

Zamani-Sabzi, Hamed; King, James Phillip; Gard, Charlotte C.; Abudu, Shalamu

Article

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Statistical and analytical comparison of multi-criteria decision-making techniques under fuzzy environment



Hamed Zamani-Sabzi^{a,*}, James Phillip King^b, Charlotte C. Gard^c, Shalamu Abudu^d

^a Department of Civil Engineering, New Mexico State University, MSC 3CE, PO Box 30001, Las Cruces, NM 88003, USA

^b Department of Civil Engineering, New Mexico State University, MSC 3CE, PO Box 30001, Las Cruces, NM, USA, 88003, and member of the Engineering Research Center for Re-inventing the Nation's Urban Water Infrastructure, Stanford University

^c Department of Economics, Applied Statistics, and International Business, New Mexico State University, MSC 3CQ, PO Box 30001, Las Cruces, NM, USA, 88003

^d Texas AgriLife Research & Extension Center at El Paso, USA, Texas A&M University System

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ABSTRACT

Different multi-criteria decision-making (MCDM) techniques require different levels of computational intensity and may produce different outputs, so selecting an appropriate technique largely determines the quality of the recommended decision and the effort required to obtain that decision. In most real environments, criteria and their constraints are not deterministic and cannot be specified precisely; therefore, those criteria are uncertain or fuzzy. To facilitate the selection of an appropriate MCDM method under a fuzzy environment, this study investigates and statistically compares the performances of ten commonly used MCDM techniques: simple additive weights (SAW), weighted product method (WPM), compromise programming (CP), technique for order preference by similarity to ideal solution (TOPSIS), four types of analytical hierarchy process (AHP), VIKOR (in Serbian: VišeKriterijumska Optimizacija I Kompromisno Rešenje), and ELECTRE (in French: ELimination Et Choix Traduisant la REalité). These techniques' performances were compared using fuzzy criteria and constraints, matching the conditions usually found in real applications. To conduct the comparisons, the 10 multi-criteria decision ranking methods were applied to 1250 simulated sets of decision matrices with fuzzy triangular values, and 12,500 sets of ranks were analyzed to compare the ranking methods. SAW and TOPSIS had statistically similar performances. ELECTRE was not preferable in providing full, sorted ranks among the alternatives. VIKOR considering its ranking process, for specific conditions, assigns identical ranks for several alternatives; when full, sorted ranks are required, VIKOR is unfavorable, although it is a powerful technique in introducing the closest alternative to the ideal condition. Types 1 and 3 of AHP and types 2 and 4 of AHP had close performances. Notably, no ranking method was significantly sensitive to uncertainty levels when uncertainty changed symmetrically.

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1. Introduction

Different multi-criteria decision-making (MCDM) techniques require different levels of effort and produce different outputs so the choice of which MCDM technique to use strongly influences the quality of the recommended decision and the amount of effort required to obtain that decision. While different decision-ranking methods may rank specific alternatives in different orders, and different decision-ranking methods have different levels of computa-

tional intensity, when a simple and a complicated decision-ranking method produce similar sorts of ranks, using the simplest method can save computation time and effort without sacrificing quality.

This paper reviews the literature on MCDM techniques that have been used in diverse engineering projects, and then it evaluates and compares the performances of those techniques in terms of similarities and dissimilarities. Through theoretical, programming, and simulation work, this study develops the extensions of each individual selected MCDM technique, and it analytically investigates and statistically compares the performances of ten commonly used MCDM techniques. Considering that, in most real environments, criteria and their constraints are not deterministic and cannot be specified precisely, the MCDM techniques are evaluated

* Corresponding author.

E-mail addresses: hzamani@nmsu.edu (H. Zamani-Sabzi), jpking@nmsu.edu (J.P. King), cgard@nmsu.edu (C.C. Gard), Shalamu.Abudu@ag.tamu.edu (S. Abudu).

under a fuzzy environment, with criteria and constraints represented as uncertain or fuzzy values.

The detailed objectives of this paper include an analytical evaluation of the different decision ranking methods and a statistical evaluation of the different ranking methods applied to decision matrices with fuzzy values. Overall, we statistically evaluated: 1) the number of alternatives, 2) the number of criteria, 3) the practice of selecting criteria weights from different distributions, 4) fuzziness level, and 5) the number of replications. These statistical evaluations allowed us to see how these five criteria affect the correlation among the final sorting of ranks when those ranks are obtained through different ranking methods applied on the same decision matrices. The developed decision matrices include triangular fuzzy numbers with both random left and right spreads and 5 levels of biased, equal, left, and right spreads. For each category, 50 sets of decision matrices were simulated. In order to investigate the role played by the size of the decision matrices (i.e., the number of alternatives and the number of criteria), 5 different decision matrices of D3,3 (3 alternatives, 3 criteria), D8,4 (8 alternatives, 4 criteria), D8,8 (8 alternatives, 8 criteria), D15,8 (15 alternatives, 8 criteria), and D15,15 (15 alternatives, 15 criteria) were defined. For each individual decision matrix, 300 sets of matrices with fuzzy values were produced (simulated) by MATLAB and designed in the form of that decision matrix (50 sets for random left and right spreads and 250 sets for 5 levels of biased symmetrical spreads). In order to show the extension of each individual decision-making method, a decision matrix with 3 alternatives and 3 criteria was ranked through all 10 decision ranking methods. The 10 selected decision ranking methods were applied on each decision matrix, and the alternatives of each decision matrix were ranked. Overall, we statistically evaluated 300 different sets of data designed for 5 different decision sizes and 10 different ranking methods. In statistical analysis, in order to analyze the effect of increasing the uncertainty level (equal increases in the left and right spread), four levels of uncertainty were selected for analysis; therefore, from 15,000 sets of produced ranks, 12,500 ranks were analyzed.

In order to show the correlation among the produced, sorted ranks, the final, sorted ranks obtained from different methods were statistically analyzed by performing Kendall's tau-b correlation test and Spearman's rho test using SPSS software. The results from this study can guide the selection of optimal decision-making processes under fuzzy environments and offer insight into detailed applications of decision-making techniques and their use in engineering projects.

This paper does not claim that any method is better than other methods across all possible circumstances, but rather it emphasizes the importance of investigating different decision-making techniques to rank the decisions of each method and the importance of finding the most appropriate method for ranking the decisions in consideration of the decision-making conditions.

Previously, not much work has been conducted to evaluate and compare the performances of MCDM methods, and most existing work has been conducted under deterministic conditions rather than the uncertain, or fuzzy, conditions that are more commonly found in real applications. In the existing literature, Celik et al. [1] conducted a comprehensive review of MCDM techniques according to interval type-2 fuzzy sets, reviewing 82 different papers developed on the basis of interval type-2 fuzzy sets (IT2FSs). They categorized the applications of MCDM techniques in the fields of transportation and logistics, technology management, risk management, manufacturing, investment management, human resources management, healthcare, environment, energy, and education. Another of the previous studies, conducted by Zanakis et al. [2], evaluated eight popular MCDM methods: Elimination Et Choix Traduisant la Réalité (ELECTRE), technique for order preference by similarity to ideal solution (TOPSIS), simple ad-

ditive weights (SAW), weighted product method (WPM), and four types of the analytical hierarchy process (AHP) – original, geometric scale, right eigenvector, and mean transformation solution [3]. Their work found that ELECTRE and VIKOR produced different rankings than the other selected MCDM methods; additionally, ELECTRE and VIKOR did not produce global rankings of the alternatives. For design parameters in the simulation of the decision-making matrix, their investigation considered the number of criteria, the number of alternatives, population distribution for selecting the weight of criteria, and number of replications. These researchers also statistically investigated the ranking effects of different methods for weighting decision criteria: 1) equal weights for all decision criteria, 2) unbiased distribution (i.e., normal distribution) of weights, and 3) a biased weight distribution (e.g., a U-shaped distribution). Several other studies in which researchers investigate specific MCDM techniques have also been conducted. Gul et al. [4] developed a literature review on VIKOR with its fuzzy extensions and applications, discussing extensions of the VIKOR method under a fuzzy environment. In total, they evaluated about 343 papers that utilized the VIKOR method in 13 different application areas. Their study showed that the major applications of VIKOR have been in the fields of mechanical engineering, manufacturing, and engineering design. Furthermore, Mardani et al. [5] developed a review study on the methodologies and applications of VIKOR method. They reviewed the studies that utilized VIKOR as a decision-making tool, reviewing 176 papers published from 2004 to 2015. Researchers from 15 different fields have utilized VIKOR, and the fields that have utilized VIKOR most have been operation management and human resource management.

Behzadian et al. [6] developed a literature review on TOPSIS applications. They studied 266 papers that applied TOPSIS to rank the alternatives. In another study, Behzadian et al. [7] developed a review on the methodologies and applications of a decision-making technique entitled "Preference Ranking Organization Method for Enrichment of Evaluations (PROMETHEE)". They reviewed and evaluated 217 papers and categorized their applications in different fields such as environment management, water resource management and hydrology, energy management, and several other decision-making areas.

A review by Behzadian et al. [6] categorized the applications of TOPSIS in multiple areas: logistics and supply chain management, manufacturing engineering, business management, health care and environment management, energy and resources management, chemical engineering management, water resources project, and several other decision-making fields. In addition, Zavadskas et al. [8] developed a review study on different applications of TOPSIS in ranking decisions in complicated decision-making projects. They reviewed 105 papers, published from 2000 to 2015, that utilized TOPSIS for ranking decisions. Their review study indicates that TOPSIS has compatibility potential with different existing conditions on decision-making environments.

Chen [9] extended the application of the ELECTRE method under a fuzzy environment for multi-criteria group decision-making. In addition, Govindan and Jepsen [10] developed a comprehensive review study on methodologies and different applications of ELECTRE. Govindan and Jepsen reviewed 686 papers, from which 544 papers considered the applications of ELECTRE in 13 major areas and several sub-areas. Their review indicates that, although ELECTRE type I is a 40-year-old method, it is still used by several decision-makers in different fields; still, ELECTRE type III has been the most popular method of the ELECTRE types. Overall, different types of ELECTRE have been utilized for decision-making in the fields of financial management, risk-related problems, energy management, and environmental and natural resources management.

Mardani et al. [11] developed a review study on MCDM techniques and their applications, based on works done from 2000 to

2014. They reviewed 393 papers, categorizing the applications of those MCDM techniques in 15 different fields.

Chen and Hwang [12] and Carlsson and Fuller [13] investigated several MCDM methods under fuzzy environments. Mendoza reviewed utilized MCDM methods in natural resources [14]. Zarghami and Szidarovszky [3] studied the application of MCDM techniques in environmental and water resource projects. Sabzi and King [15] utilized MCDM methods to find the optimal solution in a flood control system for a case study in Diez Lagos pond in southern New Mexico, assuming deterministic conditions. They ranked several flood controlling solutions (decisions) designed through one decision matrix. In this study, several of those techniques are formulated under fuzzy environment with triangular fuzzy values. Triantaphyllou and Lin [16] evaluated 5 different MCDM methods and concluded that SAW would be the simplest method to apply, but stated that fuzzy TOPSIS and fuzzy AHP would be more able to satisfy human appraisal.

Zavadskas and Turskis [17] developed a review study on different applications of the MCDM techniques, mostly under deterministic conditions in economics. They emphasized the importance of selecting optimal decisions in economics, since decisions are tied to profit or loss. Finally, they concluded that, although no one can say which model is the best model across all circumstances, wisely selecting the decision-making method and its solution is part of an optimal decision-making process.

Liou and Tzeng [18] commented on the paper “Multiple criteria decision making (MCDM) methods in economics: an overview, by Zavadskas and Turskis (2011)”. They considered the actual existing conditions in decision-making environments, in which some criteria may have different level of dependencies, whereas, in some of the traditional decision-making models, those criteria were assumed as independent variables.

The work of Zanakis et al. [2] on MCDM methods can be expanded by incorporating fuzzy values: these researchers used deterministic conditions for their decision matrix, but generally the decision-making environment is not deterministic, and the boundaries among the criteria values and selected weights are not sharp, but rather fuzzy. Therefore, in this paper, we developed the decision-making matrix under a fuzzy environment; additionally, since no previous study has performed a full pairwise comparison of the MCDM methods, this study features a full pairwise comparison of decision making methods in a fuzzy environment. In addition, in this study, the correlation analysis was performed through statistical tests to show how these methods provide similar or dissimilar sort of ranks for decision alternatives when they are applied to the same decision matrices.

Mardani et al. [19] developed a review study on fuzzy MCDM techniques and their applications. They reviewed more than 400 papers in the fields of engineering, business and management, science and technology, showing that, in the last two decades, AHP has been the most popular decision-making technique in those fields. In addition, engineering-based fields utilized the fuzzy MCDM techniques more than the other three fields did in the last two decades.

In this paper, we selected 10 common ranking methods – SAW, WPM, CP, TOPSIS, four types of AHP, ELECTRE, and VIKOR – and statistically and analytically investigated their similarities, differences, and performances in producing final, sorted ranks. The detailed objectives of our study include an analytical evaluation of the different ranking methods and a statistical evaluation of the different ranking methods applied to decision matrices with fuzzy values. In the course of the research, we statistically evaluated: 1) the number of alternatives, 2) the number of criteria, 3) the practice of selecting criteria weights from different distributions, 4) fuzziness level, and 5) the number of replications. In the final step, in order to show the correlation among the produced, sorted

ranks, the final, sorted ranks obtained from different methods were statistically analyzed by performing Kendall's tau-b correlation test and Spearman's rho test using SPSS software.

2. Materials and methods

The developed decision matrices include triangular fuzzy numbers with both random left and right spreads, and 5 levels of biased, equal, left and right spreads. For each category, 50 sets of decision matrices were simulated. In order to investigate the role played by the size of decision matrices (i.e., the number of alternatives and the number of criteria), 5 different decision matrices of $D_{3,3}$ (3 alternatives, 3 criteria), $D_{8,4}$ (8 alternatives, 4 criteria), $D_{8,8}$ (8 alternatives, 8 criteria), $D_{15,8}$ (15 alternatives, 8 criteria), and $D_{15,15}$ (15 alternatives, 15 criteria) were defined. For each individual decision matrix, 300 sets of matrices with fuzzy values were produced by MATLAB and designed in the form of that decision matrix (50 sets for random left and right spreads, and 250 sets for 5 levels of biased symmetrical spreads). In order to show the extension of each individual decision-making method, a decision matrix with 3 alternatives and 3 criteria was ranked through all 10 decision ranking methods. The 10 popular selected ranking methods were applied on each decision matrix, and the alternatives of each decision matrix were ranked. Overall, we statistically evaluated 300 different sets of data designed for 5 different decision sizes and 10 different ranking methods. In statistical analysis, in order to analyze the effect of increasing the uncertainty level (equal increases in the left and right spread), four levels of uncertainty were selected for analyzing; therefore, from 15,000 sets of produced ranks, 12,500 ranks were analyzed. Since, ELECTRE was not preferable in providing full, sorted ranks among the alternatives, in comparison of the decision ranking methods, the ELECTRE was not exploited.

Considering the heavy amount of calculation and analysis for developing the final ranks under each ranking method, a specific macro was written in Microsoft Excel. As numerical results, 15,000 final, sorted ranks were obtained by running 50 macros applied on the 300 sets of decision matrices. In the final step, in order to show the correlation among the produced, sorted ranks, the final, sorted ranks obtained from different methods were statistically analyzed by performing Kendall's tau-b correlation test and Spearman's rho test using SPSS software. The numerical result is discussed in the Sections 3 and 4.

The fuzzy environment involves the use of fuzzy sets, which have been defined by Zadeh [20] in 1965 and extended by Bellman and Zadeh [21] as a class of objects in which there is no sharp boundary between the objects that belong to the class and the objectives that do not belong to the same class [21]. When uncertainty is involved in presenting the value of x in the set A , the set will be fuzzy, and any statement regarding a number belonging to the set will have a degree of truth which can be defined by a membership function. Generally, a specific fuzzy set A in x is defined as a set of pairs as shown in Eq. (1):

$$A = \{(x, \mu_A(x))\}, \quad X = \{x\} \quad (1)$$

where $\mu_A(x)$ is the membership degree of x in A , in which, for any x , there is an associated value between 0 and 1 which represents the degree of membership of x in A .

Conceptually, membership degree is representative of the degree to which any specific number x belongs to specific data set A . The membership degrees range from 0 (completely not belonging) to 1 (completely belonging) [21]. As long as uncertainty is involved in most decision making processes, the decision-making environment will be fuzzy and the fuzzy optimization method can be used as a tool for finding the ideal and anti-ideal points in multi-objective problems.

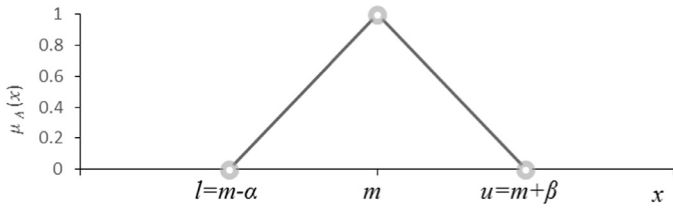


Fig. 1. Schematic of a triangular fuzzy number \tilde{A} .

All quantitative criteria can be expressed as fuzzy numbers, and qualitative criteria values can be described by linguistic terms, which can be converted to the fuzzy numbers.

2.1. Triangular fuzzy numbers

In this study, decision values were defined as triangular fuzzy numbers. Typically, there are two forms for defining triangular fuzzy numbers $\tilde{A} = (m, \alpha, \beta)$ and $\tilde{A} = (l, m, u)$, where m is the central value of the triangle that has the highest membership degree $\mu_{\tilde{A}}(x) = 1$, α and β are the extensions of the triangle to the left and right, respectively, and u and l represent the upper and lower limits, respectively, for the fuzzy number m .

Fig. 1 shows a triangular fuzzy number $\tilde{A} = (l, m, u)$. This fuzzy number is defined in Eq. (2) [22].

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq l, \\ \frac{x-l}{m-l}, & l < x \leq m, \\ \frac{u-x}{u-m}, & m < x \leq u, \\ 0, & x > u. \end{cases} \quad (2)$$

2.2. Algebraic operations of two triangular fuzzy numbers

Algebraic operations for two triangular fuzzy numbers are conducted as described below and have been widely used through several MCDM methods under fuzzy environment. Before introducing these operations, however, it should be established that a fuzzy number A is defined as a positive value if for all $x < 0$, $\mu_{\tilde{A}}(x) = 0$ [12].

Addition and subtraction of two fuzzy numbers in the form of $\tilde{A} = (l, m, u)$ and $\tilde{B} = (a, b, c)$ are defined as in Eqs. (3) and (4).

$$\tilde{A}(+) \tilde{B} = (l + a, m + b, u + c) \quad (3)$$

$$\tilde{A}(-) \tilde{B} = (l - c, m - b, u - a) \quad (4)$$

Addition and subtraction of two fuzzy numbers in the form of $\tilde{A} = (m, \alpha, \beta)$ and $\tilde{B} = (n, \gamma, \delta)$ are defined as in Eqs. (5) and (6).

$$\tilde{A}(+) \tilde{B} = (m + n, \alpha + \gamma, \beta + \delta) \quad (5)$$

$$\tilde{A}(-) \tilde{B} = (m - n, \alpha + \delta, \beta + \gamma) \quad (6)$$

Multiplication of two fuzzy numbers $\tilde{A} = (m, \alpha, \beta)$ and $\tilde{B} = (n, \gamma, \delta)$ is defined as in Eqs. (7)–(9).

$$\tilde{A}(\cdot) \tilde{B} = (mn, m\gamma + n\alpha, m\delta + n\beta) \text{ where } \tilde{A} \text{ and } \tilde{B} > 0 \quad (7)$$

$$\tilde{A}(\cdot) \tilde{B} = (mn, n\alpha - m\delta, n\beta - m\gamma) \text{ where } \tilde{A} < 0 \text{ and } \tilde{B} > 0 \quad (8)$$

$$\tilde{A}(\cdot) \tilde{B} = (mn, -n\beta - m\delta, -n\alpha - n\gamma) \text{ where } \tilde{A} < 0 \text{ and } \tilde{B} < 0 \quad (9)$$

Multiplication of two fuzzy numbers in the form of $\tilde{A} = (l, m, u)$ and $\tilde{B} = (a, b, c)$ is defined as in Eqs. (10)–(12).

$$\tilde{A}(\cdot) \tilde{B} = (la, mb, uc) \text{ where } \tilde{A} \text{ and } \tilde{B} > 0 \quad (10)$$

$$\tilde{A}(\cdot) \tilde{B} = (lc, mb, ua) \text{ where } \tilde{A} < 0 \text{ and } \tilde{B} > 0 \quad (11)$$

$$\tilde{A}(\cdot) \tilde{B} = (uc, mb, la) \text{ where } \tilde{A} < 0 \text{ and } \tilde{B} < 0 \quad (12)$$

where for scalar multiplication, for $k > 0, k \in R: k \cdot \tilde{A} = (kl, km, ku)$.

Division of two fuzzy numbers in the form of $\tilde{A} = (l, m, u)$ and $\tilde{B} = (a, b, c)$ is defined as in Eqs. (13)–(15).

$$\tilde{A}(\cdot) \tilde{B} = \left(\frac{l}{c}, \frac{m}{b}, \frac{u}{a} \right) \text{ where } \tilde{A} \text{ and } \tilde{B} > 0 \quad (13)$$

$$\tilde{A}(\cdot) \tilde{B} = \left(\frac{u}{c}, \frac{m}{b}, \frac{l}{a} \right) \text{ where } \tilde{A} < 0 \text{ and } \tilde{B} > 0 \quad (14)$$

$$\tilde{A}(\cdot) \tilde{B} = \left(\frac{u}{a}, \frac{m}{b}, \frac{l}{c} \right) \text{ where } \tilde{A} < 0 \text{ and } \tilde{B} < 0 \quad (15)$$

Division of two fuzzy numbers in the form of $\tilde{A} = (m, \alpha, \beta)$ and $\tilde{B} = (n, \gamma, \delta)$ is defined as in Eqs. (16)–(18) [23].

$$\tilde{A}(\cdot) \tilde{B} = \left(\frac{m}{n}, \frac{m\delta + n\alpha}{n^2}, \frac{m\gamma + n\beta}{n^2} \right) \text{ where } \tilde{A} \text{ and } \tilde{B} > 0 \quad (16)$$

$$\tilde{A}(\cdot) \tilde{B} = \left(\frac{m}{n}, \frac{n\alpha - m\gamma}{n^2}, \frac{n\beta - m\delta}{n^2} \right) \text{ where } \tilde{A} < 0 \text{ and } \tilde{B} > 0 \quad (17)$$

$$\tilde{A}(\cdot) \tilde{B} = \left(\frac{m}{n}, \frac{-n\beta - m\gamma}{n^2}, \frac{-n\alpha - m\delta}{n^2} \right) \text{ where } \tilde{A} < 0 \text{ and } \tilde{B} < 0 \quad (18)$$

The normalization procedure for benefit and cost criteria is developed as shown in Eqs. (19) and (20):

$$\tilde{r}_{i,j} = \left(\frac{l_{ij}}{c_j^+}, \frac{m_{ij}}{c_j^+}, \frac{u_{ij}}{c_j^+} \right) \quad (19)$$

$$\tilde{r}_{i,j} = \left(\frac{m_j^-}{l_{ij}}, \frac{m_j^-}{m_{ij}}, \frac{m_j^-}{u_{ij}} \right) \quad (20)$$

Where $c_j^+ = \max_j(u_{ij})$ and $m_j^- = \min_j(l_{ij})$

Raising a fuzzy triangular number $\tilde{A} = (l, m, u)$ to the power of another fuzzy number $\tilde{B} = (a, b, c)$ is performed as shown in Eq. (21).

$$\tilde{A}^{\tilde{B}} = (l^a, m^b, u^c) \quad (21)$$

Raising a fuzzy triangular number $\tilde{A} = (m, \alpha, \beta)$ to the power of p is performed as shown in Eq. (22):

$$\tilde{A}^p = (m^p, pm\alpha, pm\beta) \quad (22)$$

2.3. Constructing the decision matrix and developing decision ranking techniques under a fuzzy environment

In general, a decision matrix under a fuzzy environment with triangular fuzzy numbers can be established as $\tilde{x}_{i,j} = (l_{ij}, m_{ij}, u_{ij})$, in which $\tilde{x}_{i,j}$ is a representative value of alternative \tilde{A}_i against criteria C_j , by assuming that each element in a fuzzy decision matrix is a fuzzy triangular number [24]. However, different MCDMs utilize different approaches for constructing the decision matrix and developing the decision ranking techniques under a fuzzy environment.

In the following sections, we extend MCDM methods that Zamani-Sabzi and King [15] extended under deterministic conditions: TOPSIS, VIKOR, SAW, AHP, ELECTRE and CP. Sections 2.3.1 to 2.3.7.4 develop and define these methods under a fuzzy environment with fuzzy triangular values.

2.3.1. Simple additive weighting method under fuzzy environment

SAW uses linear combinations of weighted criteria for each alternative to represent and compare the overall score of the alternative, as shown in Eq. (23):

$$\tilde{F}_i = \sum_{j=1}^n \tilde{r}_{ij} \cdot \tilde{w}_j \tag{23}$$

where \tilde{r}_{ij} is the fuzzy value of i th alternative against j th criteria in the normalized fuzzy decision matrix for $i=1, 2, \dots, m$ and $j=1, 2, \dots, n$, where \tilde{w}_j is the fuzzy value of normalized weight for j th criteria for $j=1, 2, \dots, n$.

In this method, we first acquire the cumulative evaluation for each of the alternatives; then, the alternatives are ranked on the basis of these values [3].

2.3.2. Weighted product method under fuzzy environment

The rankings in the weighted product methods are calculated as shown in Eq. (24):

$$\tilde{A}_i^* = \left\{ \tilde{A}_i \prod_{j=1}^n \tilde{a}_{ij}^{\tilde{w}_j} \right\} \tag{24}$$

where \tilde{a}_{ij} are numerically in comparable scale and indicate the amount of i th alternative against j th criteria in the decision matrix for $i=1, 2, \dots, m$ and $j=1, 2, \dots, n$, where \tilde{w}_j is the normalized weight of j th criteria for $j=1, 2, \dots, n$.

Alternatives are ranked based on the \tilde{A}_i^* values in which the max (\tilde{A}_i^*) is the best rank among all alternatives.

2.3.3. Compromise programming under fuzzy environment

On the basis of how distant alternatives are from the ideal point, the compromise programming method ranks each of the alternatives. Eq. (25) shows the approach for calculating this Euclidean distance [25]:

$$\tilde{D}_i^p = \left(\sum_{j=1}^n \left(\tilde{w}_j \cdot \frac{\tilde{a}_{ij}^* - \tilde{a}_{ij}}{\tilde{a}_{ij}^* - \tilde{a}_{j*}} \right)^p \right)^{\frac{1}{p}} \tag{25}$$

where \tilde{a}_{ij} is the fuzzy value of i th alternative against j th criteria in the fuzzy decision matrix for $i=1, 2, \dots, m$ and $j=1, 2, \dots, n$, where \tilde{w}_j is the fuzzy value of normalized weight of j th criteria for $j=1, 2, \dots, n$.

For $i=1, 2, \dots, m$, \tilde{a}_{ij}^* and \tilde{a}_{j*} are defined for j th components of optimum and anti-optimum values, P is variable from 1 to $P=\infty$, and \tilde{D}_i^p represents the relative distances of alternatives from the ideal points under the fuzzy environment. Values for P show the importance of a criteria's deviation from their related ideal point. The parameter of P can be considered as a fuzzy value of \tilde{p} in the form of $\tilde{p} = (p, \alpha = 0, \beta = 0)$ or $\tilde{p} = (l = p, m = p, u = p)$. Conceptually, left and right spreads (uncertainties) of P in its fuzzy form can be considered as zero. Therefore, Eq. (21) can be utilized in the required raising a fuzzy triangular number to the power of another fuzzy triangular number.

Also, it should be noted that in compromise programming method, in Eq. (25), the normalized values of \tilde{a}_{ij} can be used, where in that condition \tilde{a}_{ij}^* and \tilde{a}_{j*} would be defined for j th components of optimum and anti-optimum values in the normalized decision matrix. The numerical results of this study showed that although \tilde{D}_i^p is affected, the final rank of alternatives does not change.

2.3.4. TOPSIS under fuzzy environment

TOPSIS as a well-known, classic ranking method, which was developed by Hwang and Yoon [26] was selected as another decision-making method and investigated under fuzzy environment. The

parametric steps of using TOPSIS to select the optimal alternative are as follows [27]: 1) show all potential decisions as different combinations of criteria in a defined mathematical model; 2) develop an objective function that recognizes all impactful quantitative and qualitative criteria; 3) quantify all impactful qualitative criteria; 4) identify each potential alternative as a final action or decision; 5) on the basis of the number of alternatives (m) and the number of criteria (n), define the decision matrix - typically, the value for m corresponds to the number of rows in the decision matrix (number of alternatives), and the value of n corresponds to the number of criteria; 6) normalize the defined decision matrix; 7) determine the optimal and anti-optimal solutions; 8) calculate the distance separating the optimal solution from each of the alternatives; 9) calculate relative closeness of each alternative to the optimal solution; and 10) rate and rank each potential alternative based on their relative closeness. TOPSIS has been widely used through several MCDM projects under fuzzy environment [28].

For any decision making problem within TOPSIS, an objective function is defined. Within this objective function, each alternative, the ideal point, the anti-ideal point, and the distance between the ideal and anti-ideal distance are derived. All alternatives are ranked and compared based on the defuzzified values of their relative closeness.

2.3.4.1. Normalizing the decision matrix with fuzzy values. Eqs. (26)–(28) can be utilized to normalize the decision matrices. In this study, we normalized all fuzzy values of benefit and cost criteria in different alternatives through Eq. (26), which, conceptually, is the same approach that we applied for normalizing the fuzzy numbers. By using Eq. (26), each individual value in each column is normalized by the maximum value of the same column. Sabzi and King [15] utilized the same Eqs. (26) to (28) for normalizing the decision matrices under deterministic values.

$$\tilde{r}_{ij} = \frac{\tilde{x}_{ij}}{\max(\tilde{x}_{ij})} \tag{26}$$

where all three elements of normalization value of $\max(\tilde{x}_{ij})$ are considered equal to its upper level as demonstrated in the numerical example in Eq. (57).

$$\tilde{r}_{ij} = \frac{\tilde{x}_{ij}}{\sqrt{\sum_{i=1}^m \tilde{x}_{ij}^2}} \tag{27}$$

$$\tilde{r}_{ij} = \frac{\tilde{x}_{ij}}{\sum_{i=1}^m \tilde{x}_{ij}} \tag{28}$$

where, in (26), (27), and (28), $i=1, 2, \dots, m$ and $j=1, 2, \dots, n$

Considering the importance preference of the criteria, different weights are proposed or defined by experts. Those fuzzy values of weights are defined in the form of matrix of weights. Then, the normalized matrix is multiplied in the matrix of weights to calculate the weighted normalized decision matrix. The matrix of weights and the weighted normalized matrix with fuzzy values are formed as follows:

$$[\tilde{W}_{i,j}] = \begin{bmatrix} \tilde{W}_{1,1} & \dots & 0 & 0 \\ 0 & \tilde{W}_{2,2} & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \tilde{W}_{n,n} \end{bmatrix}$$

$$\begin{aligned} \tilde{\vartheta}_{ij} &= [\tilde{r}_{i,j}] \otimes [\tilde{w}_{j,j}] \\ &= \begin{bmatrix} \tilde{r}_{1,1} \otimes \tilde{w}_1 & \dots & \tilde{r}_{1,n-1} \otimes \tilde{w}_{n-1} & \tilde{r}_{1,n} \otimes \tilde{w}_n \\ \tilde{r}_{2,1} \otimes \tilde{w}_1 & \dots & \tilde{r}_{2,n-1} \otimes \tilde{w}_{n-1} & \tilde{r}_{2,n} \otimes \tilde{w}_n \\ \dots & \dots & \dots & \dots \\ \tilde{r}_{m,1} \otimes \tilde{w}_1 & \dots & \tilde{r}_{m,n-1} \otimes \tilde{w}_{n-1} & \tilde{r}_{m,n} \otimes \tilde{w}_n \end{bmatrix} \end{aligned}$$

where $i=1, 2, \dots, m$ and $j=1, 2, \dots, n$.

2.3.4.2. *Determining the ideal and anti-ideal fuzzy values.* Ideal and anti-ideal points can be obtained using Eqs. (29) and (30). Additionally, several researchers, for benefit criteria, defined the optimal fuzzy value equals to $\tilde{v}_{i,j}^+ = (1,1,1)$, and the anti-ideal fuzzy value is represented by $\tilde{v}_{i,j}^- = (0, 0, 0)$.

Ideal and anti-ideal fuzzy values are determined using Eqs. (29) and (30) as follows:.

$$\tilde{A}^+ = (\tilde{\vartheta}_1^+, \dots, \tilde{\vartheta}_n^+) = \{ \max \tilde{\vartheta}_{ij}^+ | (i = 1, 2, \dots, n), j = 1, 2, \dots, n) \} \tag{29}$$

$$\tilde{A}^- = (\tilde{\vartheta}_1^-, \dots, \tilde{\vartheta}_n^-) = \{ \min \tilde{\vartheta}_{ij}^- | (i = 1, 2, \dots, n), j = 1, 2, \dots, n) \} \tag{30}$$

2.3.4.3. *Determining the ideal and anti-ideal distances.* The total measure of the distance between each alternative and the anti-ideal points is defined as shown in Eqs. (31) and (32) [27, 29–31]:

$$\tilde{s}_i^+ = \sqrt{\sum_{j=1}^n d(\tilde{\vartheta}_{ij}, \tilde{\vartheta}_{ij}^+)} \quad \text{where } i = 1, \dots, m. \tag{31}$$

$$\tilde{s}_i^- = \sqrt{\sum_{j=1}^n d(\tilde{\vartheta}_{ij}, \tilde{\vartheta}_{ij}^-)} \quad \text{where } i = 1, \dots, m. \tag{32}$$

2.3.4.4. *Calculating “the relative closeness to the ideal solution” [12].* The closeness coefficient for each alternative is calculated using Eq. (33), which is utilized to rank each individual alternative. Relative closeness indicates the relative distance of each alternative from the anti-ideal point; therefore, the highest value of closeness coefficient stands for the most preferable solution that has the farthest distance value from the anti-ideal point. Relative closeness is calculated as shown in Eq. (33).

$$\tilde{C}_i^* = \frac{\tilde{s}_i^-}{\tilde{s}_i^- + \tilde{s}_i^+} \quad \text{for } i = 1, \dots, m. \tag{33}$$

In final step, alternatives are ranked based on the values of \tilde{C}_i^* .

2.3.5. *The analytic hierarchy process under fuzzy environment*

The analytic hierarchy process was developed by Thomas L. Saaty in the 1970s, and it has been widely used in decision-making in various fields [32–34]. This method is developed based on the concept of relative importance: all defined alternatives are compared against each other versus each individual criteria to find their relative preferences. In classic AHP, linguistic variables can be used to compare all criteria and alternatives. The linguistic variables are quantified using the scalar approach, and the general steps of developing the full list of ranks using AHP is as follows:

1. In order to compare the criteria and weights with each other, all defined criteria and defined weights are simulated in the form of a matrix $n \times n$ where n is the number of the weights. For example, the comparisonwise matrix of criteria 1 and weights is obtained as shown in Equations (These two following equations can be called formulations.) 34 and 35.

$$[\tilde{C}_w] = \begin{bmatrix} \tilde{W}_1 & \dots & \tilde{W}_1 & \tilde{W}_1 \\ \tilde{W}_1 & \dots & \tilde{W}_{n-1} & \tilde{W}_n \\ \tilde{W}_2 & \dots & \tilde{W}_2 & \tilde{W}_2 \\ \tilde{W}_1 & \dots & \tilde{W}_{n-1} & \tilde{W}_n \\ \dots & \dots & \dots & \dots \\ \tilde{W}_n & \dots & \tilde{W}_n & \tilde{W}_n \\ \tilde{W}_1 & \dots & \tilde{W}_{n-1} & \tilde{W}_n \end{bmatrix} \tag{34}$$

$$[\tilde{C}_{\{\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_m \text{ against } \tilde{C}_1\}}] = \begin{bmatrix} \tilde{X}_{11} & \dots & \tilde{X}_{11} & \tilde{X}_{11} \\ \tilde{X}_{11} & \dots & \tilde{X}_{m-1,1} & \tilde{X}_{m1} \\ \tilde{X}_{21} & \dots & \tilde{X}_{21} & \tilde{X}_{21} \\ \tilde{X}_{11} & \dots & \tilde{X}_{m-1,1} & \tilde{X}_{m1} \\ \dots & \dots & \dots & \dots \\ \tilde{X}_{m1} & \dots & \tilde{X}_{m1} & \tilde{X}_{m1} \\ \tilde{X}_{11} & \dots & \tilde{X}_{m-1,1} & \tilde{X}_{m1} \end{bmatrix} \tag{35}$$

where $[\tilde{C}_w]$ is the comparisonwise matrix of weights in which $\tilde{w}_1, \tilde{w}_2, \dots, \tilde{w}_n$ are the defined weights of criteria 1 (\tilde{C}_1), criteria 2 (\tilde{C}_2), ..., criteria n (\tilde{C}_n), respectively. $\tilde{C}_{\{\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_m \text{ against } \tilde{C}_1\}}$ is the comparisonwise matrix of alternatives against criteria 1.

Against each individual criteria, the values of the alternatives are compared according to Eq. (35). All pairwise comparison matrices are reciprocal.

2. The weight vector or importance vector in comparisonwise matrices (For each criteria, a separate comparisonwise matrix is formed.), which in exact condition represents the eigenvector and can be estimated through four major methods, is calculated as follows:

- The sum of the values in first row is calculated and normalized by the sum of values of all rows. The normalized value of the first row indicates the comparative importance of the first alternative compared to the other alternatives against the associated criteria in the comparisonwise matrix.
- Within matrix \tilde{C} , the values included in each column are summed. Then their reciprocals ($1 / (\text{sum of the values of each column})$) are calculated. Next, resulting reciprocal values are normalized using (dividing) the sum of all reciprocals. The normalized values of reciprocals indicate the comparative importance of alternatives against the associated criteria in the comparisonwise matrix.
- The values of each column are normalized by the sum of the values in the same column, then the average of each row is calculated, which stands for the relative importance of alternatives against criteria 1. Average of first row stands for the comparative importance of the first alternative compared to the other alternatives against the associated criteria in the comparisonwise matrix.
- Multiplying all of the values in each row of matrix \tilde{C} and then normalizing the n th roots of those values are normalized by some of their sum (sum the n th roots of those values). The normalized values indicate the relative importance of alternatives against the associated criteria in the comparisonwise matrix.

Numerical example in Sections 2.5.5.1–2.5.5.4 clarifies the four AHP prioritizing methods.

In each of these four methods, the relative importance values, when summed, produce a value of 1. However, as compared to the other methods, the fourth method produces relative importance values that more closely align with the pairwise comparison matrix's eigenvalues.

$$[\tilde{A} \times \tilde{w}] = \begin{bmatrix} \tilde{W}_1 & \dots & \tilde{W}_1 & \tilde{W}_1 \\ \tilde{W}_1 & \dots & \tilde{W}_{n-1} & \tilde{W}_n \\ \tilde{W}_2 & \dots & \tilde{W}_2 & \tilde{W}_2 \\ \tilde{W}_1 & \dots & \tilde{W}_{n-1} & \tilde{W}_n \\ \dots & \dots & \dots & \dots \\ \tilde{W}_n & \dots & \tilde{W}_n & \tilde{W}_n \\ \tilde{W}_1 & \dots & \tilde{W}_{n-1} & \tilde{W}_n \end{bmatrix} \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \dots \\ \dots \\ \tilde{W}_{n-1} \\ \tilde{W}_n \end{bmatrix} = \lambda_{\max} \begin{bmatrix} \tilde{W}_1 \\ \tilde{W}_2 \\ \dots \\ \dots \\ \tilde{W}_{n-1} \\ \tilde{W}_n \end{bmatrix} \tag{36}$$

$$= \lambda_{\max} \tilde{w}$$

where λ_{\max} is comparison matrix C_A 's largest eigenvalue. To prioritize or weight the alternatives, eigenvector w is used [35].

Table 1
Values for random index [36].

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
RI	0	0	0.58	0.9	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.48	1.56	1.57	1.59

Within the comparison matrix, $\tilde{C}_w = \tilde{a}_{ij}$, $\tilde{a}_{ij} = \frac{\tilde{w}_i}{\tilde{w}_j}$ for $i, j = 1, 2, \dots, n$, and $\tilde{a}_{ji} = 1/\tilde{a}_{ij}$, showing the reciprocal matrix status of comparison matrix A. Furthermore, if the condition in Eq. (37) is met, comparison matrix A will be consistent [36]:

$$\tilde{a}_{jk} = \tilde{a}_{ik}/\tilde{a}_{ij} \tag{37}$$

where i, j , and $k = 1, \dots, n$.

2.3.5.1. Calculating the consistency index. The consistency index (CI) is a measure of inconsistency for a pairwise comparison, and CI can be used to determine the consistency ratio. CI is calculated as shown in Eq. (38):

$$CI = \frac{\lambda_{max} - n}{n - 1} \tag{38}$$

where λ_{max} is the largest eigenvalue of the comparisonwise matrix of \tilde{C}_A and n is the dimension of the comparisonwise matrix of \tilde{A} .

On the basis of CI, the consistency ratio can be calculated as shown in Eq. (39):

$$CR = \frac{CI}{RI} \tag{39}$$

where RI is a predefined random index that is selected from Table 1 on the basis of the comparison matrix's dimensions (n). So as long as $CR < 0.10$, there is an acceptable degree of inconsistency for using the eigenvector as a priority weight [36].

2.3.5.2. Finding the weight vector for each pairwise comparison matrix. As stated in Section 2.3.5.1, normalized eigenvectors may serve as priority weights for alternatives in a comparison matrix so long as the consistency ratio is less than 0.10. Notably, the fourth method in Section 2.3.5 gives values that are significantly near to eigenvector values; therefore, the n th roots of the multiplicative values in each row of the comparison matrix after normalization indicate the priority weights of the considered alternatives in each comparisonwise matrix of alternatives against each criteria.

2.3.5.3. Final ranking of alternatives with fuzzy values. To obtain the final ranking of alternatives, we utilize the linear combination of the products of the calculated weights of the criteria from the weight vector and the related weight component of alternatives against each individual criteria. The numerical example in Section 2.5.5 demonstrate the ranking procedure for AHP.

2.3.6. ELECTRE under fuzzy environment

ELECTRE, which originally was developed by Roy [37], develops alternatives' pairwise dominance relationships through a pairwise comparison of the alternatives and by defining and using concordance and discordance sets. The general steps of the ELECTRE for decision making under fuzzy environment are detailed in Sections 2.3.6.1 to 2.3.6.3. Extension of the ELECTRE method have been described through numerical examples in several studies under both deterministic and fuzzy environment [38].

2.3.6.1. Normalizing the decision matrix and developing weighted normalized decision matrices with fuzzy values. ELECTRE develops the normalized and weighted normalized matrices through the same method used in TOPSIS and discussed in Section 2.3.4.

2.3.6.2. Developing the concordance and discordance sets considering the fuzzy values. For any set of m alternatives, there are $m \times (m - 1)$ pairwise comparisons to be performed. In these comparisons, two subsets of concordance and discordance are recognized for each two alternatives k and l . In comparison of two fuzzy alternatives, $\tilde{A}_{lj} = (\tilde{x}_{l1}, \tilde{x}_{l2}, \dots, \tilde{x}_{ln})$ and $\tilde{A}_{kj} = (\tilde{x}_{k1}, \tilde{x}_{k2}, \dots, \tilde{x}_{kn})$, concordance and discordance sets are obtained as follows:

Concordance and Discordance sets are defined respectively as: $C_{kl} = \{\tilde{x}_{kj} \geq \tilde{x}_{lj}\}$, $D_{kl} = \{\tilde{x}_{kj} < \tilde{x}_{lj}\}$ where $\{j|j=1, 2, \dots, n\}$, k and $l = 1, 2, \dots, m$, where $k \neq l$, and m is the number of alternatives.

Conceptually and algebraically, the concordance set and the discordance set are complementary.

Developing the concordance matrix with fuzzy values. Within the concordance matrix, each particular value represents the degree of preference between two alternatives, \tilde{A}_k and \tilde{A}_l , that are being compared. The values of the concordance index can be calculated as shown in Eq. (40):

$$c_{kl} = \frac{\sum_{j \in C_{kl}} \tilde{w}_j}{\sum_{j=1}^n \tilde{w}_j} \tag{40}$$

where, for the normalized fuzzy triangular weights, the central value of $\sum_{j=1}^n \tilde{w}_j$ equals to 1. A higher value indicates the more preferable choice.

Developing the discordance matrix with fuzzy values. In the discordance matrix, each value indicates the comparative degree of inferiority between two alternatives, \tilde{A}_k and \tilde{A}_l , that are being compared. These values in the discordance index are determined as shown in Eq. (41).

$$d_{kl} = \frac{\max_{j \in D_{kl}} (|\tilde{v}_{kj} - \tilde{v}_{lj}|)}{\max_{j \in j} (|\tilde{v}_{kj} - \tilde{v}_{lj}|)} \tag{41}$$

As detailed in Section 2.3.4.1, each value of v_{lj} and v_{kj} represents the number of l th and k th alternatives and j th criteria within the weighted normalized decision matrix for $i = 1, \dots, m, j = 1, \dots, n$ and k and $l = 1, \dots, m$.

When comparing the alternatives \tilde{A}_k and \tilde{A}_l , a higher discordance index value indicates that \tilde{A}_k is less favorable compared to \tilde{A}_l .

Developing the concordance dominance matrix, ($F = [f_{kl}]$). As a necessary step in determining the concordance dominance matrix, the concordance index values of the initial concordance matrix are used to produce a threshold value, as shown in Eq. (42) [39]:

$$\bar{c} = \left(\sum_{k=1}^m \sum_{l=1}^m c_{kl} \right) / (m(m - 1)) \tag{42}$$

where $k \neq l, f_{kl} = 1$ if $c_{kl} \geq \bar{c}$, and $f_{kl} = 0$ if $c_{kl} < \bar{c}$.

Developing the discordance dominance matrix, ($G = [g_{kl}]$). As with the development of the concordance dominance matrix, to produce the discordance dominance matrix, the discordance index values of the initial discordance matrix are used to produce a threshold value, through the process shown in Eq. (43):

$$\bar{d} = \left(\sum_{k=1}^m \sum_{l=1}^m d_{kl} \right) / (m(m - 1)) \tag{43}$$

where $k \neq l, f_{kl} = 1$ if $d_{kl} \geq \bar{d}$, and $f_{kl} = 0$ if $d_{kl} < \bar{d}$.

Developing the aggregate dominance matrix, $(E=[e_{kl}])$. The intersection of the concordance and discordance dominance matrices produces the aggregate dominance matrix, which is calculated as shown in Eq. (44).

$$e_{kl} = f_{kl} \cdot g_{kl} \tag{44}$$

2.3.6.3. *Developing the final ranks of alternatives.* If $e_{kl}=1$, then, in comparing \tilde{A}_k and \tilde{A}_l , \tilde{A}_k is preferable to \tilde{A}_l .

2.3.7. VIKOR under fuzzy environment

The VIKOR method makes use of the same basic approach utilized in compromise programming (Section 2.3.3), in addition, introduces L_p -metric, which conceptually is the same equation as Eq. (25). The extension of the VIKOR method have been described through numerical examples in several researches under both deterministic and fuzzy environment [40].

2.3.7.1. *Determining the ideal and anti-ideal values of cost and benefit criteria under fuzzy environment.* Dukstein and Opricovic (1980) introduced the parameter L_p to represent the relative distance of alternatives from the ideal points. Detailed process of ranking alternatives in VIKOR method under the fuzzy environment are described through Eqs. (45) to (50) [41].

The parameter of L_p is defined in Eq. (45) as follows:

$$L_{p,i} = \left(\sum_{j=1}^n \left(\tilde{w}_j \cdot \frac{\tilde{f}_j^* - \tilde{f}_{ij}}{\tilde{f}_j^* - \tilde{f}_{j-}} \right)^p \right)^{\frac{1}{p}} \tag{45}$$

where \tilde{w}_j is the normalized fuzzy value of weight of j th criteria for $j=1, \dots, n$. In Eqs. (45) to (49), and each individual \tilde{f}_{ij} is the fuzzy value of the i th alternative against j th criteria in the decision matrix. Similar to TOPSIS, the ideal and anti-ideal fuzzy values (points), \tilde{f}_j^* and \tilde{f}_{j-} for $j=1, \dots, m$ are obtained using Eqs. (46) and (47):

$$\tilde{f}_j^* = \{ (\max \tilde{f}_{ij} | j \in J), (\min \tilde{f}_{ij} | j \in J') | i = 1, 2, \dots, m \} \\ = \{ \tilde{f}_{1}^*, \tilde{f}_{2}^*, \dots, \tilde{f}_{j}^*, \tilde{f}_{n}^* \} \tag{46}$$

$$\tilde{f}_{j-} = \{ (\min \tilde{f}_{ij} | j \in J), (\max \tilde{f}_{ij} | j \in J') | i = 1, 2, \dots, m \} \\ = \{ \tilde{f}_{1-}, \tilde{f}_{2-}, \dots, \tilde{f}_{j-}, \tilde{f}_{n-} \} \tag{47}$$

where $J=\{j=1, 2, \dots, n | j, \text{ related to the benefit criteria}\}$; $J'=\{j=1, 2, \dots, n | j, \text{ related to the cost criteria}\}$; for $i=1, 2, \dots, m$ and $j=1, 2, \dots, n$, each individual \tilde{f}_{ij} is the amount of i th alternative against j th criteria in the decision matrix for $i=1, 2, \dots, m, j=1, 2, \dots, n$, and $\{\tilde{f}_1^*, \tilde{f}_2^*, \dots, \tilde{f}_j^*, \tilde{f}_n^*\}$; and $\{\tilde{f}_{1-}, \tilde{f}_{2-}, \dots, \tilde{f}_{j-}, \tilde{f}_{n-}\}$ are the ideal and anti-ideal fuzzy values of alternatives in the fuzzy decision matrix versus each individual criteria.

2.3.7.2. *Calculating the \tilde{S}_i and \tilde{R}_i under fuzzy environment.* \tilde{S}_i and \tilde{R}_i are calculated through Eqs. (48) and (49) as follows:

$$\tilde{S}_i = \sum_{j=1}^n \tilde{w}_j \cdot \frac{\tilde{f}_j^* - \tilde{f}_{ij}}{\tilde{f}_j^* - \tilde{f}_{j-}} \tag{48}$$

where \tilde{f}_j^* , \tilde{f}_{j-} , and \tilde{f}_{ij} are defined parameters derived through Eqs. (45) to (47).

$$\tilde{R}_i = \max \left(\tilde{w}_j \cdot \frac{\tilde{f}_j^* - \tilde{f}_{ij}}{\tilde{f}_j^* - \tilde{f}_{j-}} \right) \tag{49}$$

2.3.7.3. *Calculating the \tilde{Q}_i .* Q_i is calculated as shown in Eq. (50).

$$\tilde{Q}_i = \nu \cdot \left(\frac{\tilde{S}_j - \tilde{S}^*}{\tilde{S}^- - \tilde{S}^*} \right) + (1 - \nu) \cdot \left(\frac{\tilde{R}_j - \tilde{R}^*}{\tilde{R}^- - \tilde{R}^*} \right) \tag{50}$$

where $\tilde{S}^* = \min (\tilde{S}_j)$, $\tilde{S}^- = \max (\tilde{S}_j)$, $\tilde{R}^- = \max (\tilde{R}_j)$, $\tilde{R}^* = \min (\tilde{R}_j)$, ν is the defined weight for maximum group utility (majority of the criteria), and $(1 - \nu)$ is the defined weight for individual regret.

2.3.7.4. *Developing final ranks of alternatives based on the fuzzy values of \tilde{S} , \tilde{R} and \tilde{Q} .* Typically, alternatives are ranked using three different ways on the basis of the \tilde{S} , \tilde{R} and \tilde{Q} values. Different conditions affect the rating process, as detailed below.

The alternative A_1 with the minimum \tilde{Q} can hold the best rank if the two subsequent conditions are met:

1. $\tilde{Q}(A_2) - \tilde{Q}(A_1) \geq D\tilde{Q}$, where A_2 is the alternative that is ranked second and $D\tilde{Q} = \frac{1}{J-1}$, where J is the number of alternatives.
2. When alternative A_1 holds the best rank on the basis of \tilde{Q} (i.e., A_1 has the minimum \tilde{Q}), A_1 also should have the best rank on the basis of \tilde{S} , \tilde{R} , or both. Furthermore, for the compromise solution to be stable, ν must be greater than 0.5.

If either condition 1 or 2 is not satisfied, the ranking will be performed subject to the following rules:

1. Alternatives A_1 and A_2 may share the same rank if only condition 2 is not satisfied.
2. Alternatives $\tilde{A}_1, \tilde{A}_2, \dots, \tilde{A}_m$ may share the same rank if only condition 1 is not satisfied. The value for \tilde{A}_m is specified through the consideration that $\tilde{Q}(\tilde{A}_m) - \tilde{Q}(\tilde{A}_1) < D\tilde{Q}$.

2.4. Numerical examples

In this section, we develop an example to show the extension of defuzzification process.

2.4.1. Transforming fuzzy numbers to crisp values through defuzzification methods

In order to compare two fuzzy values, they should be defuzzified. Several defuzzification methods have been developed to transfer a fuzzy value to a crisp value. In order to have consistency in the defuzzification process, Yager's centroid index was utilized throughout the required defuzzification for all ranking methods.

2.4.2. Ranking of fuzzy numbers based on centroid index

Based on Yager's centroid index, the geometric center of fuzzy number \tilde{x} on the horizontal axis is calculated as follows:

$$x_0 = \frac{\int_0^1 g(x) \mu_x dx}{\int_0^1 \mu_x dx} \tag{51}$$

where $g(x)$ can be considered as a weight function of x values. Usually, $g(x)$ is assumed equal to x , μ_x is the membership degree of x values, and the denominator is considered as a normalizing factor equal to the total area under the membership degree function in Fig. 1 [12,42]. Fuzzy numbers can be ranked based on x_0 values, with higher x_0 values standing for better rank. According to Eq. (51), Yager's centroid index can be considered as a weighted mean of fuzzy number $\tilde{x} = (l, m, u)$. Eqs. (52) to (55) offer a numerical example of calculating the centroid index for ranking two fuzzy numbers, $\tilde{x}_1 = (0.2917, 0.3194, 0.3472)$ and $\tilde{x}_2 =$

(0.3056, 0.3264, 0.3444).

$$\mu_{\tilde{x}_1} = \begin{cases} 0 & x \leq 0.2917 \\ \frac{x - 0.2917}{0.3194 - 0.2917} & 0.2917 < x \leq 0.3194 \\ \frac{0.3472 - x}{x - 0.3194} & 0.3194 < x \leq 0.3472 \\ 0 & x > 0.3472 \end{cases} \quad (52)$$

$$x_{0_1} = \frac{\int_{0.2917}^{0.3194} x * \left(\frac{x-0.2917}{0.3194-0.2917}\right) dx + \int_{0.3194}^{0.3472} x * \left(\frac{0.3472-x}{0.3472-0.3194}\right) dx}{\int_{0.2917}^{0.3194} \left(\frac{x-0.2917}{0.3194-0.2917}\right) dx + \int_{0.3194}^{0.3472} \left(\frac{0.3472-x}{0.3472-0.3194}\right) dx} = 0.3134 \quad (53)$$

$$\mu_{\tilde{x}_2} = \begin{cases} 0 & x \leq 0.3056 \\ \frac{x - 0.3056}{0.3264 - 0.3056} & 0.3056 < x \leq 0.3264 \\ \frac{0.3444 - x}{x - 0.3264} & 0.3264 < x \leq 0.3444 \\ 0 & x > 0.3444 \end{cases} \quad (54)$$

$$x_{0_2} = \frac{\int_{0.3056}^{0.3264} x * \left(\frac{x-0.3056}{0.3264-0.3056}\right) dx + \int_{0.3264}^{0.3444} x * \left(\frac{0.3444-x}{0.3444-0.3264}\right) dx}{\int_{0.3056}^{0.3264} \left(\frac{x-0.3056}{0.3264-0.3056}\right) dx + \int_{0.3264}^{0.3444} \left(\frac{0.3444-x}{0.3444-0.3264}\right) dx} = 0.3248 \quad (55)$$

where, by ranking based on the calculated x_{0_1} and x_{0_2} , $\tilde{x}_2 > \tilde{x}_1$.

2.5. Numerical examples for developing the final rankings through different ranking methods

In this section, we use a decision matrix with three alternatives against three criteria to present examples of how different ranking methods develop final rankings. In the provided numerical example, in order to the simplification, for all alternatives, all the criteria have been considered as benefit criteria. All calculations required through these numerical examples are developed based on the algebraic and mathematical computations of fuzzy numbers that have been explained in Eqs. (3)–(21).

Decision Matrix

$$= \begin{bmatrix} (21, 23, 25) & (28, 30, 32) & (61.5, 63, 64.5) \\ (22, 23.5, 24.8) & (25, 27, 29) & (85.5, 87, 88.5) \\ (47.5, 48, 48.5) & (81, 83, 85) & (14.5, 16, 17.5) \end{bmatrix} \quad (56)$$

The normalized decision matrix values are calculated by introduced method in Eq. (26) as follows:

Normalized Decision Matrix

$$= \begin{bmatrix} \frac{(21, 23, 25)}{\max\{25, 24.8, 48.5\}} & \frac{(28, 30, 32)}{\max\{32, 29, 85\}} & \frac{(61.5, 63, 64.5)}{\max\{64.5, 88.5, 17.5\}} \\ \frac{(22, 23.5, 24.8)}{\max\{25, 24.8, 48.5\}} & \frac{(25, 27, 29)}{\max\{32, 29, 85\}} & \frac{(85.5, 87, 88.5)}{\max\{64.5, 88.5, 17.5\}} \\ \frac{(47.5, 48, 48.5)}{\max\{25, 24.8, 48.5\}} & \frac{(81, 83, 85)}{\max\{32, 29, 85\}} & \frac{(14.5, 16, 17.5)}{\max\{64.5, 88.5, 17.5\}} \end{bmatrix} \quad (57)$$

$$\text{Normalized Decision Matrix} = \begin{bmatrix} \frac{(21, 23, 25)}{48.5} & \frac{(28, 30, 32)}{85} & \frac{(61.5, 63, 64.5)}{88.5} \\ \frac{(22, 23.5, 24.8)}{48.5} & \frac{(25, 27, 29)}{85} & \frac{(85.5, 87, 88.5)}{88.5} \\ \frac{(47.5, 48, 48.5)}{48.5} & \frac{(81, 83, 85)}{85} & \frac{(14.5, 16, 17.5)}{88.5} \end{bmatrix}$$

$$= \begin{bmatrix} (0.4330, 0.4742, 0.5155) & (0.3294, 0.3529, 0.3765) & (0.6949, 0.7119, 0.7288) \\ (0.4536, 0.4845, 0.5113) & (0.2941, 0.3176, 0.3412) & (0.9661, 0.9831, 1.0000) \\ (0.9794, 0.9897, 1.0000) & (0.9529, 0.9765, 1.0000) & (0.1638, 0.1808, 0.1977) \end{bmatrix} \quad (58)$$

The evaluated weights for three criteria are as follows: $\tilde{w}_1 = (0.799, 0.8, 0.801)$, $\tilde{w}_2 = (0.4, 0.5, 0.6)$, and $\tilde{w}_3 = (0.3, 0.4, 0.5)$.

The normalized weights are calculated by Eq. (59) as follows:

$$\tilde{w} = \frac{\tilde{w}_1}{\sum \tilde{w}_j}, \frac{\tilde{w}_2}{\sum \tilde{w}_j}, \frac{\tilde{w}_3}{\sum \tilde{w}_j} \quad (59)$$

where, $\sum \tilde{w}_j = (0.799, 0.8, 0.801) + (0.4, 0.5, 0.6) + (0.3, 0.4, 0.5) = (1.499, 1.700, 1.901)$.

Therefore, the normalized weights are calculated as follows:

$$\text{Normalized } \tilde{w}_1 = \frac{(0.799, 0.8, 0.801)}{(1.499, 1.700, 1.901)}$$

$$= \left(\frac{0.799}{1.901}, \frac{0.8}{1.700}, \frac{0.801}{1.499}\right) = (0.4203, 0.4706, 0.5344)$$

$$\text{Normalized } \tilde{w}_2 = \frac{(0.799, 0.8, 0.801)}{(1.499, 1.700, 1.901)}$$

$$= \left(\frac{0.4}{1.901}, \frac{0.5}{1.700}, \frac{0.6}{1.499}\right) = (0.2104, 0.2941, 0.4003)$$

$$\text{Normalized } \tilde{w}_3 = \frac{(0.799, 0.8, 0.801)}{(1.499, 1.700, 1.901)}$$

$$= \left(\frac{0.3}{1.901}, \frac{0.4}{1.700}, \frac{0.5}{1.499}\right) = (0.1578, 0.2353, 0.3336)$$

In order to calculate the weighted and normalized decision matrix, matrices of weights are formed as shown in Eq. (60).

$$\tilde{w} = \begin{bmatrix} \tilde{w}_{1,1}(0, 0, 0) & (0, 0, 0) \\ (0, 0, 0) & \tilde{w}_{2,2}(0, 0, 0) \\ (0, 0, 0) & (0, 0, 0) & \tilde{w}_{3,3} \end{bmatrix} = \begin{bmatrix} (0.4203, 0.4706, 0.5344), & (0, 0, 0), & (0, 0, 0) \\ (0, 0, 0), & (0.2104, 0.2941, 0.4003), & (0, 0, 0) \\ (0, 0, 0), & (0, 0, 0), & (0.1578, 0.2353, 0.3336) \end{bmatrix} \quad (60)$$

The weighted, normalized matrix is calculated as follows:

$$\tilde{d}_{ij} = [\tilde{r}_{i,j}] \otimes [\tilde{w}_{j,j}] = \begin{bmatrix} \tilde{r}_{1,1} \otimes \tilde{w}_1, & \tilde{r}_{1,2} \otimes \tilde{w}_2, & \tilde{r}_{1,3} \otimes \tilde{w}_3 \\ \tilde{r}_{2,1} \otimes \tilde{w}_1, & \tilde{r}_{2,2} \otimes \tilde{w}_2, & \tilde{r}_{2,3} \otimes \tilde{w}_3 \\ \tilde{r}_{3,1} \otimes \tilde{w}_1, & \tilde{r}_{3,2} \otimes \tilde{w}_2, & \tilde{r}_{3,3} \otimes \tilde{w}_3 \end{bmatrix} = \begin{bmatrix} (0.4330, 0.4742, 0.5155) \otimes (0.4203, 0.4706, 0.5344); & & \\ (0.3294, 0.3529, 0.3765) \otimes (0.2104, 0.2941, 0.4003); & & \\ (0.6949, 0.7119, 0.7288) \otimes (0.1578, 0.2353, 0.3336) & & \\ (0.4536, 0.4845, 0.5113) \otimes (0.4203, 0.4706, 0.5344); & & \\ (0.2941, 0.3176, 0.3412) \otimes (0.2104, 0.2941, 0.4003); & & \\ (0.9661, 0.9831, 1.0000) \otimes (0.1578, 0.2353, 0.3336) & & \\ (0.9794, 0.9897, 1.0000) \otimes (0.4203, 0.4706, 0.5344); & & \\ (0.9529, 0.9765, 1.0000) \otimes (0.2104, 0.2941, 0.4003); & & \\ (0.1638, 0.1808, 0.1977) \otimes (0.1578, 0.2353, 0.3336) & & \\ (0.1820, 0.2232, 0.2754); & (0.0693, 0.1038, 0.1507); & \\ (0.1097, 0.1675, 0.2431) & & \\ (0.1907, 0.2280, 0.2732); & (0.0619, 0.0934, 0.1366); & \\ (0.1525, 0.2313, 0.3336) & & \\ (0.4116, 0.4657, 0.5344); & (0.2005, 0.2872, 0.4003); & \\ (0.0259, 0.0425, 0.0660) & & \end{bmatrix}$$

where, $i = 1, 2, \dots, m$, and $j = 1, 2, \dots, n$.

2.5.1. Final ranking based on SAW

The overall score of each individual alternative is obtained through the sum of weighted normalized values as follows:

$$\begin{aligned} \tilde{F}_1 &= \sum_{j=1}^3 \tilde{r}_{1j} \cdot \tilde{w}_j = \tilde{w}_j \cdot \tilde{r}_{1j} \\ &= \{((0.4203, 0.4706, 0.5344) \times (0.4330, 0.4742, 0.5155)) \\ &\quad + ((0.2104, 0.2941, 0.4003) \times (0.3294, 0.3529, 0.3765)) \\ &\quad + ((0.1578, 0.2353, 0.3336) \times (0.6949, 0.7119, 0.7288))\} \\ &= (0.3610, 0.4945, 0.6692) \\ \tilde{F}_2 &= \sum_{j=1}^3 \tilde{r}_{2j} \cdot \tilde{w}_j = \tilde{w}_j \cdot \tilde{r}_{2j} \\ &= \{(((0.4203, 0.4706, 0.5344) \times (0.4536, 0.4845, 0.5113))) \\ &\quad + ((0.2104, 0.2941, 0.4003) \times (0.2941, 0.3176, 0.3412))) \\ &\quad + ((0.1578, 0.2353, 0.3336) \times (0.9661, 0.9831, 1.0000))\} \\ &= (0.4049, 0.5527, 0.7434) \\ \tilde{F}_3 &= \sum_{j=1}^3 \tilde{r}_{3j} \cdot \tilde{w}_j = \tilde{w}_j \cdot \tilde{r}_{3j} \\ &= \{(((0.4203, 0.4706, 0.5344) \times (0.9794, 0.9897, 1.0000)) \\ &\quad + ((0.2104, 0.2941, 0.4003) \times (0.9529, 0.9765, 1.0000)) \\ &\quad + ((0.1578, 0.2353, 0.3336) \times (0.1638, 0.1808, 0.1977)))\} \\ &= (0.6380, 0.7955, 1.0006) \end{aligned}$$

In SAW, the final ranking is developed based on the centroids values of \tilde{F}_i , which are calculated as follows:

$$\begin{aligned} \tilde{F}_1 &= \frac{0.3610 + 0.4945 + 0.6692}{3} = 0.5082, \\ \tilde{F}_2 &= \frac{0.4049 + 0.5527 + 0.7434}{3} = 0.5670, \text{ and} \\ \tilde{F}_3 &= \frac{0.6380 + 0.7995 + 1.0006}{3} = 0.8114. \end{aligned}$$

Therefore, $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$.

2.5.2. Final ranking based on WPM

According to Eq. (24), $\tilde{A}_i^* = \prod_{j=1}^n \tilde{a}_{ij}^{\tilde{w}_j}$, \tilde{A}_1^* , \tilde{A}_2^* , and \tilde{A}_3^* are calculated as follows:

$$\begin{aligned} \tilde{A}_1^* &= \prod_{j=1}^3 \tilde{a}_{1j}^{\tilde{w}_j} = (21, 23, 25)^{(0.4203, 0.4706, 0.5344)} \\ &\quad \times (28, 30, 32)^{(0.2104, 0.2941, 0.4003)} \\ &\quad \times (61.5, 63, 64.5)^{(0.1578, 0.2353, 0.3336)} = (13.88, 31.52, 89.79) \\ \tilde{A}_2^* &= \prod_{j=1}^3 \tilde{a}_{2j}^{\tilde{w}_j} = (22, 23.5, 24.8)^{(0.4203, 0.4706, 0.5344)} \\ &\quad \times (25, 27, 29)^{(0.2104, 0.2941, 0.4003)} \times (85.5, 87, 88.5)^{(0.1578, 0.2353, 0.3336)} \\ &= (14.56, 33.31, 95.51) \\ \tilde{A}_3^* &= \prod_{j=1}^3 \tilde{a}_{3j}^{\tilde{w}_j} = (47.5, 48, 48.5)^{(0.4203, 0.4706, 0.5344)} \\ &\quad \times (81, 83, 85)^{(0.2104, 0.2941, 0.4003)} \times (14.5, 16, 17.5)^{(0.1578, 0.2353, 0.3336)} \\ &= (19.48, 43.54, 122.37) \end{aligned}$$

Ranking is developed based on the centroid values of \tilde{A}_i^* which are calculated as follows:

$$\begin{aligned} \tilde{A}_1^* &= \frac{13.88 + 31.52 + 89.79}{3} = 45.06, \\ \tilde{A}_2^* &= \frac{14.56 + 33.31 + 95.48}{3} = 47.79, \text{ and} \\ \tilde{A}_3^* &= \frac{19.48 + 43.54 + 122.37}{3} = 61.80. \end{aligned}$$

Therefore, $\tilde{A}_1^* < \tilde{A}_2^* < \tilde{A}_3^*$.

2.5.3. Final ranking based on CP

By setting $P=2$ and considering Eq. (24), \tilde{D}_1^p , \tilde{D}_2^p , and \tilde{D}_3^p are calculated as follows:

$$\begin{aligned} \text{for } i = 1, \tilde{D}_1^p &= \left(\sum_{j=1}^3 \left(\tilde{w}_j \cdot \frac{\tilde{a}_{1j}^* - \tilde{a}_{1j}}{\tilde{a}_{1j}^* - \tilde{a}_{j^*}} \right)^p \right)^{\frac{1}{p}} \\ &= \left(\left(\tilde{w}_1 \times \frac{\tilde{a}_{1j}^* - \tilde{a}_{12}}{\tilde{a}_{1j}^* - \tilde{a}_{j^*}} \right)^2 + \left(\tilde{w}_2 \times \frac{\tilde{a}_{1j}^* - \tilde{a}_{12}}{\tilde{a}_{1j}^* - \tilde{a}_{j^*}} \right)^2 + \left(\tilde{w}_3 \times \frac{\tilde{a}_{1j}^* - \tilde{a}_{13}}{\tilde{a}_{1j}^* - \tilde{a}_{j^*}} \right)^2 \right)^{\frac{1}{2}} \\ &= \left(\left((0.4203, 0.4706, 0.5344) \times \frac{(48.5, 48.5, 48.5) - (28, 30, 32)}{(48.5, 48.5, 48.5) - (21.0, 21.0, 21.0)} \right)^2 \right. \\ &\quad \left. + \left((0.2104, 0.2941, 0.4003) \times \frac{(85, 85, 85) - (61.5, 63, 64.5)}{(85, 85, 85) - (25, 25, 25)} \right)^2 \right. \\ &\quad \left. + \left((0.1578, 0.2353, 0.3336) \times \frac{(88.5, 88.5, 88.5) - (61.5, 63, 64.5)}{(88.5, 88.5, 88.5) - (14.5, 14.5, 14.5)} \right)^2 \right)^{\frac{1}{2}} \\ &= (0.4076, 0.5193, 0.6770). \\ \text{for } i = 2, \tilde{D}_2^p &= \left(\sum_{j=1}^3 \left(\tilde{w}_j \cdot \frac{\tilde{a}_{2j}^* - \tilde{a}_{2j}}{\tilde{a}_{2j}^* - \tilde{a}_{j^*}} \right)^p \right)^{\frac{1}{p}} \\ &= \left(\left(\tilde{w}_1 \times \frac{\tilde{a}_{2j}^* - \tilde{a}_{22}}{\tilde{a}_{2j}^* - \tilde{a}_{j^*}} \right)^2 + \left(\tilde{w}_2 \times \frac{\tilde{a}_{2j}^* - \tilde{a}_{22}}{\tilde{a}_{2j}^* - \tilde{a}_{j^*}} \right)^2 + \left(\tilde{w}_3 \times \frac{\tilde{a}_{2j}^* - \tilde{a}_{23}}{\tilde{a}_{2j}^* - \tilde{a}_{j^*}} \right)^2 \right)^{\frac{1}{2}} \\ &= \left(\left((0.4203, 0.4706, 0.5344) \times \frac{(48.5, 48.5, 48.5) - (22, 23.5, 24.8)}{(48.5, 48.5, 48.5) - (21.0, 21.0, 21.0)} \right)^2 \right. \\ &\quad \left. + \left((0.2104, 0.2941, 0.4003) \times \frac{(85, 85, 85) - (25, 27, 29)}{(85, 85, 85) - (25, 25, 25)} \right)^2 \right. \\ &\quad \left. + \left((0.1578, 0.2353, 0.3336) \times \frac{(88.5, 88.5, 88.5) - (85.5, 87, 88.5)}{(88.5, 88.5, 88.5) - (14.5, 14.5, 14.5)} \right)^2 \right)^{\frac{1}{2}} \\ &= (0.4120, 0.5137, 0.6523) \\ \text{for } i = 3, \tilde{D}_3^p &= \left(\sum_{j=1}^3 \left(\tilde{w}_j \cdot \frac{\tilde{a}_{3j}^* - \tilde{a}_{3j}}{\tilde{a}_{3j}^* - \tilde{a}_{j^*}} \right)^p \right)^{\frac{1}{p}} \\ &= \left(\left(\tilde{w}_1 \times \frac{\tilde{a}_{3j}^* - \tilde{a}_{32}}{\tilde{a}_{3j}^* - \tilde{a}_{j^*}} \right)^2 + \left(\tilde{w}_2 \times \frac{\tilde{a}_{3j}^* - \tilde{a}_{32}}{\tilde{a}_{3j}^* - \tilde{a}_{j^*}} \right)^2 + \left(\tilde{w}_3 \times \frac{\tilde{a}_{3j}^* - \tilde{a}_{33}}{\tilde{a}_{3j}^* - \tilde{a}_{j^*}} \right)^2 \right)^{\frac{1}{2}} \\ &= \left(\left((0.4203, 0.4706, 0.5344) \times \frac{(48.5, 48.5, 48.5) - (47.5, 48, 48.5)}{(48.5, 48.5, 48.5) - (21.0, 21.0, 21.0)} \right)^2 \right. \\ &\quad \left. + \left((0.2104, 0.2941, 0.4003) \times \frac{(85, 85, 85) - (81, 83, 85)}{(85, 85, 85) - (25, 25, 25)} \right)^2 \right. \\ &\quad \left. + \left((0.1578, 0.2353, 0.3336) \times \frac{(88.5, 88.5, 88.5) - (14.5, 16, 17.5)}{(88.5, 88.5, 88.5) - (14.5, 14.5, 14.5)} \right)^2 \right)^{\frac{1}{2}} \\ &= (0.1514, 0.2309, 0.3352) \end{aligned}$$

For the ideal and anti-ideal points, in order to avoid having negative values for lower and upper levels through the calculation process, their upper and lower values have been assumed equal to their central values. Ranking is developed based on the centroid values of \tilde{D}_i^p , which are calculated as follows:

$$\begin{aligned} \tilde{D}_1 &= \frac{0.4076 + 0.5193 + 0.6770}{3} = 0.5313, \\ \tilde{D}_2 &= \frac{0.4120 + 0.5137 + 0.6523}{3} = 0.5260, \text{ and} \\ \tilde{D}_3 &= \frac{0.1514 + 0.2309 + 0.3352}{3} = 0.2392. \end{aligned}$$

In the compromise programming method, a lower value of \tilde{D}_i^p is ranked as $\tilde{D}_1 > \tilde{D}_2 > \tilde{D}_3$. Therefore, $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$.

2.5.4. Final ranking based on TOPSIS

After calculating the weighted normalized decision matrix, geometrical distances of alternatives from both ideal $\tilde{A}_1^+ = (1, 1, 1)$ and anti-ideal $\tilde{A}_1^- = (0, 0, 0)$ distances are calculated as follows:

$$\begin{aligned}
 s_1^+ &= \sqrt{\sum_{j=1}^n d(\tilde{\vartheta}_{ij}, \tilde{\vartheta}_{1j}^+)} \\
 &= \sqrt{((1, 1, 1) - (0.1820, 0.2232, 0.2754))^2 + ((1, 1, 1) - (0.0693, 0.1038, 0.1507))^2 + ((1, 1, 1) - (0.1097, 0.1675, 0.2431))^2} \\
 &= \sqrt{(1-0.2754)^2 + (0.2232)^2 + (1-0.1820)^2 + (1-0.1507)^2 + (1-0.1038)^2 + (1-0.0693)^2 + (1-0.2431)^2 + (1-0.1675)^2 + (1-0.1097)^2} \\
 &= 2.58
 \end{aligned}$$

With the same approach: $s_2^+ = 2.28$, $s_3^+ = 2.20$, $s_1^- = 0.45$, $s_2^- = 0.78$, and $s_3^- = 0.81$.

In order to develop the final ranking, the relative closeness coefficients are calculated as follows:

$$C_1 = \frac{s_1^-}{s_1^- + s_1^+} = \frac{0.45}{0.45 + 2.58} = 0.149, \quad C_2 = \frac{s_2^-}{s_2^- + s_2^+} = \frac{0.78}{0.78 + 2.28} = 0.255, \quad \text{and} \quad C_3 = \frac{s_3^-}{s_3^- + s_3^+} = \frac{0.81}{0.81 + 2.20} = 0.269$$

$C_1 > C_2 > C_3$. Therefore, $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$

2.5.5. Final ranking based on AHP

In order to develop the final ranking of alternatives through the AHP method, Saaty’s four major prioritizing methods are calculated and evaluated. These methods are described in Sections 2.5.5.1–2.5.5.4 [36].

2.5.5.1. Final ranking based on the AHP prioritizing method 1. General form of comparisonwise matrix as explained in Eq. (35), for three factors is developed as shown in Eq. (61):

$$[\tilde{C}] = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_1 & \tilde{C}_1 \\ \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 \\ \tilde{C}_2 & \tilde{C}_2 & \tilde{C}_2 \\ \tilde{C}_3 & \tilde{C}_3 & \tilde{C}_3 \\ \tilde{C}_1 & \tilde{C}_2 & \tilde{C}_3 \end{bmatrix} \tag{61}$$

In the above comparisonwise matrix \tilde{c} , for three factors in the first row, $\frac{\tilde{c}_1}{\tilde{c}_1}$, $\frac{\tilde{c}_1}{\tilde{c}_2}$, and $\frac{\tilde{c}_1}{\tilde{c}_3}$ are representatives for comparing \tilde{c}_1 against \tilde{c}_1 , \tilde{c}_2 , and \tilde{c}_3 .

The priority vector according to method AHP1 is calculated as follows:

As explained in the Section 2.3.5, the sum of each row is normalized by the sum of all elements of the comparisonwise matrix. Then, the importance of each alternative compared to the other alternatives against the associated criteria in the comparisonwise matrix is obtained.

It should be noted that, division of two same fuzzy values is assumed as 1 through all computation process, and it does not follow division rules for two fuzzy values as shown in Eq. (15). Considering the Eq. (35) and Eq. (15), for $\tilde{A}_{11} = (21, 23, 25)$, $\tilde{A}_{21} = (22, 23.5, 24.8)$, and $\tilde{A}_{31} = (47.5, 48, 48.5)$, priority vectors of alternatives $\tilde{c}_{\tilde{a}_{i1}}$, $\tilde{c}_{\tilde{a}_{i2}}$, and $\tilde{c}_{\tilde{a}_{i3}}$ are calculated as follows:

$$\begin{aligned}
 \tilde{c}_{\tilde{a}_{i1}} &= \begin{bmatrix} \frac{\tilde{a}_{11}}{\tilde{a}_{11}} & \frac{\tilde{a}_{11}}{\tilde{a}_{21}} & \frac{\tilde{a}_{11}}{\tilde{a}_{31}} \\ \frac{\tilde{a}_{21}}{\tilde{a}_{11}} & \frac{\tilde{a}_{21}}{\tilde{a}_{21}} & \frac{\tilde{a}_{21}}{\tilde{a}_{31}} \\ \frac{\tilde{a}_{31}}{\tilde{a}_{11}} & \frac{\tilde{a}_{31}}{\tilde{a}_{21}} & \frac{\tilde{a}_{31}}{\tilde{a}_{31}} \end{bmatrix} = \begin{bmatrix} \frac{(21, 23, 25)}{(21, 23, 25)} & \frac{(21, 23, 25)}{(22, 23.5, 24.8)} & \frac{(21, 23, 25)}{(47.5, 48, 48.5)} \\ \frac{(22, 23.5, 24.8)}{(21, 23, 25)} & \frac{(22, 23.5, 24.8)}{(22, 23.5, 24.8)} & \frac{(22, 23.5, 24.8)}{(47.5, 48, 48.5)} \\ \frac{(47.5, 48, 48.5)}{(21, 23, 25)} & \frac{(47.5, 48, 48.5)}{(22, 23.5, 24.8)} & \frac{(47.5, 48, 48.5)}{(47.5, 48, 48.5)} \end{bmatrix} \\
 &= \begin{bmatrix} (1.0000, 1.0000, 1.0000)(0.8468, 0.9787, 1.1364)(0.4330, 0.4792, 0.5263) \\ (0.8800, 1.0217, 1.1810)(1.0000, 1.0000, 1.0000)(0.4536, 0.4896, 0.5221) \\ (1.9000, 2.0870, 2.3095)(1.9153, 2.0426, 2.2045)(1.0000, 1.0000, 1.0000) \end{bmatrix} \tag{62}
 \end{aligned}$$

According to the method 1, priority vector for $\tilde{c}_{\tilde{a}_{i1}}$ (for $i = 1, 2,$ and 3) is calculated as follows:

$$\text{Priority of } \tilde{C}_{\tilde{a}_{11}} = \left[\begin{array}{c} \frac{\tilde{a}_{11}}{\tilde{a}_{11} + \tilde{a}_{21} + \tilde{a}_{31}} + \frac{\tilde{a}_{11}}{\tilde{a}_{12} + \tilde{a}_{21} + \tilde{a}_{31}} + \frac{\tilde{a}_{11}}{\tilde{a}_{21} + \tilde{a}_{31} + \tilde{a}_{11}} \\ \frac{\tilde{a}_{21}}{\tilde{a}_{11} + \tilde{a}_{21} + \tilde{a}_{31}} + \frac{\tilde{a}_{21}}{\tilde{a}_{12} + \tilde{a}_{21} + \tilde{a}_{31}} + \frac{\tilde{a}_{21}}{\tilde{a}_{21} + \tilde{a}_{31} + \tilde{a}_{11}} \\ \frac{\tilde{a}_{31}}{\tilde{a}_{11} + \tilde{a}_{21} + \tilde{a}_{31}} + \frac{\tilde{a}_{31}}{\tilde{a}_{12} + \tilde{a}_{21} + \tilde{a}_{31}} + \frac{\tilde{a}_{31}}{\tilde{a}_{21} + \tilde{a}_{31} + \tilde{a}_{11}} \end{array} \right] = \left[\begin{array}{c} (0.2095, 0.2434, 0.2824) \\ (0.2145, 0.2487, 0.2867) \\ (0.4426, 0.5079, 0.5848) \end{array} \right]$$

For $\tilde{c}_{12} = (28, 30, 32)$, $\tilde{c}_{22} = (25, 27, 29)$, and $\tilde{c}_{32} = (81, 83, 85)$

$$\tilde{C}_{\tilde{a}_{12}} = \left[\begin{array}{c} \frac{\tilde{a}_{12}}{\tilde{a}_{12}} , \frac{\tilde{a}_{12}}{\tilde{a}_{22}} , \frac{\tilde{a}_{12}}{\tilde{a}_{32}} \\ \frac{\tilde{a}_{22}}{\tilde{a}_{12}} , \frac{\tilde{a}_{22}}{\tilde{a}_{22}} , \frac{\tilde{a}_{22}}{\tilde{a}_{32}} \\ \frac{\tilde{a}_{32}}{\tilde{a}_{12}} , \frac{\tilde{a}_{32}}{\tilde{a}_{22}} , \frac{\tilde{a}_{32}}{\tilde{a}_{32}} \end{array} \right] = \left[\begin{array}{c} \frac{(28, 30, 32)}{(28, 30, 32)} , \frac{(28, 30, 32)}{(25, 27, 29)} , \frac{(28, 30, 32)}{(81, 83, 85)} \\ \frac{(25, 27, 29)}{(28, 30, 32)} , \frac{(25, 27, 29)}{(25, 27, 29)} , \frac{(25, 27, 29)}{(81, 83, 85)} \\ \frac{(81, 83, 85)}{(28, 30, 32)} , \frac{(81, 83, 85)}{(25, 27, 29)} , \frac{(81, 83, 85)}{(81, 83, 85)} \end{array} \right]$$

$$= \left[\begin{array}{c} (1.0000, 1.0000, 1.0000)(0.9655, 1.1111, 1.2800)(0.3294, 0.3614, 0.3951) \\ (0.7813, 0.9000, 1.0357)(1.0000, 1.0000, 1.0000)(0.2941, 0.3253, 0.3580) \\ (2.5313, 2.7667, 3.0357)(2.7931, 3.0741, 3.4000)(1.0000, 1.0000, 1.0000) \end{array} \right]$$

$$\text{Priority of } \tilde{C}_{\tilde{a}_{12}} = \left[\begin{array}{c} (0.1835, 0.2143, 0.2501) \\ (0.1660, 0.1929, 0.2238) \\ (0.5058, 0.5929, 0.6953) \end{array} \right]$$

For $\tilde{c}_{13} = (61.5, 63, 64.5)$, $\tilde{c}_{23} = (85.5, 87, 88.5)$, and $\tilde{c}_{33} = (14.5, 16, 17.5)$

$$\tilde{C}_{\tilde{a}_{13}} = \left[\begin{array}{c} \frac{\tilde{a}_{13}}{\tilde{a}_{13}} , \frac{\tilde{a}_{13}}{\tilde{a}_{23}} , \frac{\tilde{a}_{13}}{\tilde{a}_{33}} \\ \frac{\tilde{a}_{23}}{\tilde{a}_{13}} , \frac{\tilde{a}_{23}}{\tilde{a}_{23}} , \frac{\tilde{a}_{23}}{\tilde{a}_{33}} \\ \frac{\tilde{a}_{33}}{\tilde{a}_{13}} , \frac{\tilde{a}_{33}}{\tilde{a}_{23}} , \frac{\tilde{a}_{33}}{\tilde{a}_{33}} \end{array} \right] \text{ Priority of } \tilde{C}_{\tilde{a}_{13}} = \left[\begin{array}{c} (0.3209, 0.3795, 0.4492) \\ (0.4442, 0.5241, 0.6186) \\ (0.0855, 0.0964, 0.1078) \end{array} \right]$$

$\tilde{w}_1 = (0.4203, 0.4706, 0.5344)$, $\tilde{w}_2 = (0.2104, 0.2941, 0.4003)$, and $\tilde{w}_3 = (0.1578, 0.2353, 0.3336)$

$$\tilde{C}_{\tilde{w}} = \left[\begin{array}{c} \frac{\tilde{w}_1}{\tilde{w}_1} , \frac{\tilde{w}_1}{\tilde{w}_2} , \frac{\tilde{w}_1}{\tilde{w}_3} \\ \frac{\tilde{w}_2}{\tilde{w}_1} , \frac{\tilde{w}_2}{\tilde{w}_2} , \frac{\tilde{w}_2}{\tilde{w}_3} \\ \frac{\tilde{w}_3}{\tilde{w}_1} , \frac{\tilde{w}_3}{\tilde{w}_2} , \frac{\tilde{w}_3}{\tilde{w}_3} \end{array} \right]$$

$$= \left[\begin{array}{c} \frac{(0.4203, 0.4706, 0.5344)}{(0.4203, 0.4706, 0.5344)} , \frac{(0.4203, 0.4706, 0.5344)}{(0.2104, 0.2941, 0.4003)} , \frac{(0.4203, 0.4706, 0.5344)}{(0.1578, 0.2353, 0.3336)} \\ \frac{(0.2104, 0.2941, 0.4003)}{(0.4203, 0.4706, 0.5344)} , \frac{(0.2104, 0.2941, 0.4003)}{(0.2104, 0.2941, 0.4003)} , \frac{(0.2104, 0.2941, 0.4003)}{(0.1578, 0.2353, 0.3336)} \\ \frac{(0.1578, 0.2353, 0.3336)}{(0.4203, 0.4706, 0.5344)} , \frac{(0.1578, 0.2353, 0.3336)}{(0.2104, 0.2941, 0.4003)} , \frac{(0.1578, 0.2353, 0.3336)}{(0.1578, 0.2353, 0.3336)} \end{array} \right]$$

$$= \left[\begin{array}{c} (1, 1, 1) \left(\frac{0.4203}{0.4003} , \frac{0.4706}{0.2941} , \frac{0.5344}{0.2104} \right) \left(\frac{0.4203}{0.3336} , \frac{0.4706}{0.2353} , \frac{0.5344}{0.1578} \right) \\ \left(\frac{0.2104}{0.5344} , \frac{0.2941}{0.4706} , \frac{0.4003}{0.4203} \right) (1, 1, 1) \left(\frac{0.2104}{0.3336} , \frac{0.2941}{0.2353} , \frac{0.4003}{0.1578} \right) \\ \left(\frac{0.1578}{0.5344} , \frac{0.2353}{0.4706} , \frac{0.3336}{0.4203} \right) \left(\frac{0.1578}{0.4003} , \frac{0.2353}{0.2941} , \frac{0.3336}{0.2104} \right) (1, 1, 1) \end{array} \right]$$

$$= \begin{bmatrix} (1.0000, 1.0000, 1.0000)(1.3317, 1.6000, 2.0025)(1.5980, 2.0000, 2.6700) \\ (0.4994, 0.6250, 0.7509)(1.0000, 1.0000, 1.0000)(0.8000, 1.2500, 2.000) \\ (0.3745, 0.5000, 0.6258)(0.5000, 0.8000, 1.2500)(1.0000, 1.0000, 1.0000) \end{bmatrix}$$

Priority of $\tilde{C}_{\tilde{w}}$ = $\begin{bmatrix} (0.3195, 0.4706, 0.7000) \\ (0.1870, 0.2941, 0.4629) \\ (0.1524, 0.2353, 0.3549) \end{bmatrix}$

The final priority of alternatives is calculated as follows:

$$[\tilde{C}_{\tilde{a}_{11}}, \tilde{C}_{\tilde{a}_{12}}, \tilde{C}_{\tilde{a}_{13}}] \otimes [\tilde{C}_{\tilde{w}}] = \begin{bmatrix} (0.2095, 0.2434, 0.2824)(0.2145, 0.2487, 0.2867)(0.4426, 0.5079, 0.5848) \\ (0.1835, 0.2143, 0.2501)(0.1660, 0.1929, 0.2238)(0.5058, 0.5929, 0.6953) \\ (0.3209, 0.3795, 0.4492)(0.4442, 0.5241, 0.6186)(0.0855, 0.0964, 0.1078) \end{bmatrix}$$

$$\otimes \begin{bmatrix} (0.3195, 0.4706, 0.7000) \\ (0.1870, 0.2941, 0.4629) \\ (0.1524, 0.2353, 0.3549) \end{bmatrix} = \begin{bmatrix} (0.1502, 0.2669, 0.4729) \\ (0.1673, 0.2971, 0.5238) \\ (0.2490, 0.4361, 0.7695) \end{bmatrix}$$

In order to compare the final ranking, the centroids of the final priority ranking values are calculated as follows:

$$\tilde{F}_1 = \frac{0.1502 + 0.2669 + 0.4729}{3} = 0.2966, \tilde{F}_2 = \frac{0.1673 + 0.2971 + 0.5238}{3} = 0.3294, \text{ and } \tilde{F}_3 = \frac{0.2490 + 0.4361 + 0.7695}{3} = 0.4848.$$

Therefore, $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$.

2.5.5.2. *Final ranking based on the AHP prioritizing method 2.* As explained in Section 2.3.5, the sum of the values in each column of matrix \tilde{c} is calculated, and then the reciprocals, which are $1/(\text{Sum of the values of each column})$, are calculated. Next, the reciprocal values are normalized by the sum of all reciprocals. The normalized values of the reciprocals indicate the relative importance values of alternatives against the considered criteria in comparisonwise matrix.

Considering $\tilde{c}_{\tilde{w}}$, $\tilde{c}_{\tilde{a}_{11}}$, $\tilde{c}_{\tilde{a}_{12}}$, and $\tilde{c}_{\tilde{a}_{13}}$ in the above comparisonwise matrix \tilde{c} , for three factors in the first row, $\frac{\tilde{c}_1}{\tilde{c}_1}$, $\frac{\tilde{c}_1}{\tilde{c}_2}$, and $\frac{\tilde{c}_1}{\tilde{c}_3}$ are representatives for comparing \tilde{c}_1 against \tilde{c}_1 , \tilde{c}_2 , and \tilde{c}_3 .

For $\tilde{c}_{11} = (21, 23, 25)$, $\tilde{c}_{21} = (22, 23.5, 24.8)$, and $\tilde{c}_{31} = (47.5, 48, 48.5)$, the comparison matrix of alternatives against criteria 1 is obtained as follows:

$$\tilde{C}_{\tilde{a}_{11}} = \begin{bmatrix} (1.0000, 1.0000, 1.0000)(0.8468, 0.9787, 1.1364)(0.4330, 0.4792, 0.5263) \\ (0.8800, 1.0217, 1.1810)(1.0000, 1.0000, 1.0000)(0.4536, 0.4896, 0.5221) \\ (1.9000, 2.0870, 2.3095)(1.9153, 2.0426, 2.2045)(1.0000, 1.0000, 1.0000) \end{bmatrix}$$

According to method 2, the priority vector for $\tilde{c}_{\tilde{a}_{1j}}$ (for $j = 1, 2,$ and 3) is calculated as follows:

- By considering the comparison matrix against criteria 1, the sum of each column is calculated as follows:

$$(1.0000, 1.0000, 1.0000) + (0.8800, 1.0217, 1.1810) + (1.9000, 2.0870, 2.3095) = (3.7800, 4.1087, 4.4905)$$

$$(0.8468, 0.9787, 1.1364) + (1.0000, 1.0000, 1.0000) + (1.9153, 2.0426, 2.2045) = (3.7621, 4.0213, 4.3409)$$

$$(0.4330, 0.4792, 0.5263) + (0.4536, 0.4896, 0.5221) + (1.0000, 1.0000, 1.0000) = (1.8866, 1.9688, 2.0484)$$

Reciprocal of a triangular fuzzy number $\tilde{A} = (l, m, u)$ can be considered as $\frac{1}{\tilde{A}} = \tilde{A}^{-1}$ which equals to its inverse, which is calculated using Eq. (63) as follows:

$$\tilde{A}^{-1} = \left(\frac{1}{u}, \frac{1}{m}, \frac{1}{l} \right) \tag{63}$$

then,

- The reciprocals of the sums of the columns are calculated as follows:

$$(3.7800, 4.1087, 4.4905)^{-1} = (1/4.4905, 1/4.1087, 1/3.7800) = (0.2227, 0.2434, 0.2646)$$

$$(3.7621, 4.0213, 4.3409)^{-1} = (1/4.3409, 1/4.0213, 1/3.7621) = (0.2304, 0.2487, 0.2658)$$

$$(1.8866, 1.9688, 2.0484)^{-1} = (1/2.0484, 1/1.9688, 1/1.8866) = (0.4882, 0.5079, 0.5301)$$

then,

- The final priority weights of alternatives against criteria 1 is calculated based on normalizing the reciprocals of the sums of each column by sum of the reciprocals of all columns in a comparisonwise matrix of alternatives against criteria 1, as follows:

$$\frac{(0.2227, 0.2434, 0.2646)}{(0.2227, 0.2434, 0.2646) + (0.2304, 0.2487, 0.2658) + (0.4882, 0.5079, 0.5301)} = (0.2100, 0.2434, 0.2811)$$

$$\frac{(0.2304, 0.2487, 0.2658)}{(0.2227, 0.2434, 0.2646) + (0.2304, 0.2487, 0.2658) + (0.4882, 0.5079, 0.5301)} = (0.2172, 0.2487, 0.2824)$$

$$\frac{(0.4882, 0.5079, 0.5301)}{(0.2227, 0.2434, 0.2646) + (0.2304, 0.2487, 0.2658) + (0.4882, 0.5079, 0.5301)} = (0.4604, 0.5079, 0.5631)$$

Therefore,

$$\begin{aligned} \text{Priority of } \tilde{C}_{\tilde{a}_{11}} &= \begin{bmatrix} (0.2100, 0.2434, 0.2811) \\ (0.2172, 0.2487, 0.2824) \\ (0.4604, 0.5079, 0.5631) \end{bmatrix} & \text{Priority of } \tilde{C}_{\tilde{a}_{12}} &= \begin{bmatrix} (0.1864, 0.2143, 0.2457) \\ (0.1664, 0.1929, 0.2227) \\ (0.5392, 0.5929, 0.6527) \end{bmatrix} \\ \text{Priority of } \tilde{C}_{\tilde{a}_{13}} &= \begin{bmatrix} (0.3542, 0.3795, 0.4067) \\ (0.4925, 0.5241, 0.5580) \\ (0.0835, 0.0964, 0.1103) \end{bmatrix} & \text{Priority of } \tilde{C}_{\tilde{w}} &= \begin{bmatrix} (0.3562, 0.4706, 0.6412) \\ (0.1991, 0.2941, 0.4243) \\ (0.1493, 0.2353, 0.3536) \end{bmatrix} \end{aligned}$$

Final priority of alternatives is calculated as follows:

$$\begin{aligned} [\tilde{C}_{\tilde{a}_{11}} \tilde{C}_{\tilde{a}_{12}} \tilde{C}_{\tilde{a}_{13}}] \otimes [\tilde{C}_{\tilde{w}}] &= \begin{bmatrix} (0.2095, 0.2434, 0.2824)(0.2145, 0.2487, 0.2867)(0.4426, 0.5079, 0.5848) \\ (0.1835, 0.2143, 0.2501)(0.1660, 0.1929, 0.2238)(0.5058, 0.5929, 0.6953) \\ (0.3209, 0.3795, 0.4492)(0.4442, 0.5241, 0.6186)(0.0855, 0.0964, 0.1078) \end{bmatrix} \\ &\otimes \begin{bmatrix} (0.3195, 0.4706, 0.7000) \\ (0.1870, 0.2941, 0.4629) \\ (0.1524, 0.2353, 0.3549) \end{bmatrix} = \begin{bmatrix} (0.1648, 0.2669, 0.4283) \\ (0.1841, 0.2971, 0.4729) \\ (0.2838, 0.4361, 0.6771) \end{bmatrix} \end{aligned}$$

In order to compare the final ranking, the centroids of the final priority ranking values are calculated as follows:

$$\tilde{F}_1 = \frac{0.1648 + 0.2669 + 0.4283}{3} = 0.2867, \tilde{F}_2 = \frac{0.1841 + 0.2971 + 0.4729}{3} = 0.3180, \text{ and } \tilde{F}_3 = \frac{0.2838 + 0.4361 + 0.6771}{3} = 0.4657.$$

Therefore, $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$.

2.5.5.3. *Final ranking based on the AHP prioritizing method 3.* After developing the comparisonwise matrix of alternative values against each criteria, the values of each column are normalized by the sum of the values in the same column, then the average of each row is calculated, which stands for the relative importance of alternatives against that criteria. For example, based on the developed priority weights for the comparisonwise matrix of alternative values compared against criteria 1, the average of the first row stands for the relative importance of alternative 1 against criteria 1, the average of the second row stands for the relative importance of alternative 2 against the criteria 1, and so on.

Then, the sum of each column is calculated, and each value in each column is normalized by the sum of the values in the same column, as follows:

$$\begin{aligned} &\begin{bmatrix} \frac{(1.0000, 1.0000, 1.0000)}{(3.7800, 4.1087, 4.4905)}, & \frac{(0.8800, 1.0217, 1.1810)}{(3.7621, 4.0213, 4.3409)}, & \frac{(1.9000, 2.0870, 2.3095)}{(1.8866, 1.9688, 2.0484)} \\ \frac{(0.8468, 0.9787, 1.1364)}{(3.7800, 4.1087, 4.4905)}, & \frac{(1.0000, 1.0000, 1.0000)}{(3.7621, 4.0213, 4.3409)}, & \frac{(1.9153, 2.0426, 2.2045)}{(1.8866, 1.9688, 2.0484)} \\ \frac{(0.4330, 0.4792, 0.5263)}{(3.7800, 4.1087, 4.4905)}, & \frac{(0.4536, 0.4896, 0.5221)}{(3.7621, 4.0213, 4.3409)}, & \frac{(1.0000, 1.0000, 1.0000)}{(1.8866, 1.9688, 2.0484)} \end{bmatrix} \\ &= \begin{bmatrix} (0.2227, 0.2434, 0.2646)(0.1951, 0.2434, 0.3021)(0.2114, 0.2434, 0.2790) \\ (0.1960, 0.2487, 0.3124)(0.2304, 0.2487, 0.2658)(0.2214, 0.2487, 0.2767) \\ (0.4231, 0.5079, 0.6110)(0.4412, 0.5079, 0.5860)(0.4882, 0.5079, 0.5301) \end{bmatrix} \end{aligned}$$

Then, in order to find the final priority weights of each alternative in comparison with other alternatives against the same criteria, the average of each row is calculated as follows:

$$\text{Priority of } \tilde{C}_{\tilde{a}_{11}} = \begin{bmatrix} \frac{(0.2227, 0.2434, 0.2646)+(0.1951, 0.2434, 0.3021)+(0.2114, 0.2434, 0.2790)}{3} \\ \frac{(0.1960, 0.2487, 0.3124)+(0.2304, 0.2487, 0.2658)+(0.2214, 0.2487, 0.2767)}{3} \\ \frac{(0.4231, 0.5079, 0.6110)+(0.4412, 0.5079, 0.5860)+(0.4882, 0.5079, 0.5301)}{3} \end{bmatrix} = \begin{bmatrix} (0.2097, 0.2434, 0.2819) \\ (0.2159, 0.2487, 0.2850) \\ (0.4508, 0.5079, 0.5757) \end{bmatrix}$$

With the same approach:

$$\begin{aligned} \text{Priority of } \tilde{C}_{\tilde{a}_{12}} &= \begin{bmatrix} (0.1850, 0.2143, 0.2481) \\ (0.1660, 0.1929, 0.2236) \\ (0.5204, 0.5929, 0.6781) \end{bmatrix} & \text{Priority of } \tilde{C}_{\tilde{a}_{13}} &= \begin{bmatrix} (0.3420, 0.3795, 0.4237) \\ (0.4734, 0.5241, 0.5838) \\ (0.0842, 0.0964, 0.1094) \end{bmatrix} \\ \text{Priority of } \tilde{C}_{\tilde{w}} &= \begin{bmatrix} (0.3386, 0.4706, 0.6755) \\ (0.1955, 0.2941, 0.4475) \\ (0.1505, 0.2353, 0.3566) \end{bmatrix} \end{aligned}$$

Final priorities of alternatives are calculated as follows:

$$[\tilde{C}_{\tilde{a}_{11}} \tilde{C}_{\tilde{a}_{12}} \tilde{C}_{\tilde{a}_{13}}] \otimes [\tilde{C}_{\tilde{w}}] = \begin{bmatrix} (0.2097, 0.2434, 0.2819)(0.1850, 0.2143, 0.2481)(0.3420, 0.3795, 0.4237) \\ (0.2159, 0.2487, 0.2850)(0.1660, 0.1929, 0.2236)(0.4734, 0.5241, 0.5838) \\ (0.4508, 0.5079, 0.5757)(0.5204, 0.5929, 0.6781)(0.0842, 0.0964, 0.1094) \end{bmatrix}$$

$$\otimes \begin{bmatrix} (0.3386, 0.4706, 0.6755) \\ (0.1955, 0.2941, 0.4475) \\ (0.1505, 0.2353, 0.3566) \end{bmatrix} = \begin{bmatrix} (0.1586, 0.2669, 0.4525) \\ (0.1768, 0.2971, 0.5008) \\ (0.2670, 0.4361, 0.7313) \end{bmatrix}$$

In order to compare the final ranking, the centroids of the final priority ranking values are calculated as follows:

$$\tilde{F}_1 = \frac{0.1586 + 0.2669 + 0.4525}{3} = 0.2927, \tilde{F}_2 = \frac{0.1768 + 0.2971 + 0.5008}{3} = 0.3249, \text{ and } \tilde{F}_3 = \frac{0.2670 + 0.4361 + 0.7313}{3} = 0.4782.$$

Therefore, $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$.

2.5.5.4. *Final ranking based on the AHP prioritizing method 4.* The product of all values of each row in matrix \tilde{C} is calculated and their n^{th} roots are normalized. The resultant normalized values stand for the relative importance of \tilde{A}_1, \tilde{A}_2 , and \tilde{A}_3 .

In comparisonwise matrix \tilde{C} , the product of the values in each row is calculated as follows:

$$\begin{bmatrix} (1.0000, 1.0000, 1.0000) \otimes (0.8468, 0.9787, 1.1364) \otimes (0.4330, 0.4792, 0.5263) \\ (0.8800, 1.0217, 1.1810) \otimes (1.0000, 1.0000, 1.0000) \otimes (0.4536, 0.4896, 0.5221) \\ (1.9000, 2.0870, 2.3095) \otimes (1.9153, 2.0426, 2.2045) \otimes (1.0000, 1.0000, 1.0000) \end{bmatrix}$$

$$= \begin{bmatrix} (1.0000 \otimes 0.8468 \otimes 0.4330, 1.0000 \otimes 0.9787 \otimes 0.4792, 1.0000 \otimes 1.1364 \otimes 0.5263) \\ (0.8800 \otimes 1.0000 \otimes 0.4536, 1.0217 \otimes 1.0000 \otimes 0.4896, 1.1810 \otimes 1.0000 \otimes 0.5221) \\ (1.9000 \otimes 1.9153 \otimes 1.0000, 2.0870 \otimes 2.0426 \otimes 1.0000, 2.3095 \otimes 2.2045 \otimes 1.0000) \end{bmatrix} = \begin{bmatrix} (0.3666, 0.4690, 0.5981) \\ (0.3992, 0.5002, 0.6166) \\ (3.6391, 4.2627, 5.0915) \end{bmatrix}$$

As explained in Section 2.3.3, power of $\frac{1}{3}$ can be considered as fuzzy value of $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, which prevents producing negative fuzzy values. The n^{th} roots of the product values according to the Eq. (21) are calculated as follows:

$$P = \begin{bmatrix} (0.3666, 0.4690, 0.5981)^{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})} \\ (0.3992, 0.5002, 0.6166)^{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})} \\ (3.6391, 4.2627, 5.0915)^{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})} \end{bmatrix} = \begin{bmatrix} (0.3666^{\frac{1}{3}}, 0.4690^{\frac{1}{3}}, 0.5981^{\frac{1}{3}}) \\ (0.3992^{\frac{1}{3}}, 0.5002^{\frac{1}{3}}, 0.6166^{\frac{1}{3}}) \\ (3.6391^{\frac{1}{3}}, 4.2627^{\frac{1}{3}}, 5.0915^{\frac{1}{3}}) \end{bmatrix} = \begin{bmatrix} (0.7157, 0.7769, 0.8425) \\ (0.7363, 0.7938, 0.8511) \\ (1.5381, 1.6214, 1.7203) \end{bmatrix}$$

Then, the final priority values of alternatives compared against the same criteria are calculated by normalizing the matrix P by the sum of the values, calculated as follows:

$$\begin{bmatrix} \frac{(0.7157, 0.7769, 0.8425)}{(0.7157, 0.7769, 0.8425) + (0.7363, 0.7938, 0.8511) + (1.5381, 1.6214, 1.7203)} \\ \frac{(0.7363, 0.7938, 0.8511)}{(0.7157, 0.7769, 0.8425) + (0.7363, 0.7938, 0.8511) + (1.5381, 1.6214, 1.7203)} \\ \frac{(1.5381, 1.6214, 1.7203)}{(0.7157, 0.7769, 0.8425) + (0.7363, 0.7938, 0.8511) + (1.5381, 1.6214, 1.7203)} \end{bmatrix} = \begin{bmatrix} (0.2096, 0.2434, 0.2818) \\ (0.2157, 0.2487, 0.2846) \\ (0.4505, 0.5079, 0.5753) \end{bmatrix}$$

where the centroid values of priority weights of alternatives against criteria 1, is obtained as: $0.2434 + 0.2487 + 0.5079 \approx 1$. with the same approach:

$$\text{Priority of } \tilde{C}_{\tilde{a}_{12}} = \begin{bmatrix} (0.1849, 0.2143, 0.2479) \\ (0.1659, 0.1929, 0.2235) \\ (0.5198, 0.5929, 0.6774) \end{bmatrix} \quad \text{Priority of } \tilde{C}_{\tilde{a}_{13}} = \begin{bmatrix} (0.3411, 0.3795, 0.4225) \\ (0.4721, 0.5241, 0.5823) \\ (0.0843, 0.0964, 0.1094) \end{bmatrix}$$

$$\text{Priority of } \tilde{C}_{\tilde{w}} = \begin{bmatrix} (0.3371, 0.4706, 0.6739) \\ (0.1930, 0.2941, 0.4413) \\ (0.1500, 0.2353, 0.3551) \end{bmatrix}$$

where the centroid values of priority weights of alternatives against criteria 2, is obtained as: $0.2143 + 0.199 + 0.5929 \approx 1$.

where the centroid values of priority weights of alternatives against criteria 3, is obtained as: $0.3795 + 0.5241 + 0.0964 = 1$.

where the centroid values of priority of weights is obtained as: $0.4706 + 0.2941 + 0.2353 = 1$.

Final priorities of alternatives are calculated as follows:

$$[\tilde{C}_{\tilde{a}_{11}} \tilde{C}_{\tilde{a}_{12}} \tilde{C}_{\tilde{a}_{13}}] \otimes [\tilde{C}_{\tilde{w}}] = \begin{bmatrix} (0.2096, 0.2434, 0.2818)(0.1849, 0.2143, 0.2479)(0.3411, 0.3795, 0.4225) \\ (0.2157, 0.2487, 0.2846)(0.1659, 0.1929, 0.2235)(0.4721, 0.5241, 0.5823) \\ (0.4505, 0.5079, 0.5753)(0.5198, 0.5929, 0.6774)(0.0843, 0.0964, 0.1094) \end{bmatrix}$$

$$\otimes \begin{bmatrix} (0.3371, 0.4706, 0.6739) \\ (0.1930, 0.2941, 0.4413) \\ (0.1500, 0.2353, 0.3551) \end{bmatrix} = \begin{bmatrix} (0.1575, 0.2669, 0.4493) \\ (0.1755, 0.2971, 0.4972) \\ (0.2649, 0.4361, 0.7255) \end{bmatrix}$$

In order to compare the final ranking, the centroids of the final priority ranking values are calculated as follows:

$$\tilde{F}_1 = \frac{0.1575 + 0.2669 + 0.4493}{3} = 0.2912, \tilde{F}_2 = \frac{0.1755 + 0.2971 + 0.4972}{3} = 0.3233, \text{ and } \tilde{F}_3 = \frac{0.2649 + 0.4361 + 0.7255}{3} = 0.4755.$$

Therefore, $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$.

Table 2

Numerical results of Kendall's tau-b test of correlation significance of rank between the compared methods as an example for decision matrix with 15 alternatives and 15 criteria.

Kendall's tau-b		SAW	WPM	CP	TOPSIS	AHP1	AHP2	AHP3	AHP4	VIKOR
SAW	Correlation Coefficient	1.000	.600**	.619**	.886**	.619**	.695**	.638**	.676**	.637**
	Sig. (2-tailed)		.002	.001	.000	.001	.000	.001	.000	.004
	N	15	15	15	15	15	15	15	15	15
WPM	Correlation Coefficient	.600**	1.000	.562**	.676**	.676**	.790**	.657**	.771**	.292
	Sig. (2-tailed)	.002		.004	.000	.000	.000	.001	.000	.184
	N	15	15	15	15	15	15	15	15	15
CP	Correlation Coefficient	.619**	.562**	1.000	.619**	.429*	.543**	.410*	.486*	.558*
	Sig. (2-tailed)	.001	.004		.001	.026	.005	.033	.012	.011
	N	15	15	15	15	15	15	15	15	15
TOPSIS	Correlation Coefficient	.886**	.676**	.619**	1.000	.695**	.810**	.676**	.752**	.531*
	Sig. (2-tailed)	.000	.000	.001		.000	.000	.000	.000	.016
	N	15	15	15	15	15	15	15	15	15
AHP1	Correlation Coefficient	.619**	.676**	.429*	.695**	1.000	.695**	.943**	.790**	.266
	Sig. (2-tailed)	.001	.000	.026	.000		.000	.000	.000	.228
	N	15	15	15	15	15	15	15	15	15
AHP2	Correlation Coefficient	.695**	.790**	.543**	.810**	.695**	1.000	.714**	.905**	.319
	Sig. (2-tailed)	.000	.000	.005	.000	.000		.000	.000	.148
	N	15	15	15	15	15	15	15	15	15
AHP3	Correlation Coefficient	.638**	.657**	.410*	.676**	.943**	.714**	1.000	.810**	.319
	Sig. (2-tailed)	.001	.001	.033	.000	.000	.000		.000	.148
	N	15	15	15	15	15	15	15	15	15
AHP4	Correlation Coefficient	.676**	.771**	.486*	.752**	.790**	.905**	.810**	1.000	.266
	Sig. (2-tailed)	.000	.000	.012	.000	.000	.000	.000		.228
	N	15	15	15	15	15	15	15	15	15
VIKOR	Correlation Coefficient	.637**	.292	.558*	.531*	.266	.319	.319	.266	1.000
	Sig. (2-tailed)	.004	.184	.011	.016	.228	.148	.148	.228	
	N	15	15	15	15	15	15	15	15	15

Table 3

Numerical results of Spearman's rho test of correlation significance of rank between the compared methods as an example for decision matrix with 15 alternatives and 15 criteria.

Spearman's rho		SAW	WPM	CP	TOPSIS	AHP1	AHP2	AHP3	AHP4	VIKOR
SAW	Correlation Coefficient	1.000	.768**	.811**	.964**	.779**	.832**	.793**	.811**	.777**
	Sig. (2-tailed)		.001	.000	.000	.001	.000	.000	.000	.001
	N	15	15	15	15	15	15	15	15	15
WPM	Correlation Coefficient	.768**	1.000	.739**	.843**	.796**	.925**	.811**	.907**	.362
	Sig. (2-tailed)	.001		.002	.000	.000	.000	.000	.000	.185
	N	15	15	15	15	15	15	15	15	15
CP	Correlation Coefficient	.811**	.739**	1.000	.814**	.604*	.739**	.579*	.682**	.706**
	Sig. (2-tailed)	.000	.002		.000	.017	.002	.024	.005	.003
	N	15	15	15	15	15	15	15	15	15
TOPSIS	Correlation Coefficient	.964**	.843**	.814**	1.000	.825**	.918**	.829**	.889**	.665**
	Sig. (2-tailed)	.000	.000	.000		.000	.000	.000	.000	.007
	N	15	15	15	15	15	15	15	15	15
AHP1	Correlation Coefficient	.779**	.796**	.604*	.825**	1.000	.850**	.986**	.925**	.323
	Sig. (2-tailed)	.001	.000	.017	.000		.000	.000	.000	.240
	N	15	15	15	15	15	15	15	15	15
AHP2	Correlation Coefficient	.832**	.925**	.739**	.918**	.850**	1.000	.850**	.979**	.411
	Sig. (2-tailed)	.000	.000	.002	.000	.000		.000	.000	.128
	N	15	15	15	15	15	15	15	15	15
AHP3	Correlation Coefficient	.793**	.811**	.579*	.829**	.986**	.850**	1.000	.918**	.364
	Sig. (2-tailed)	.000	.000	.024	.000	.000	.000		.000	.182
	N	15	15	15	15	15	15	15	15	15
AHP4	Correlation Coefficient	.811**	.907**	.682**	.889**	.925**	.979**	.918**	1.000	.351
	Sig. (2-tailed)	.000	.000	.005	.000	.000	.000	.000		.200
	N	15	15	15	15	15	15	15	15	15
VIKOR	Correlation Coefficient	.777**	.362	.706**	.665**	.323	.411	.364	.351	1.000
	Sig. (2-tailed)	.001	.185	.003	.007	.240	.128	.182	.200	
	N	15	15	15	15	15	15	15	15	15

Notably, comparison of the calculated values for \tilde{F}_1 , \tilde{F}_2 , and \tilde{F}_3 using four different AHP prioritizing methods shows that those priority values are significantly close together. In addition, in all four types of AHP prioritizing methods, the sum of the centroid values in final priority values of comparisonwise matrices of alternatives against three different criteria ($\tilde{c}_{a_{11}}$, $\tilde{c}_{a_{12}}$, and $\tilde{c}_{a_{13}}$) yields 1.

2.5.6. Final ranking based on ELECTRE

Based on the same numerical example that was investigated above in order to develop the concordance and discordance sets

and matrices, the weighted and normalized decision matrix should be defuzzified. Therefore, all fuzzy values are defuzzified through the selected defuzzification method. In order to compare the final rankings of different alternative through several decision ranking techniques, the same defuzzification method should be used throughout all decision ranking techniques. Here, the centroid method is applied to defuzzify the fuzzy values. The defuzzified form of the weighted normalized decision-making matrix and

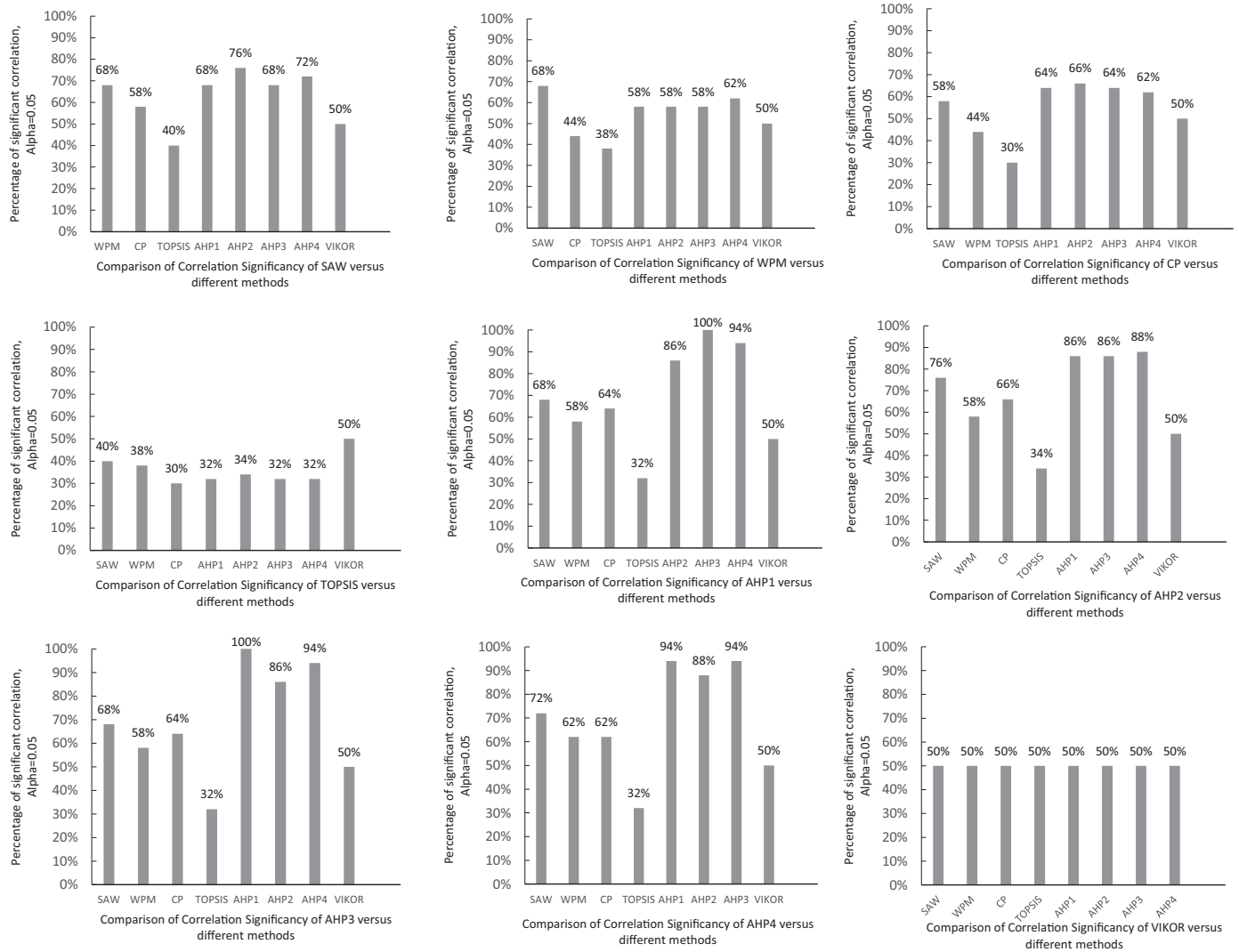


Fig. 2. Statistical results of correlation significance among different outranking methods applied on decision matrix with 3 alternatives versus 3 criteria.

weights of criteria is as follows:

$$\text{weighted normalized decision matrix} = \begin{bmatrix} 0.2269, 0.1079, 0.1734 \\ 0.2306, 0.0973, 0.2391 \\ 0.4706, 0.2960, 0.0448 \end{bmatrix}$$

$$w_1 = 0.4751, w_2 = 0.3016, \text{ and } w_3 = 0.2422$$

where $A_1 = \{0.2269, 0.1079, 0.1734\}$, $A_2 = \{0.2306, 0.0973, 0.2391\}$, and $A_3 = \{0.4706, 0.2960, 0.0448\}$ are the values of alternatives A_1 , A_2 , and A_3 against criteria 1, 2, and 3 after the defuzzification.

In comparisonwise matrices, among the alternatives $A_1 = \{0.2269, 0.1079, 0.1734\}$, $A_2 = \{0.2306, 0.0973, 0.2391\}$, and $A_3 = \{0.4706, 0.2960, 0.0448\}$, concordance and discordance sets are developed as follows:

In the comparison of alternatives 1 and 2, if $a_{1j} \geq a_{2j}$, then 1, otherwise 0. For example, (if $0.2269 \leq 0.2306$, then 0), (if $0.1097 > 0.0973$, then 1), and (if $0.1734 < 0.2391$, then 0).

2.5.6.1. Developing the concordance matrix. The concordance sets and related values in the concordance matrix for C_{ij} for $i=1, 2$, and 3 and $j=1, 2$, and 3 are as follows:

$$\begin{aligned} C_{12} &= \{0, 1, 0\}, & C_{12} &= 0 \otimes w_1 + 1 \otimes w_2 + 0 \otimes w_3 = 0.3016 \\ C_{13} &= \{0, 0, 1\}, & C_{13} &= 0 \otimes w_1 + 0 \otimes w_2 + 1 \otimes w_3 = 0.2422 \\ C_{21} &= \{1, 0, 1\}, & C_{21} &= 1 \otimes w_1 + 0 \otimes w_2 + 1 \otimes w_3 = 0.7173 \\ C_{23} &= \{0, 0, 1\}, & C_{23} &= 0 \otimes w_1 + 0 \otimes w_2 + 1 \otimes w_3 = 0.2422 \\ C_{31} &= \{1, 1, 0\}, & C_{31} &= 1 \otimes w_1 + 1 \otimes w_2 + 0 \otimes w_3 = 0.7766 \\ C_{32} &= \{1, 1, 0\}, & C_{32} &= 1 \otimes w_1 + 1 \otimes w_2 + 0 \otimes w_3 = 0.7766 \end{aligned}$$

Finally, the concordance matrix is developed as follows:

$$\begin{bmatrix} - & C_{12} & C_{13} \\ C_{21} & - & C_{23} \\ C_{31} & C_{32} & - \end{bmatrix} = \begin{bmatrix} - & 0.3016 & 0.2422 \\ 0.7173 & - & 0.2422 \\ 0.7766 & 0.7766 & - \end{bmatrix}$$

The discordance set is the complementary of concordance sets; therefore:

$$D_{12} = \{1, 0, 1\}, D_{13} = \{1, 1, 0\}, D_{21} = \{0, 1, 0\}, D_{23} = \{1, 1, 0\}, D_{31} = \{0, 0, 1\}, \text{ and } D_{32} = \{0, 0, 1\}$$

2.5.6.2. Developing the discordance matrix. Each value in the discordance matrix indicates the relative degree of inferiority of two compared alternatives A_k and A_l . Discordance index values are calculated as follows:

By considering Eq. (41) for calculating the values of discordance matrix, discordance sets, and using the defuzzified values in

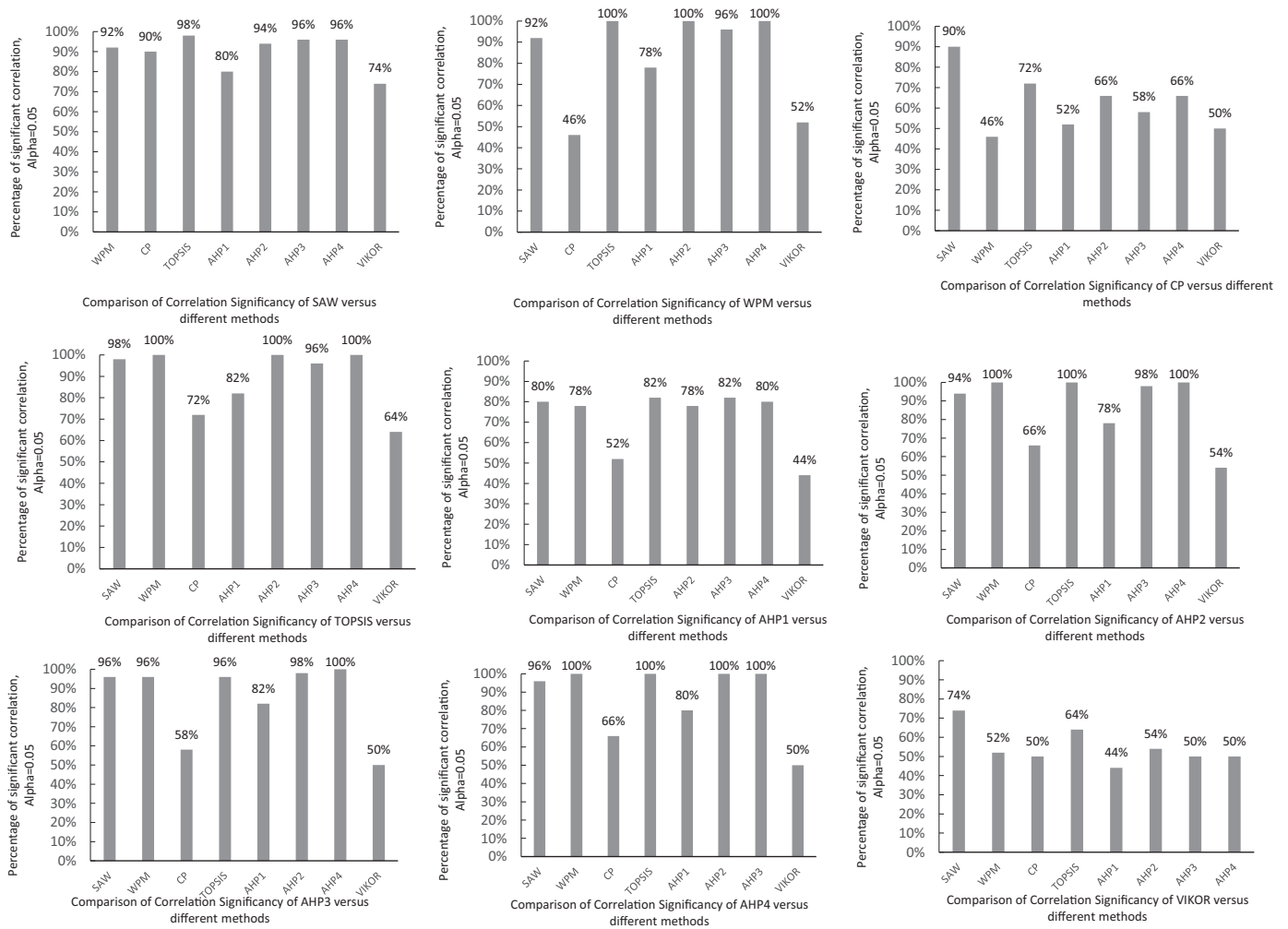


Fig. 3. Statistical results of correlation significance among different outranking methods applied on decision matrix with 8 alternatives versus 4 criteria.

weighted normalized decision matrix, all discordance values of the discordance matrix are calculated as follows:

$$d_{12} = \frac{\max_{j \in D_{12}} (|v_{1j} - v_{2j}|)}{\max_{j \in J} (|v_{1j} - v_{2j}|)}$$

$$= \frac{\max \text{ of } \{0.2269 - 0.2306\}, \text{ and } \{(0.1734 - 0.2391)\}}{\max \text{ of } \{0.2269 - 0.2306\}, \{0.1079 - 0.0973\}, \text{ and } \{(0.1734 - 0.2391)\}}$$

$$= \frac{0.0657}{0.0657} = 1$$

With the same approach, d_{13} , d_{21} , d_{23} , d_{31} , and d_{32} are calculated as follows:

$$d_{13} = \frac{\max_{j \in D_{13}} (|\tilde{v}_{1j} - \tilde{v}_{3j}|)}{\max_{j \in J} (|\tilde{v}_{1j} - \tilde{v}_{3j}|)} = 1, \quad d_{21} = \frac{\max_{j \in D_{21}} (|\tilde{v}_{2j} - \tilde{v}_{1j}|)}{\max_{j \in J} (|\tilde{v}_{2j} - \tilde{v}_{1j}|)} = 0.1621,$$

$$d_{23} = \frac{\max_{j \in D_{23}} (|\tilde{v}_{2j} - \tilde{v}_{3j}|)}{\max_{j \in J} (|\tilde{v}_{2j} - \tilde{v}_{3j}|)} = 1, \quad d_{31} = \frac{\max_{j \in D_{31}} (|\tilde{v}_{3j} - \tilde{v}_{1j}|)}{\max_{j \in J} (|\tilde{v}_{3j} - \tilde{v}_{1j}|)} = 0.5278,$$

$$\text{and } d_{32} = \frac{\max_{j \in D_{32}} (|\tilde{v}_{3j} - \tilde{v}_{2j}|)}{\max_{j \in J} (|\tilde{v}_{3j} - \tilde{v}_{2j}|)} = 0.8099$$

Finally, the discordance matrix is developed as follows:

$$\begin{bmatrix} - & d_{12} & d_{13} \\ d_{21} & - & d_{23} \\ d_{31} & d_{32} & - \end{bmatrix} = \begin{bmatrix} - & 1.0000 & 1.0000 \\ 0.1621 & - & 1.0000 \\ 0.5278 & 0.8099 & - \end{bmatrix}$$

In the comparison of two alternatives A_k and A_l , a higher value of the concordance index represents the more preferable alternative, and a higher discordance index value represents a less favorable A_k in comparison with A_l .

In order to develop the concordance dominance matrix, a threshold value is calculated based on the concordance index values of the concordance matrix. This threshold value is calculated as follows:

$$\bar{c} = \left(\sum_{k=1}^m \sum_{l=1}^m c_{kl} \right) / (m(m-1)) = \left(\sum_{k=1}^3 \sum_{l=1}^3 c_{kl} \right) / (3(3-1))$$

$$= \frac{0.3016 + 0.2422 + 0.7173 + 0.2422 + 0.7766 + 0.7766}{6}$$

$$= 0.5095$$

where $k \neq l$, $f_{kl} = 1$ if $c_{kl} \geq \bar{c}$, and $f_{kl} = 0$ if $c_{kl} < \bar{c}$.

Therefore, dominance concordance matrix is developed as follows:

$$\text{dominance concordance matrix} = f_{kl} = \begin{bmatrix} - & 0 & 0 \\ 1 & - & 0 \\ 1 & 1 & - \end{bmatrix}$$

In order to develop the discordance dominance matrix, a threshold value is calculated based on the discordance index values of concordance matrix. This threshold value is calculated as follows:

$$\bar{d} = \left(\sum_{k=1}^m \sum_{l=1}^m d_{kl} \right) / (m(m-1)) = \left(\sum_{k=1}^3 \sum_{l=1}^3 d_{kl} \right) / (3(3-1))$$

$$= \frac{1 + 1 + 0.1621 + 1 + 0.5278 + 0.8099}{6} = 0.7500$$

where $k \neq l$: $f_{kl} = 1$ if $d_{kl} \geq \bar{d}$, and $f_{kl} = 0$ if $d_{kl} < \bar{d}$.

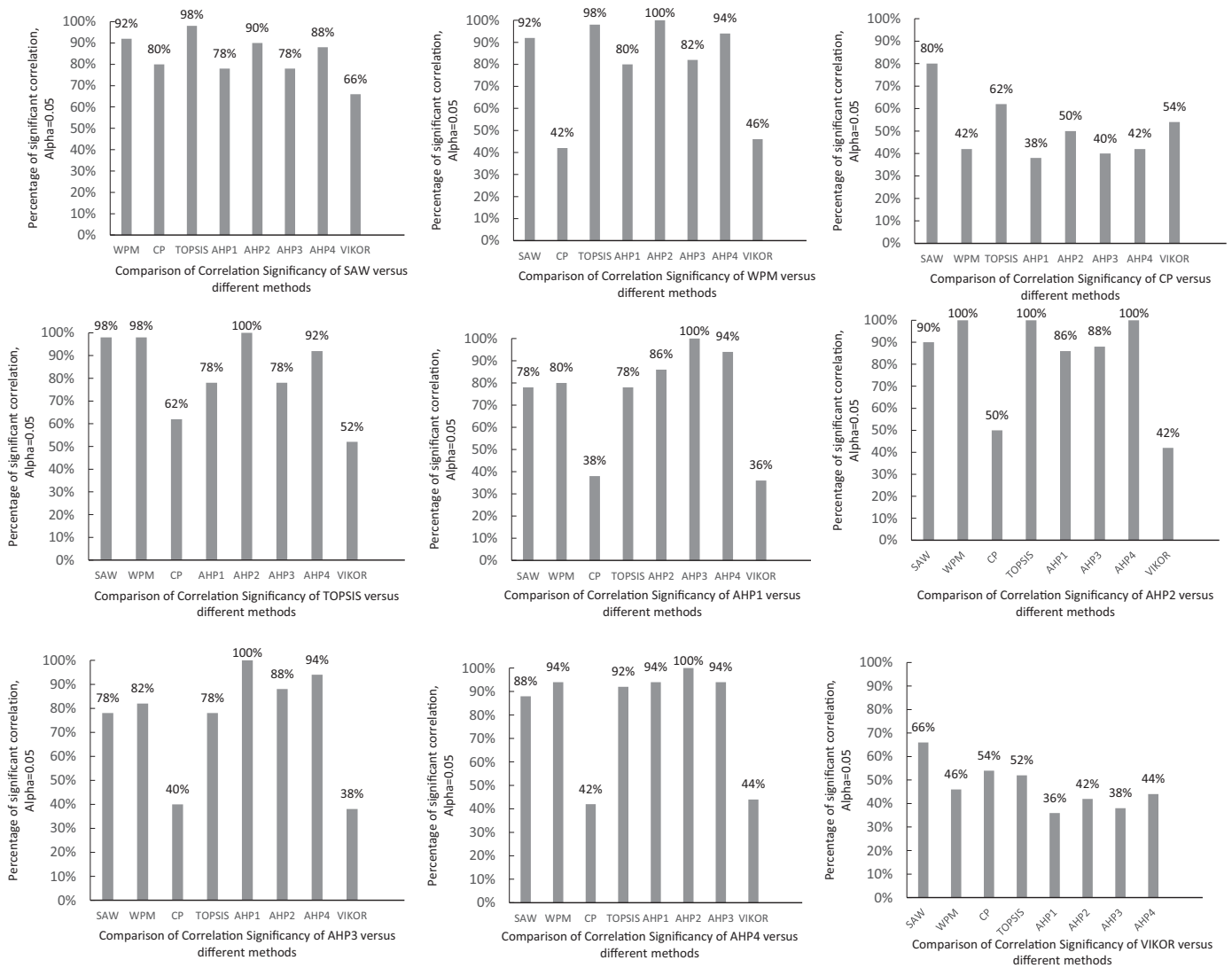


Fig. 4. Statistical result of correlation significance among different outranking methods applied on decision matrix with 8 alternatives versus 8 criteria.

Therefore, the dominance concordance matrix is developed as follows:

$$\text{dominance discordance matrix} = g_{kl} = \begin{bmatrix} - & 1 & 1 \\ 0 & - & 1 \\ 0 & 1 & - \end{bmatrix}$$

In order to determine the aggregate dominance matrix, the intersection of the two concordance and discordance dominance matrices is calculated as follows:

$$e_{kl} = f_{kl} \cdot g_{kl} = \begin{bmatrix} - & (0 \times 1) & (0 \times 1) \\ (1 \times 0) & - & (0 \times 1) \\ (1 \times 0) & (1 \times 1) & - \end{bmatrix} = \begin{bmatrix} - & 0 & 0 \\ 0 & - & 0 \\ 0 & 1 & - \end{bmatrix}$$

where each individual array value in the matrix e_{kl} is the product of arrays of concordance and discordance matrices with the same positions.

In developing the final ranking, if $e_{kl} = 1$, A_k is preferred to A_l . The aggregate matrix A_3 has dominance on A_2 , but A_1 and A_2 can be categorized as the same rank, because none of them shows clear dominance to the other one. Therefore, $\tilde{A}_1 \approx \tilde{A}_2$ and $\tilde{A}_2 < \tilde{A}_3$.

2.5.7. Final ranking based on VIKOR

Considering the basic decision matrix, the ideal and anti-ideal points are developed as follows:

The maximum and minimum values of alternatives for the same criteria are taken for finding the ideal and anti-ideal points as follows:

$$\begin{aligned} \text{max of } \tilde{x}_{ij} \text{ for } i \text{ and } j = 1, 2, \text{ and } 3 \\ = (48.5, 48.5, 48.5), (85, 85, 85), \text{ and } (88.5, 88.5, 88.5) \end{aligned}$$

$$\begin{aligned} \text{min of } \tilde{x}_{ij} \text{ for } i \text{ and } j = 1, 2, \text{ and } 3 \\ = (21, 21, 21), (25, 25, 25), \text{ and } (14, 14, 14) \end{aligned}$$

In VIKOR, \tilde{S}_i and \tilde{R}_i are calculated as shown below.

$$\begin{aligned} \tilde{S}_1 = \sum_{i=1}^n \tilde{w}_j \cdot \frac{\tilde{f}_j^* - \tilde{f}_{1j}}{\tilde{f}_i^* - \tilde{f}_{i-}} = (0.5962, 0.7871, 0.9907) \\ \tilde{S}_2 = (0.5586, 0.7169, 0.8928) \\ \tilde{S}_3 = (0.1514, 0.2489, 0.3632) \end{aligned}$$

where f_i^* , f_{i-} , and f_{ij} are parameters derived through Eqs. (45) and (47).

$$\tilde{R}_1 = \max \left(\tilde{w}_j \cdot \frac{\tilde{f}_j^* - \tilde{f}_{1j}}{\tilde{f}_i^* - \tilde{f}_{i-}} \right) = 0.0824, \tilde{R}_1 = 0.0038, \text{ and } \tilde{R}_1 = 0.0061$$

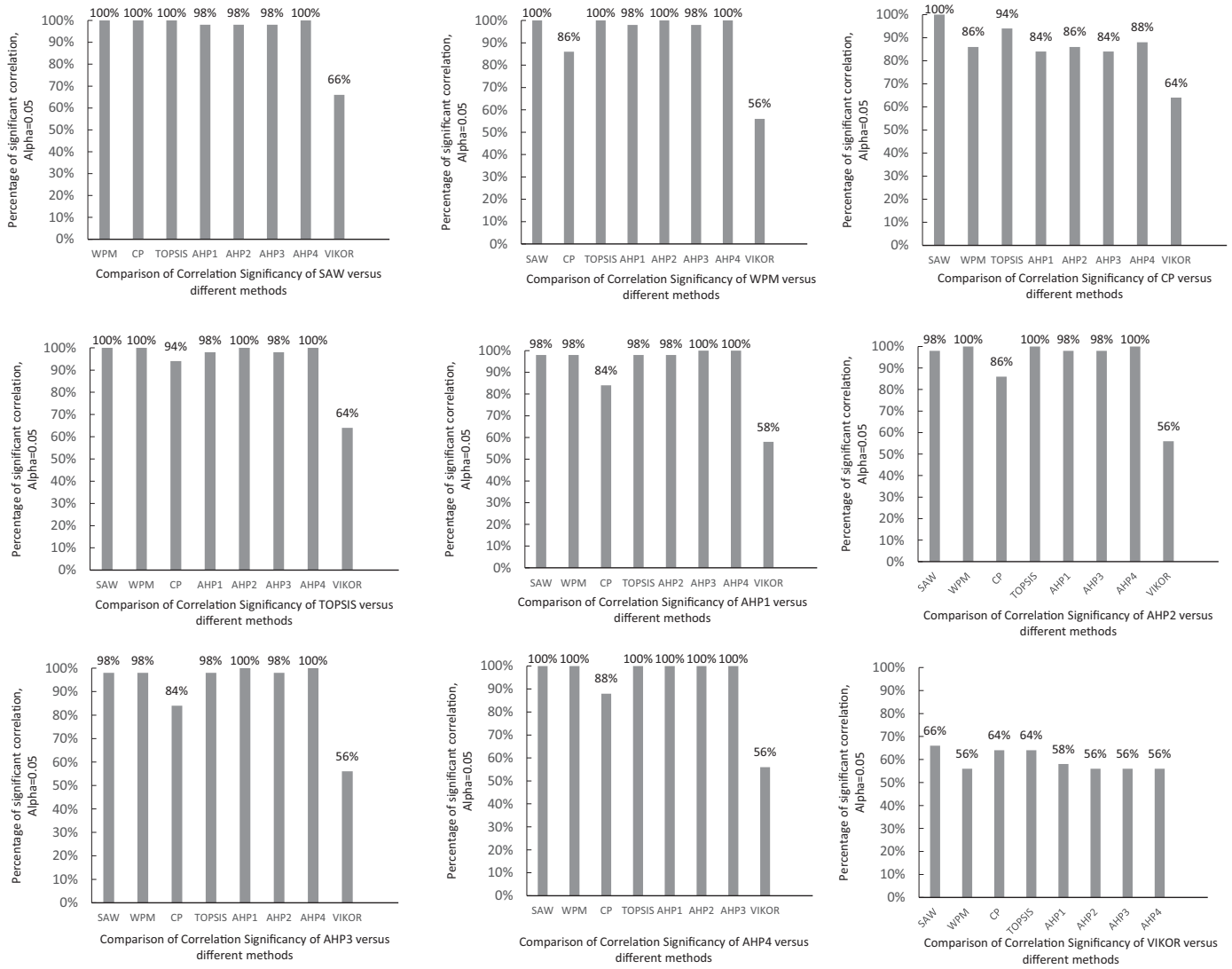


Fig. 5. Statistical result of correlation significance among different outranking methods applied on decision matrix with 15 alternatives versus 8 criteria.

For calculating Q_i , considering Eq. (50), $\tilde{Q}_1 = 1.0000$, $\tilde{Q}_2 = 0.9140$, and $\tilde{Q}_3 = 0.0000$. Therefore, based on the VIKOR method, $\tilde{A}_1 \approx \tilde{A}_2 < \tilde{A}_3$.

2.6. Statistical comparison of ranking methods using Kendall's tau-b and Spearman's rho

While several alternatives in a decision matrix are ranked by several decision ranking techniques, the produced ranks for those alternatives would have different levels of similarities. Conceptually, in order to investigate the similarities of the produced ranks, the Kendall's tau-b and Spearman's rho were selected to analyze the produced ranks through different decision-ranking methods in terms of their pairwise correlations. These two statistical tests are non-parametric tests that are used to measure the ordinal association between the two measured quantities. The Kendall's tau-b represents the similarities in the ordering of ranked quantities. For identical produced ranks, the Kendall's correlation coefficient would be 1, and for full differentiated produced ranks (completely dissimilar), the Kendall's correlation coefficient would be -1 . Basically, the Kendall's tau-b and Spearman's rho are considered as two accepted measures of non-parametric rank correlations that are used for bivariate analysis of the values' ranks. In more detail, Spearman's rank correlation coefficient represents a monotonic

function describing the strength of the linear relationship between the produced ranks. Conceptually, Spearman's rho is equal to the Pearson correlation coefficient between the obtained ranks of two measured variables, which here are two sets of alternatives.

Since the same sort of rank ranges are produced for all decision-making techniques (except VIKOR and ELECTRE); therefore, the mean of the ranks from the application of all techniques will be same. As a result, non-parametric tests should be used to see how the final ranks compare among the methods. In both developed tests on ranked value, the objective is to show how different decision-making methods lead to similar and dissimilar ranks. These two nonparametric tests were performed to describe the strength of correlation between the rank orders of two groups. Both Kendall's tau-b and Spearman's rho are performed for investigating the correlation of rank orders for each pair of two groups.

2.6.1. Kendall's tau-b

Kendall's tau-b test is a coefficient indicates the concordant and discordant association between the ranks of two compared groups of ranks. Kendall's tau-b coefficient is calculated using Eq. (64) as follows [43]:

$$\tau_B = \frac{n_c - n_d}{\sqrt{(n_0 - n_1)(n_0 - n_2)}} \tag{64}$$

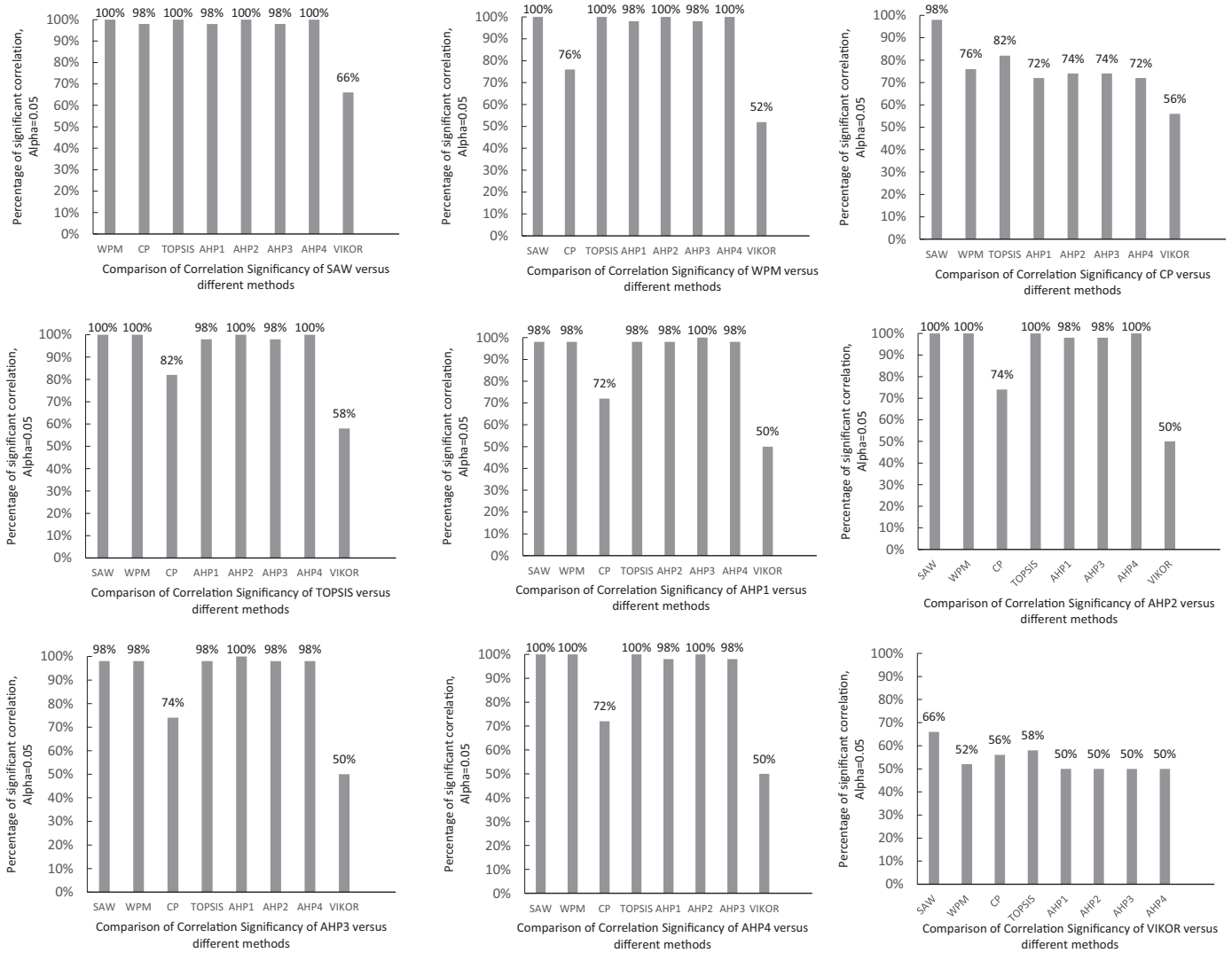


Fig. 6. Statistical result of correlation significance among different outranking methods applied on decision matrix with 15 alternatives versus 15 criteria.

where $n_0 = n(n-1)/2$, $n_1 = \sum_i t_i(t_i - 1)/2$, $n_2 = \sum_j u_j(u_j - 1)/2$, n_c is the number of concordant pairs, n_d is the number of discordant pairs, t_i is the number of tied values in the i^{th} group of ties for the first quantity, and u_j is the number of tied values in the j^{th} group of ties for the second quantity [44].

This formulation yields τ_B between -1 and $+1$. The value of -1 stands for 100% negative association, and the value of $+1$ stands for 100% positive associations. The value of zero stands for the absence of any association.

2.6.2. Spearman's rho [43]

In order to assess the correlation between the bivariate pairs of data in the form of rank values, Spearman's rho is calculated. The Spearman rank correlation coefficient, denoted by ρ , is defined using Eq. (65) as follows:

$$\rho = \frac{\sum_{i=1}^n \{(x_i - \bar{x})(y_i - \bar{y})\}}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}} \quad (65)$$

where x_i and y_i are the ranks of each variable in group 1 and group 2, and \bar{x} and \bar{y} are the averages of rank values in groups 1 and 2, respectively. This formulation yields ρ between -1 and $+1$. Positive values of ρ indicate that higher values of one group are associated with higher values of the other group, and vice versa.

Negative values of ρ indicate that higher values of one group are associated with lower values of the other group. Higher absolute values indicate stronger associations between the two compared sets of ranks.

2.6.3. Sensitivity of final ranks to the selected fuzziness intervals

The fuzziness interval is related to the uncertainty level, degree of bias, and random nature of the values. The fuzziness intervals are selected from different biased and unbiased distributions. The main concept behind selecting the fuzziness values from different distributions is to investigate the sensitivity of final rankings to the fuzziness levels.

2.6.4. Sensitivity of the similarities and dissimilarities of different decision ranking methods to dimensions of the decision matrix

In this section, the same decision-making matrices are simulated in the form of different matrix sizes, and the final rankings are compared to each other statistically. Statistical analysis is performed to investigate the role of matrix size on the final rankings. In the simulation process, five different decision matrices with different combinations of alternatives and criteria of $D_{3,3}$ (3 alternatives and 3 criteria), $D_{8,4}$, $D_{8,8}$, $D_{15,8}$, $D_{15,15}$ were defined.

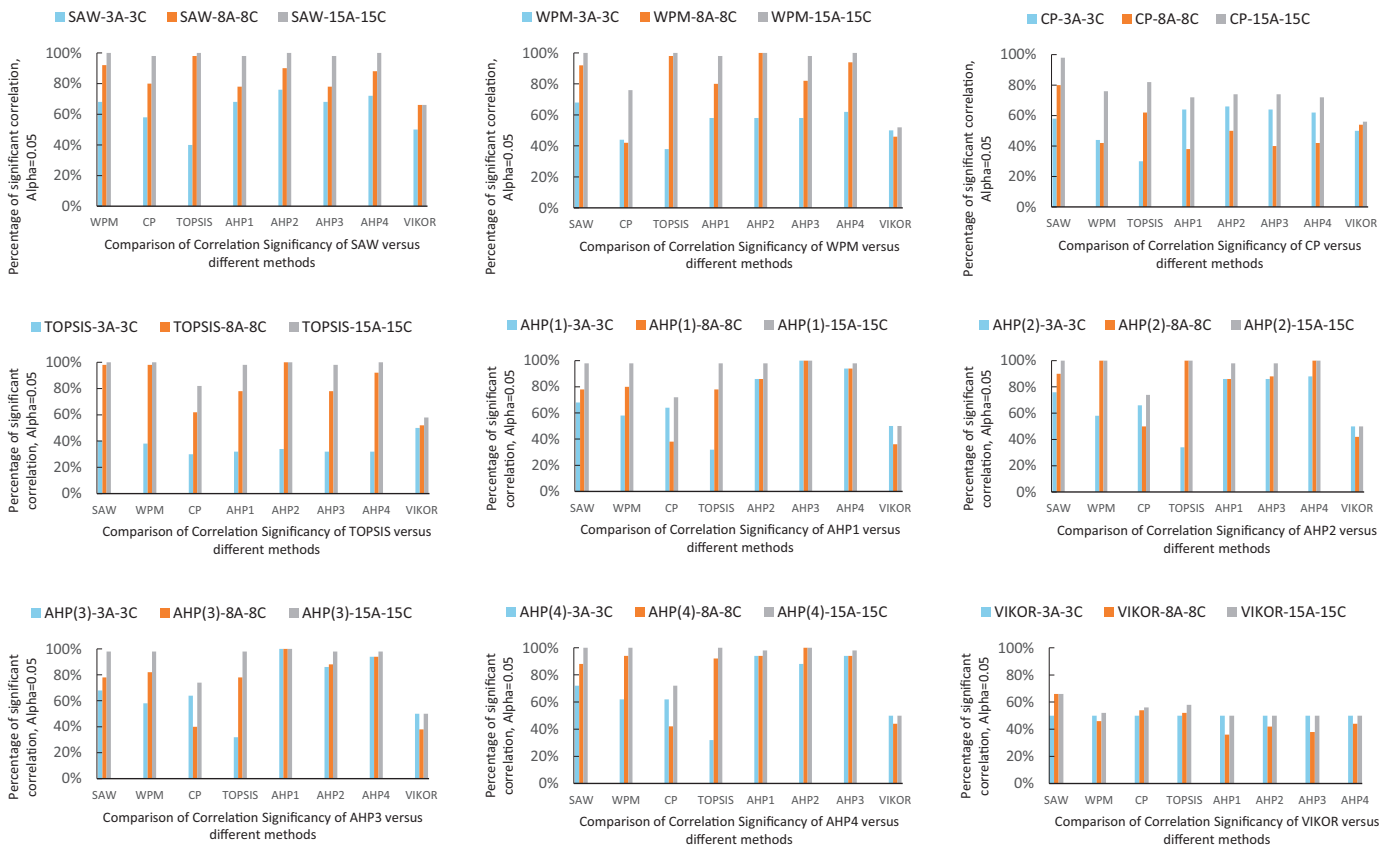


Fig. 7. Multiple comparison of statistical results of correlation significance percentage among different outranking methods applied on symmetrical decision matrices with 3, 8, and 15 alternatives.

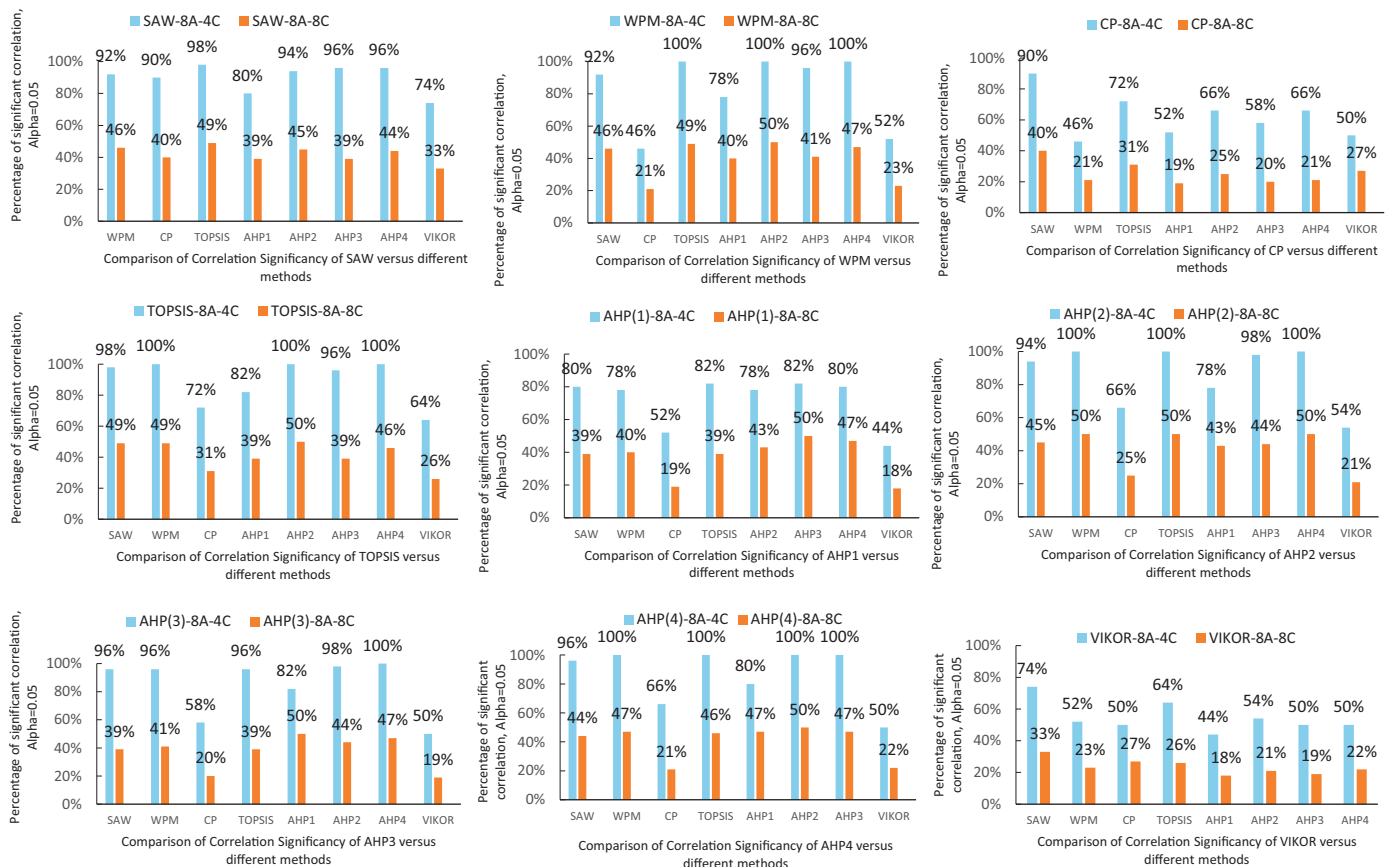


Fig. 8. Multiple comparison of statistical results of correlation significance percentage among different outranking methods applied on decision matrices with D8,4 (Decision matrix with 8 alternatives and 4 criteria) and D8,8.

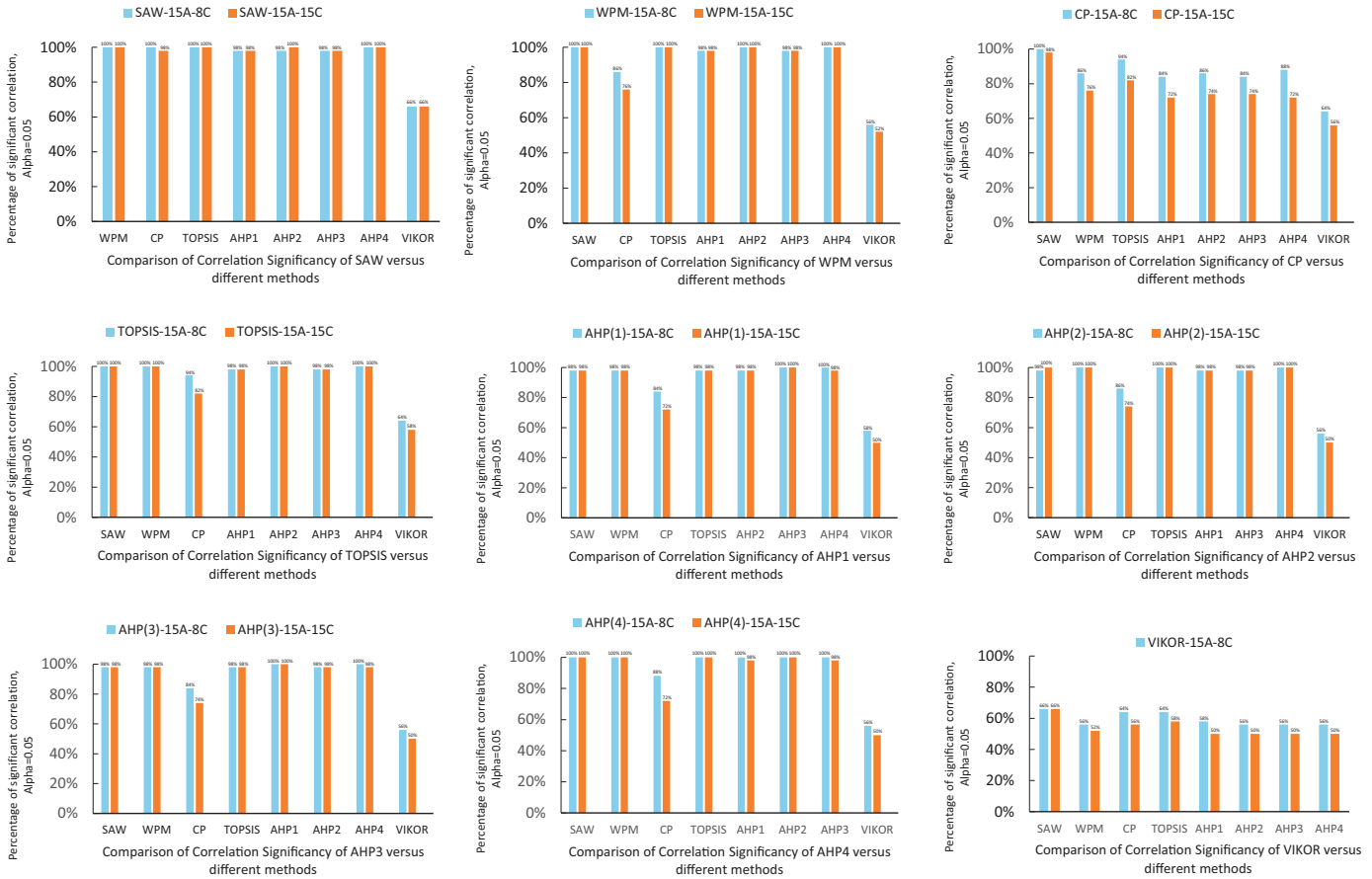


Fig. 9. Multiple comparison of statistical results of correlation significance percentage among different outranking methods applied on decision matrices with D15,8, and D15,15.

3. Numerical results and discussion

Tables 2 and 3 represent a sample of the statistical results of the Kendall's tau-b and Spearman's rho tests developed to investigate the significant correlation percentage among the order of the ranks developed by 9 different methods on the same decision matrix with 15 alternatives and 15 criteria (D_{15,15}). The significant values in Tables 2 and 3 represent statistically indistinguishable correlation rates between both Kendall's tau-b and Spearman's rho tests. By assuming that α -level = 0.05, the results of statistical analysis in Tables 2 and 3 indicate that there is extremely similar performances between two tests of Kendall's tau-b and Spearman's rho; therefore, Kendall's tau-b has been selected to evaluate the similarity and dissimilarity of the final ranks produced by different decision-ranking techniques. Whenever reported p-values less than α -level=0.05, which indicates that there is a statistically significant correlation among the produced ranks of the two compared methods.

Figs. 2–6 illustrate the statistical results of the correlation significance percentages among 9 different ranking methods applied on decision matrices with the random left and right spreads of triangular fuzzy values and sizes of D_{3,3} (3 alternative and 3 criteria), D_{8,4} (8 alternative and 4 criteria), D_{8,8} (8 alternative and 8 criteria), D_{15,8} (15 alternative and 8 criteria), and D_{15,15} (15 alternative and 15 criteria). Generally, the ELECTRE method did not produce a complete sorting of alternatives; therefore, we excluded it from the statistical comparison of the performances of the ranking methods. Fig. 7 illustrates multiple comparisons of statistical results of correlation significance percentages among different ranking methods applied on symmetrical decision matrices with 3, 8, and 15 alternatives.

Fig. 8 illustrates multiple comparisons of statistical results of correlation significance percentage among different ranking methods applied on decision matrices with D_{8,4} (8 alternatives and 4 criteria), and D_{8,8}. Fig. 9 illustrates multiple comparisons of statistical results of correlation significance percentage among different ranking methods applied on decision matrices with D_{15,8}, and D_{15,15}. Fig. 10 illustrates multiple comparisons of statistical results of correlation significance percentage among different ranking methods applied on decision matrices of D_{3,3}, D_{8,4}, D_{8,8}, D_{15,8}, and D_{15,15}. Fig. 11 illustrates multiple comparisons of statistical results of correlation significance percentage applied on D_{8,8} with 4 uncertainty levels. Significantly similar patterns were observed for other evaluated sizes of matrices D_{3,3}, D_{8,4}, D_{15,8}, and D_{15,15} with 4 uncertainty levels. Fig. 12 shows the agreement percentage between the methods in selecting the first rank. In order to compare the performance of ranking methods in selecting the first rank, 8 methods were compared with SAW.

The numerical results in Fig. 2 show that, for the 50 sets of decision matrices of D_{3,3} (3 alternative and 3 criteria), SAW, in comparison with the other 8 methods, had the highest significant correlation percentage (76%) with the AHP2; WPM had the highest significant correlation percentage (68%) with SAW; CP had the highest significant correlation percentage (66%) with the AHP2; TOPSIS has the highest significant correlation percentage (50%) with the VIKOR; and AHP1, AHP2, AHP3 and AHP4 behave similarly and have the highest significant correlation percentage with each other. AHP1 and AHP3 produced 100% correlation percentages. AHP1 had the lowest similar behavior with TOPSIS. VIKOR had similarly significant correlation percentages with the other methods. The same interpretative approach applies for Figs. 3 through 6.

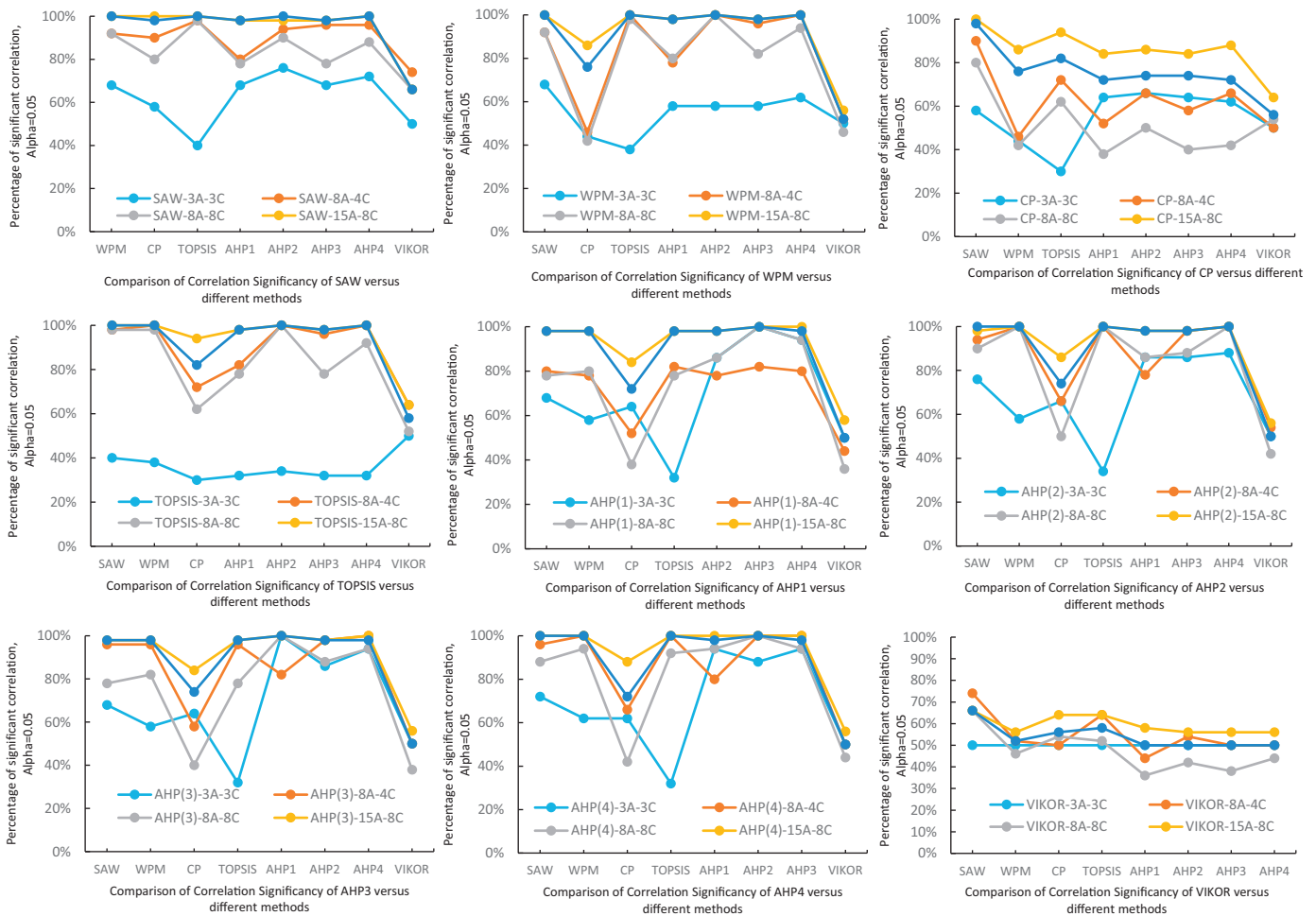


Fig. 10. Multiple comparison of statistical results of correlation significance percentage among different outranking methods applied on decision matrices of $D_{3,3}$, $D_{8,4}$, $D_{8,8}$, $D_{15,8}$, and $D_{15,15}$.

Numerical results in Fig. 7 indicate that, as the size of the decision matrices (number of alternatives) increases, the percentage of statistically significant correlations increases.

Graphical illustration of the statistical analysis in Fig. 8 shows that, statistically, for matrices with an equal number of alternatives, as the number of criteria increased, the statistically significant correlation percentage decreased as investigated for 50 sets of decision matrices of $D_{8,4}$ and $D_{8,8}$.

The graphical illustration of the statistical analysis in Fig. 9 shows that, statistically, for matrices with an equal number of alternatives, as the number of criteria increases, the statistically significant correlation percentage decreases as investigated for 50 sets of decision matrices of $D_{15,8}$ and $D_{15,15}$. However, the difference in the significant correlation percentage for matrices with 15 alternatives with a different number of criteria is less than matrices with 8 alternatives, because, as shown in Figs. 8 and 9, with an increasing number of alternatives, the percentage of statistically significant correlation increased.

4. Conclusion

The findings rank MCDMs under fuzzy environment by the performances and show when simple MCDMs match the performance of complicated MCDMs, making it possible to optimize results while minimizing computational effort. The findings also reveal several general results. Statistical analysis of the decision matrices $D_{8,4}$, $D_{8,8}$, $D_{15,8}$, and $D_{15,15}$, shown in Figs. 7 and 10, indicates that SAW, TOPSIS, WPM, AHP1, AHP2, AHP3, and AHP4 have similar

performances; AHP1 and AHP3 produced identical performances, and AHP2 and AHP4 produced very similar performances. In contrast, as compared to the other methods, CP and VIKOR had a less significant correlation percentage. Notably, Fig. 11 shows that when the uncertainty levels are raised through equal increases in the left and right spreads (i.e., when the uncertainty is changed symmetrically) there was no significant impact on the final ranking. Fig. 12, for its part, shows the performance of the ranking methods compared with SAW in finding the first rank among all alternatives. Since VIKOR, in comparison with the other methods, categorizes several alternatives with the same ranks, it exhibits more similarity to SAW. However, when the other ranking methods are compared to SAW, SAW and TOPSIS exhibit higher similarity for choosing the first rank from decision matrices $D_{8,4}$, $D_{8,8}$, $D_{15,8}$, and $D_{15,15}$.

The numerical results, along with the examples for each individual method, show that SAW, WPM, CP, and TOPSIS are computationally simple to apply; in contrast, ELECTRE, VIKOR, and the four types of AHP are computationally large and elaborate. In comparison with the other evaluated methods, SAW proved to be an especially simple method to understand and apply in ranking the alternatives of a decision matrix.

The graphical representation of the results of statistical analysis, shown in Fig. 7, indicates that in most of the evaluated methods except VIKOR, by increasing the size of the decision matrix (number of alternatives), the percentage of significant correlation among the ranks of pairwise compared methods increases regularly for SAW versus WPM, CP, TOPSIS, AHP1, AHP2, AHP3, AHP4, and VIKOR; WPM versus SAW, TOPSIS, AHP1, AHP2, AHP3, and

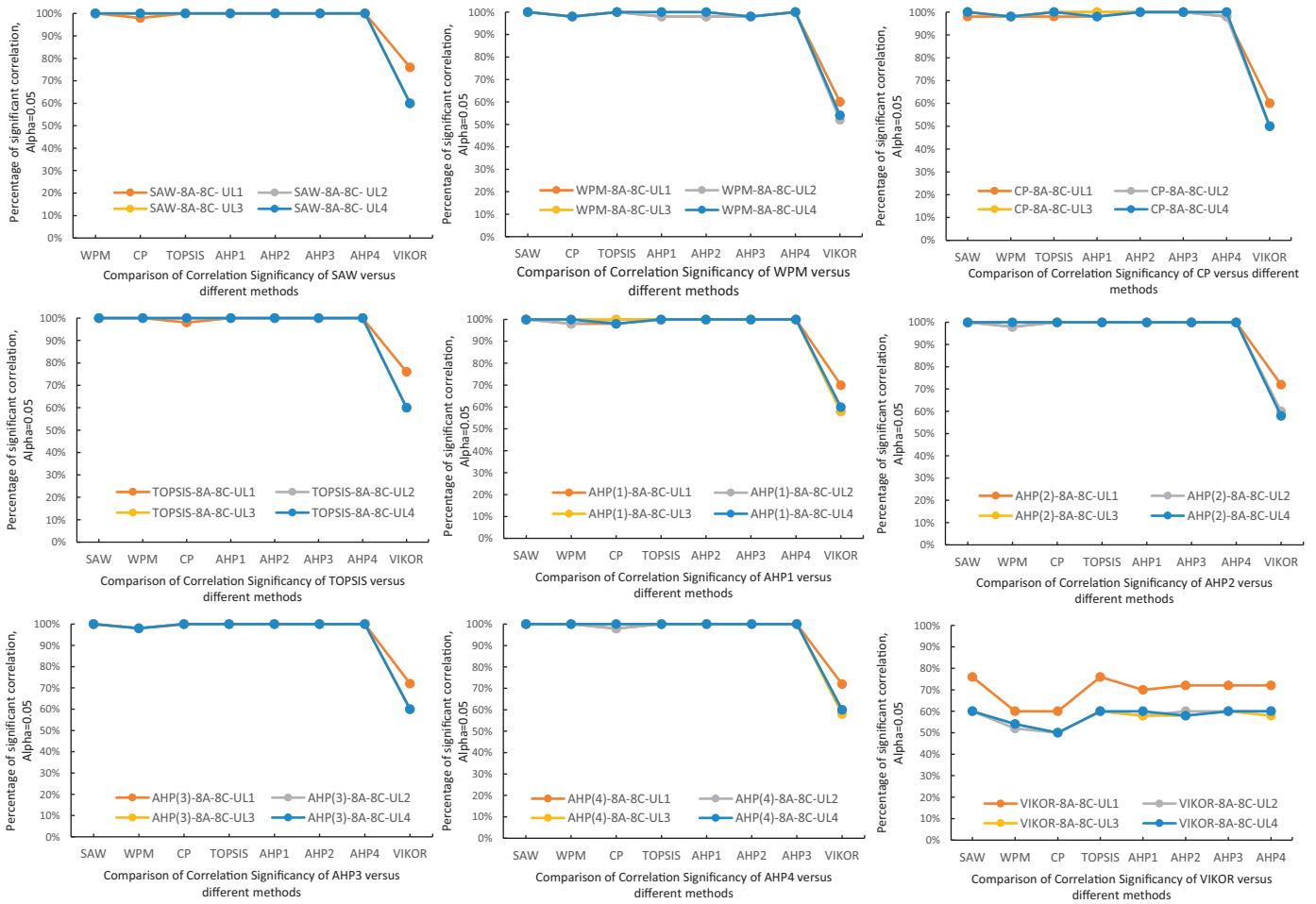


Fig. 11. Multiple comparison of statistical results of correlation significance percentage applied on $D_{8,8}$ with 4 uncertainty levels.

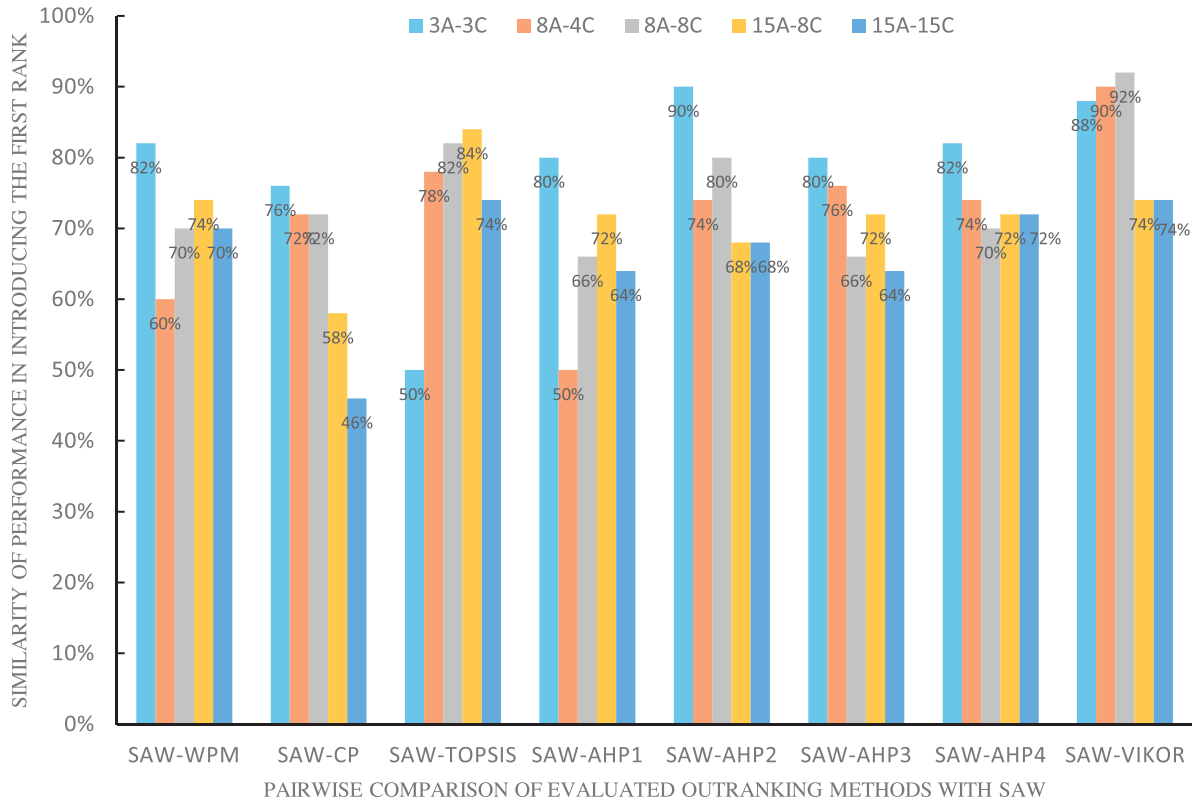


Fig. 12. Pairwise comparison of outranking methods with SAW in introducing the first rank alternatives among all potential alternatives.

AHP4; TOPSIS versus SAW, WPM, AHP1, AHP2, AHP3, and AHP4; AHP1 versus SAW, WPM, AHP2, AHP3, and AHP4; AHP2 versus SAW, WPM, AHP1, AHP3, and AHP4; AHP3 versus SAW, WPM, AHP1, and AHP4; and AHP4 versus SAW, WPM, AHP1, AHP3, and AHP4. Statistical results show that VIKOR has the lowest significant correlation percentage with the other methods. Significant correlation among the methods demonstrates strong similarity of the ranks' orders between pairwise compared methods applied on the same sort of alternatives. As the size of matrices increased, the similarity of rank orders increased. Numerical results in Figs. 8 and 9 demonstrate that increasing the number of criteria in decision matrices with equal numbers of alternatives led to a lower percentage of significant correlation among the ranks obtained from different methods. Since some methods produce similar ranks, and considering that different methods have different levels of difficulty, it is rational to use the easiest and simplest method for developing the full range of ranks on the same decision matrices.

For the investigated decision matrix sizes, the Kendall's tau-b correlation coefficients have been provided as supplementary data sets through the excel files, in which they provide reliable sources for investigating the correlation strengths between the produced ranks by different MCDM techniques applied on the same decision matrices. In addition to the multiple comparison of statistical results of correlation significance percentage, it is recommended to investigate the magnitude of the correlation coefficients among the produced ranks by different techniques when those techniques are applied on the same decision matrices.

The methods selected and discussed through this paper are classic, but still in use; for example, the Analytical Hierarchy Process has been in continuous use since the 1970s. However, there is a need to discuss and investigate recently developed techniques, such as stepwise weight assessment ratio analysis (SWARA), the weighted aggregated sum product assessment (WASPAS), additive ratio assessment (ARAS), the method of complex proportional assessment (COPRAS), multi-objective optimization by ratio analysis (MOORA), and MOORA plus a full multiplicative form (MULTI-MOORA). The evaluation of these techniques, absent in this paper, is suggested as a future direction of research.

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