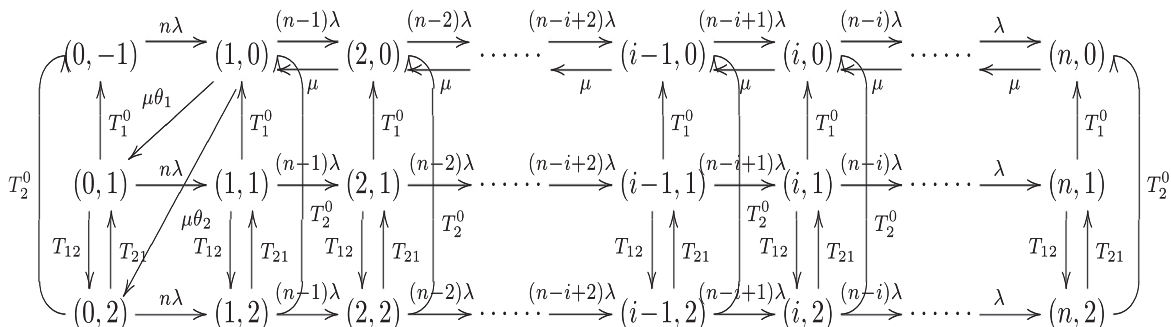


Fig. 1. State transition diagram of the system with $m = 2$.

$$\begin{aligned} P_0(t) &= (P_{0,-1}(t), P_{0,1}(t), \dots, P_{0,m}(t)), \\ P_i(t) &= (P_{i,0}(t), P_{i,1}(t), \dots, P_{i,m}(t)), \quad i = 1, 2, \dots, n. \end{aligned}$$

By a straightforward probability arguments, the transition equations of the Markov model are formed as ordinary differential equations with initial condition

$$\begin{cases} \frac{d}{dt} \mathbf{P}(t) = \mathbf{P}(t) \mathbf{Q}, \\ \mathbf{P}(0) = \boldsymbol{\omega}_{1 \times (n+1)(m+1)}(r), \end{cases} \quad (2)$$

where $\mathbf{P}(t) = (P_0(t), P_1(t), \dots, P_n(t))$, $\boldsymbol{\omega}_{1 \times (n+1)(m+1)}(r)$ is a row vector of dimension $(n+1)(m+1)$ with 1 in the r th position and 0 others.

The solution of the above ordinary differential equations can be written as

$$\mathbf{P}(t) = \mathbf{P}(0) e^{\mathbf{Q}t}. \quad (3)$$

From Eq. (3), we can get the system state probabilities $P_{i,j}(t)$ ($(i, j) \in \Omega$). Then the system availability and the rate of occurrence of failures of the system at time t are easily obtained.

- The system availability at time t

$$A(t) = P_{0,-1}(t) + \sum_{j=1}^m P_{0,j}(t) + \sum_{i=1}^{n-k} \sum_{j=0}^m P_{i,j}(t).$$

- The rate of occurrence of failures at time t

$$m_f(t) = k\lambda \sum_{j=0}^m P_{n-k,j}(t).$$

5. Steady-state analysis

This section will derive some system performance measures in stationary regime. We first define the stationary system state probabilities

$$\begin{aligned} P_{i,j} &= \lim_{t \rightarrow \infty} P_{i,j}(t) = \lim_{t \rightarrow \infty} P\{L(t) = i, J(t) = j\}, \quad (i, j) \in \Omega, \\ P_0 &= (P_{0,-1}, P_{0,1}, \dots, P_{0,m}), \quad P_i = (P_{i,0}, P_{i,1}, \dots, P_{i,m}), \quad i = 1, 2, \dots, n, \\ P &= (P_0, P_1, \dots, P_n). \end{aligned}$$

Then, it follows from the matrix equation $\mathbf{PQ} = \mathbf{0}$ that

$$P_0 \mathbf{A}_0 + P_1 \mathbf{B}_1 = \mathbf{0}_{1 \times (m+1)}, \quad (4)$$

$$P_i \mathbf{C}_i + P_{i+1} \mathbf{A}_{i+1} + P_{i+2} \mathbf{B}_{i+2} = \mathbf{0}_{1 \times (m+1)}, \quad i = 0, 1, \dots, n-2, \quad (5)$$

$$P_{n-1} \mathbf{C}_{n-1} + P_n \mathbf{A}_n = \mathbf{0}_{1 \times (m+1)}. \quad (6)$$

Further, the following normalizing equation should be satisfied

$$\sum_{i=0}^n P_i \mathbf{e}_{m+1} = \mathbf{1}. \quad (7)$$

After some algebraic manipulation, we have

$$P_i = P_0 \prod_{r=1}^i \xi_r, \quad i = 1, 2, \dots, n, \quad (8)$$

$$P_0 (\mathbf{A}_0 + \xi_1 \mathbf{B}_1) = \mathbf{0}_{1 \times (m+1)}, \quad (9)$$

where $\xi_i = -\mathbf{C}_{i-1} (\mathbf{A}_i + \xi_{i+1} \mathbf{B}_{i+1})^{-1}$, $i = 1, 2, \dots, n-1$, $\xi_n = -\mathbf{C}_{n-1} \mathbf{A}_n^{-1}$ are all square matrices of order $m+1$. Once P_0 being obtained, then the steady-state probability vector $P = (P_0, P_1, \dots, P_n)$ are determined.

With Eqs. (7) and (8), we get

$$P_0 \left(\mathbf{e}_{m+1} + \sum_{i=1}^n \prod_{r=1}^i \xi_r \mathbf{e}_{m+1} \right) = \mathbf{1}. \quad (10)$$

Solving Eqs (9) and (10) simultaneously would get the stationary solution P_0 . Further, we obtain the steady-state system state probabilities

$$P_{0,-1} = P_0 \mathbf{e}_{m+1}(1),$$

$$P_{0,j} = P_0 \mathbf{e}_{m+1}(j+1), \quad j = 1, 2, \dots, m,$$

$$P_{i,j} = P_i \mathbf{e}_{m+1}(j+1), \quad i = 1, 2, \dots, n, \quad j = 0, 1, \dots, m.$$

Having computed the system state probabilities, we give various steady-state system performance measures straightforwardly.

- The steady-state system availability

$$A = P_{0,-1} + \sum_{j=1}^m P_{0,j} + \sum_{i=1}^{n-k} \sum_{j=0}^m P_{i,j}.$$

- The steady-state rate of occurrence of failure

$$m_f = k\lambda \sum_{j=0}^m P_{n-k,j}.$$

- The expected number of failed components in the system

$$E[L] = \sum_{j=0}^m \sum_{i=1}^n i P_{i,j}.$$

- The probability that the repairman is on vacation

$$P_v = \sum_{i=0}^n \sum_{j=1}^m P_{i,j}.$$

6. System reliability and the mean time to the first failure

In this section, we will employ the probabilistic properties of pH distribution to derive the expressions of the system reliability and the mean time to the first failure (MTTFF). To achieve this, we lump all failure states together to make one absorbing state, say “*”. Then, consider an absorbing Markov process with state space

$$(0, -1), (0, 1), \dots, (0, m), (1, 0), \dots, (1, m), \dots, (n-k, 0), \dots, (n-k, m), *$$

where only the state “*” is absorbing and all other states are transient.

The corresponding infinitesimal generator Θ of the absorbing Markov process has the following form

$$\Theta = \begin{pmatrix} \mathbf{S}_{(n-k+1)(m+1) \times (n-k+1)(m+1)} & \mathbf{S}_{(n-k+1)(m+1) \times 1}^0 \\ \mathbf{0}_{1 \times (n-k+1)(m+1)} & \mathbf{0}_{1 \times 1} \end{pmatrix},$$

where

$$\mathbf{S} = \begin{pmatrix} \mathbf{A}_0 & \mathbf{C}_0 & & & \\ \mathbf{B}_1 & \mathbf{A}_1 & \mathbf{C}_1 & & \\ & \ddots & \ddots & \ddots & \\ & & \mathbf{B}_{n-k-1} & \mathbf{A}_{n-k-1} & \mathbf{C}_{n-k-1} \\ & & & \mathbf{B}_{n-k} & \mathbf{A}_{n-k} \end{pmatrix},$$

and initial probability vector $(\sigma_{1 \times (n-k+1)(m+1)}, \mathbf{0}_{1 \times 1})$. Here $\sigma = \boldsymbol{\omega}_{1 \times (n-k+1)(m+1)}(r)$ ($r = 1, 2, \dots, (n-k+1)(m+1)$) is determined by the initial state of the system.

According to the definition of the pH distribution, we know that the reliability of the system is

$$\begin{aligned} R(t) &= \sigma_{1 \times (n-k+1)(m+1)} \exp(\mathbf{S}_{(n-k+1)(m+1) \times (n-k+1)(m+1)} t) \\ &\quad \times \mathbf{e}_{(n-k+1)(m+1) \times 1}. \end{aligned}$$

The mean time to the first failure of the system is

$$\text{MTTF} = -\sigma_{1 \times (n-k+1)(m+1)} \mathbf{S}_{(n-k+1)(m+1) \times (n-k+1)(m+1)}^{-1} \mathbf{e}_{(n-k+1)(m+1) \times 1}.$$

7. The waiting time of an arbitrary failed component

In this section, employing the theory of pH distribution and the probability decomposition technique, we will discuss the waiting time distribution and the mean waiting time of an arbitrary failed component.

First, let $L^-(t) = i (i = 0, 1, \dots, n-1)$ be the number of failed components at an instant just prior to time t which is a failed component arrival epoch. Then, the new process $\mathcal{L} = \{L^-(t), J(t) : t \geq 0\}$ forms a Markov process with state space

$$\tilde{\Omega} = \{(0, -1)\} \cup \{(0, j) : j = 1, 2, \dots, m\} \cup \{(i, j) : i = 1, 2, \dots, n-1, j = 0, 1, \dots, m\}.$$

Moreover, define the following system state probabilities

$$P_{0,-1}^- = \lim_{t \rightarrow \infty} P\{L^-(t) = 0, J(t) = -1\},$$

$$P_{0,j}^- = \lim_{t \rightarrow \infty} P\{L^-(t) = 0, J(t) = j\}, j = 1, 2, \dots, m,$$

$$P_{i,j}^- = \lim_{t \rightarrow \infty} P\{L^-(t) = i, J(t) = j\}, i = 1, 2, \dots, n-1, j = 0, 1, \dots, m.$$

Denote by $A(t, t + \Delta t)$ the event that one of the working components fails during the time interval $(t, t + \Delta t]$. We have that

$$\begin{aligned} P_{0,-1}^- &= \lim_{t \rightarrow \infty} \lim_{\Delta t \rightarrow 0} P\{L(t) = 0, J(t) = -1 | A(t, t + \Delta t)\} \\ &= \lim_{t \rightarrow \infty} \lim_{\Delta t \rightarrow 0} \frac{P\{A(t, t + \Delta t) | L(t) = 0, J(t) = -1\} P_{0,-1}(t)}{\left\{ n\lambda \Delta t P_{0,-1}(t) + \sum_{j=1}^m P\{A(t, t + \Delta t) | L(t) = 0, J(t) = j\} P_{0,j}(t) \right.} \\ &\quad \left. + \sum_{i=1}^{n-1} \sum_{j=0}^m P\{A(t, t + \Delta t) | L(t) = i, J(t) = j\} P_{i,j}(t) \right\}} \\ &= \lim_{t \rightarrow \infty} \lim_{\Delta t \rightarrow 0} \frac{n\lambda \Delta t P_{0,-1}(t) + o(\Delta t)}{n\lambda \Delta t P_{0,-1}(t) + \sum_{j=1}^m n\lambda \Delta t P_{0,j}(t) + \sum_{i=1}^{n-1} \sum_{j=0}^m (n-i)\lambda \Delta t P_{i,j}(t) + o(\Delta t)} \\ &= \frac{nP_{0,-1}}{nP_{0,-1} + \sum_{j=1}^m nP_{0,j} + \sum_{i=1}^{n-1} \sum_{j=0}^m (n-i)P_{i,j}}. \end{aligned}$$

Similarly, we get

$$\begin{aligned} P_{0,j}^- &= \frac{nP_{0,j}}{nP_{0,-1} + \sum_{r=1}^m nP_{0,r} + \sum_{i=1}^{n-1} \sum_{r=0}^m (n-i)P_{i,r}}, j = 1, 2, \dots, m, \\ P_{i,j}^- &= \frac{(n-i)P_{i,j}}{nP_{0,-1} + \sum_{r=1}^m nP_{0,r} + \sum_{i=1}^{n-1} \sum_{r=0}^m (n-i)P_{i,r}}, i = 1, 2, \dots, n-1, \\ &j = 0, 1, \dots, m. \end{aligned}$$

Let $W_q^{(i,j)}$ be the actual waiting time of an arbitrary failed component in the system, given that the system state is $(L^-(t), J(t)) = (i, j)$, $(i, j) \in \tilde{\Omega}$ prior to its arrival instant. Further, setting

$$\begin{aligned} W_q^{(i,j)}(t) &= P\{W_q^{(i,j)} \leq t\}, t \geq 0, E[W_q^{(i,j)}] \\ &= \int_0^\infty t dW_q^{(i,j)}(t), (i, j) \in \tilde{\Omega}. \end{aligned}$$

We now derive the expressions of $W_q^{(i,j)}(t)$ and $E[W_q^{(i,j)}]$ by the following four cases. Here, we take the arriving failed component as the tagged one, and assume its position in the waiting line is $i (i = 0, 1, \dots, n-1)$. Based on the model description, its position may gradually move as the failed components ahead of it complete the repair.

- Case 1: $(L^-(t), J(t)) = (i, 0)$, $i = 1, 2, \dots, n-1$

In this case, there are i failed components ahead of the tagged one, and the repairman is busy with components. We observe the behavior of the tagged component since the arrival instant. Because the future failed components arrival after the tagged one can not change its position in the waiting line, its position only occurs to one direction, namely, $(i, 0) \rightarrow (i-1, 0) \rightarrow \dots \rightarrow (2, 0) \rightarrow (1, 0) \rightarrow \otimes$, where \otimes is an absorbing state in the sense that the tagged component is taken for repair by the repairman. Thus, the actual waiting time $W_q^{(i,0)}$ of the tagged component is the time till absorbing in the Markov chain \mathcal{L} . The infinitesimal generator matrix is given by

$$\begin{pmatrix} \Lambda_{i \times i} & \Lambda_{i \times 1}^0 \\ \mathbf{0}_{1 \times i} & \mathbf{0}_{1 \times 1} \end{pmatrix},$$

where

$$\Lambda = \begin{pmatrix} -\mu & \mu & & \\ & \ddots & \ddots & \\ & & -\mu & \mu \\ & & & -\mu \end{pmatrix}, \Lambda^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mu \end{pmatrix}.$$

The initial probability vector is $(\omega_{1 \times i}(1), \mathbf{0}_{1 \times 1})$. Therefore, we have that

$$\begin{aligned} W_q^{(i,0)}(t) &= 1 - \omega_{1 \times i}(1) \exp(\Lambda_{i \times i} t) \mathbf{e}_{i \times 1}, E[W_q^{(i,0)}] \\ &= -\omega_{1 \times i}(1) \Lambda_{i \times i}^{-1} \mathbf{e}_{i \times 1}, i = 1, 2, \dots, n-1. \end{aligned}$$

Remark 2. From the definition of the Erlangian distribution, we can rewrite

$$\begin{aligned} W_q^{(i,0)}(t) &= 1 - \sum_{r=0}^{i-1} \frac{1}{r!} e^{-\mu x} (\mu x)^r, E[W_q^{(i,0)}] = \frac{i}{\mu}, \\ &i = 1, 2, \dots, n-1. \end{aligned}$$

In order to give an uniform expression for $W_q^{(i,j)}(t)$ and $E[W_q^{(i,j)}]$ under four cases, the well-structured matrix expressions are chosen in this paper.

- Case 2: $(L^-(t), J(t)) = (i, j)$, $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, m$
In this case, there are i failed components ahead of the tagged one, and the repairman is on vacation. So, the tagged component receives repair until the repairman returns from the vacation and all the i failed components complete the repair. We enumerate all possible states of this situation in lexicographic order

$$(i, 1), (i, 2), \dots, (i, m), (i, 0), (i-1, 0), (i-2, 0), \dots, (2, 0), (1, 0), \otimes,$$

where the states $(i, 1), \dots, (i, m), (i, 0), (i-1, 0), \dots, (1, 0)$ are transient, the state \otimes is absorbing. Thus, the actual waiting time $W_q^{(i,j)}$ of the tagged component is the time till absorbing in the Markov chain \mathcal{L} . The infinitesimal generator matrix is

$$\begin{pmatrix} \Upsilon_{(m+i) \times (m+i)} & \Upsilon_{(m+i) \times 1}^0 \\ \mathbf{0}_{1 \times (m+i)} & \mathbf{0}_{1 \times 1} \end{pmatrix},$$

where

$$\Upsilon = \begin{pmatrix} \mathbf{T}_{m \times m} & \begin{pmatrix} \mathbf{T}_{m \times 1}^0 & \mathbf{0}_{m \times (i-1)} \end{pmatrix} \\ \mathbf{0}_{i \times m} & \Lambda_{i \times i} \end{pmatrix}, \Upsilon^0 = \begin{pmatrix} \mathbf{0}_{(m+i-1) \times 1} \\ \mu \end{pmatrix}.$$

From Definition 2, the equilibrium distribution of the repairman's vacation time with pH representation $(\theta_{1 \times m}, \mathbf{T}_{m \times m})$ is also a pH distribution with an explicit pH representation $(\beta_{1 \times m}, \mathbf{T}_{m \times m})$, where $\beta_{1 \times m} = \frac{1}{E[V]} (-\theta_{1 \times m} \mathbf{T}_{m \times m}^{-1})$.

The initial probability vector is $(\omega_{1 \times (m+i)}^{(i,j)}, 0_{1 \times 1})$, where $\omega_{1 \times (m+i)}^{(i,j)} = (\beta_{1 \times m}(j), 0_{1 \times i})$, $j = 1, 2, \dots, m$. Therefore, we know that

$$W_q^{(i,j)}(t) = 1 - \omega_{1 \times (m+i)}^{(i,j)} \exp(\Upsilon_{(m+i) \times (m+i)} t) \mathbf{e}_{(m+i) \times 1},$$

$$E[W_q^{(i,j)}] = -\omega_{1 \times (m+i)}^{(i,j)} \Upsilon_{(m+i) \times (m+i)}^{-1} \mathbf{e}_{(m+i) \times 1},$$

$$i = 1, 2, \dots, n-1, \quad j = 1, 2, \dots, m.$$

- Case 3: $(L^-(t), J(t)) = (0, j)$, $j = 1, 2, \dots, m$

In this case, there is no failed component in the system and the repairman is on vacation. Thus, the tagged component is being repaired while the repairman comes back from his/her vacation. According to the results derived in case 2, the distribution and the mean of the actual waiting time $W_q^{(0,j)}$ of the tagged component is

$$W_q^{(0,j)}(t) = 1 - \beta_{1 \times m}(j) \exp(\mathbf{T}_{m \times m} t) \mathbf{e}_{m \times 1},$$

$$E[W_q^{(0,j)}] = -\beta_{1 \times m}(j) \mathbf{T}_{m \times m}^{-1} \mathbf{e}_{m \times 1}, \quad j = 1, 2, \dots, m.$$

- Case 4: $(L^-(t), J(t)) = (0, -1)$

In this case, there is no failed component in the system and the repairman is idle. So, the tagged component receives repair immediately, and its actual waiting time $W_q^{(0,-1)}$ is zero. Thus, the distribution and the mean of $W_q^{(0,-1)}$ of the tagged component is

$$W_q^{(0,-1)}(t) = 1, \quad E[W_q^{(0,-1)}] = 0.$$

Let W_q be the waiting time of an arbitrary failed component in the system. By using the probability decomposition technique, we get

$$\begin{aligned} W_q(t) &= P\{W_q \leq t\} \\ &= \sum_{i=1}^{n-1} P\{W_q \leq t, L^-(t) = i, J(t) = 0\} \\ &\quad + \sum_{i=1}^{n-1} \sum_{j=1}^m P\{W_q \leq t, L^-(t) = i, J(t) = j\} \\ &\quad + \sum_{j=1}^m P\{W_q \leq t, L^-(t) = 0, J(t) = j\} \\ &\quad + P\{W_q \leq t, L^-(t) = 0, J(t) = -1\} \\ &= \sum_{i=1}^{n-1} P\{W_q^{(i,0)} \leq t\} P_{i,0}^- + \sum_{i=1}^{n-1} \sum_{j=1}^m P\{W_q^{(i,j)} \leq t\} P_{i,j}^- \\ &\quad + \sum_{j=1}^m P\{W_q^{(0,j)} \leq t\} P_{0,j}^- + P_{0,-1}^- \\ &= \sum_{i=1}^{n-1} (1 - \omega_{1 \times i}(1) \exp(\Lambda_{i \times i} t) \mathbf{e}_{i \times 1}) P_{i,0}^- \\ &\quad + \sum_{i=1}^{n-1} \sum_{j=1}^m (1 - \omega_{1 \times (m+i)}^{(i,j)} \exp(\Upsilon_{(m+i) \times (m+i)} t) \mathbf{e}_{(m+i) \times 1}) P_{i,j}^- \\ &\quad + \sum_{j=1}^m (1 - \beta_{1 \times m}(j) \exp(\mathbf{T}_{m \times m} t) \mathbf{e}_{m \times 1}) P_{0,j}^- + P_{0,-1}^- \\ &= 1 - \sum_{i=1}^{n-1} \omega_{1 \times i}(1) \exp(\Lambda_{i \times i} t) \mathbf{e}_{i \times 1} P_{i,0}^- \\ &\quad + \sum_{i=1}^{n-1} \sum_{j=0}^m \omega_{1 \times (m+i)}^{(i,j)} \exp(\Upsilon_{(m+i) \times (m+i)} t) \mathbf{e}_{(m+i) \times 1} P_{i,j}^- \\ &\quad + \sum_{j=1}^m \beta_{1 \times m}(j) \exp(\mathbf{T}_{m \times m} t) \mathbf{e}_{m \times 1} P_{0,j}^-. \end{aligned}$$

The mean waiting time is given by

$$\begin{aligned} E[W_q] &= \sum_{i=1}^{n-1} E[W_q; L^-(t) = i, J(t) = 0] \\ &\quad + \sum_{i=1}^{n-1} \sum_{j=1}^m E[W_q; L^-(t) = i, J(t) = j] \\ &\quad + \sum_{j=1}^m E[W_q; L^-(t) = 0, J(t) = j] \\ &\quad + E[W_q; L^-(t) = 0, J(t) = -1] \\ &= \sum_{i=1}^{n-1} E[W_q^{(i,0)}] P_{i,0}^- + \sum_{i=1}^{n-1} \sum_{j=1}^m E[W_q^{(i,j)}] P_{i,j}^- + \sum_{j=1}^m E[W_q^{(0,j)}] P_{0,j}^- \\ &= - \sum_{i=1}^{n-1} (\omega_{1 \times i}(1) \Lambda_{i \times i}^{-1} \mathbf{e}_{i \times 1}) P_{i,0}^- \\ &\quad - \sum_{i=1}^{n-1} \sum_{j=1}^m (\omega_{1 \times (m+i)}^{(i,j)} \Upsilon_{(m+i) \times (m+i)}^{-1} \mathbf{e}_{(m+i) \times 1}) P_{i,j}^- \\ &\quad - \sum_{j=1}^m (\beta_{1 \times m}(j) \mathbf{T}_{m \times m}^{-1} \mathbf{e}_{m \times 1}) P_{0,j}^- \end{aligned}$$

8. Numerical examples

This section provides a variety of numerical examples to demonstrate the applicability of the theoretical results.

Example 1. This example discusses the time-dependent behavior of the system availability $A(t)$, the transient rate of occurrence of failures $m_f(t)$, the system reliability $R(t)$, and the waiting time distribution of an arbitrary failed component $W_q(t)$. First, we let $k = 5$, $n = 14$, $\mu = 3.5$. The vacation time follows a pH distribution with representation (θ, \mathbf{T}) of order 4 and mean value 3.8125, where

$$\theta = (0.5, 0.2, 0.15, 0.15), \quad \mathbf{T} = \begin{pmatrix} -\frac{4}{5} & \frac{4}{5} & 0 & 0 \\ 0 & -\frac{4}{5} & \frac{4}{5} & 0 \\ 0 & 0 & -\frac{4}{5} & \frac{4}{5} \\ 0 & 0 & 0 & -\frac{4}{5} \end{pmatrix},$$

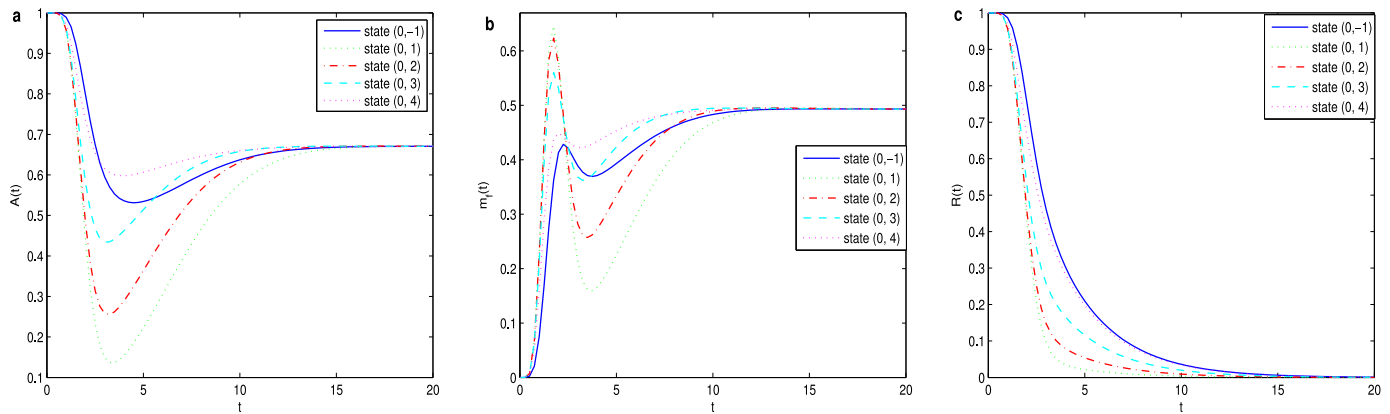
$$\mathbf{T}^0 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \frac{4}{5} \end{pmatrix}.$$

Table 1, Fig. 2 and Table 2 perform the above specific parameters with $\lambda = 0.6$. Fig. 3 performs the above specific parameters by collocating with different values of $\lambda = 0.5, 0.6, 0.7, 0.8$.

We observe from Fig. 2 that these curves display violent fluctuations in the early stage. The higher the probability that the repairman returns from his/her vacation, the bigger the system availability and system reliability are, and the smaller the rate of occurrence of failures is. But after some time units these fluctuations of them tend to disappear. Further, one checks easily that the convergence of these reliability measures are independent of initial conditions, which is as we have anticipated. It appears from Table 2 that the mean time to the first failure is a monotonically decreasing function of the number of failed components initially in the system. Table 2 shows it is also sensitive to system initial conditions. Fig. 3 indicates the waiting time distribution decreases with the increase of λ . This is what we had expected due to the fact that having larger value of λ would increase the number of failed component in the system.

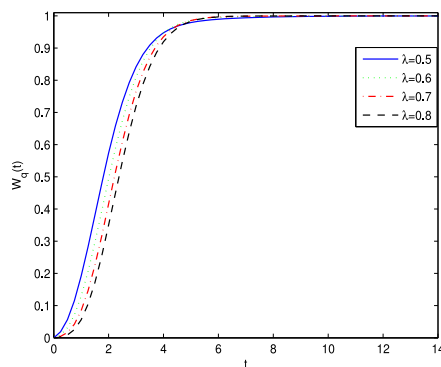
Table 1System state probabilities at time t under initial condition $(0, -1)$.

(i, j)	$P_{i,j}(0.5)$	$P_{i,j}(1.0)$	$P_{i,j}(3.0)$	$P_{i,j}(5.0)$	$P_{i,j}(10)$	$P_{i,j}(15)$
$(0, -1)$	0.123480	0.000495	0.000020	0.000011	0.000012	0.000013
$(0, 1)$	0.047637	0.005397	0.000598	0.000358	0.000374	0.000411
$(0, 2)$	0.022140	0.002842	0.000294	0.000175	0.000182	0.000200
$(0, 3)$	0.015678	0.001979	0.000206	0.000123	0.000128	0.000141
$(0, 4)$	0.015284	0.001869	0.000198	0.000118	0.000123	0.000136
$(1, 0)$	0.236247	0.020596	0.002981	0.001859	0.001972	0.002162
$(1, 1)$	0.033635	0.007924	0.000621	0.000354	0.000363	0.000401
$(1, 2)$	0.016803	0.005282	0.000368	0.000206	0.000211	0.000233
$(1, 3)$	0.011631	0.003622	0.000252	0.000141	0.000144	0.000159
$(1, 4)$	0.011184	0.003233	0.000231	0.000130	0.000133	0.000147

**Fig. 2.** Illustration of system reliability measures under different initial conditions.**Table 2**

The mean time to the first failure under different initial conditions.

(i, j)	MTTF	(i, j)	MTTF	(i, j)	MTTF	(i, j)	MTTF
$(0, -1)$	3.71099503	$(2, 2)$	1.98033721	$(4, 4)$	2.69774882	$(7, 1)$	0.86486112
$(0, 1)$	2.08838396	$(2, 3)$	2.41058385	$(5, 0)$	3.65577663	$(7, 2)$	0.90953956
$(0, 2)$	2.32032006	$(2, 4)$	3.20101062	$(5, 1)$	1.28460834	$(7, 3)$	1.06760738
$(0, 3)$	2.79970655	$(3, 0)$	4.02254891	$(5, 2)$	1.38262802	$(7, 4)$	1.56765258
$(0, 4)$	3.49342452	$(3, 1)$	1.63848568	$(5, 3)$	1.66813633	$(8, 0)$	2.23890172
$(1, 0)$	3.59194741	$(3, 2)$	1.79364313	$(5, 4)$	2.37129314	$(8, 1)$	0.61825570
$(1, 1)$	1.94724718	$(3, 3)$	2.18386977	$(6, 0)$	3.31478324	$(8, 2)$	0.64133617
$(1, 2)$	2.15561659	$(3, 4)$	2.97610558	$(6, 1)$	1.08490172	$(8, 3)$	0.73426543
$(1, 3)$	2.61459056	$(4, 0)$	3.89616641	$(6, 2)$	1.15514530	$(8, 4)$	1.08381328
$(1, 4)$	3.35365590	$(4, 1)$	1.46816357	$(6, 3)$	1.37877976		
$(2, 0)$	3.97515589	$(4, 2)$	1.59482778	$(6, 4)$	1.99581410		
$(2, 1)$	1.79767084	$(4, 3)$	1.93632603	$(7, 0)$	2.85780890		

**Fig. 3.** The waiting time distribution under different values of λ .

Example 2. This example investigates an 4-out-of-7:G repairable system with $\mu = 3.5$, and λ from 0.2 to 1.0. We specify the following three cases of pH vacation time with representation (θ, T) of order 3.

- Case 1. Phase type distribution (PH):

$$\theta = (1, 0, 0), \quad T = \begin{pmatrix} -13 & 8 & 3 \\ 2 & -15 & 10 \\ 1 & 3 & -18 \end{pmatrix},$$

$$T^0 = \begin{pmatrix} 2 \\ 3 \\ 14 \end{pmatrix}, \quad E[V] = 0.1926.$$

- Case 2. Coxian distribution (COX):

$$\theta = (1, 0, 0), \quad T = \begin{pmatrix} -3 & 2 & 0 \\ 0 & -6 & 5 \\ 0 & 0 & -7 \end{pmatrix},$$

$$T^0 = \begin{pmatrix} 1 \\ 1 \\ 7 \end{pmatrix}, \quad E[V] = 0.5238.$$

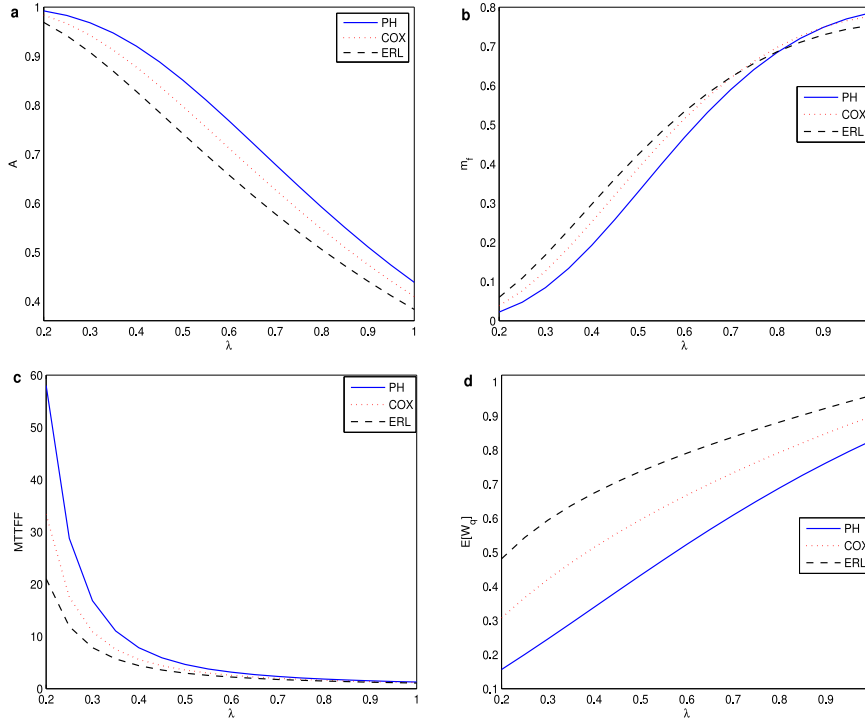


Fig. 4. Different vacation time distributions on the system performance measures.

- Case 3. Erlangian distribution (ERL):

$$\theta = (1, 0, 0), \quad T = \begin{pmatrix} -3.3 & 3.3 & 0 \\ 0 & -3.3 & 3.3 \\ 0 & 0 & -3.3 \end{pmatrix},$$

$$T^0 = \begin{pmatrix} 0 \\ 0 \\ 3.3 \end{pmatrix}, \quad E[V] = 0.9091.$$

The numerical computation results of system performance measures are displayed in Fig. 4(a)–(d). It appears from Fig. 4 that the system availability and the mean time to the first failure are monotonically decreasing function of $E[V]$, and the rate of occurrence of failures and the mean waiting time of failed component go up with the increasing values of $E[V]$. On the other hand, for fixed value of $E[V]$, the availability and the mean time to the first failure decrease with the increase of λ . Fig. 4(b) and (d) show the reverse pattern, namely, the rate of occurrence of failures and the mean waiting time of failed component increase as λ increases. This is because the larger value of λ , the working components are more easily fail.

Example 3. In this numerical example, we select $k = 2$, $n = 3$, $\mu = 5.5$, $\theta = 1$, $T = (-10)$, and λ from 0.2 to 0.8. The Monte Carlo simulation is carried out by 500,000 trials. It appears from Table 3 and Fig. 5 that the numerical results obtained by the matrix analytical method are close to those obtained by the Monte Carlo simulation. Moreover, this example shows that the validity of these theoretical results of system reliability measures in this paper.

Example 4. Let $n = 2$, $k = 1$, $P\{V = 0\} = 1$, thus our model can be reduced to the special case 2-component parallel system. Employing the Markov analysis method, Cao and Cheng [3, pp. 216–219] deduced the transient system reliability measures which are given as follows.

$$A^a(t) = \frac{2\lambda\mu + \mu^2}{2\lambda^2 + 2\lambda\mu + \mu^2} - \frac{2\lambda^2(s_2e^{s_1t} - s_1e^{s_2t})}{s_1s_2(s_1 - s_2)},$$

$$m_f^a(t) = \frac{2\lambda^2\mu}{2\lambda^2 + 2\lambda\mu + \mu^2} + \frac{2\lambda^2(s_1 + \mu)}{s_1(s_1 - s_2)}e^{s_1t} + \frac{2\lambda^2(s_2 + \mu)}{s_2(s_2 - s_1)}e^{s_2t},$$

$$R^a(t) = \frac{c_1e^{c_2t} - c_2e^{c_1t}}{c_1 - c_2},$$

where $s_1, s_2 = \frac{1}{2}[-(3\lambda + 2\mu) \pm \sqrt{\lambda^2 + 4\lambda\mu}]$, $c_1, c_2 = \frac{1}{2}[-(3\lambda + \mu) \pm \sqrt{\lambda^2 + 6\lambda\mu + \mu^2}]$.

To validate the correctness of the analytical results of our model, we select $k = 1$, $n = 2$, $\lambda = 0.9$, $\mu = 2.5$, $\theta = 1$, $T = (-10^3)$. The numerical results presented in Table 4 indicate that the expressions of reliability measures derived in this paper exactly match the one reported by Cao and Cheng [3].

Example 5. Set $k = 1$, $P\{V = 0\} = 1$, then our model reduces to the classical n components parallel repairable system. Applying the Markov analysis method, Cao and Cheng [3, pp. 212–216] derived the steady-state system reliability measures which provided below.

$$A^b = \frac{\sum_{i=1}^n \frac{1}{i!} \left(\frac{\mu}{\lambda}\right)^i}{\sum_{i=0}^n \frac{1}{i!} \left(\frac{\mu}{\lambda}\right)^i}, \quad m_f^b = \frac{\mu}{\sum_{i=0}^n \frac{1}{i!} \left(\frac{\mu}{\lambda}\right)^i},$$

$$MTTF^b = \frac{1}{\mu} \sum_{i=1}^n \sum_{j=0}^{i-1} \frac{j!}{i!} \left(\frac{\mu}{\lambda}\right)^{i-j}.$$

To verify the correctness of the formulae obtained in the present paper, we choose $k = 1$, $\lambda = 0.85$, $\mu = 2.0$, $\theta = 1$, $T = (-10^3)$. The computation results tabulated in Table 5 show that the formulae obtained in this paper exactly agree with that given in Cao and Cheng [3].

9. Conclusions

This paper discussed a repairable k -out-of- n :G system with CO rule and repairman's single vacation, the duration of which follows

Table 3
Reliability measures for different values of λ .

λ	A			m_f		
	Analytical	Simulation	Absolute error	Analytical	Simulation	Absolute error
0.20	0.99232461	0.99276747	0.00044286	0.04009295	0.03863633	0.00145662
0.25	0.98814412	0.98919622	0.00105210	0.06118701	0.05808302	0.00310399
0.30	0.98314101	0.98458279	0.00144179	0.08598835	0.08096614	0.00502221
0.35	0.97736308	0.97960425	0.00224118	0.11413672	0.10650742	0.00762931
0.40	0.97086038	0.97417715	0.00331677	0.14527884	0.13399007	0.01128877
0.45	0.96368439	0.96763623	0.00395183	0.17907155	0.16574597	0.01332558
0.50	0.95588727	0.96067108	0.00478381	0.21518442	0.19658309	0.01860134
0.55	0.94752115	0.95414978	0.00662864	0.25330183	0.22831478	0.02498705
0.60	0.93863762	0.94680956	0.00817194	0.29312446	0.26303828	0.03008618
0.65	0.92928726	0.93863565	0.00934839	0.33437037	0.29777606	0.03659431
0.70	0.91951922	0.92987761	0.01035839	0.37677571	0.33349949	0.04327621
0.75	0.90938092	0.92187423	0.01249331	0.42009499	0.36926743	0.05082757
0.80	0.89891782	0.91358741	0.01466959	0.46410122	0.40393407	0.06016714

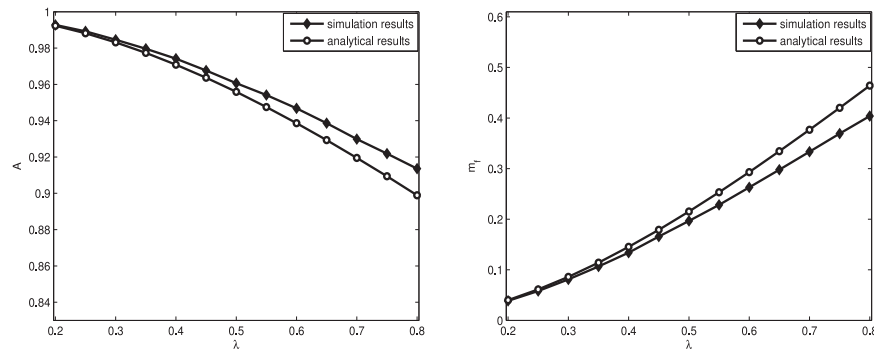


Fig. 5. System reliability measures for different values of λ .

Table 4
Illustration of transient system reliability measures with time t .

	$A(t)$	$A^a(t)$	$m_f(t)$	$m_f^a(t)$	$R(t)$	$R^a(t)$
$t=0.5$	0.93494618	0.93495468	0.29326997	0.32445578	0.90239092	0.90240148
$t=1.0$	0.89166465	0.89168549	0.32124910	0.33115186	0.76890619	0.76894079
$t=1.5$	0.87634720	0.87637398	0.32580197	0.32891352	0.65142769	0.65148258
$t=2.0$	0.87135768	0.87138669	0.32696320	0.32790735	0.55157217	0.55164088
$t=3.0$	0.86924738	0.86927747	0.32742519	0.32745673	0.39538359	0.39546483
$t=5.0$	0.86901025	0.86904048	0.32747646	0.32740555	0.20316293	0.20323756
$t=10.0$	0.86900777	0.86903799	0.32747699	0.32740501	0.03845112	0.03848081
$t=15.0$	0.86900777	0.86903799	0.32747699	0.32740501	0.00727735	0.00728592

Table 5
System reliability measures for different values of n .

n	A	A^b	m_f	m_f^b	MTTF	MTTF ^b
2	0.83663061	0.83663086	0.33035367	0.32673827	3.11268564	3.14878893
3	0.87940484	0.87940500	0.24418102	0.24118999	5.01461489	5.08786213
4	0.89549951	0.89549961	0.21074691	0.20900079	6.45158206	6.52261108
5	0.90167483	0.90167488	0.19750714	0.19665023	7.38346237	7.43308117
6	0.90390183	0.90390184	0.19255636	0.19219630	7.95671879	7.98620669
7	0.90462792	0.90462793	0.19087439	0.19074415	8.32340445	8.34019847
8	0.90483939	0.90483940	0.19036219	0.19032121	8.58126407	8.59137252
9	0.90489453	0.90489453	0.19022231	0.19021095	8.78098817	8.78775789
10	0.90490749	0.90490749	0.19018783	0.19018502	8.94660062	8.95161327
11	0.90491026	0.90491026	0.19018011	0.19017948	9.08964155	9.09361443

a pH distribution. Utilizing the Markov process theory and the matrix analytical method, a variety of system performance measures are derived in transient regime and in stationary regime. Further, some numerical examples are given to illustrate how the various system parameters influence the behavior of the system. Employing the theory of pH distribution, we deduced the well-structured matrix expressions of the system reliability, the mean time to the first failure, and the waiting time of an arbitrary failed compo-

nent. Compared with previous theoretical results, our expressions are more concise and numerically tractable.

In the future, an interesting extension is to study such a system with repairman's vacation policy and non-identical components subject to repair priorities. Once a working component fails, it is immediately replaced and exchanged with a good component taken from the spare parts inventory. The failed components are sent to a central repair center for repair considering priority rule.

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