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**Article**

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# Portfolio performance evaluation in Mean-CVaR framework: A comparison with non-parametric methods value at risk in Mean-VaR analysis



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## ABSTRACT

As we know, there is a belief in the finance literature that Value at Risk (VaR) and Conditional Value at Risk (CVaR) are new approaches to manage and control the risk. Regard to, value at risk is not a coherent risk measure and it is not sub-additive and convex, so, we have considered conditional value at risk as a risk measure by different confidence level in the Mean-CVaR and multi objective proportional change Mean-CVaR models and compared these models with our previous mean-VaR models. This paper focuses on the performance evaluation process and portfolios selection by using Data Envelopment Analysis (DEA). Conventional DEA models assume non-negative values for inputs and outputs, but many of data take the negative value. Therefore, we have used our models based on Range Directional Measure (RDM) that can take positive and negative values. Here value at risk is obtained by non-parametric methods such as historical simulation and Monte Carlo simulation. Finally, a numerical example in Iran's market is presented.

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## 1. Introduction

Portfolio selection and portfolio management are the most important problems from the past that have attracted the attention of investors. To solve this problems, Markowitz [14] proposed his model that was named Markowitz or mean-variance (MV) model. He believed that all investors want maximum return and minimum risk in their investment. This model results in an area with a frontier called efficient frontier. Morey and Morey [15] proposed mean-variance framework based on Data Envelopment Analysis, in which variance of the portfolio is used as an input and expected return is used as an output to DEA models. Data Envelopment Analysis has proved the efficiency for assessing the relative efficiency of Decision Making Units (DMUs) that employs multiple inputs to produce multiple outputs [8]. Briec et al. [7] tried to project points in a preferred direction on efficient frontier and evaluate points' efficiencies by their distances. Demonstrated model by Briec et al., which is also known as a shortage function, has some advantages. For example optimization can be done in

any direction of a mean-variance space according to the investors' ideal. Furthermore, in shortage function, efficiency of each security is defined as the distance between the asset and its projection in a pre-assumed direction. Based on this definition if distance equals to zero, that security is on the frontier area and its efficiency equals to 1. This number, in fact is the result of shortage function which tries to summarize value of efficiency by a number. In those papers, variance was considered as a risk measure. Measures of risk have a crucial role in optimization under uncertainty, especially in coping with the losses that might be incurred in finance of the insurance industry. value at risk, or VaR for short, is one of the most popular measures due to its simplicity, which has achieved the high status of being written into industry regulations. But this risk measure is not sub-additive, nor convex. This risk measure is proposed by Baumol [6]. Duffie and Pan [10] used VaR to measure the risk of firms. Glasserman et al. [12] use the Monte Carlo method along with quadratic estimation to measure the portfolio's VaR. Chen and Tang [9] verified other nonparametric approximation of VaR for related financial returns. Artzner et al. [1,2] proposed the main properties that a risk measures must satisfy, thus establishing the notion of coherent risk measure. Rockfeller and Uryasev [20,21], expressed another risk measure which was named Conditional Value at Risk (CVaR). CVaR is also

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called Expected shortfall (ES), Average Value at Risk (AVaR) and expected tail loss (ETL). CVaR is defined as the weighted average of VaR and losses strictly exceeding VaR for general distribution. The CVaR risk measure has been proved to be a coherent risk measure (Pflug [18], Ogryczak and Ruszczyński [16]) and researcher use CVaR as a risk measure for portfolio and financial problems. Hong and Liu [13] used the Monte Carlo simulation method to calculate CVaR for portfolio optimization. The group of fully non parametric estimators based on the empirical conditional quantile function are considered in Peracchi and Tanase [17]. For using DEA models, must defined inputs and outputs (For example risk can be considered as input and return as output). Majority of DEA models cannot be used for the case in which DMUs include both negative and positive inputs/outputs. For example, CCR model (Charnes, Cooper, Rhodes [8]) and BCC model (Banker, Charnes, Cooper, [5]). Portela et al. [19] represented a DEA model by name Range Directional Measure (RDM) model which can be used in cases where input/output data take positive and negative values. In 2014, [4] proposed non-linear mean-variance and modified mean-variance-skewness based on RDM model for portfolio performance evaluation. In 2016, regard to this subject that variance is not a good measure of risk, they replaced variance by value at risk and tried to decrease it in a mean-value at risk framework with negative data by using mean-value at risk efficiency (MVE) model and multi objective mean-value at risk (MOMV) model. Investors have different attitudes, but always the main concern of investors are the return and risk. Maximizing return and minimizing risk are two opposite target that the investor wants to focus on both of them at the same time. To achieve compromise solutions in this context, the Multi Objective Decision Making (MODM) models are used. Markowitz [14] was the first one who expressed the portfolio management as a MODM problem with two objectives (return and risk). Zopounidis [25] was the researcher who has noted the multidimensional nature of financial decisions and considered all relevant factors involved. Steuer and Paul [23] augmented a general review of MODM for financial problems. Subbu et al. [24] proposed a model that maximizes the return and minimizes the variance and VaR of portfolio. In this paper, regard to VaR is a very popular risk measure but it is not a coherent risk measure and it has undesirable mathematical characteristic such as a lack of sub-additivity and convexity, we proposed Mean-CVaR model and multi objective proportional change Mean-CVaR by mean of return as output and risk measure CVaR as input. we used MODM to maximize the mean of return and minimize the risk measure of CVaR.

The remainder of this paper is organized as follows. Mathematical definitions and formulations are explained in Section 2. Methodology is described in Section 3. The experimental testing of methodology and comparing the different results of different risk measures in Iran's market is represented in Section 4. The conclusion is represented in Section 5.

## 2. Mathematical definition and formulation

### 2.1. Mean-variance models consist of Markowitz model, Morey and Morey model and Brier model

First portfolio theory for investing was published by Markowitz [14]. The model he introduced, was known as Markowitz or mean-variance (MV) model, tries to decrease variance as a risk parameter in all levels of mean. This model results in an area with a frontier calling efficient frontier. Assume that,  $n$  is the number of total assets,  $r_{R_i R_j}$   $\sigma_{R_i} \sigma_{R_j}$  is the covariance between returns of asset  $i$  and  $j$ ,  $\mu_i$  is the expected return of asset  $i$  and  $R_f$  is the riskless return. The decision variable  $\lambda_i$  represents the proportion of capital to be invested in asset  $i$ . The MV model or Markowitz

model is description as follow:

$$\begin{aligned} \text{Min } z &= \sum_{i=1}^n \sum_{j=1}^n \lambda_i \lambda_j r_{R_i R_j} \sigma_{R_i} \sigma_{R_j} \\ \text{s.t. } \sum_{j=1}^n \lambda_j \mu_j &\geq R_f \\ \sum_{j=1}^n \lambda_j &= 1 \\ 0 \leq \lambda_j &\leq 1, \quad j = 1, 2, \dots, n \end{aligned} \tag{1}$$

The objective is finding a portfolio with the minimum standard deviation (as a risk) under the situation that the corresponding expected return must be greater than riskless return ( $R_f$ ). The sum of the proportions of capital allocated to all stocks must be equal to 1 and they should be in the range of [0, 1].

Based on Markowitz [14] theory, it is required to characterize the whole efficient frontier, which for large number of assets is cumbersome. In contrast Morey and Morey [15] measured efficiency of under evaluation assets through DEA models. Data Envelopment Analysis (DEA) is a nonparametric method for evaluating the efficiency of systems with multiple inputs or outputs. In this section we present, not discussing in details, some basic definitions, models and concepts that will be used in other sections. Consider DMU <sub>$j$</sub>  ( $j = 1, \dots, n$ ) where each DMU consumes  $m$  inputs to produce  $s$  outputs. Also, Suppose that the observed input and output vectors of DMU <sub>$j$</sub>  are  $X_j = (x_{1j}, \dots, x_{mj})$  and  $Y_j = (y_{1j}, \dots, y_{sj})$  respectively, and let  $X_j \geq 0$ ,  $X_j \neq 0$  and  $Y_j \geq 0$ ,  $Y_j \neq 0$ . A basic DEA formulation in input orientation is as follows:

$$\begin{aligned} \text{min } \theta - \varepsilon & \left( \sum_{r=1}^s s_r^+ + \sum_{i=1}^m s_i^- \right) \\ \text{s.t. } \sum_{j=1}^n \lambda_j x_{ij} + s_i^- &= \theta x_{io} \quad i = 1, \dots, m, \\ \sum_{j=1}^n \lambda_j y_{rj} - s_r^+ &= y_{ro} \quad r = 1, \dots, s, \\ \lambda &\in \Lambda, \\ s^+, s^- &\geq 0, \\ \varepsilon &\geq 0 \end{aligned} \tag{2}$$

where  $\lambda$  is a  $n$ -vector of variables  $\lambda_i$ ,  $s^+$  is a  $s$ -vector of output slacks,  $s^-$  is a  $m$ -vector of input slacks,  $\varepsilon$  is a non-Archimedes factor, and the set  $\Lambda$  is defined as follows:

$$\Lambda = \begin{cases} \lambda \in R_+^n & \text{with constant returns to scale,} \\ \lambda \in R_+^n, 1^T \lambda \leq 1 & \text{with non - increasing returns to scale,} \\ \lambda \in R_+^n, 1^T \lambda = 1 & \text{with variable returns to scale.} \end{cases} \tag{3}$$

Note that subscript 'o' refers to the under evaluation unit. A Decision Making Unit (DMU) is efficient if and only if  $\theta = 1$  and all slack variables ( $s^+$  and  $s^-$ ) are equal to zero otherwise it is inefficient, [8]. In the DEA formulation (5), the left-hand-sides of constraints define an efficient unit, while, the scalars in the right-hand sides are the inputs and outputs of the under evaluation unit and the theta is a multiplier that defines the distance from the efficient frontier. The slack variables are also used to ensure that the efficient points are fully efficient.

In solving DEA models three different attitudes can be considered. DEA models can be input, output or combined oriented, where, each orientation has its own interpretation in financial fields. In recent years following models have been widely used to evaluate portfolio efficiency that have DEA-like framework.

Morey and Morey [15] used this model to measure efficiency of under evaluation assets only by characterizing projection points. Based on this model, efficiency measure of an asset ( $\theta$ ) is the distance between an asset and its projection. In Morey and Morey [15] model, there are  $n$  assets, and  $\lambda_j$  is the weight of asset  $j$  in the projection point.  $r_j$  is the expected return of asset  $j$ .  $s_1$  is a  $s$ -vector of output slacks and  $s_2$  is a  $m$ -vector of input slacks. Also,  $\varepsilon$  is a non-Archimedes factor and  $\mu_o$  and  $\sigma_o^2$  are expected return and variance of under evaluation asset respectively. Efficiency measure ( $\theta$ ) can be determined by following model:

$$\begin{aligned} \min \quad & \theta - \varepsilon(s_1 + s_2) \\ \text{s.t.} \quad & E \left[ \sum_{j=1}^n \lambda_j r_j \right] - s_1 = \mu_o, \\ & E \left[ \left( \sum_{j=1}^n \lambda_j (r_j - \mu_j) \right)^2 \right] + s_2 = \theta \sigma_o^2, \\ & \sum_{j=1}^n \lambda_j \leq 1 \quad \forall \lambda \geq 0 \end{aligned} \tag{4}$$

Model (4) is achieved by the non-parametric efficiency analysis Data Envelopment Analysis. Bricc et al. [7] used direction in optimization. They tried to project the under evaluation assets on the efficient frontier via maximizing return and minimizing variance simultaneously in the direction of the vector  $g = (|\mu_o|, -|\sigma_o^2|)$  using the following model:

$$\begin{aligned} \max \quad & \beta \\ \text{s.t.} \quad & E \left[ \sum_{j=1}^n \lambda_j r_j \right] \geq \mu_o + \beta \mu_o, \\ & \text{var}[r(\lambda)] \leq \sigma_o^2 - \beta \sigma_o^2, \\ & \sum_{j=1}^n \lambda_j = 1 \quad \lambda \geq 0. \end{aligned} \tag{5}$$

Where  $\text{var}[r(\lambda)]$  can be defined as following:

$$\text{var}[r(\lambda)] = E[(r(\lambda) - E[r(\lambda)])^2] = \sum_{i,j=1}^n \lambda_i \lambda_j \Omega_{ij}. \tag{6}$$

When model (5) equals zero, the under evaluation unit is on the efficient frontier. In Eq. (6)  $n$  is number of assets in the portfolio.  $\lambda_j$  is proportion of portfolio's initial value invested in asset  $j$  and  $\lambda$  is a  $n$ -vector of variables  $\lambda_j$ . Also  $r_j$  is return of asset  $j$  and  $\Omega_{ij}$  is covariance of returns between asset  $i$  and asset  $j$ .

### 2.2. Range Directional Measure (RDM) model

In the conventional DEA models, each DMU $_j$  ( $j = 1, \dots, n$ ) is specified by a pair of non-negative input and output vectors  $(x_i, y_j) \in R_+^{m+s}$ , in which inputs  $x_{ij}$  ( $i = 1, \dots, m$ ) are utilized to produce outputs,  $y_{rj}$  ( $r = 1, \dots, s$ ). These models cannot be used for the cases in which DMUs include both negative and positive inputs and/or outputs. Portela et al. [19] considered a DEA model which can be applied in cases where input/output data take positive and negative values. Range Directional Measure (RDM) model proposed by Portela et al. [19] is as follow:

$$\begin{aligned} \text{Max} \quad & \beta \\ \text{s.t.} \quad & \sum_{j=1}^n \lambda_j x_{ij} \leq x_{io} - \beta R_{io} \quad i = 1, \dots, m \\ & \sum_{j=1}^n \lambda_j y_{rj} \geq y_{ro} + \beta R_{ro} \quad r = 1, \dots, s \end{aligned}$$

$$\begin{aligned} \sum_{j=1}^n \lambda_j &= 1 \\ \lambda_j &\geq 0 \quad j = 1, \dots, n \end{aligned} \tag{7}$$

Where, directions can be defined as following:

$$R_{io} = x_{io} - \min_j \{x_{ij} : j = 1, \dots, n\}, \quad i = 1, \dots, m \tag{8}$$

$$R_{ro} = \max_j \{y_{rj} : j = 1, \dots, n\} - y_{ro}, \quad r = 1, \dots, s \tag{9}$$

Ideal point (I) within the attendance of negative data is:

$$I = \left( \max_j \{y_{rj} : r = 1, \dots, s\}, \min_j \{x_{ij} : i = 1, \dots, m\} \right) \tag{10}$$

and the purpose is to project each under evaluation asset's point to this ideal point. Other models that use negative data are modified slacks-based measure model (MSBM), Emrouznejad [11] and semi-oriented radial measure (SORM), Sharp et al. [22].

In the later sections, the measure of risk consists of Value at Risk (VaR) and Conditional Value at Risk (CVaR) are defined.

### 2.3. Value at Risk (VaR)

VaR is defined as maximum quantity of invest that one may loss in a specified time interval. In the other words, VaR can answer this question: how much one can expect to loss in the specified time (a day, week, month, ...). VaR defined as the quantile of a distribution. Suppose that  $P_t$  is the initial wealth and  $P_{t+k}$  is the Secondary wealth after  $k$  period time, probability of loss is:

$$p(-\Delta_k P_t < VaR) = \alpha \tag{11}$$

where  $\Delta_k P_t = P_{t+k} - P_t$  and  $1 - \alpha$  is the margin of error so  $\alpha$  is the confidence level.

There are different methods for computing the VaR, such as Variance-Covariance method, Historical simulation and Monte Carlo simulation. Variance-covariance method only uses for normal distribution data. Since the price of stock have not normal distribution, so we cannot use this method for calculating the VaR. There is no need for normal distribution data in Historical simulation and Monte Carlo simulation methods, thus we can use these methods for computing the VaR.

#### 2.3.1. Historical simulation

One of nonparametric methods for calculating the VaR is Historical simulation. In this method there is no need to know distribution of data. In fact, VaR is computed by attention of an assumptive time series of returns and supposition that changes of future data are based on historical changes. The convenience of this method is no variance and covariance need to calculate. This method believes that behavior of returns is the same as before.

#### 2.3.2. Monte Carlo simulation

Another nonparametric method for calculating the VaR is Monte Carlo simulation. This method is based on stronger supposition about distribution of returns in comparison with historical simulation method. This method specifies possibility distribution of returns. First distribution most determines, then a lot of samples of returns will simulate and parameters will calculate based on those samples.

For using Monte Carlo method to calculate the VaR, distribution of stock companies must be known. Because of the fluctuations of stock price, it is hard to obtain distributions. Thus, we used sampling methods. First, we specified the margin of error and number of needed samples that is shown the whole population. Then, we used boot strapping method. We repeat sampling procedure for

1000 times and calculate VaR of each stock companies. At the end, the  $\overline{VaR}$  is:

$$\overline{VaR} = \frac{1}{1000} \sum_{i=1}^{1000} VaR_i \tag{12}$$

Where  $VaR_i$  is the value at risk of stock company  $i$  and  $\overline{VaR}$  is the estimate of value at risk of population.

**Definition 2.1.** Assume that a portfolio is going to be selected from  $n$  financial assets,  $\lambda_i$  is the proportion of invested money in asset  $i$ . The set of our acceptable portfolios is:

$$\phi = \left\{ \lambda_i \in \mathbb{R}^n; \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0 \right\} \tag{13}$$

Return of portfolio  $r(\lambda)$  is:

$$r(\lambda) = \sum_{i=1}^n \lambda_i r_i \tag{14}$$

Expected return of this portfolio is:

$$E(r(\lambda)) = \sum_{i=1}^n \lambda_i E(r_i) \tag{15}$$

2.4. Conditional Value at Risk (CVaR)

Let  $\lambda \in \phi \subset \mathbb{R}^n$  be a decision vector,  $r \in \mathbb{R}^n$  be the random vector representing the value of under lying risk factors, and  $f(\lambda, r)$  be the corresponding loss. For simplicity, we assume that  $r \in \mathbb{R}^n$  is a continuous random vector. For a given portfolio  $\lambda$ , the probability of the loss not exceeding a threshold  $\Gamma$  is given by the probability function  $\mathbb{P}(\cdot)$

$$\psi(\lambda, \Gamma) := \mathbb{P}(f(\lambda, r) \leq \Gamma) \tag{16}$$

The VaR associated with a portfolio  $\lambda$  and a specified confidence level  $\alpha$  ( $0 < \alpha < 1$ ) is the minimal  $\Gamma$  satisfying  $(\lambda, \Gamma) \geq \alpha$ , that is:

$$VaR_\alpha(\lambda) := \inf\{\Gamma \in \mathbb{R}, \psi(\lambda, \Gamma) \geq \alpha\} \tag{17}$$

Since  $\psi(\lambda, \Gamma)$  is continuous by assumption, we have:

$$\mathbb{P}(f(\lambda, r) \leq VaR_\alpha(\lambda)) = \psi(\lambda, VaR_\alpha(\lambda)) = \alpha \tag{18}$$

CVaR is defined as the conditional expectation of the portfolio loss exceeding or equal to VaR

$$CVaR_\alpha(\lambda) := E[f(\lambda, r) | f(\lambda, r) \geq VaR_\alpha(\lambda)] \\ = \frac{1}{1-\alpha} \int_{VaR_\alpha(\lambda)}^{+\infty} xp(x)dx \tag{19}$$

where  $E$  is the expectation operator and  $p(x)$  is the probability density function of the loss  $f(\lambda, r)$ .

Rockfeller and Uryasev [20,21] prove that CVaR has an equivalent definition as follows:

$$CVaR_\alpha(\lambda) = \min_{\Gamma} F_\alpha(\lambda, \Gamma) \tag{20}$$

where  $F_\alpha(\lambda, \Gamma)$  is defined as:

$$F_\alpha(\lambda, \Gamma) := \Gamma + \frac{1}{1-\alpha} E[(f(\lambda, r) - \Gamma)^+] \tag{21}$$

with  $(x)^+ = \max\{x, 0\}$ . They also show that minimizing CVaR over  $\lambda \in \phi \subset \mathbb{R}^n$  is equivalent to minimizing  $F_\alpha(\lambda, \Gamma)$  over  $(\lambda, \Gamma) \in \phi \times \mathbb{R}$ , i.e.,

$$\min_{\lambda \in \phi} CVaR_\alpha(\lambda) = \min_{(\lambda, \Gamma) \in \phi \times \mathbb{R}} F_\alpha(\lambda, \Gamma). \tag{22}$$

Furthermore, when  $\phi$  is a convex set and  $f(\lambda, r)$  is convex with respect to  $\lambda$ , the problem is a convex programming problem.

**Definition 2.2.** Weakly efficient frontier described as:

$$fr^w(\phi) = \{(\mu, R) \in S; (-\mu', CVaR') \\ < (-\mu, CVaR) \Rightarrow (\mu', CVaR') \notin S\} \tag{23}$$

This frontier is a part of the boundary of the disposal region set ( $S$ ). The weakly frontier can contain points that are not reachable by real portfolios.

**Definition 2.3.** Strongly efficient frontier described as:

$$fr^s(\phi) = \{(\mu, CVaR) \in S; (-\mu', CVaR') \\ \leq (-\mu, CVaR) \text{ and } (-\mu', CVaR') \neq (-\mu, CVaR) \\ \Rightarrow (\mu', CVaR') \notin S\} \tag{24}$$

In Definition 2.2 and 2.3,  $\mu$  and  $CVaR$  are expected return (mean) and risk measure of a point in disposal region. Similarly,  $\mu'$  and  $CVaR'$  are expected return (mean) and risk measure of an optional point in Mean-CVaR space. As we know, weakly efficient frontier is included in the strongly efficient frontier

2.5. Non-linear mean-variance RDM model and Mean-VaR model

In the Last papers, we have proposed two models based on RDM model. Banihashemi et al. [3] have presented following non-linear mean-variance RDM model on the negative data. Let

$$g = (R_{\mu_o}, R_{\delta^2_o}) \in [0, +\infty) \times [0, +\infty) \tag{25}$$

be a vector that shows direction in which  $\beta$  is going to be maximized. Non-linear mean-variance RDM model is defined as:

$$\xi : \mathbb{R}^2 \rightarrow (0, 1], \\ \xi(y) = \sup\{ \beta; y + \beta g \in S | \beta \in \mathbb{R}_+ \}. \tag{26}$$

Based on vector  $g$ , definition and mentioned set of  $\beta$ , it is obvious that the aim is to increase mean of return and to reduce variance of portfolio as risk of a portfolio in direction of vector  $g$  simultaneously. One of them should care about directions in interpretation of model while directions affect non-linear mean-variance RDM model. Vector of direction can be chosen as following:

$$g = \left( \begin{array}{l} (\max_j(\mu_j : j = 1, \dots, n) - \mu_o) = R_{\mu_o} \\ ([\delta^2_o - \min(\delta^2_j : j = 1, \dots, n)]) = R_{\delta^2_o} \end{array} \right). \tag{27}$$

**Definition 2.4.** Consider a vector with specified direction  $g = (R_{\mu_o}, R_{\delta^2_o})$  and an under evaluation asset  $y, y = (\mu_o, \delta^2_o)$ , the non-linear mean-variance RDM model is description as follow:

$$\text{Max } \beta \\ \text{s.t. } E(r(\lambda)) \geq \mu_o + \beta R_{\mu_o} \\ \text{var}[r(\lambda)] \leq \sigma_o^2 - \beta \sigma_o^2, \\ \sum_{j=1}^n \lambda_j = 1 \\ \beta \geq 0, \quad 0 \leq \lambda_j \leq 1 \quad j = 1, \dots, n \tag{28}$$

In 2016, regard to this subject that variance is not a good measure of risk, Banihashemi et al. replaced variance by value at risk and tried to decrease it in a mean-value at risk framework with negative data by using mean-value at risk efficiency (MVE) model and multi objective mean-value at risk (MOVm) model. The MVE model can be obtained through solving following linear model:



$$\begin{aligned}
 & \max \beta \\
 & \text{s.t. } E[r(\lambda)] \geq \mu_o + \beta R_{\mu_o}, \\
 & \quad \text{VaR}[r(\lambda)] \leq \text{VaR}_o + \beta R_{\text{VaR}_o}, \\
 & \quad \sum_{i=1}^n \lambda_j = 1, \\
 & \quad \beta \geq 0, 0 \leq \lambda_j \leq 1 \quad \text{for } j \in \{1, \dots, n\}.
 \end{aligned} \tag{29}$$

Also MOMV model can be obtained through solving following linear model:

$$\begin{aligned}
 & \max \frac{1}{2}(\beta_1 + \beta_2) \\
 & \text{s.t. } E[r(\lambda)] \geq \mu_o + \beta_1 R_{\mu_o}, \\
 & \quad \text{VaR}[r(\lambda)] \leq \text{VaR}_o + \beta_2 R_{\text{VaR}_o}, \\
 & \quad \sum_{j=1}^n \lambda_j = 1, \\
 & \quad \beta_1, \beta_2 \geq 0, 0 \leq \lambda_j \leq 1 \quad \text{for } j \in \{1, \dots, n\}.
 \end{aligned} \tag{30}$$

In the later section we have proposed Mean-CVaR and multi objective proportional change Mean-CVaR models.

### 3. Modeling and analyzing MVaR and MCVaR portfolio

Based on the RDM model provided by Portela et al. [19], we propose the Mean-CVaR model and the multi objective proportional change Mean-CVaR model. After using our proposed models, the efficient stock companies will select for making the portfolio. let

$$\begin{aligned}
 & g = (R_{\mu_o}, R_{\text{CVaR}_o}) \in [0, +\infty) \times [0, +\infty) \\
 & \mathbf{g} = \left( \begin{array}{l} (\max_j(\mu_j : j = 1, \dots, n) - \mu_o) = R_{\mu_o} \\ ([\text{CVaR}_o^2 - \min(\text{CVaR}_j^2 : j = 1, \dots, n)]) = R_{\text{CVaR}_o^2} \end{array} \right)
 \end{aligned} \tag{31}$$

be a vector shows direction in which  $\beta$  is going to be maximized.

**Definition 3.1.** Consider a vector with specified direction  $g = (R_{\mu_o}, R_{\text{CVaR}_o})$  and an under evaluation asset  $= (\mu_o, \text{CVaR}_o)$ , the Mean-CVaR model is:

$$\begin{aligned}
 & \text{Max } \beta \\
 & \text{s.t. } E(r(\lambda)) \geq \mu_o + \beta R_{\mu_o} \\
 & \quad \text{CVaR}(r(\lambda)) \leq \text{CVaR}_o + \beta R_{\text{CVaR}_o}, \\
 & \quad \sum_{i=1}^n \lambda_i = 1 \\
 & \quad \beta \geq 0, 0 \leq \lambda_i \leq 1 \quad i = 1, \dots, n
 \end{aligned} \tag{32}$$

The efficient projected point in the direction of vector  $g$  is the point in Mean-CVaR space with coordinates determined by the right-hand sides of the inequality constraints of above model evaluated at the optimal solution (i.e.,  $(\mu_o + \beta^* R_{\mu_o}, \text{CVaR}_o + \beta^* R_{\text{CVaR}_o})$ ). Mechanism of the Mean-CVaR model is just like the RDM model. When amount of  $\beta$  for under evaluation asset equals to zero, it will be understood that this asset is efficient and Mean-CVaR point is part of the weakly efficient frontier. Otherwise, as can be seen from the right-hand-sides of the inequality constraints of the above model, the optimal  $\beta$  indicates a change in mean of return and risk measure CVaR that results in a projection of the evaluated Mean-CVaR point onto the weakly efficient frontier. In the other words,  $1 - \beta$  is amount of the efficiency. The Mean-CVaR model seeks simultaneously to improve mean of return and to reduce risk in the direction of the vector  $g$ . The use of this model guarantees that a projected Mean-CVaR point is part of the weakly efficient subset. To ensure that the projection of

the Mean-CVaR point is part of the strongly efficient subset, one should change proportional in all dimension. Therefore, we should introduce another model that projects the point proportionally.

**Definition 3.2.** Consider a vector with specified direction  $g = (R_{\mu_o}, R_{\text{CVaR}_o})$  and an under evaluation asset  $DMU_o = (\mu_o, \text{CVaR}_o)$ , by using multi objective function for the Mean-CVaR model, the model will be:

$$\begin{aligned}
 & \text{Max } (\beta_1, \beta_2) \\
 & \text{s.t. } E(r(\lambda)) \geq \mu_o + \beta_1 R_{\mu_o} \\
 & \quad \text{CVaR}(r(\lambda)) \leq \text{CVaR}_o + \beta_2 R_{\text{CVaR}_o}, \\
 & \quad \sum_{i=1}^n \lambda_i = 1 \\
 & \quad \beta_1, \beta_2 \geq 0, 0 \leq \lambda_i \leq 1 \quad i = 1, \dots, n
 \end{aligned} \tag{33}$$

Multi objective proportional change functions are more flexible than single objective functions in determination of optimal directions. Multi objective functions try to maximize the average of objects (because of having more than one parameter to maximize). If the multi objective proportional change model equals zero, then the Mean-CVaR point is part of the strongly efficient frontier. If it is nonzero, then the optimal  $\beta_i$  indicate the proportional change per expected return, CVaR that guaranties a projection of the evaluated Mean-CVaR point onto the strongly efficient frontier. Multi objective functions in here try to maximize  $\beta$  in directions of mean and risk measure separately. Mechanism of the multi objective proportional change Mean-CVaR model is just like the Mean-CVaR model. When amount of  $\beta$  for the under evaluation asset equals to zero, means that the under evaluation asset is efficient. In the other words,  $1 - \beta$  is amount of the efficiency. But there is a fundamental difference between these models; the under evaluation assets which are efficient in the Mean-CVaR model locate on the weakly efficient frontier but the under evaluation assets which are efficient in the multi objective proportional change Mean-CVaR model locate on the strongly efficient frontier. We want to compare different results of these models by using different risk measures. The risk measures are VaR and CVaR. Section 4 includes the practical work and comparing the results.

### 4. Application in Iranian stock companies

#### 4.1. Data collection

The dataset was randomly collected from the stock's price of the 15 Iranian stock companies, from 25/04/2015 till 25/04/2016. Also missing data over holidays estimated through interpolation. The dataset was obtained from <http://www.irvex.ir/index>.

All of the stock companies are shown by company symbol in Table 1.

The price volatility of the stock companies is shown in Fig. 1.

#### 4.2. Constructing the portfolio and calculating the efficiency

In this section a comparison study is conducted to compare models introduced in previous section and Mean-VaR model and multi objective mean-VaR model that were introduced in previous papers. To do this a sample of 15 corporations from Tehran stock is randomly selected. Efficiency of each asset is going to be evaluated and methods of computing efficiencies compared. The software Matlab was used to calculate value at risk and conditional value at risk stock companies. In these Table 2 reveals input and output that include amounts of expected return, value at risk which has calculated by historical simulation and Monte Carlo simulation methods, or conditional value at risk as risk measures for the stock companies.

**Table 1**  
Symbol of the stock companies that were used.

Company symbol	Company symbol	Company symbol	Company symbol	Company symbol
CONT1	NAFT1	TRIR1	RENA1	PSIR1
DJBR1	SHND1	TRNS1	GHAT1	KRTI1
DSIN1	KHAZ1	AZAB1	IPAR1	PASH1

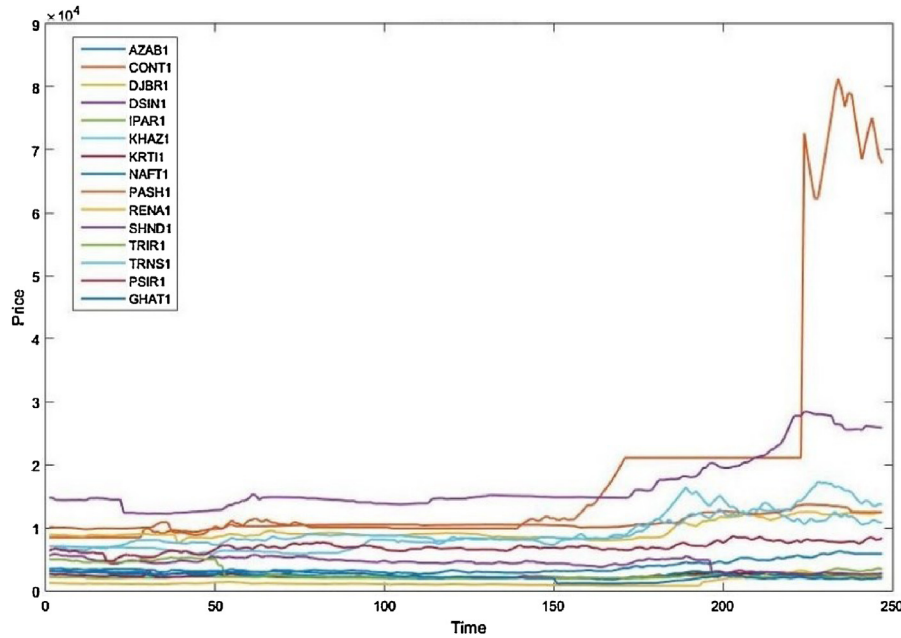


Fig. 1. The price volatility of the stock companies.

**Table 2**  
Input and output consist of expected return and value at risk or conditional value at risk as risk measures.

Number of asset	Stock companies	Expected return	Value at risk historical simulation			Value at risk Monte Carlo simulation			Conditional value at risk (CVaR)		
			%90	%95	%95	%90	%95	%99	%90	%95	%99
1	AZAB1	0.0026	0.0315	0.0392	0.0469	0.0285	0.0352	0.0374	0.0392	0.0430	0.0476
2	CONT1	0.0085	0.0225	0.0364	0.0513	0.0195	0.0310	0.0348	0.0361	0.0452	0.0513
3	DJBR1	0.0013	0.0074	0.0137	0.0315	0.0088	0.0222	0.0267	0.0231	0.0348	0.0901
4	DSIN1	0.0023	0.0046	0.0080	0.0500	0.0066	0.0217	0.0267	0.0195	0.0328	0.0941
5	IPAR1	0.0019	0.0101	0.0197	0.0470	0.0115	0.0233	0.0273	0.0265	0.0396	0.0681
6	KHAZ1	0.0017	0.0385	0.0469	0.0503	0.0321	0.0418	0.0451	0.0471	0.0516	0.0653
7	KRTI1	-0.0003	0.0303	0.0442	0.1302	0.0299	0.0557	0.0643	0.0586	0.0802	0.1945
8	NAFT1	-0.0006	0.0364	0.0453	0.0512	0.0321	0.0402	0.0429	0.0455	0.0499	0.0574
9	PASH1	0.0009	0.0040	0.0077	0.0243	0.0046	0.0135	0.0165	0.0150	0.0245	0.0749
10	RENA1	0.0030	0.0339	0.0447	0.0497	0.0320	0.0392	0.0416	0.0433	0.0471	0.0506
11	SHND1	-0.0029	0.0304	0.0393	0.0734	0.0296	0.0723	0.0866	0.0755	0.1163	0.3914
12	TRIR1	-0.0035	0.0220	0.0397	0.0570	0.0260	0.0686	0.0828	0.0680	0.1062	0.3497
13	TRNS1	0.0027	0.0216	0.0343	0.0466	0.0212	0.0302	0.0332	0.0343	0.0422	0.0476
14	PSIR1	0.0011	0.0322	0.0397	0.0499	0.0300	0.0442	0.0489	0.0481	0.599	0.1227
15	GHAT1	-0.0023	0.0353	0.0456	0.1074	0.0326	0.0683	0.0802	0.0717	0.1021	0.3059

The return volatility of the stock companies is shown in Fig. 2. As mentioned before, we have used the Mean-VaR and Mean-CVaR models and the Multi objective proportional change Mean-VaR and multi objective proportional change Mean-CVaR models to calculate the efficiency of the stock companies. The software GAMS was used to measure the relative efficiency of selected stock companies. In these models  $\beta$  shows amount of inefficiency. Therefore, when amount of  $\beta$  for the stock company equal to zero, means that the stock company is efficient. Now based on values at risk in Table 2, and using Mean-VaR model, efficiency of each asset is calculated. Table 3 reveals amount of inefficiency of the stock companies by using Mean-VaR and Mean-CVaR models. Based on data in Table 3, assets 2,11,15 in all levels of historical and Monet Carlo simulation in VaR and all levels of CVaR are efficient. How-

ever, asset 13 in highest level of CVaR is efficient. For all assets it can be interpreted, as the confidence level of risk increases, assets get less amount of efficiency and amount of efficiencies are accurate. Same data is used and efficiency of assets is calculated by using multi objective proportional change models. Results are provided in Table 4. Table 4 reports values of inefficiency ( $\beta$ ). Interpretations are same as before. Based on multi objective proportional change models assets 2,11,15 are efficient. Also same as single objective models in higher levels of confidence, assets get less amount of efficiency and amount of efficiencies are accurate. Also by comparing results of Tables 3 and 4, it can be concluded that results of multi objective proportional change models generally greater than results of single objective models. It is a general characteristic of multi objective proportional change models.

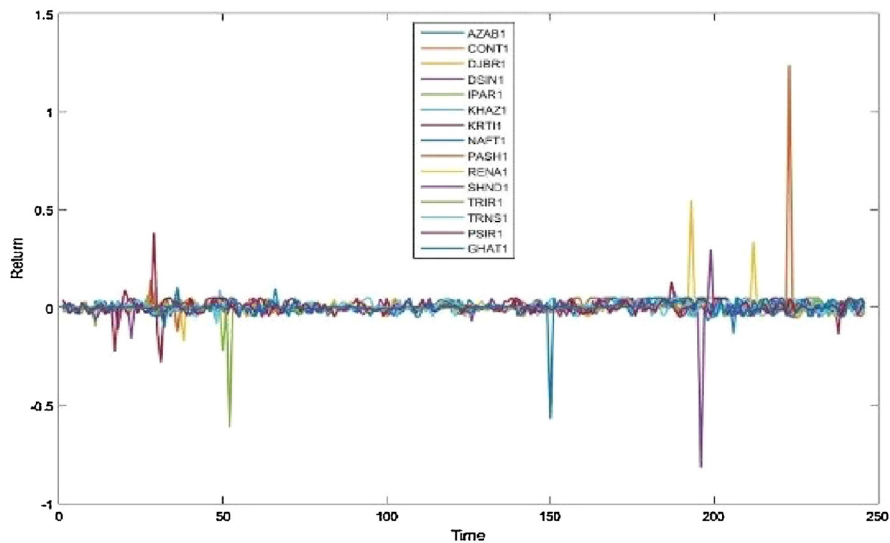


Fig. 2. The Return volatility of the stock companies.

**Table 3**  
Inefficiency of the stock companies by using the Mean-VaR and Mean-CVaR models.

Number of asset	Stock companies	$\beta$ with VaR historical simulation			$\beta$ with VaR Monte Carlo simulation			$\beta$ with CVaR		
		90%	95%	99%	90%	95%	99%	90%	95%	99%
1	AZAB1	0.45	0.36	0.19	0.46	0.35	0.31	0.34	0.20	0.00
2	CONT1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	DJBR1	0.10	0.13	0.00	0.10	0.11	0.15	0.11	0.11	0.32
4	DSIN1	0.00	0.00	0.26	0.00	0.09	0.14	0.00	0.04	0.28
5	IPAR1	0.12	0.16	0.22	0.12	0.12	0.14	0.12	0.16	0.23
6	KHAZ1	0.47	0.40	0.23	0.45	0.36	0.35	0.36	0.32	0.34
7	KRTI1	0.36	0.31	0.55	0.44	0.51	0.52	0.49	0.52	0.54
8	NAFT1	0.48	0.41	0.27	0.47	0.39	0.38	0.38	0.35	0.38
9	PASH1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.18
10	RENA1	0.39	0.28	0.06	0.45	0.34	0.30	0.29	0.16	0.08
11	SHND1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12	TRIR1	0.05	0.17	0.09	0.20	0.35	0.36	0.30	0.35	0.39
13	TRNS1	0.29	0.30	0.20	0.30	0.23	0.22	0.23	0.22	0.00
14	PSIR1	0.40	0.31	0.20	0.41	0.36	0.35	0.35	0.52	0.39
15	GHAT1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

**Table 4**  
Inefficiency of the stock companies by using the multi objective proportional change models.

Number of asset	Stock companies	$\beta$ with VaR historical simulation			$\beta$ with VaR Monte Carlo simulation			$\beta$ with CVaR		
		90 %	95 %	99 %	90 %	95 %	99 %	90 %	95 %	99 %
1	AZAB1	0.52	0.40	0.26	0.55	0.46	0.42	0.42	0.30	0.06
2	CONT1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
3	DJBR1	0.10	0.14	0.00	0.10	0.12	0.15	0.12	0.12	0.56
4	DSIN1	0.00	0.00	0.38	0.00	0.10	0.15	0.00	0.06	0.54
5	IPAR1	0.12	0.18	0.31	0.15	0.12	0.14	0.16	0.22	0.51
6	KHAZ1	0.55	0.49	0.33	0.54	0.52	0.51	0.51	0.47	0.51
7	KRTI1	0.43	0.41	0.57	0.48	0.52	0.53	0.52	0.53	0.54
8	NAFT1	0.56	0.48	0.36	0.57	0.53	0.51	0.51	0.45	0.55
9	PASH1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.53
10	RENA1	0.43	0.37	0.19	0.46	0.38	0.36	0.35	0.27	0.11
11	SHND1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
12	TRIR1	0.20	0.27	0.25	0.30	0.37	0.37	0.35	0.37	0.40
13	TRNS1	0.35	0.34	0.27	0.43	0.35	0.33	0.33	0.29	0.00
14	PSIR1	0.50	0.39	0.31	0.51	0.50	0.50	0.49	0.55	0.52
15	GHAT1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

This means that multi objective proportional change models are accurate because proportional change in input and output.

Table 4 reveals amount of inefficiency of the stock companies by using multi objective proportional change models.

Here we have used the same inputs and outputs for the Mean-VaR and Mean-CVaR models and the multi objective proportional change models. By comparing the results of Tables 3 and 4, we figure out:

- In calculating VaR, the results of Monte Carlo simulation method are more accurate than historical simulation method.
- In calculating VaR and CVaR, the higher confidence levels are more accurate than lower levels.
- CVaR is the most accurate risk measure.



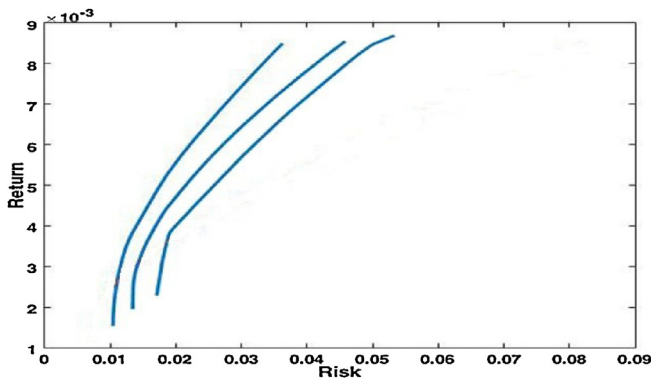


Fig. 3. Mean-CVaR frontiers by different confidence levels.

d. We can derive that results of the multi objective proportional change models are generally better and more accurate than results of the Mean-Risk models.

Since the CVaR is the more accurate than VaR, we compare Mean-CVaR frontiers with different confidence levels.

Frontiers are shown in Fig. 3 in three different confidence. In this figure frontiers are made using conditional value at risk and expected return. Fig. 3 illustrates as the risk's confidence level, increases the whole efficient frontier moves rightward. Therefore, as the confidence level increases, investors get surer about the amount of risk that may face on a predefined level of return. Also, the curve of above segment of efficient frontiers increase by confidence levels increasing. In higher levels of confidence levels, risk of an under evaluation asset is calculated more precisely. In fact, by comparing efficient frontiers in Fig. 3, we find out the higher confidence levels are more accurate than lower levels. This means that, in Fig. 3, 99% confidence level is better than 90% and 95% confidence levels.

## 5. Conclusion

In this paper, we have compared two risk measures such as value at risk (historical simulation and Monte Carlo simulation) and conditional value at risk to find the best one for portfolio optimization. By comparing the results of Tables 3 and 4, we figure out CVaR is the more accurate than VaR. As you see in the figure and the results of tables, we find out the higher confidence levels are more accurate than lower levels. For calculating the efficiency of the stock companies, we must use DEA models. So we used the Mean-Risk model and the multi objective mean-risk proportional change model by different risk measures consist of VaR and CVaR to calculate the efficiency of the stock companies. Multi objective proportional change functions are more accurate, so the general results of the multi objective proportional change Mean-Risk model are generally better than results of the Mean-Risk model. Finally, the method was applied to the Iran's market and the

results were shown in the tables and figure. As it is said, CVaR is more accurate than other risk measures such as, variance, semi variance and VaR because these risk measures except variance are downside risk measures. In this paper, one of Data Envelopment Analysis models named RDM model is considered. In other papers, other models are presented.

For future studies, other risk measure can be compared to find the best one.

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