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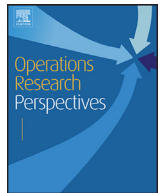
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Project management under uncertainty beyond beta: The generalized bicubic distribution

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ABSTRACT

The beta distribution has traditionally been employed in the PERT methodology and generally used for modeling bounded continuous random variables based on expert's judgment. The impossibility of estimating four parameters from the three values provided by the expert when the beta distribution is assumed to be the underlying distribution has been widely debated. This paper presents the generalized bicubic distribution as a good alternative to the beta distribution since, when the variance depends on the mode, the generalized bicubic distribution approximates the kurtosis of the Gaussian distribution better than the beta distribution. In addition, this distribution presents good properties in the PERT methodology in relation to moderation and conservatism criteria. Two empirical applications are presented to demonstrate the adequateness of this new distribution.

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1. Introduction

The beta distribution has traditionally been applied in different fields as a bounded distribution alternative to the normal distribution. In addition, its use has been proposed when no data are available but only the information provided by an expert about the optimistic (a), most likely (m) and pessimistic (b) values. This methodology, originally proposed in the context of the Project Evaluation and Review Technique (PERT), has been widely applied in important projects, such as the Concorde plane [1,2], and has even been included as a recommended practice in the Project Management Book of Knowledge [3]. Some authors have even stated that PERT was applied in the Polaris Missile system [4–7], although this fact has been questioned by many authors, such as Engwall [8] based on an updated study of Sapolsky' book [9].

The need to work with distributions defined on a bounded domain and with different degrees of asymmetry has justified the use of the beta distribution rather than the Gaussian distribution in the PERT methodology and generally when working under ex-

pert's judgment [10,11]. However, the use of the beta distribution encounters some drawbacks since it is not possible to estimate the four parameters of the beta distribution from the three values provided by the expert. Consequently, expressions (1) and (2), proposed by Malcolm et al. [12] cannot be directly obtained from the probability density function (pdf) of a beta distribution:

$$E[X] = \frac{a + 4m + b}{6} \quad (1)$$

and

$$\text{var}[X] = \frac{(b - a)^2}{36}. \quad (2)$$

These facts have generated a great discussion that continues today. First, Sasieni [13] questioned the historically repeated formulas presented by Malcolm et al. [12] and provided a new expression for the expected value (see expression (3)), which is followed by the expression of the variance provided by Golenko–Ginzburg [14] (see expression (4)):

$$E[X] = \frac{a + km + b}{k + 2} \quad (3)$$

and

$$\text{var}[X] = \frac{k^2(m - a)(b - m) + (k + 1)(b - a)^2}{(k + 3)(k + 2)^2}. \quad (4)$$

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Sasieni [13] also raised several questions related to the assumptions considered by Malcolm et al. [12]. He not only showed that expressions (1) and (2) cannot be obtained from the pdf of the beta distribution but also reported that the variance ignores the modal value. Note that expression (1) is equal to expression (3) when $k = 4$. The question posed by Sasieni [13] was how to obtain and justify this concrete value for the parameter k . Littlefield and Randolph [15] and Gallagher [16] provided some of the first answers to these questions.

A definitive answer to the question raised by Sasieni was provided by Herrerías et al. [17], in the same vein as Kamburowski [18]. These authors showed that the value of $k = 4$ originates from the intersection between the constant variance beta subfamily (a set of beta distributions with the same variance as a normal distribution truncated at ± 2.66 , $\sigma^2 = 1/36$, [5,11]) and the mesokurtic beta subfamily (a set of beta distributions with the same coefficient of kurtosis as the normal distribution, $\beta_2 = 3$). In a separate study, Xinghua [19] obtained the classic PERT expressions through interpolation using the Chebyshev quadratic polynomial. This concrete type of beta distribution has been proposed as a good candidate for modeling bounded continuous random variables only with the information provided by an expert not only in project management but also in different fields, such as probabilistic resource assessment [20], construction duration [21,22], and engineering [23], among others. However, the presence of heavy tails in some areas, such as physics, hydrology, meteorology, engineering and finance (see, e.g., [24–30], requires the use of alternative distributions. Thus, many authors have noted that the use of the beta distribution is only an assumption and not a definitive conclusion.

Clark [5] reported that *the author has no information concerning distributions of activity times; in particular, it is not suggested that the beta or any other distribution is appropriate*. Following MacCrimmon and Ryaec [31] and Moder and Rodgers [32], Perry and Greig [33] affirmed that *the original PERT formulae employ the mode and endpoints to estimate the mean and standard deviation of subjective probability distributions. Though widely used, they have met with criticism for their inaccuracy, and for their being limited to the beta distribution when there is not reason why the distribution should be beta*. Ajiboye [34] added that *the assumption of a beta distribution is just an assumption to approximate the distribution of activity durations, which can be dispensed with if necessary or validated through empirical study*. As stated by Clark [5], the analysis requires a distribution, and the beta distribution is the first to come to the author's mind, but it is required to use its constant variance version to allow the estimation of the four parameters of the beta distribution from the three points provided by the expert. This fact has been widely criticized ([35,36]) because the constant variance beta distribution ignores the most likely value, which could be understood as the most committed value provided by the expert. Hahn [35] even stated that the constant variance assumption *may be in conflict with reality*. For this reason, there are many studies focused on finding an alternative underlying distribution in the area of the PERT methodology.

In this sense, several authors have suggested some alternative distributions (see Table 1). In contrast, Hadju and Bokor [37] recently concluded that the precise estimation of the 'three points' has considerably greater significance on the project duration than the applied activity distributions. However, this statement is based only on the comparison of the beta, uniform, lognormal and triangular distributions. Moreover, Tesfaye et al. [38] and Peters [39] recommended special care regarding the selection and motivation of the underlying distribution.

In this paper, we present the generalized bicubic distribution as an alternative to the beta distribution for modeling bounded random variables under uncertainty. The primary motivation for this work is to prove that the generalized bicubic distribution is more

Table 1

Studies about distributions in the PERT methodology.

Author/s	Research
Parks and Ramsing [40] Kotiah and Wallance [41] Kumaraswamy [42] Sculli and Wong [43]	Introduced the Compound Poisson distribution Used a maximum entropy approach that leads to a truncated normal distribution Introduced the Kumaraswamy distribution Explored the errors involved in the approximation of both the maximum and sum of two independent beta random variables
Berny [44]	Introduced a new distribution with four parameters
Dodin and Sirvanci [45] Johnson [46]	Proposed an extreme value distribution Used the triangular distribution as a proxy for the beta distribution
Johnson [47]	Studied the accuracy of a number of approximations using the gamma, lognormal and F distributions
Cottrell [48]	Used an approximation of the normal distribution to determine expected time and variance using two time estimates
Kotz and van Dorp [49] Abdelkader [50]	Provided a discussion of novel bounded distributions alternative to the beta one Developed the moment's method when activities are Weibull distributed
García et al. [51]	Suggested the application of the two-sided power distribution presented by van Dorp and Kotz [52]
Mohan et al. [53]	Suggested a two parameter lognormal approximation
Hahn [35] García et al. [54] Hahn and López-Martín [55]	Proposed the beta rectangular distribution Proposed the generalized biparabolic distribution Proposed the tilted beta distribution

appropriate for the purpose of PERT than the beta and other distributions used in PERT. Among other properties, the generalized bicubic distribution approximates the kurtosis of the Gaussian distribution better than the beta distribution, and it also verifies the moderation and conservatism criteria, which will be presented in Section 5.

The remainder of this paper is organized as follows. Section 2 introduces the generalized bicubic distribution, denoted as GBC, and some of its stochastic characteristics together with the standard version of the generalized bicubic distribution (SGBC). Section 3 shows that the standard generalized bicubic distribution is consistent with the original motivation of Malcolm and that it can be a good alternative in the PERT methodology. Section 4 presents the different subfamilies of the standard generalized bicubic distribution and the classical subfamily distribution. Section 5 analyzes the behavior of the standard generalized bicubic distribution in relation to the moderation and conservatism criteria. The obtained results are compared with the standard two-sided power (denoted as STSP) and the standard generalized biparabolic distributions (denoted as SGBP). In Section 6, two illustrative examples using Monte Carlo simulations are provided. In Section 7, we discuss the main conclusions.

2. The generalized bicubic distribution

Based on the generating density presented by van Dorp and Kotz [56], López [57] introduced the bicubic distribution, whose probability density function is defined as

$$f(x) = \begin{cases} \frac{-2}{(m-a)^3(b-a)}(x-a)^2(2x-3m+a) & \text{if } a < x \leq m, \\ \frac{-2}{(m-b)^3(b-a)}(x-b)^2(2x-3m+b) & \text{if } m < x < b. \end{cases} \quad (5)$$

Considering $t = \frac{x-a}{b-a}$ and $M = \frac{m-a}{b-a}$, the standard bicubic distribution is obtained, whose probability density function is given by ex-

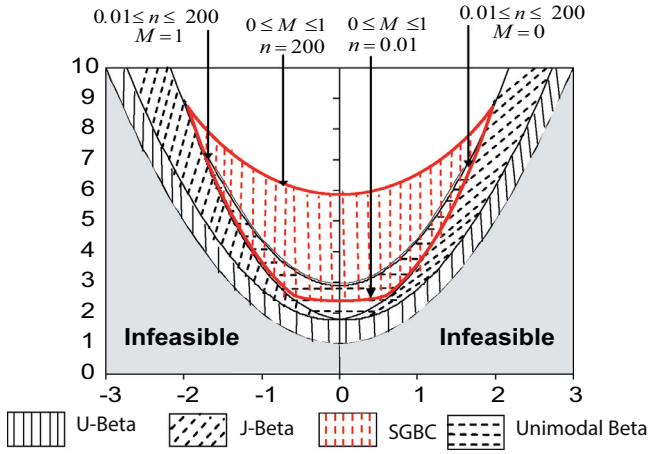


Fig. 1. $(\sqrt{\beta_1}, \beta_2)$ moment ratio diagram.

pression (6):

$$g(t) = \begin{cases} 6 \left[\left(\frac{t}{M} \right)^2 - \frac{2}{3} \left(\frac{t}{M} \right)^3 \right] & \text{if } 0 < t \leq M, \\ 6 \left[\left(\frac{1-t}{1-M} \right)^2 - \frac{2}{3} \left(\frac{1-t}{1-M} \right)^3 \right] & \text{if } M < t < 1, \end{cases} \quad (6)$$

A new parameter n is introduced with the goal of increasing the flexibility of the distribution in relation to asymmetry and kurtosis. In this case, it is stated that the random variable T is distributed according to a standard generalized bicubic distribution, denoted as $\text{SGBC}(0, M, 1, n)$ with $n \geq 0$, if and only if its probability density function is given by:

$$g(t) = \frac{3(n+3)(n+4)}{n+6} \times \begin{cases} \left(\frac{t}{M} \right)^{n+2} - \frac{2}{3} \left(\frac{t}{M} \right)^{n+3} & \text{if } 0 < t \leq M, \\ \left(\frac{1-t}{1-M} \right)^{n+2} - \frac{2}{3} \left(\frac{1-t}{1-M} \right)^{n+3} & \text{if } M < t < 1. \end{cases} \quad (7)$$

From (7), the expressions of the expected value and variance are obtained as:

$$E[T] = \frac{Mn^2 + (9M+1)n + 12M + 9}{(n+5)(n+6)} \quad (8)$$

and

$$\text{var}[T] = \frac{(2M^2 - 2M + 1)n^2 + (30M^2 - 30M + 16)n + (36M^2 - 36M + 39)}{(n+5)^2(n+6)^2}. \quad (9)$$

Fig. 1 analyzes the effect of the distribution parameters (M and n) on the skewness and coefficient of kurtosis [58]. Note that the SGBC distribution increases the coverage of the beta distribution over the area of unimodal distributions in a manner similar to the standard two-sided power (STSP) [56] and the standard generalized biparabolic distributions (SGBP) [54]. For this reason, we consider that the SGBC distribution could serve as an alternative to the beta distribution in all fields where it is applied.

3. The generalized bicubic distribution and the PERT assumptions

It is possible to interpret that in his original idea, Malcolm [12] pretended to use an underlying distribution similar to the normal distribution but asymmetric. Thus, the underlying distribution should have the same coefficient of kurtosis as the Gaussian distribution and the same variance as a normal distribution truncated at ± 2.66 . This last assumption was recently questioned by Hahn [35] and Herrerías-Velasco et al. [36], who argued that a constant variance may be in direct conflict with reality.

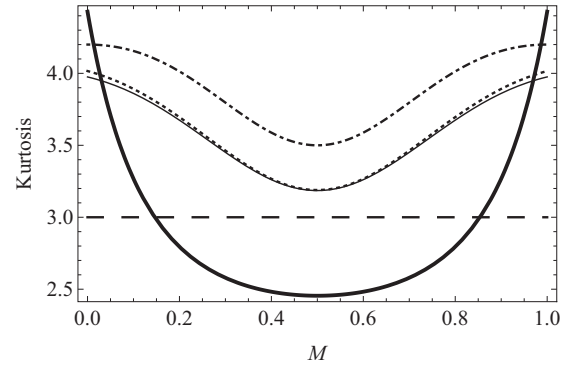


Fig. 2. Kurtosis of beta PERT Eq. (2) (thick solid line); STSP($M, 5$) [52] (dot-dashed line); SGBP($0, M, 1, 4.9367$) (thin solid line); SGBC($0, M, 1, 3$) (dotted line); and kurtosis of normal distribution (dashed line).

Except for the constant variance assumption, the SGBC distribution satisfies the remainder of the assumptions for $n = 3$. Fig. 2 shows the kurtosis of the SGBC distribution compared with the SGBP, STSP and the beta PERT distributions. Note that the SGBP($0, M, 1, 4.9367$) and SGBC($0, M, 1, 3$) distributions have a kurtosis close to 3 (which is the Gaussian kurtosis) when $M \in (0.32, 0.68)$.

Conversely, note that for $n = 3$, expression (8) becomes expression (1), and the variance depends on M and is given by the following expression:

$$\text{var}[T] = \frac{3M^2 - 3M + 2}{108}. \quad (10)$$

In this case, the variance of the SGBC($0, M, 1, 3$) distribution has its minimum at $M = \frac{1}{2}$, whereas the variance of the beta distribution proposed by [36] has a maximum at this point. We believe that this fact deserves to be analyzed in more detail.

Recall that the goal of the PERT methodology is to estimate $E[X]$ and $\text{var}[X]$ from the classical values (a, m, b). In addition, it is satisfied that $E[X]$ will be always between $\frac{a+b}{2}$ and m or m and $\frac{a+b}{2}$ ($\frac{1}{2}$ and M or M and $\frac{1}{2}$ with the standard variable). Thus, if the expert provides¹ a value of $m = \frac{a+b}{2}$ ($M = \frac{1}{2}$), then the expected value of PERT has been completely fixed with this information. From another perspective, within the PERT context, the expert is asked about the optimistic (a), most likely (m) and pessimistic (b) values, or by standardizing ($0, M, 1$). Regardless, the expert provides three values that can be interpreted as a discrete distribution if the expression (3) is accepted as the expected value. The variance of this discrete distribution is the expression (11) that presents a minimum for $M = 1/2$.

$$\text{var}[T] = \frac{1 + k(1 - 2(1 - M)M)}{(k + 2)^2}. \quad (11)$$

Once the optimistic and pessimistic values are provided, the expert can select any value between them as the most likely value. If the selected most likely value is the average point of the interval ($M = \frac{1}{2}$), then the corresponding discrete distribution will present a minimum variance. Thus, the information provided by the expert can be interpreted as a discrete distribution, and the confidence of this information will be greater as the variance is lower. Then, the continuous underlying distribution in the PERT methodology should also have a minimum variance for $M = \frac{1}{2}$, which occurs with the triangular, beta rectangular, STSP, SGBP and SGBC distributions, among others. Regardless, this could become a philosophical question depending on the interpretation given to the intentions of the expert.

¹ Recall that the expert is questioned in relation to the mode and not to the expected value.

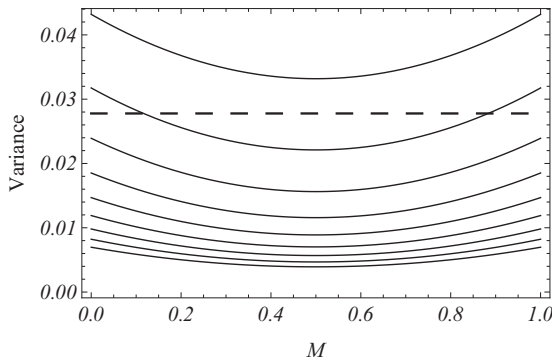


Fig. 3. Variance of SGBC distribution for $n = 0.01, n = 1, n = 2, n = 3, n = 4, n = 5, n = 6, n = 7, n = 8$, (solid lines, from top to bottom) and variance suggested in the classical PERT [12] (dashed line).

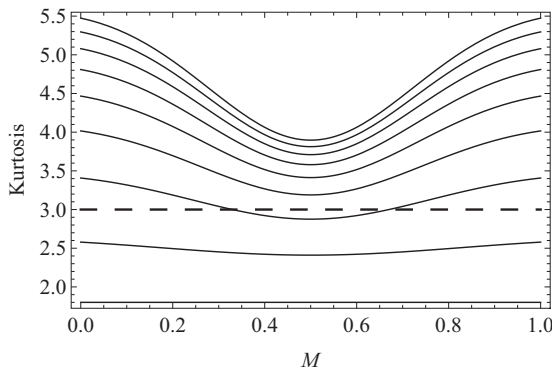


Fig. 4. Kurtosis of SGBC distribution for $n = 0.01, n = 1, n = 2, n = 3, n = 4, n = 5, n = 6, n = 7, n = 8$, (solid lines, from bottom to top) and the kurtosis of normal distribution (dashed line).

In conclusion, we consider that, despite the criticism, the assumption of constant variance appears to be consistent when the beta distribution is the underlying distribution in PERT since the expression of the non-constant variance for the beta distribution takes a maximum at the point $M = 1/2$ ($m = \frac{a+b}{2}$), which is when the expert completely determines the expected value based only on the mode. Consequently, we consider it appropriate to assume constant variance for the beta distribution in PERT, and we will continue with the SGBC with constant variance to follow with the initial motivation of Malcolm [12].

Conversely, it is possible to analyze the behavior of the variance and kurtosis of SGBC for different values of n , attempting to express expression (8) in line with Sasieni [13], expression (3). Thus, the following values for n and k are obtained

$$n = \lambda - 9,$$

$$k = \frac{(\lambda - 9)^2 + 9(\lambda - 9) + 12}{\lambda},$$

for $\lambda > 9$. Fig. 3 shows that the variance of SGBC decreases as n increases, and Fig. 4 shows that the kurtosis increases as n increases, ‘hovering’ around three (Gaussian kurtosis) for $n = 2$.

4. The subfamilies of the SGBC distribution in the PERT methodology.

Kamburowski [18] and Herreras et al. [17] are the only authors that based their response to the question raised by Sasieni [13] about the ‘mysterious’ formula of PERT on the similarity between the beta and the normal distributions regarding their variance and kurtosis. In fact, Herreras et al. [17] described the *classical*

distribution of PERT methodology as the intersection of the constant variance and the mesokurtic subfamilies of the beta distribution. Recall that the subfamily of constant variance is defined as the set composed by the distributions with the same variance as the normal distribution when the random variable is standardized. Conversely, the mesokurtic subfamily is defined as the set of distributions with a coefficient of kurtosis equal to three.

Now, we perform a comparative study of the subfamilies of the SGBC, SGBP and STSP distributions. For this comparison, it is important to note that the relation between parameter k in the beta distribution and parameter n in the STSP distribution is given by $n = k + 1$ [51]. This same relation holds between the SGBP and SGBC distributions. In this situation, each probabilistic model has the same range.

Fig. 5 shows the relation between n and M for all the studied subfamilies. As shown, the mesokurtic beta distribution has no results for values of M on the interval (0.27, 0.72). We observe similar behavior when comparing the constant variance subfamilies of the STSP, SGBP and SGBC distributions, where the parameter n is higher when the mode becomes closer to 0 and 1 endpoints. The opposite occurs when comparing the mesokurtic subfamilies of the STSP, SGBP and SGBC distributions, i.e., the parameter n is lower when the mode is close to 0 and 1. Note that the SGBC distribution shows lower values of n for every value of M defined in the interval (0, 1) in the constant variance and the mesokurtic cases, as shown in Fig. 5. However, note that the graphical behavior of the three probabilistic models is very similar.

Table 2 presents the intersection points between the subfamilies. Note that the constant variance subfamily of the SGBC distribution, denoted as CV-SGBC, intersects only with the mesokurtic subfamily of the SGBC distribution, denoted as M-SGBC. As a special situation, the constant variance STSP distribution, CV-STSP, intersects with all the mesokurtic subfamilies.

As mentioned above, the classical beta distribution [17] is the intersection between the mesokurtic and the constant variance subfamilies of the beta distribution, and it is obtained when $k = 4$. Table 3 presents the expressions of the expected values of the classical distributions supported on a bounded interval $[a, b]$.

From the values shown in Table 2, Table 4 presents the values for M and n required for obtaining the classical STSP [51], SGBP [54] and SGBC distributions. Table 4 also includes the expected value from which we conclude that the classical STSP distribution is the one that presents the most moderate mean.

5. Moderation and conservation criteria

In the context of PERT, it is assumed that if the probabilistic model presents a centered expected value, i.e., an expected value close to 0.5, then it will be more moderate in its estimations [59,60]. Thus, we will compare the expected value of each of the distributions.

Fig. 6 (A), (B) and (C) show the comparison between the estimated mean of each subfamily considering that $M \in (0, 0.3]$, $M \in [0.3, 0.7]$, and $M \in [0.7, 1]$, respectively, and that n is obtained from the conditions of constant variance equal to $1/36$ and a kurtosis equal to three. It is clear that each distribution, except for the mesokurtic beta distribution, has a common point in $M = 0.5$, as shown in Fig. 6(B).

From these results, we have determined an order of moderation among the models presented in Table 5. As shown, when the value of M becomes closer to 0 and 1 ($M < 0.21$ or $M > 0.79$), the mesokurtic STSP distribution (M-STSP) is the most moderate. However, when the mode obtains moderate values (between 0.21 and 0.79), the constant variance SGBP and SGBC distributions exhibit similar behavior and are the most moderate in mean. Normally, the expert will not provide a standard mode extreme value since it

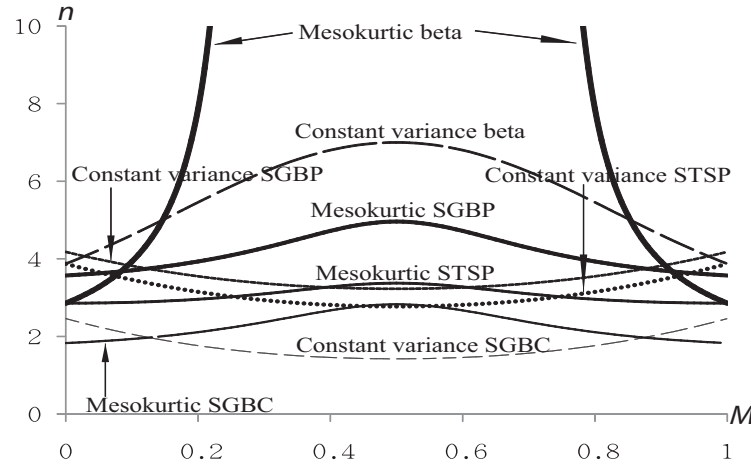


Fig. 5. Relationship between n and M in all constant variance and mesokurtic subfamilies of the STSP, SGBP and SGBC distributions.

Table 2
Intersection points (M, n) among different mesokurtic and constant variance subfamilies.

	CV-Beta	CV-STSP	CV-SGBP	CV-SGBC
M-Beta	(0.1464, 5) (0.8535, 5)	(0.0746, 3.55560) (0.9254, 3.55560)	(0.0942, 3.83618) (0.9058, 3.83618)	No solution
M-STSP	No solution	(0.2529, 3.02344) (0.7471, 3.02344)	(0.3937, 3.26526) (0.6063, 3.26526)	No solution
M-SGBP	No solution	(0.0552, 3.63282) (0.9448, 3.63282)	(0.1235, 3.74669) (0.8765, 3.74669)	No solution
M-SGBC	No solution	(0.4493, 2.78220) (0.5507, 2.78220)	No solution	(0.1256, 1.97037) (0.8743, 1.97037)

Table 3

Expected value of the classical distributions with support $[a, b]$.

Distribution	Expected value and variance
Classical beta	$E[X] = \frac{a+4m+b}{6}$
Classical TSP	$E[X] = 0.2885a + 0.5029m + 0.2885b$
Classical GBP	$E[X] = 0.2192a + 0.5160m + 0.2192b$
Classical GBC	$E[X] = 0.2396a + 0.5208m + 0.2396b$

Table 4

Values for M, n and expected value for the classical STSP, SGBP and SGBC distributions.

Distribution	n	M	$E(T)$
Classical STSP	3.02344	0.25287 or 0.74713	0.375715 or 0.624285
Classical SGBP	2.74669	0.12835 or 0.87165	0.308218 or 0.691782
Classical SGBC	0.97037	0.12565 or 0.87435	0.305025 or 0.694975

will be a risky position. Thus, the CV-SGBP and CV-SGBC could be considered to be the most moderate alternative distributions when the standard mode varies from 0.21 to 0.79.

Similar to the above section, we will now study the behavior of each probabilistic model with respect to their conservatism. It is stated that a probabilistic model is more conservative when it has a maximum value in its variance [59,60]. Thus, if we have two probabilistic models with the same mean, it appears to be better to work with the distribution that has a greater estimated variance to minimize the risk of concluding with optimistic results [59,60].

Fig. 7 shows the estimated variance of each distribution when it belongs to the constant variance and mesokurtic subfamilies. From these results, we can describe the order of conservatism of each probabilistic model summarized in Table 6. Note that working with the constant variance subfamily, the different dis-

Table 5

Order of moderation of the mesokurtic and constant variance subfamilies of the STSP, SGBP and SGBC distributions.

Intervals of mode (M)	Ordered distributions
$0 < M < 0.123526$	M-STSP > M-SGBP ~ M-SGBC
$0.123526 < M < 0.21$	M-STSP > CV-SGBP ~ CV-SGBC
$0.21 < M < 0.252867$	CV-SGBP ~ CV-SGBC > M-STSP
$0.252867 < M < 0.5$	CV-SGBP ~ CV-SGBC > CV-STSP
$0.5 < M < 0.747133$	CV-SGBP ~ CV-SGBC > CV-STSP
$0.747133 < M < 0.79$	CV-SGBP ~ CV-SGBC > M-STSP
$0.79 < M < 0.876474$	M-STSP > CV-SGBP ~ CV-SGBC
$0.876474 < M < 1$	M-STSP > M-SGBP ~ M-SGBC

tributions have the same value since the variance of all distributions is equal to $1/36$. This is why we have used the notation 'CV subfamilies' in Table 6. We can conclude that the mesokurtic STSP distribution is the most conservative model when $M \in (0, 0.252867) \cup (0.747133, 1)$. In the other cases, when the standard mode varies from 0.252867 to 0.747133, the CV subfamilies have the maximum estimated variance.

To summarize, note that the expert will generally provide a moderate standard mode, and in this case, the CV-SGBC distribution will be the most moderate (together with the CV-SGBP distribution) and the most conservative (together with the remaining constant variance subfamilies).

From another perspective, the intersection between the constant variance and the mesokurtic subfamilies will make stochastic sense if the corresponding estimations are similar between them. The Chow structural change test allows the statistical homogeneity to be studied. If we fail to reject the null hypothesis, then there is no structural change and the estimations are similar. In the case of the subfamilies CV-SGBC and M-SGBC, we have that $F_{\text{exp}} = 0.04932$, and then the estimations obtained by these sub-

Table 6

Order of conservatism of the mesokurtic and constant variance subfamilies of the beta, STSP, SGBP and SGBC distributions.

Intervals of mode (M)	Ordered distributions
$0 < M < 0.125649$	M-STSP>M-Beta>M-SGBC~ M-SGBP> CV subfamilies
$0.125649 < M < 0.146437$	M-STSP>M-Beta>CV subfamilies>M-SGBC>M-SGBP
$0.146437 < M < 0.252867$	M-STSP>CV subfamilies>M-SGBP>M-SGBC>M-Beta
$0.252867 < M < 0.747133$	CV subfamilies>M-STSP>M-SGBC>M-SGBP
$0.747133 < M < 0.853553$	M-STSP>CV subfamilies>M-SGBC>M-SGBP>M-Beta
$0.853553 < M < 0.874351$	M-STSP>M-Beta>CV subfamilies>M-SGBC>M-SGBP
$0.874351 < M < 1$	M-STSP>M-Beta>M-SGBC~ M-SGBP>CV subfamilies

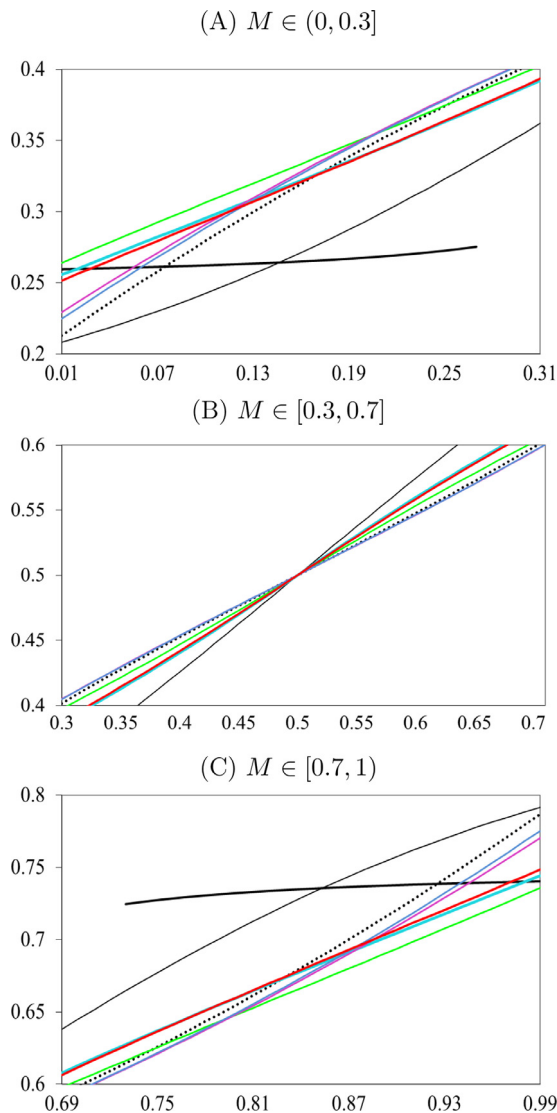


Fig. 6. Estimation of the expected values for M-Beta (thick black solid), CV-Beta (thin black solid), M-STSP (green), CV-STSP (dotted), M-SGBP (cyan), CV-SGBP (purple), M-SGBC (red), and CV-SGBC (blue). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

families come from the same distribution (with a 95% of confidence), and this fact makes the intersection between the constant variance and the mesokurtic subfamilies more natural.

6. Empirical applications

We will now analyze the behavior of the classical beta, classical STSP, classical SGBP and classical SGBC distributions in two em-

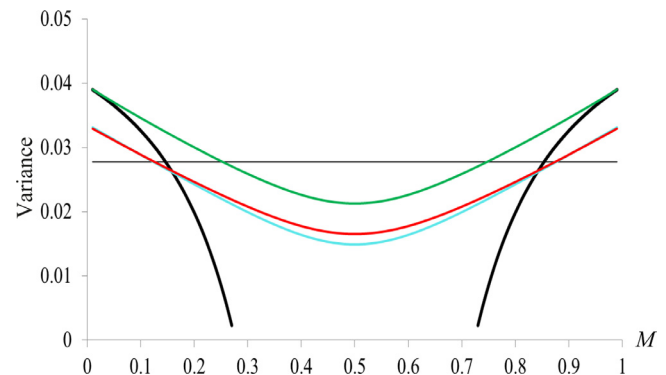


Fig. 7. Estimation of the variance values for M-Beta (thick black solid), CV-Beta (thin black solid), M-STSP (green), CV-STSP (dotted), M-SGBP (cyan), CV-SGBP (purple), M-SGBC (red), and CV-SGBC (blue). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 7

Empirical application data proposed by Render [61].

Activity	Description	Predecessor	a	m	b
A	Build internal components	–	1	2	3
B	Modify roof and floor	–	2	3	4
C	Construct collection stack	A	1	2	3
D	Pour concrete and install frame	B	2	4	6
E	Build high-temperature burner	C	1	4	7
F	Install control system	C	1	2	9
G	Install air pollution device	D,E	3	4	1
H	Inspection and testing	F,G	1	2	3

pirical PERT applications previously proposed by Render [61] and Neuwirth and Zelnick [62], respectively. Following Hadju and Bokor [37], we have included in the comparison the classical beta using -10% and $+10\%$ durations for pessimistic, most likely and optimistic durations.

6.1. Empirical application 1

This problem presented by Render [61] is based on a metal-works plant in Milwaukee, General Foundry, Inc., that is required to install a complex air filter system on its main smokestack. The activities involved in the project, the precedence relations among them, and the durations in weeks are shown in Table 7.

The critical path is given by A-C-E-G-H with an expected time of 15 weeks to finish the activity. We will simulate the distribution of the total time required to perform the project one hundred thousand times by applying Monte Carlo simulations.

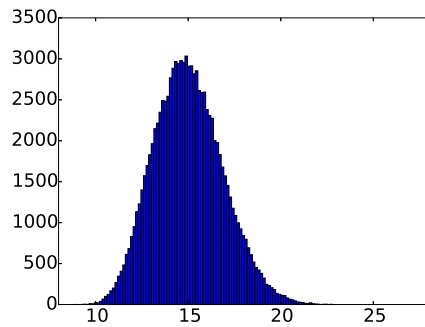
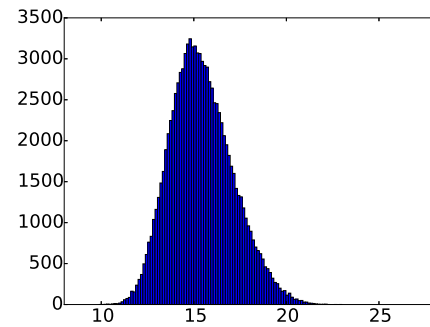
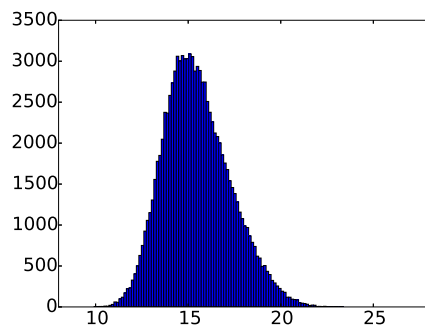
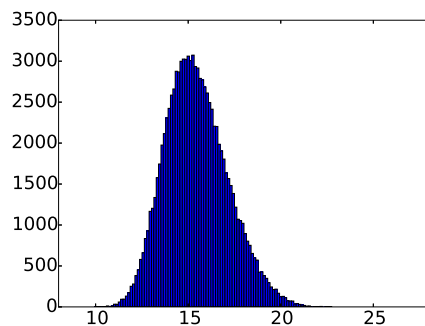
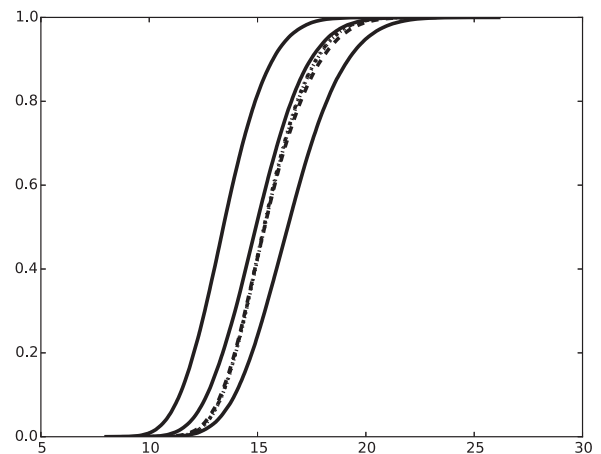
Figs. 8, 9, 10 and 11 show the total variable project duration by using the classical distributions. Note that the behavior of each probabilistic model is very similar.

The main stochastic characteristics are shown in Table 8. The results obtained using the classical SGBP and the classical SGBC

Table 8

Stochastic characteristics of the total project time variable for the classical beta, classical STSP, classical SGBP and classical SGBC distributions obtained by Monte Carlo simulations.

Distribution	Mean	S. Deviation	Skewness	Kurtosis	Interval 95%
Classical beta	15.008165	1.855174	0.270186	2.931070	(11.660234, 18.870979)
Classical beta –10%	13.501358	1.664028	0.274479	2.939714	(10.490845, 16.996131)
Classical beta +10%	16.495701	2.034742	0.282465	2.963557	(12.812710, 20.753840)
Classical STSP	15.487874	1.801529	0.417302	3.007731	(12.382658, 19.395115)
Classical SGBP	15.452421	1.692248	0.365241	2.983243	(12.482779, 19.085543)
Classical SGBC	15.429422	1.687322	0.376980	2.992578	(12.481305, 19.050390)

**Fig. 8.** Distribution of project duration using the classical beta ($k = 4$) distribution.**Fig. 11.** Distribution of project duration using the classical SGBC ($n = 0.97037$) distribution.**Fig. 9.** Distribution of project duration using the classical STSP ($n = 3.02344$) distribution.**Fig. 10.** Distribution of project duration using the classical SGBP ($n = 2.74669$) distribution.**Fig. 12.** Cumulative density function of the classical beta, classical beta –10% and classical beta +10% (solid), classical STSP (dashed), classical SGBP (dotted) and classical SGBC (dot-dashed) distributions.

distributions are quite similar. Note that the estimations of the standard deviation and coefficient of kurtosis of the four distributions are close to the values of the normal distribution, as shown in Table 8, and consequently, the distribution used to estimate the total project duration retains the principal properties of the Gaussian model.

Note that the classical STSP, classical SGBP and classical SGBC distributions are more prudent because the estimations of the average durations are higher than in the case of the classical beta distribution. Furthermore, the classical STSP distribution is more

conservative in variance than the classical SGBP and classical SGBC distributions.

However, if we study the 95% confidence interval, we observe that the beta distribution presents the largest interval. The range of this interval is 7.21. Conversely, the classical SGBC distribution has the smallest interval.

Fig. 12 shows the comparison among the four classical distributions studied in this paper together with the classical beta distribution using –10% and +10% durations for pessimistic, most probable and optimistic durations by following [37]. It is shown that the classical beta distribution is always above the other distributions. This result implies that, for the same probability, in the classical STSP, classical SGBP and classical SGBC distributions, the estimation of the total project time is higher than the estimation provided by the classical beta distribution. This is why we can conclude that these probabilistic models provide more prudent estimations.

Table 9
Empirical application data proposed by Neuwirth and Zelnick [62].

Activity	Description	Predecessor	a	m	b
0	Reorganization approved	–	–	–	–
1	Administrator's appointment announced	0	0.1	1	2
2	Progress reporting system installed	1	1	3	4
3	Personnel evaluations updated	2	4	6	12
4	New position descriptions completed	2	0.1	2	4
5	New personnel specifications completed	4	1	1.5	3
6	New salary classifications determined	5	1	4	8
7	Manpower forecasts completed to 1962	5	1	3	4
8	Manpower forecasts completed to 1967	7	1	2	4
9	Recruitment program launched	6	0.1	1	4
9	Recruitment program launched	7	0.1	2	3
10	Organization costs estimated	6	0.1	1	3
10	Organization costs estimated	7	0.1	1	2
11	Procedural system started	2	0.1	1	2
12	Existing procedures documented	11	6	13	30
13	Essential new procedures documented	12	10	12	40
14	First stage completed	9, 10, 11, 13, 3, 8	0	0	0

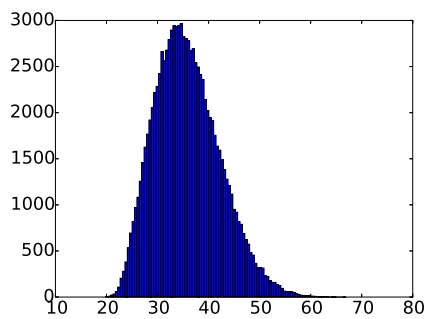


Fig. 13. Distribution of project duration using the classical beta ($k = 4$) distribution.

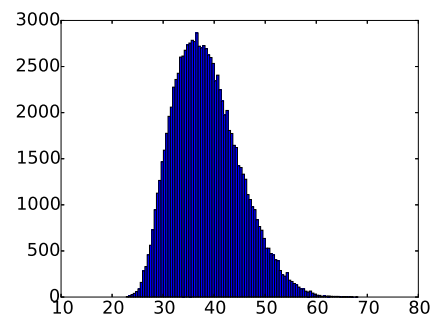


Fig. 15. Distribution of project duration using the classical SGBP ($n = 2.74669$) distribution.

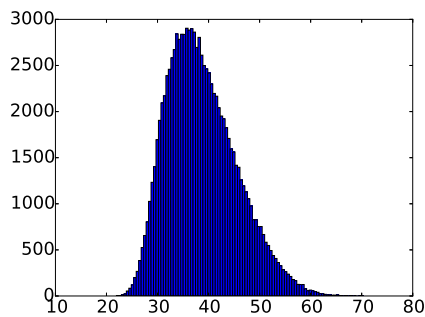


Fig. 14. Distribution of project duration using the classical STSP ($n = 3.02344$) distribution.

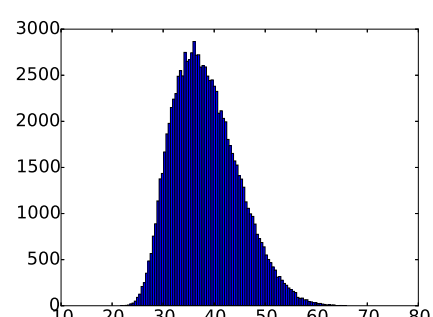


Fig. 16. Distribution of project duration using the classical SGBC ($n = 0.97037$) distribution.

6.2. Empirical application 2

We will now analyze the behavior of the classical beta, classical STSP, classical SGBP and classical SGBC distributions in an empirical PERT application originally proposed by Neuwirth and Zelnick [62] in relation to the installation of a new organizational structure. Table 9 shows the PERT network. We will simulate the distribution of the total time required to conduct the project one hundred thousand times by applying Monte Carlo simulations.

Figs. 13, 14, 15 and 16 show the *total variable project duration* by using the classical distributions. Note that the behavior of each probabilistic model is very similar.

The main stochastic characteristics are shown in Table 10. Similar to the first application, the results from the classical SGBP and the classical SGBC distributions are quite similar. The four distributions provide estimations of the standard deviation and kurtosis coefficient that are similar to the normal distribution, as shown

in Table 10, and consequently, the distribution used to estimate the total project duration retains the principal properties of the Gaussian model. Note that the classical STSP and classical SGBP distributions are more prudent because the estimations of the average durations are higher than in the case of the classical beta distribution. Furthermore, the classical STSP distribution is more conservative in variance.

However, if we study the 95% confidence interval, we observe that the classical STSP distribution presents the largest interval. The range of this interval is 26.08. Conversely, the classical SGBC distribution has the smallest interval.

Fig. 17 shows the comparison among the four classical distributions studied in this paper together with the classical beta distribution using -10% and $+10\%$ durations for pessimistic, most probable and optimistic durations by following [37]. It is shown that the classical beta distribution is always above the other distributions. This result implies that, for the same probability, in the classical

Table 10

Stochastic characteristics of the total project time variable for the classical beta, classical STSP, classical SGBP and classical SGBC distributions obtained by Monte Carlo simulations.

Distribution	Mean	S. Deviation	Skewness	Kurtosis	Interval 95%
Classical beta	35.869323	6.393357	0.470557	3.026389	(25.132198, 49.763173)
Classical beta –10%	32.302552	5.764584	0.476537	3.033078	(22.641757, 44.953618)
Classical beta +10%	39.457435	7.036563	0.472908	3.038709	(27.661468, 54.780050)
Classical STSP	38.727221	6.826805	0.577564	3.052636	(27.864311, 53.945726)
Classical SGBP	38.526419	6.398061	0.507454	2.998448	(28.075792, 52.500611)
Classical SGBC	38.426060	6.390328	0.519158	2.991826	(28.086665, 52.493270)

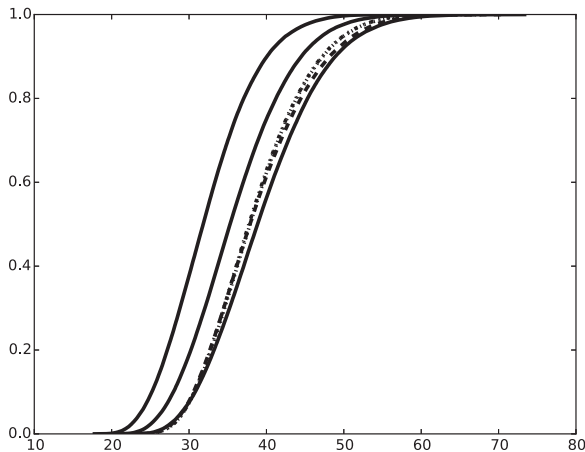


Fig. 17. Cumulative density function of classical beta, classical beta –10% and classical beta +10% (solid), classical TSP (dashed), classical GBP (dotted) and classical GBC (dot-dashed) distributions.

STSP, classical SGBP and classical SGBC distributions, the estimate for a total project time is higher than that estimated by the classical beta distribution. This is why we can conclude that these probabilistic models provide more prudent estimations.

7. Conclusions

The original assumption of the PERT methodology was that the activity time follows a beta distribution with expressions (1) and (2). These expressions cannot be obtained from the probability density function of the beta distribution since the four parameters of the beta distribution cannot be estimated from the three values provided by the expert. Herrerías et al. [17] showed that the distribution obtained by the intersection between the constant variance and mesokurtic subfamilies of beta distributions is the one that presents the same mean and variance used in the traditional PERT methodology and that can be estimated by using only the information provided by the expert. This fact answers the question raised by Sasieni [13] regarding the formula of PERT, justifying the value of $k = 4$ due to the similarity with the normal distribution.

Beyond the beta distribution, numerous authors have proposed alternative distributions to be applied in this methodology. This paper presents the standard generalized bicubic distribution and shows that for $n = 3$, it has an expected value similar to the one initially proposed for the PERT methodology [14] and a kurtosis close to the Gaussian one when $M \in (0.32, 0.68)$. Since the constant variance assumption has been recently criticized, an explicit expression is also presented for the variance of the SGBC for $n = 3$. This non-constant variance has a minimum for $M = 0.5$ and $n \geq 0$, which we consider to be consistent with the original motivation of Malcolm. In contrast, the beta distribution has a maximum for $M = 0.5$, and this fact could justify the assumption of constant variance for the beta distribution.

Moreover, we have also analyzed the subfamilies of the SGBC distribution in relation to the criteria of conservatism and moderation showing very appropriated characteristics. Finally, we illustrate all the results with two empirical examples, confirming the suitability of the generalized bicubic distribution. Tables 8 and 10 show the estimations of the expected value with the classical distributions for the first and second applications, respectively. Note that the difference between the expected value obtained from the classical beta and the expected value obtained from any of the other classical distributions is approximately 3% in the first application, increasing to 8% in the second application.

This fact contrasts with the statement provided by Hadju and Bokor [37], who considered irrelevant the applied distribution based only on the comparison of the beta, uniform, lognormal and triangular distributions. In both examples, we have shown that to use alternative distributions to beta, one can be almost as relevant as an upward systematic error in the estimations with the great difference that estimation errors are in both directions. For this reason, we agree with Hadju and Bokor [37] that it is necessary to incorporate many more projects and alternative distributions to draw a universal conclusion. In conclusion, we consider that the GBC distribution can be an alternative to the beta distribution not only in project management but also in the treatment of uncertainty in general.

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