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Article

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Application of Generalized Hukuhara derivative approach in an economic production quantity model with partial trade credit policy under fuzzy environment



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ABSTRACT

In this present study, a production inventory model with partial trade credit is formulated and solved in fuzzy environment via Generalized Hukuhara derivative approach. To capture the market, a supplier offers a trade credit period to its retailers. Due to this facility, retailer also offers a partial trade credit period to his/her customer to boost the demand of the item. In practical life situation, demands are generally dependent upon time. Constant demand of an item varies time to time. In this vague situation, demands are taken as time dependent, where its constant part is taken as Left Right - type fuzzy number. In this paper, Generalized Hukuhara derivative approach is used to solve the fuzzy inventory model. Four different cases are considered by using Generalized Hukuhara-(i) differentiability and Generalized Hukuhara-(ii) differentiability. The objective of this paper is to find out the optimal time so as the total inventory cost is minimum. Finally the model is solved by generalized reduced gradient method. The proposed model and technique are illustrated by numerical examples. Some sensitivity analyses both in tabular and graphical forms are presented and the effects of minimum cost with respect to various inventory parameters are discussed.

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1. Introduction

The classical economic order quantity (EOQ) model was first developed by Harris [16]. Lot of researcher works on inventory control system are available in the literature (e.g., Chang and Dye [27], Mandal and Maiti [38,39] etc). Lots of works has been done in the area of inventory control by considering multi items and multi period such as Mousavi et al. [56,57,60–62], Pasandideh et al. [58], Mousavi and Pasandideh [59], Roozbeh [2], Nobil [3] etc.

Demand has been always one of the most effective factors in the decisions relating to EOQ model as well as Economic Production Quantity (EPQ) model. Due to this reason, various forms of consumption tendency have been studied by inventory control practitioners, such as constant demand (Wee et al. [28]), customer credit period dependent demand (Maiti, [39]), etc. All of them developed their models in crisp environment, i.e., demand coefficients are considered as crisp number. But, it is better to estimate

the demand coefficients as imprecise with fuzzy numbers. A notable work was done by Sadeghi and Niaki [34] by taking demand as a trapezoidal fuzzy number.

The presence of fuzzy demand and fuzzy production rate lead to fuzzy differential equation (FDE) of instantaneous state of inventory level. Till now fuzzy differential equation is little used to solve fuzzy inventory models though the topics on fuzzy differential equations have been rapidly growing in the recent years. The first impetus on solving fuzzy differential equation was made by Kandel and Byatt [1]. After that different approaches have been made by several authors to solve fuzzy differential equations (Kaleva [44]; Buckley and Feuring, [32]; Chalco-Cano and Roman-Flores [65]).

The first publication accommodating the uncertainty in non-stochastic sense was in 1965 by Prof. Zadeh [36]. But applications of fuzzy sets in inventory control problems are around 15–20 years back. Among these works, one can refer the works of Maiti [41], Mandal and Maiti [38], Wee et al. [28], Harish [19–22,25], Harish et al. [23,24] etc. Till now, FDE and fuzzy integration are little used to solve fuzzy inventory models [51,54], though the topics on fuzzy differential equations have been rapidly growing in the recent years. The first impetus on solving FDE was made by Kandel and Byatt [1]. One of the notable paper in this direction is

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due to Buckley and Feuring [32]. Wu [26] introduced the concept of fuzzy Riemann integral and its numerical integration. Generally, fuzzy inventory models are developed considering some of the inventory parameters as fuzzy in nature [28,31,45,52,53]. To reduce the objective function, they defuzzified the fuzzy parameters to a crisp one by either defuzzification methods or following possibility/ necessity measure of fuzzy events. Finally they solve the reduced crisp model to derive the decisions for the DM. In the existing literature, little attention has been paid on fuzzy demand and fuzzy production rate. Recently, Sarkar and Chakraborti [53] developed an EPQ model, where demand is considered as time dependent fuzzy number and followed FDE approach to formulate the model. They found the α -cut of total variable cost and formulated the problem as multi-objective minimization problem by considering the two components of α -cut of total variable cost as two objectives.

Partial trade credit financing is one of central features in supply chain management. Since in most business transactions, as the one level trade credit financing is unrealistic, we want to investigate the situation in a supply chain in which the supplier is willing to provide the retailer a full trade credit period for payments and the retailer offers the partial trade to his/her customers. Some researcher's like as Mahata and Mahata [17], Soni and Joshi [18], Jaggi and Mona [11] Yan et al. [43] investigate the inventory model under partial trade credit financing.

The concept of the fuzzy derivative was first initiated by Chang and Zadeh [54]. The concept of differential equations in a fuzzy environment was first formulated by Kaleva [44]. In fuzzy differential equation, all derivatives are deliberated as either Hukuhara or generalized derivatives. The Hukuhara differentiability has a deficiency (cf. [7]). The solution turns imprecise as time goes by. Bede et al. [8] exhibited that a large class of Boundary Value Problems (BVPs) has no solution if the Hukuhara derivative is applied. To remove this difficulty, the concept of a generalized derivative was i.e. Generalized Hukuhara (gH) developed [7,9] and fuzzy differential equations were smeared using this concept. Recently, Bede and Stefanini [6] used the concept of generalization of Hukuhara difference for compact convex set [35]. They introduced generalized Hukuhara differentiability for fuzzy valued function and displayed that, this concept of differentiability have relationships with weakly generalized differentiability and strongly generalized differentiability.

There are many approaches for solving FDE. Some researchers transform the FDE into equivalent fuzzy integral equation and then solve this [33]. Another one is Zadeh extension principle method. In this method first solve the associated Ordinary Differential Equation (ODE) and lastly fuzzify the solution and check whether it is satisfied or not [63,64]. In the third approach, the fuzzy problem is converted to a crisp problem. Hüllermeier [15] uses the concept of differential inclusion. In this way, by taking a α -cut of the initial value and the solution, the given differential equation is converted to a differential inclusion and the solution is accepted as the α -cut of the fuzzy solution. Laplace transform method is used in linear FDE (cf. [10]). Recently, Mondal and Roy solved the first order Linear FDE by Lagrange multiplier method [55]. Using generalized Hukuhara differentiability concept, we transform the given FDE into two ODEs, these ODEs are also differential equations involving the parametric forms of a fuzzy number.

Recently Mondal et al. [40] considered a production-recycling model with variable demand, demand-dependent fuzzy return rate and solved the corresponding fuzzy differential equation by using α -cut methods. First α -cut of the differential equation is taken, then the equation is converted to two crisp differential equations. Guchhait et al. [49] considered a production inventory model with fuzzy production and demand which are fuzzy differential equations and solved by an interval compared genetic algorithm

approach. Different evolutionary methods have been used for different models [46,47], which can also be used for inventory control systems. A two storage production-repairing model with fuzzy defective rate and displayed inventory dependent demand has formulated on the concept of FDE and solved by Mondal et al. [42]. Inventory model of a deteriorating item with price and credit linked fuzzy demand also considered in the same concept and solved by Guchhait et al. [50]. Jana et al. [14] considered a two plants production-recycling-disposal inventory problem and solved the corresponding fuzzy differential equation via genetic algorithms. For multi objective optimization technique, weighted sum method [29] is one of the important technique for solving multi objective model.

1.1. Novelities

Although the contribution of above mentioned development to the production inventory model in fuzzy environment is satisfactory, but in our present study we point out some of the fundamental issue that probably need to be considered to make worldwide development of this inventory model.

- (i) Though there were some publications [c.f. 16, 40, 49, 50, etc.] of fuzzy inventory models which were solved by fuzzy differential equation approach, till now none has solved inventory problem with fuzzy differential equation approach by generalized Hukuhara derivative concepts. The difference between the above mentioned works and our work is that earlier workers solved the fuzzy differential equations without considering the gH derivative approach. Here in the above mentioned works, crisp solution is replaced by fuzzy solution which is not realistic. In this paper, we considered an EPQ model with partial trade credit financing in fuzzy environment, where demand is taken as a left right type of fuzzy number and gh derivative approach is used to solve the fuzzy differential equation.
- (ii) In real-life inventory system sometimes demands are taken as time dependent. Generally constant part of the demand is not crisp. The constant part varies due to stock, advertisement of the store, etc. So fuzzy demand is more realistic than the crisp demand. There are some research publications of various types of demand [c.f. 28,43,etc],but till now none has considered constant part of the demand of a item as Left-Right type fuzzy number and solved by gH derivative approach. This situation motivated us to formulate one model where constant part of the demand of an item is taken as Left-Right type fuzzy number and gH derivative approach is used to solve the fuzzy differential equation.
- (iii) In practice, partial trade credit financing to a retailer is more matched to real life supply chains. For example, in India, the TATA Company can delay the full amount of purchasing cost until the end of the delay period offered by his supplier. But the TATA Company only offers partial delay payment to his dealership on the permissible credit period and the rest of the total amount is payable at the time when the dealer places a replenishment order. There are some research publications about partial trade credit financing [c.f. 11,17,18,43 etc.].In the above mentioned works the concept of partial trade credit financing is also applied. But we claim that our work is more realistic than the earlier works. It is due to the fact that, along with the partial trade credit financing, we have considered the fuzzy EPQ model where demand is taken as left right type fuzzy number. Moreover fuzzy derivative approach like gH derivative approach is used in our work. For solving any fuzzy differential equation, fuzzy derivation is very much important. So we can claim that our

work is more novel than the others. Till now none has investigated the inventory level under partial trade credit financing in fuzzy environment using gH derivative approach for solving the fuzzy differential equation. This situation motivated us to formulate one model where partial trade credit financing is considered along with gH derivative approach. Four different cases are considered with respect to the gH-(i) differentiability and gH-(ii) differentiability.

In this paper, we formulated the inventory problem with first order fuzzy differential equation (FDE). We solve the inventory model using FDE concepts. Presence of fuzzy demand and quantity the model leads to a FDE. Now the question arise in the readers, mind that, “what is new approach for solving this fuzzy inventory model?”. In the last few decay’s many researchers considered the inventory models in fuzzy environment. They solved the inventory model with crisp number and then substitute the concerned parameter by a fuzzy number. Here the concept of fuzzy differential equation is missing. Moreover the application of fuzzy concept is violated. We consider four cases where the inventory model is considered with fuzzy differential equation. The solution procedure is used here namely generalized Hukuhara derivative approach, which is more recent general concept for solving fuzzy differential equation. Two objectives are converted to a single objective using weighted sum method and then the single objective is solved by the generalized reduced gradient method using the LINGO 13.0 software.

The paper is organized as follows: Section 2 describes about the multi-objective optimization technique followed by Weighted Sum Method. In Section 3 assumptions and notations of the proposed inventory model are listed. Section 4 contains mathematical formulation of the model. Numerical examples are taken in Section 5 and corresponding results are expressed in both tabular and graphical forms. Results and some sensitivity analyses are discussed in Section 6. Managerial insights of the current model is discussed in Section 7. Brief conclusions and future research works are drawn in Section 8. Lastly, preliminaries and basic concepts on fuzzy number, left right type fuzzy number, fuzzy derivative are given in last section named as appendix.

2. Multi-Objective Programming (MOP) problem in crisp environment

The world has become more complex and almost every important real-world problem involves more than one objective which may be linear or non-linear in nature. In such cases, decision makers find imperative to evaluate best possible approximate solution alternatives according to multiple criteria.

A general multi-objective non-linear programming problem is of the following form [29]:

$$\left. \begin{aligned} &\text{Find } x = (x_1, x_2, x_3, \dots, x_n)^T \\ &\text{which minimizes } F(x) = (f_1(x), f_2(x), f_3(x), f_4(x), \dots, f_k(x))^T \\ &\quad \text{subject to } x \in X \\ &\left\{ \begin{array}{l} x: \quad g_j(x) \leq 0, \quad j = 1, 2, 3, \dots, m \\ \quad \quad x_i \geq 0, \quad i = 1, 2, 3, \dots, n \end{array} \right. \end{aligned} \right\} \quad (1)$$

where $f_1(x), f_2(x), f_3(x), f_4(x), \dots, f_k(x)$ are $k(\geq 2)$ objectives.

2.1. Weighted sum method

The weighted sum method scalarizes a set of objectives into a single objective by multiplying each objective with user’s supplied weights. The weights of an objective are usually chosen in proportion to the objective’s relative importance in the problem. However

setting up an appropriate weight vector depends on the scaling of each objective function. It is likely that different objectives take different orders of magnitude. When such objectives are weighted to form a composite objective function, it would be better to scale them appropriately so that each objective possesses more or less the same order of magnitude. This process is called normalization of objectives. After the objectives are normalized, a composite objective function $F(x)$ can be formed by summing the weighted normalized objectives and the MOP given in Eq. (1) is then converted to a single-objective optimization problem as follows [29]:

$$\text{Minimize } F(x) = \sum_{i=1}^k \lambda_i f_i(x), \quad \lambda_i \in [0, 1], \quad x \in X \quad (2)$$

Here, λ_i is the weight of the i -th objective function. Since the minimum of the above problem does not change if all the weights are multiplied by a constant, it is the usual practice to choose weights such that their sum is one, i.e. $\sum_{i=1}^k \lambda_i = 1$.

3. Notations and assumptions

The following parameters are taken as notations

$\tilde{D}(t)$: $\tilde{a} - bt$; the annual demand as a decreasing function of time where $\tilde{a} = (a_1, a_2, a_3)$ be a triangular fuzzy number, is fixed demand and b ($0 < b < 1$) denotes the rate of change of demand(variable).

- C_p : Unit purchase cost.
- C_s : Unit selling cost.
- h : Inventory holding cost per year excluding interest charges.
- A : Ordering cost per order.
- θ : Constant deterioration rate where $0 < \theta < 1$.
- Q : Ordered quantity (variable).
- α_1 : Customer’s fraction of the total amount owed payable at the time of placing an order offered by the retailer $0 < \alpha_1 < 1$.
- M : Retailer’s trade credit period offered by the supplier in years(decision variable).
- N : Customer’s trade period offered by the retailer in years (decision variable).
- I_c : Interest charges payable per \$ per year to the supplier.
- I_e : Interest earned per \$ per year.
- TVC : Total inventory cost per time period for fuzzy inventory model (decision variable).
- $TVC_R(\alpha)$: Upper α – cut of total inventory cost TVC . Let $Z_1 = TVC_R(\alpha)$ (decision variable).
- $TVC_L(\alpha)$: Lower α – cut of total inventory cost TVC . Let $Z_2 = TVC_L(\alpha)$ (decision variable).
- K : Production rate per year.
- t : variable time.
- t_1 : Production time in years (decision variable).
- T : Cycle time in years (decision variable).
- T^* : Optimal cycle time (decision variable).
- α : A real number which lies between 0 & 1.

The following assumptions are taken to formulate the present inventory model

Rate of replenishment is finite. Annual demand is a decreasing function of time. $q_1(t)$ is the inventory level that changes with time t during production period and $q_2(t)$ is the inventory level that changes with time during nonproduction period. $I_c \geq I_e$. Time horizon is infinite. Shortages are not allowed.

4. Mathematical formulation of the model of deteriorating items

A constant production starts at $t = 0$ and continues up to $t = t_1$ when the inventory level reaches maximum. Production then stops at $t = t_1$ and the inventory gradually is depleted to zero at the end of the production cycle $t = T$ due to deterioration and consumption. Therefore, during the time interval $(0, t_1)$, the system is subjected to the effect of production, demand and deterioration.

Then the change in the inventory level can be described by the following differential equation:

$$\frac{d\tilde{q}_1(t)}{dt} + \theta\tilde{q}_1(t) = k - (\tilde{a} - bt), \quad 0 \leq t \leq t_1 \tag{3}$$

with the initial condition

$$\tilde{q}_1(0) = 0 \tag{4}$$

On the other hand, in the interval (t_1, T) , the system is effected by the combined effect of demand and deterioration.

Hence, the change in the inventory level is governed by the following differential equation:

$$\frac{d\tilde{q}_2(t)}{dt} + \theta\tilde{q}_2(t) = -(\tilde{a} - bt), \quad t_1 \leq t \leq T \tag{5}$$

with the ending condition

$$\tilde{q}_2(T) = 0 \tag{6}$$

The Generalized Hukuhara derivative [65] of a fuzzy valued function $f : (a, b) \rightarrow \mathfrak{R}_F$ at t_0 is defined as

$$f'(t_0) = \lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus_g f(t_0)}{h}$$

In parametric form we say that

- $f(t)$ is gH-(i) differentiable at t_0 if $[f'(t_0)]_\alpha = [f'_L(t_0, \alpha), f'_R(t_0, \alpha)]$ and
- $f(t)$ is gH-(ii) differentiable at t_0 if $[f'(t_0)]_\alpha = [f'_R(t_0, \alpha), f'_L(t_0, \alpha)]$.

Four different cases arise

Case 1: $\tilde{q}_1(t)$ and $\tilde{q}_2(t)$ is gH-(i) differentiable

Then differential Eq. (3) is converted to the following differential equations

$$\frac{dq_{1L}(t, \alpha)}{dt} = -\theta q_{1R}(t, \alpha) - k - a_R(\alpha) + bt, \quad 0 \leq t \leq t_1 \tag{7}$$

$$\frac{dq_{1R}(t, \alpha)}{dt} = -\theta q_{1L}(t, \alpha) + k - a_L(\alpha) + bt, \quad 0 \leq t \leq t_1 \tag{8}$$

with initial condition

$$q_{1L}(0, \alpha) = 0 \text{ and } q_{1R}(0, \alpha) = 0 \tag{9}$$

The differential Eq. (5) is converted to the following differential equations

$$\frac{dq_{2L}(t, \alpha)}{dt} = -\theta q_{2R}(t, \alpha) - a_R(\alpha) + bt, \quad t_1 \leq t \leq T \tag{10}$$

$$\frac{dq_{2R}(t, \alpha)}{dt} = -\theta q_{2L}(t, \alpha) - a_L(\alpha) + bt, \quad t_1 \leq t \leq T \tag{11}$$

with initial condition

$$q_{2L}(T, \alpha) = 0 \text{ and } q_{2R}(T, \alpha) = 0 \tag{12}$$

and the α -cut of $\tilde{q}_1(t)$ is $[q_{1L}(t, \alpha), q_{1R}(t, \alpha)]$ and \tilde{a} is $[a_L(\alpha), a_R(\alpha)]$

The solutions of above differential equations are given by

$$q_{1L}(t, \alpha) = \frac{1}{2\theta} [L_1 e^{\theta t} + L_2 e^{-\theta t}] - \frac{a_L(\alpha)}{\theta} - \frac{k}{\theta} (e^{-\theta t} - 1) + \frac{b}{\theta^2} + \frac{bt}{\theta} \tag{13}$$

$$q_{1R}(t, \alpha) = \frac{1}{2\theta} [-L_1 e^{\theta t} + L_2 e^{-\theta t}] - \frac{a_R(\alpha)}{\theta} - \frac{k}{\theta} (e^{-\theta t} - 1) + \frac{b}{\theta^2} + \frac{bt}{\theta} \tag{14}$$

$$q_{2L}(t, \alpha) = \frac{1}{2\theta} [L_1 e^{(t-T)\theta} + L_2 e^{(T-t)\theta}] - \frac{a_L(\alpha)}{\theta} + \frac{b}{\theta} (t - T e^{(T-t)\theta}) + \frac{b}{\theta^2} (e^{(T-t)\theta} - 1) \tag{15}$$

$$q_{2R}(t, \alpha) = \frac{1}{2\theta} [-L_1 e^{(t-T)\theta} + L_2 e^{(T-t)\theta}] - \frac{a_R(\alpha)}{\theta} + \frac{b}{\theta} (t - T e^{(T-t)\theta}) + \frac{b}{\theta^2} (e^{(T-t)\theta} - 1) \tag{16}$$

where $L_1 = a_R(\alpha) - a_L(\alpha)$, $L_2 = a_R(\alpha) + a_L(\alpha)$.

Total relevant costs are given by

- (i) Lower α -cut and upper α -cut of ordering cost (OC_L, OC_R) are given by

$$OC_L = OC_R = c_2 \tag{17}$$

- (ii) Lower α -cut and upper α -cut of stock holding cost ($THC_L(\alpha), THC_R(\alpha)$) are given by

$$THC_L(\alpha) = h \left[\int_0^{t_1} q_{1L}(t, \alpha) dt + \int_{t_1}^T q_{2L}(t, \alpha) dt \right] = \frac{h}{\theta^2} \left[\frac{1}{2} \{ L_1 (1 - e^{-\theta T}) e^{\theta t_1} - L_2 (1 - e^{\theta T}) e^{-\theta t_1} \} - \theta T a_L(\alpha) + k(e^{-\theta t_1} + \theta t_1 - 1) + bT(1 - e^{(T-t_1)\theta}) + \frac{b}{2} \theta T^2 - \frac{b}{\theta} (1 - e^{(T-t_1)\theta} + \theta T) \right] \tag{18}$$

$$THC_R(\alpha) = h \left[\int_0^{t_1} q_{1R}(t, \alpha) dt + \int_{t_1}^T q_{2R}(t, \alpha) dt \right] = \frac{h}{\theta^2} \left[\frac{1}{2} \{ -L_1 (1 - e^{-\theta T}) e^{\theta t_1} - L_2 (1 - e^{\theta T}) e^{-\theta t_1} \} - \theta T a_R(\alpha) + k(e^{-\theta t_1} + \theta t_1 - 1) + bT(1 - e^{(T-t_1)\theta}) + \frac{b}{2} \theta T^2 - \frac{b}{\theta} (1 - e^{(T-t_1)\theta} + \theta T) \right] \tag{19}$$

- (iii) Lower α -cut and upper α -cut of annual cost due to deteriorated units ($DC_L(\alpha), DC_R(\alpha)$) are given by

$$DC_L(\alpha) = c_p \theta \left[\int_0^{t_1} q_{1L}(t, \alpha) dt + \int_{t_1}^T q_{2L}(t, \alpha) dt \right] = \frac{c_p}{\theta} \left[\frac{1}{2} \{ L_1 (1 - e^{-\theta T}) e^{\theta t_1} - L_2 (1 - e^{\theta T}) e^{-\theta t_1} \} - \theta T a_L(\alpha) + k(e^{-\theta t_1} + \theta t_1 - 1) + bT(1 - e^{(T-t_1)\theta}) + \frac{b}{2} \theta T^2 - \frac{b}{\theta} (1 - e^{(T-t_1)\theta} + \theta T) \right] \tag{20}$$

$$DC_R(\alpha) = c_p \theta \left[\int_0^{t_1} q_{1R}(t, \alpha) dt + \int_{t_1}^T q_{2R}(t, \alpha) dt \right] = \frac{c_p}{\theta} \left[\frac{1}{2} \{ -L_1 (1 - e^{-\theta T}) e^{\theta t_1} - L_2 (1 - e^{\theta T}) e^{-\theta t_1} \} \right]$$

$$\begin{aligned}
 & -\theta T a_R(\alpha) + k(e^{-\theta t_1} + \theta t_1 - 1) + bT(1 - e^{(T-t_1)\theta}) \\
 & + \left[\frac{b}{2}\theta T^2 - \frac{b}{\theta}(1 - e^{(T-t_1)\theta} + \theta T) \right] \quad (21)
 \end{aligned}$$

(iv) According to given assumptions, there are three sub cases to occur in the interest charged for the items kept in stock per year.

Sub case 1.1: $N \leq M \leq t_1 \leq T$

Lower α -cut and upper α -cut of interest payable ($TIP_L(\alpha), TIP_R(\alpha)$) are given by

$$TIP_R(\alpha) = \left[c_p I_c \int_M^T q_{2R}(t, \alpha) dt \right] = c_p I_c A_1$$

where,

$$\begin{aligned}
 A_1 = & \frac{1}{\theta^2} \left[\frac{1}{2} \{ -L_1(1 - e^{(M-T)\theta}) - L_2(1 - e^{(T-M)\theta}) \} \right. \\
 & - \theta(T-M)a_R(\alpha) + bT(1 - e^{(T-M)\theta}) + \frac{b}{2}\theta(T^2 - M^2) \\
 & \left. - \frac{b}{\theta}(1 - e^{(T-M)\theta} + \theta(T-M)) \right] \quad (22)
 \end{aligned}$$

$$TIP_L(\alpha) = \left[c_p I_c \int_M^T q_{2L}(t, \alpha) dt \right] = c_p I_c A_2$$

where,

$$\begin{aligned}
 A_2 = & \frac{1}{\theta^2} \left[\frac{1}{2} \{ L_1(1 - e^{(M-T)\theta}) - L_2(1 - e^{(T-M)\theta}) \} \right. \\
 & - \theta(T-M)a_L(\alpha) + bT(1 - e^{(T-M)\theta}) + \frac{b}{2}\theta(T^2 - M^2) \\
 & \left. - \frac{b}{\theta}(1 - e^{(T-M)\theta} + \theta(T-M)) \right] \quad (23)
 \end{aligned}$$

Sub case 1.2: $N \leq t_1 \leq T \leq M$

In this case Lower α -cut and upper α -cut of interest payable ($TIP_L(\alpha), TIP_R(\alpha)$) are zero.

Sub case 1.3: $t_1 \leq T \leq N \leq M$

In this case Lower α -cut and upper α -cut of interest payable ($TIP_L(\alpha), TIP_R(\alpha)$) are zero.

(v) According to given assumptions, three cases will occur in interest earned per year.

Sub case 1.1: $N \leq M \leq t_1 \leq T$

Lower α -cut and upper α -cut of annual interest earned by the retailer ($TIE_L(\alpha), TIE_R(\alpha)$) are given by

$$\begin{aligned}
 TIE_R(\alpha) = & c_s I_e \left[\alpha_1(T - t_1) \int_0^{t_1} (a_R(\alpha) - bt) dt + \alpha_1(T - N) \right. \\
 & \times \int_{t_1}^N (a_R(\alpha) - bt) dt + (T - M) \int_N^M (a_R(\alpha) - bt)(M - t) dt \\
 & \left. + \int_M^T (a_R(\alpha) - bt)(T - t) dt \right] \\
 = & c_s I_e [\alpha_1(T - t_1)B_1 + \alpha_1(T - N)B_2 + (T - M)B_3 + B_4] \quad (24)
 \end{aligned}$$

where,

$$\begin{aligned}
 B_1 = & \int_0^{t_1} (a_R(\alpha) - bt) dt = \alpha_1(T - t_1) \left(a_R(\alpha)t_1 - b\frac{t_1^2}{2} \right) \\
 B_2 = & \int_{t_1}^N (a_R(\alpha) - bt) dt = \left(a_R(\alpha)(N - t_1) - b\left(\frac{N^2}{2} - \frac{t_1^2}{2}\right) \right) \\
 B_3 = & \int_N^M (a_R(\alpha) - bt)(M - t) dt = (a_R(\alpha)(M - N)^2
 \end{aligned}$$

$$- b\left(\frac{M}{2}(M^2 - N^2) - \frac{1}{3}(M^3 - N^3)\right)$$

$$\begin{aligned}
 B_4 = & \int_M^T (a_R(\alpha) - bt)(T - t) dt = (a_R(\alpha)(T - M)^2 \\
 & - b\left(\frac{T}{2}(T^2 - M^2) - \frac{1}{3}(T^3 - M^3)\right))
 \end{aligned}$$

$$\begin{aligned}
 TIE_L(\alpha) = & c_s I_e \left[\alpha_1(T - t_1) \int_0^{t_1} (a_L(\alpha) - bt) dt + \alpha_1(T - N) \right. \\
 & \int_{t_1}^N (a_L(\alpha) - bt) dt + (T - M) \int_N^M (a_L(\alpha) - bt)(M - t) dt \\
 & \left. + \int_M^T (a_L(\alpha) - bt)(T - t) dt \right] \\
 = & c_s I_e [\alpha_1(T - t_1)B_1 + \alpha_1(T - N)B_2 + (T - M)B_3 + B_4] \quad (25)
 \end{aligned}$$

where,

$$\begin{aligned}
 B_5 = & \int_0^{t_1} (a_L(\alpha) - bt) dt = \alpha_1(T - t_1) \left(a_L(\alpha)t_1 - b\frac{t_1^2}{2} \right) \\
 B_6 = & \int_{t_1}^N (a_L(\alpha) - bt) dt = \left(a_L(\alpha)(N - t_1) - b\left(\frac{N^2}{2} - \frac{t_1^2}{2}\right) \right) \\
 B_7 = & \int_N^M (a_L(\alpha) - bt)(M - t) dt = (a_L(\alpha)(M - N)^2 \\
 & - b\left(\frac{M}{2}(M^2 - N^2) - \frac{1}{3}(M^3 - N^3)\right)) \\
 B_8 = & \int_M^T (a_L(\alpha) - bt)(T - t) dt = (a_L(\alpha)(T - M)^2 \\
 & - b\left(\frac{T}{2}(T^2 - M^2) - \frac{1}{3}(T^3 - M^3)\right))
 \end{aligned}$$

Sub Case 1.2: $N \leq t_1 \leq T \leq M$

Lower α -cut and upper α -cut of annual interest earned by the retailer ($TIE_L(\alpha), TIE_R(\alpha)$) are given by

$$\begin{aligned}
 TIE_R(\alpha) = & c_s I_e \left[\alpha_1(M - t_1) \int_0^{t_1} (a_R(\alpha) - bt) dt + \alpha_1(M - N) \right. \\
 & \int_{t_1}^N (a_R(\alpha) - bt) dt + (M - T) \int_N^T (a_R(\alpha) - bt)(T - t) dt \left. \right] \\
 = & c_s I_e [\alpha_1(M - t_1)B_1 + \alpha_1(M - N)B_2 + (M - T)B_9] \quad (26)
 \end{aligned}$$

Where,

$$\begin{aligned}
 B_9 = & (a_R(\alpha)(T - N)^2 - b\left(\frac{T}{2}(T^2 - N^2) - \frac{1}{3}(T^3 - N^3)\right)) \\
 TIE_L(\alpha) = & c_s I_e \left[\alpha_1(M - t_1) \int_0^{t_1} (a_L(\alpha) - bt) dt + \alpha_1(M - N) \right. \\
 & \times \int_{t_1}^N (a_L(\alpha) - bt) dt + (M - T) \int_N^T (a_L(\alpha) - bt)(T - t) dt \left. \right] \\
 = & c_s I_e [\alpha_1(M - t_1)B_5 + \alpha_1(M - N)B_6 + (M - T)B_{10}] \quad (27)
 \end{aligned}$$

where,

$$B_{10} = (a_L(\alpha)(T - N)^2 - b\left(\frac{T}{2}(T^2 - N^2) - \frac{1}{3}(T^3 - N^3)\right))$$

Sub Case 1.3: $t_1 \leq T \leq N \leq M$

Lower α -cut and upper α -cut of annual interest earned by the retailer ($TIE_L(\alpha), TIE_R(\alpha)$) are given by

$$\begin{aligned}
 TIE_R(\alpha) = & c_s I_e \left[\alpha_1(M - t_1) \int_0^{t_1} (a_R(\alpha) - bt) dt + \alpha_1(M - T) \right. \\
 & \left. \int_{t_1}^T (a_R(\alpha) - bt) dt \right] \\
 = & c_s I_e [\alpha_1(M - t_1)B_1 + \alpha_1(M - T)B_{11}] \quad (28)
 \end{aligned}$$

where,

$$B_{11} = \left(a_R(\alpha)(T - t_1) - b \left(\frac{T^2}{2} - \frac{t_1^2}{2} \right) \right)$$

$$TIE_L(\alpha) = c_s I_e \left[\alpha_1(M - t_1) \int_0^{t_1} (a_L(\alpha) - bt) dt + \alpha_1(M - T) \int_{t_1}^T (a_L(\alpha) - bt) dt \right]$$

$$= c_s I_e [\alpha_1(M - t_1)B_5 + \alpha_1(M - T)B_{12}] \tag{29}$$

where,

$$B_{12} = \left(a_L(\alpha)(T - t_1) - b \left(\frac{T^2}{2} - \frac{t_1^2}{2} \right) \right)$$

The annual total cost incurred by the retailer for all sub cases of all cases is

$$TVC = \frac{1}{T} (THC + DC + OC + TIP - TIE) \tag{30}$$

Therefore total variable cost per unit time is a fuzzy quantity and is defined by

$$TVC = \begin{cases} TVC_R(\alpha) \\ TVC_L(\alpha) \end{cases} \tag{31}$$

where

$$TVC_R(\alpha) = \sup\{x \in R : \mu_{TVC}(x) \geq \alpha\} \quad \& \quad TVC_L(\alpha) = \inf\{x \in R : \mu_{TVC}(x) \geq \alpha\}$$

Here

$$TVC_R(\alpha) = \frac{1}{T} (THC_R(\alpha) + DC_R(\alpha) + OC_R(\alpha) + TIP_R(\alpha) - TIE_R(\alpha))$$

and

$$TVC_L(\alpha) = \frac{1}{T} (THC_L(\alpha) + DC_L(\alpha) + OC_L(\alpha) + TIP_L(\alpha) - TIE_L(\alpha))$$

Therefore this model mathematically can be written as

$$\text{Minimize } \{TVC_R(\alpha), \quad TVC_L(\alpha)\} \tag{32}$$

subject to $0 \leq \alpha_1 \leq 1$

Therefore the problem is a multi-objective optimization problem. **Weighted sum method** is used to solve this multi-objective optimization problem.

By using **Weighted sum method**, the above problem reduces to

$$\text{Minimize } Z = \lambda_1 Z_1 + \lambda_2 Z_2 \tag{33}$$

subject to $0 \leq \alpha \leq 1$ where

$$Z_1 = TVC_L(\alpha), \quad Z_2 = TVC_R(\alpha), \quad (\lambda_1 + \lambda_2) = 1$$

Case 2: $\tilde{q}_1(t)$ is gH-(i) differentiable and $\tilde{q}_2(t)$ is gH-(ii) differentiable

Then differential Eq. (3) is converted to the following differential equation

$$\frac{dq_{1L}(t, \alpha)}{dt} = -\theta q_{1R}(t, \alpha) - k - a_R(\alpha) + bt, \quad 0 \leq t \leq t_1 \tag{34}$$

$$\frac{dq_{1R}(t, \alpha)}{dt} = -\theta q_{1L}(t, \alpha) - k - a_L(\alpha) + bt, \quad 0 \leq t \leq t_1 \tag{35}$$

with initial condition

$$q_{1L}(0, \alpha) = 0 \text{ and } q_{1R}(0, \alpha) = 0 \tag{36}$$

Then differential Eq. (5) is converted to the following differential equation

$$\frac{dq_{2L}(t, \alpha)}{dt} = -\theta q_{2L}(t, \alpha) - a_L(\alpha) + bt, \quad t_1 \leq t \leq T \tag{37}$$

$$\frac{dq_{2R}(t, \alpha)}{dt} = -\theta q_{2R}(t, \alpha) - a_R(\alpha) + bt, \quad t_1 \leq t \leq T \tag{38}$$

with initial condition

$$q_{2L}(T, \alpha) = 0 \text{ and } q_{2R}(T, \alpha) = 0 \tag{39}$$

and the α -cut of $\tilde{q}_1(t)$ is $[q_{1L}(t, \alpha), q_{1R}(t, \alpha)]$ and \tilde{a} is $[a_L(\alpha), a_R(\alpha)]$

The solutions of above differential equations are given by

$$q_{1L}(t, \alpha) = \frac{1}{2\theta} [(a_R(\alpha) - a_L(\alpha))e^{\theta t} + (a_R(\alpha) + a_L(\alpha))e^{-\theta t}] - \frac{a_L(\alpha)}{\theta} - \frac{k}{\theta} (e^{-\theta t} - 1) + \frac{b}{\theta^2} + \frac{bt}{\theta} \tag{40}$$

$$q_{1R}(t, \alpha) = \frac{1}{2\theta} [-(a_R(\alpha) - a_L(\alpha))e^{\theta t} + (a_R(\alpha) + a_L(\alpha))e^{-\theta t}] - \frac{a_R(\alpha)}{\theta} - \frac{k}{\theta} (e^{-\theta t} - 1) + \frac{b}{\theta^2} + \frac{bt}{\theta} \tag{41}$$

$$q_{2L}(t, \alpha) = \left[\frac{a_L(\alpha)}{\theta} + \frac{b}{\theta^2} \right] (e^{(T-t)\theta} - 1) - \frac{bT}{\theta} e^{(T-t)\theta} + \frac{bt}{\theta} \tag{42}$$

$$q_{2R}(t, \alpha) = \left[\frac{a_R(\alpha)}{\theta} + \frac{b}{\theta^2} \right] (e^{(T-t)\theta} - 1) - \frac{bT}{\theta} e^{(T-t)\theta} + \frac{bT}{\theta} \tag{43}$$

Total relevant costs are given by

(i) Lower α -cut and upper α -cut of ordering cost (OC_L, OC_R) are given by

$$OC_L = OC_R = c_2 \tag{44}$$

(ii) Lower α -cut and upper α -cut of stock holding cost ($THC_L(\alpha), THC_R(\alpha)$) are given by

$$THC_R(\alpha) = h \left[\frac{1}{\theta^2} \left(\frac{1}{2} \{ -L_1(e^{\theta t_1} - 1) - L_2(e^{-\theta t_1} - 1) \} - \theta t_1 a_R(\alpha) + k(e^{-\theta t_1} + \theta t_1 - 1) + bt_1 \right) - \frac{1}{\theta^3} (\theta a_R(\alpha) + b) (1 - e^{(T-t_1)\theta} + \theta(T - t_1)) + \frac{bT}{\theta^2} (1 - e^{(T-t_1)\theta}) + \frac{b}{2\theta} T^2 \right] \tag{45}$$

$$THC_L(\alpha) = h \left[\frac{1}{\theta^2} \left(\frac{1}{2} \{ -L_1(e^{\theta t_1} - 1) - L_2(e^{-\theta t_1} - 1) \} - \theta t_1 a_L(\alpha) + k(e^{-\theta t_1} + \theta t_1 - 1) + bt_1 \right) - \frac{1}{\theta^3} (\theta a_L(\alpha) + b) (1 - e^{(T-t_1)\theta} + \theta(T - t_1)) + \frac{bT}{\theta^2} (1 - e^{(T-t_1)\theta}) + \frac{b}{2\theta} T^2 \right] \tag{46}$$

(iii) Lower α -cut and upper α -cut of annual cost due to deteriorated units ($DC_L(\alpha), DC_R(\alpha)$) are given by

$$DC_R(\alpha) = c_p \theta \left[\frac{1}{\theta^2} \left(\frac{1}{2} \{ -L_1(e^{\theta t_1} - 1) - L_2(e^{-\theta t_1} - 1) \} - \theta t_1 a_R(\alpha) + k(e^{-\theta t_1} + \theta t_1 - 1) + bt_1 \right) - \frac{1}{\theta^3} (\theta a_R(\alpha) + b) (1 - e^{(T-t_1)\theta} + \theta(T - t_1)) + \frac{bT}{\theta^2} (1 - e^{(T-t_1)\theta}) + \frac{b}{2\theta} T^2 \right] \tag{47}$$

$$DC_L(\alpha) = c_p \theta \left[\frac{1}{\theta^2} \left(\frac{1}{2} \{ -L_1(e^{\theta t_1} - 1) - L_2(e^{-\theta t_1} - 1) \} - \theta t_1 a_L(\alpha) + k(e^{-\theta t_1} + \theta t_1 - 1) + bt_1 \right) \right]$$

$$-\frac{1}{\theta^3}(\theta a_L(\alpha) + b) \left(1 - e^{(T-t_1)\theta} + \theta(T - t_1)\right) + \frac{bT}{\theta^2}(1 - e^{(T-t_1)\theta}) + \frac{b}{2\theta}T^2 \quad (48)$$

(iv) According to given assumptions, there are three sub cases to occur in interest charged for the items kept in stock per year.

Sub case 2.1: $N \leq M \leq t_1 \leq T$

Lower α -cut and upper α -cut of interest payable ($TIP_L(\alpha)$, $TIP_R(\alpha)$) are given by

$$TIP_R(\alpha) = \left[c_p I_c \int_M^T q_{2R}(t, \alpha) dt \right] = [c_p I_c A_3] \quad (49)$$

where,

$$A_3 = \left[-\frac{1}{\theta^3}(\theta a_R(\alpha) + b)(1 - e^{(T-M)\theta} + \theta(T - M)) + \frac{bT}{\theta^2}(1 - e^{(T-M)\theta}) + \frac{b}{2\theta}(T^2 - M^2) \right]$$

$$TIP_L(\alpha) = \left[c_p I_c \int_M^T q_{2L}(t, \alpha) dt \right] = [c_p I_c A_4] \quad (50)$$

where,

$$A_4 = \left[-\frac{1}{\theta^3}(\theta a_L(\alpha) + b)(1 - e^{(T-M)\theta} + \theta(T - M)) + \frac{bT}{\theta^2}(1 - e^{(T-M)\theta}) + \frac{b}{2\theta}(T^2 - M^2) \right]$$

Sub case 2.2: $N \leq t_1 \leq T \leq M$

In this case lower α -cut and upper α -cut of interest payable ($TIP_L(\alpha)$, $TIP_R(\alpha)$) are zero.

Sub case 2.3: $t_1 \leq T \leq N \leq M$

In this case lower α -cut and upper α -cut of interest payable ($TIP_L(\alpha)$, $TIP_R(\alpha)$) are zero.

(v) According to given assumptions, three cases will occur in interest earned per year.

In this case, upper and lower α -cuts of interest earned in all sub cases are same as the all sub cases of case-1.

Case 3: $\tilde{q}_1(t)$ is gH-(ii) differentiable and $\tilde{q}_2(t)$ is gH-(i) differentiable

If $\tilde{q}_1(t)$ is gH-(ii) differentiable and $\tilde{q}_2(t)$ is gH-(i) differentiable, then proceeding as in the previous cases, we get the corresponding solutions as

$$q_{1L}(t, \alpha) = \left[\frac{k - a_L(\alpha)}{\theta} - \frac{b}{\theta^2} \right] (1 - e^{-\theta t}) + \frac{bt}{\theta} \quad (51)$$

$$q_{1R}(t, \alpha) = \left[\frac{k - a_R(\alpha)}{\theta} - \frac{b}{\theta^2} \right] (1 - e^{-\theta t}) + \frac{bt}{\theta} \quad (52)$$

$$q_{2L}(t, \alpha) = \left[\frac{1}{2\theta}(a_R(\alpha) - a_L(\alpha))e^{(t-T)\theta} + (a_R(\alpha) + a_L(\alpha))e^{(T-t)\theta} - \frac{a_L(\alpha)}{\theta} + \frac{b}{\theta}(t - Te^{(T-t)\theta}) + \frac{b}{\theta^2}(e^{(T-t)\theta} - 1) \right] \quad (53)$$

$$q_{2R}(t, \alpha) = \frac{1}{2\theta} \left[-(a_R(\alpha) - a_L(\alpha))e^{(t-T)\theta} + (a_R(\alpha) + a_L(\alpha))e^{(T-t)\theta} - \frac{a_R(\alpha)}{\theta} + \frac{b}{\theta}(t - Te^{(T-t)\theta}) + \frac{b}{\theta^2}(e^{(T-t)\theta} - 1) \right] \quad (54)$$

Total relevant costs are given by

(i) Lower α -cut and upper α -cut of ordering cost (OC_L , OC_R) are given by

$$OC_L = OC_R = c_2 \quad (55)$$

(ii) Lower α -cut and upper α -cut of stock holding cost ($THC_R(\alpha)$, $THC_L(\alpha)$) are given by

$$THC_R(\alpha) = \left[\left(\frac{k - a_R(\alpha)}{\theta} - \frac{b}{\theta^2} \right) \left(t_1 + \frac{1}{\theta}(e^{-\theta t_1} - 1) \right) + \frac{bt_1^2}{\theta} + \frac{1}{\theta^2} \left\{ \frac{1}{2} \{-L_1(1 - e^{(T-t_1)\theta}) - L_2(1 - e^{(T-t_1)\theta})\} - \theta(T - t_1)a_R(\alpha) + \frac{b}{2}(\theta(T^2 - t_1^2)) + 2T(1 - e^{(T-t_1)\theta}) - \frac{b}{\theta}((1 - e^{(T-t_1)\theta})) + \theta(T - t_1) \right\} \right] \quad (56)$$

$$THC_L(\alpha) = h \left[\left(\frac{k - a_L(\alpha)}{\theta} - \frac{b}{\theta^2} \right) \left(t_1 + \frac{1}{\theta}(e^{-\theta t_1} - 1) \right) + \frac{bt_1^2}{\theta} + \frac{1}{\theta^2} \left\{ \frac{1}{2} \{-L_1(1 - e^{(t_1-T)\theta}) - L_2(1 - e^{(T-t_1)\theta})\} - \theta(T - t_1)a_L(\alpha) + \frac{b}{2}(\theta(T^2 - t_1^2)) + 2T(1 - e^{(T-t_1)\theta}) - \frac{b}{\theta}((1 - e^{(T-t_1)\theta})) + \theta(T - t_1) \right\} \right] \quad (57)$$

(iii) Lower α -cut and upper α -cut of annual cost due to deteriorated units ($DC_L(\alpha)$, $DC_R(\alpha)$) are given by

$$DC_R(\alpha) = c_p \theta \left[\left(\frac{k - a_R(\alpha)}{\theta} - \frac{b}{\theta^2} \right) \left(t_1 + \frac{1}{\theta}(e^{-\theta t_1} - 1) \right) + \frac{bt_1^2}{\theta} + \frac{1}{\theta^2} \left\{ \frac{1}{2} \{-L_1(1 - e^{(t_1-T)\theta}) - L_2(1 - e^{(T-t_1)\theta})\} - \theta(T - t_1)a_R(\alpha) + \frac{b}{2}(\theta(T^2 - t_1^2)) + 2T(1 - e^{(T-t_1)\theta}) - \frac{b}{\theta}((1 - e^{(T-t_1)\theta})) + \theta(T - t_1) \right\} \right] \quad (58)$$

$$DC_L(\alpha) = c_p \theta \left[\left(\frac{k - a_L(\alpha)}{\theta} - \frac{b}{\theta^2} \right) \left(t_1 + \frac{1}{\theta}(e^{-\theta t_1} - 1) \right) + \frac{bt_1^2}{\theta} + \frac{1}{\theta^2} \left\{ \frac{1}{2} \{-L_1(1 - e^{(t_1-T)\theta}) - L_2(1 - e^{(T-t_1)\theta})\} - \theta(T - t_1)a_L(\alpha) + \frac{b}{2}(\theta(T^2 - t_1^2)) + 2T(1 - e^{(T-t_1)\theta}) - \frac{b}{\theta}((1 - e^{(T-t_1)\theta})) + \theta(T - t_1) \right\} \right] \quad (59)$$

(iv) According to given assumptions, there are three sub cases to occur in interest charged for the items kept in stock per year.

Sub case 3.1: $N \leq M \leq t_1 \leq T$

Lower α -cut and upper α -cut of interest payable ($TIP_L(\alpha)$, $TIP_R(\alpha)$) are given by

$$\begin{aligned} \text{TIP}_R(\alpha) &= \left[c_p I_c \int_M^T q_{2R}(t, \alpha) dt \right] \\ &= [c_p I_c A_1] \end{aligned} \tag{60}$$

$$\begin{aligned} \text{TIP}_L(\alpha) &= \left[c_p I_c \int_M^T q_{2L}(t, \alpha) dt \right] \\ &= [c_p I_c A_2] \end{aligned} \tag{61}$$

Sub case 3.2: $N \leq t_1 \leq T \leq M$

In this case lower α -cut and upper α -cut of interest payable ($\text{TIP}_L(\alpha)$, $\text{TIP}_R(\alpha)$) are zero.

Sub case 3.3: $t_1 \leq T \leq N \leq M$

In this case lower α -cut and upper α -cut of interest payable ($\text{TIP}_L(\alpha)$, $\text{TIP}_R(\alpha)$) are zero.

(v) According to given assumptions, three cases will occur in interest earned per year.

In this case upper and lower α -cut of interest earned in all sub cases are same as the all sub cases of case 1.

Case 4: $\tilde{q}_1(t)$ and $\tilde{q}_2(t)$ is gH-(ii) differentiable

If $\tilde{q}_1(t)$ is gH-(ii) differentiable and $\tilde{q}_2(t)$ is gH-(i) differentiable, then proceeding as in the previous cases we get the corresponding solutions as

$$q_{1L}(t, \alpha) = \left[\frac{k - a_L(\alpha)}{\theta} - \frac{b}{\theta^2} \right] (1 - e^{-\theta t}) + \frac{bt}{\theta} \tag{62}$$

$$q_{1R}(t, \alpha) = \left[\frac{k - a_R(\alpha)}{\theta} - \frac{b}{\theta^2} \right] (1 - e^{-\theta t}) + \frac{bt}{\theta} \tag{63}$$

$$q_{2L}(t, \alpha) = \left[\frac{a_L(\alpha)}{\theta} + \frac{b}{\theta^2} \right] (e^{(T-t)\theta} - 1) - \frac{bT}{\theta} e^{(T-t)\theta} + \frac{bt}{\theta} \tag{64}$$

$$q_{2R}(t, \alpha) = \left[\frac{a_R(\alpha)}{\theta} + \frac{b}{\theta^2} \right] (e^{(T-t)\theta} - 1) - \frac{bT}{\theta} e^{(T-t)\theta} + \frac{bt}{\theta} \tag{65}$$

(i) Lower α -cut and upper α -cut of ordering cost (OC_L , OC_R) are given by

$$OC_L = OC_R = c_2 \tag{66}$$

(ii) Lower α -cut and upper α -cut of stock holding cost ($\text{THC}_L(\alpha)$, $\text{THC}_R(\alpha)$) are given by

$$\begin{aligned} \text{THC}_R(\alpha) &= h \left[\int_0^{t_1} q_{1R}(t, \alpha) dt + \int_{t_1}^T q_{2R}(t, \alpha) dt \right] \\ &= \frac{h}{\theta} \left(kt_1 - a_R(\alpha) T + \frac{bT^2}{2} \right) \end{aligned} \tag{67}$$

$$\begin{aligned} \text{THC}_L(\alpha) &= h \left[\int_0^{t_1} q_{1L}(t, \alpha) dt + \int_{t_1}^T q_{2L}(t, \alpha) dt \right] \\ &= \frac{h}{\theta} \left(kt_1 - a_L(\alpha) T + \frac{bT^2}{2} \right) \end{aligned} \tag{68}$$

(iii) Lower α -cut and upper α -cut of annual cost due to deteriorated units ($\text{DC}_L(\alpha)$, $\text{DC}_R(\alpha)$) are given by

$$\text{DC}_R(\alpha) = c_p \left(kt_1 - a_R(\alpha) T + \frac{bT^2}{2} \right) \tag{69}$$

$$\text{DC}_L(\alpha) = c_p \left(kt_1 - a_L(\alpha) T + \frac{bT^2}{2} \right) \tag{70}$$

(iv) According to given assumption, there are three cases to occur in interest charged for the items kept in stock per year.

Case 4.1. $N \leq M \leq t_1 \leq T$

Lower α -cut and upper α -cut of interest payable ($\text{TIP}_L(\alpha)$, $\text{TIP}_R(\alpha)$) are given by

$$\begin{aligned} \text{TIP}_R(\alpha) &= \left[c_p I_c \int_M^T q_{2R}(t, \alpha) dt \right] \\ &= [c_p I_c A_3] \end{aligned} \tag{71}$$

$$\begin{aligned} \text{TIP}_L(\alpha) &= \left[c_p I_c \int_M^T q_{2L}(t, \alpha) dt \right] \\ &= [c_p I_c A_4] \end{aligned} \tag{72}$$

Case 4.2. $N \leq t_1 \leq T \leq M$

In this case Lower α -cut and upper α -cut of interest payable ($\text{TIP}_L(\alpha)$, $\text{TIP}_R(\alpha)$) are zero.

Case 4.3. $t_1 \leq T \leq N \leq M$

In this case Lower α -cut and upper α -cut of interest payable ($\text{TIP}_L(\alpha)$, $\text{TIP}_R(\alpha)$) are zero.

(v) According to given assumption, three cases will occur in interest earned per year. In this case upper and lower α -cut of interest earned in all sub cases are same as the all sub cases of case 1.

Using Eq. (33) we can find out the optimal cycle time so as to minimize annual inventory cost for all sub cases of all cases.

5. Numerical examples

To illustrate the results of the proposed model we solve the following numerical examples

Let $c_p = \text{Rs } 12/\text{unit}$, $c_s = \text{Rs } 15/\text{unit}$, $c_2 = \text{Rs } 500/\text{order}$, $k = 2900 \text{ units/year}$, $h = \text{Rs } 6/\text{unit}$, $I_e = 0.11$, $I_c = 0.31$, $\alpha = 0.97$, $\theta = 0.2$, $b = 0.5$, $\lambda_1 = 0.4$, $\lambda_2 = 0.6$, $a = (500, 1000, 1500)$, $\alpha_1 = 0.4$ then the optimal results for different cases are given in Table 1.

6. Result and some sensitivity analyses

Table 1 describes the optimal results of fuzzy inventory model. In this proposed model, we observe that in sub case 1.3 of case 1, the annual inventory cost is minimum because in sub case 1.1, retailer's trade credit period is much greater than total cycle time T. From Table 2, we observe that if the demand parameter b increases, the total demand decreases and as a result annual inventory cost (lower and upper α -cut) increases. Usually with the increase of deterioration, the total deterioration cost increases and as a result, the annual inventory cost increases. This real fact is depicted from Table 3.

Sensitivity analyses are carried out with respect to the inventory parameters to show the feasibility of the model. Fig. 1 shows that with the increase of interest earned (i_e), the upper and lower α -cut of inventory cost of retailer decreases for sub case 3.1 of case 3. Fig. 2 depicts that with the increase of holding cost (h), the upper and lower α -cuts of inventory cost of retailer increases for sub 1.1 of case 1. Fig. 3 shows that with the increase of α_1 , the upper and lower α -cuts of inventory cost of retailer decrease for case 1.1 of Case 1. Because total interest earned by the retailer increases with the increase of α_1 and as a result, inventory cost decreases. Usually with the increase of α , lower α -cut of annual inventory cost decreases and upper α cut of annual inventory cost increases. This real fact is depicted from Fig. 4. Fig. 5 shows that with the increase of selling price (c_s), the upper and lower α -cuts of inventory cost of retailer decreases for sub 2.3 of case 2. Since with the increase of selling price the annual interest earned by the retailer increases and annual inventory cost decreases. Fig. 6 depicts that

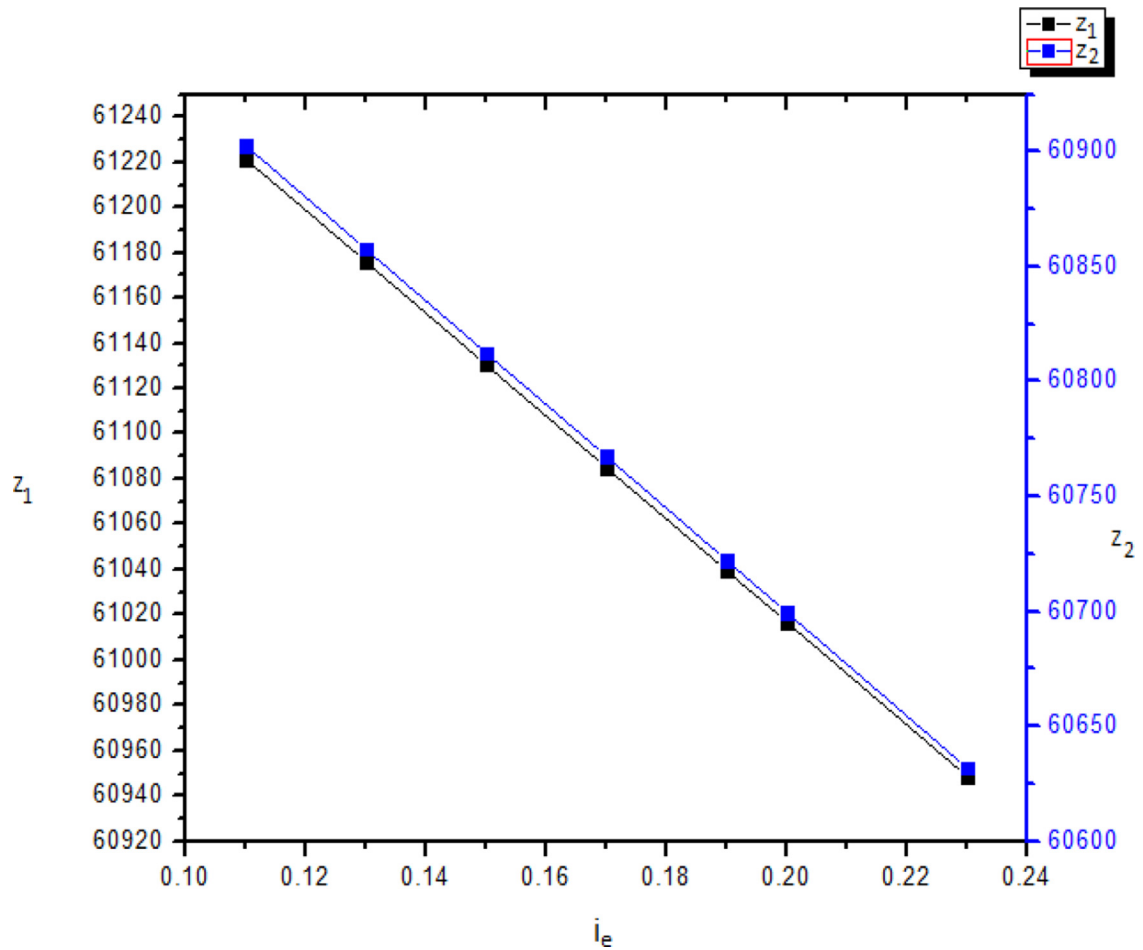


Fig. 1. Effect of interest earned (i_e) on Z_1 and Z_2 of Sub case 3.1 of Case 3.

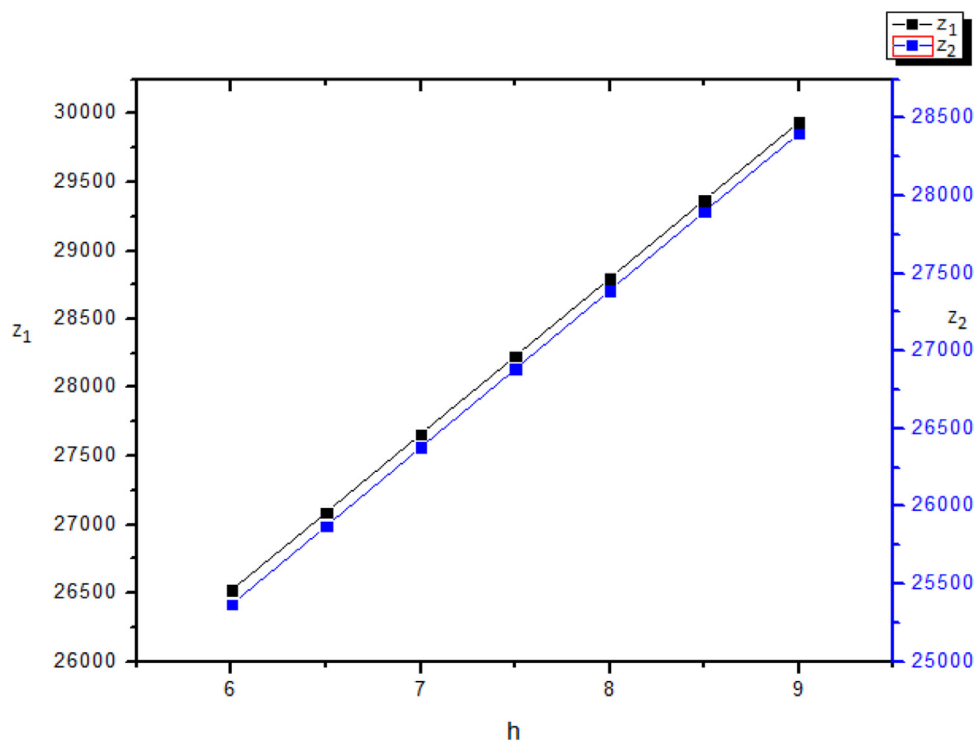


Fig. 2. Effect of unit holding cost (h) on Z_1 and Z_2 of Sub case 1.1 of Case 1.

Table 1
Optimal results for different cases.

Different Cases	Different Sub cases	t_1	T	M	N	Z	Z_1	Z_2
Case 1	Sub case 1.1	1.51	3.02	2.516667	2.013333	25,830.12	26,509.99	25,376.87
	Sub case 1.2	0.99	2.0012	2.16	1.11	22,858.45	23,491.38	22,436.49
	Sub case 1.3	1.102	2.2009	3.96	2.86	22,629.70	23,256.18	22,212.05
Case 2	Sub case 2.1	5.062250	10.12450	7.788077	6.749667	29,574.76	45,685.26	18,834.42
	Sub case 2.2	4.950495	10.0143	10.80	5.55	26,576.05	42,666.80	15,848.88
	Sub case 2.3	5.493250	10.98650	21.34128	15.38110	26,424.60	41,949.53	16,074.65
Case 3	Sub case 3.1	.5428553	1.031425	0.8595208	0.6876167	62,634.59	62,981.79	62,403.13
	Sub case 3.2	.5154743	1.041258	1.124559	0.5784767	60,902.86	61,221.53	60,690.41
	Sub case 3.3	0.5073921	1.014784	1.826611	1.319219	58,812.39	59,124.79	58,604.12
Case 4	Sub case 4.1	2.843728	5.687456	4.374966	3.791637	68,077.24	68,129.47	68,042.42
	Sub case 4.2	4.26205	7.2455	7.317955	4.830333	48,856.23	49,163.27	48,651.53
	Sub case 4.3	3.602353	6.1240	11.63560	7.9612	45,664.68	45,947.48	45,476.15

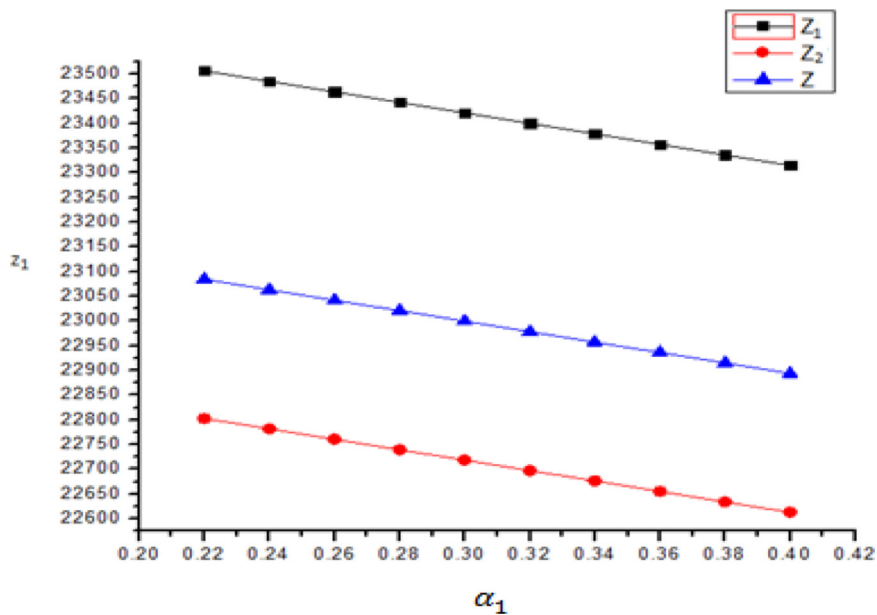


Fig. 3. Effect of partial trade credit (α_1) on Z_1 , Z_2 and Z for case 1.1 of Case 1.

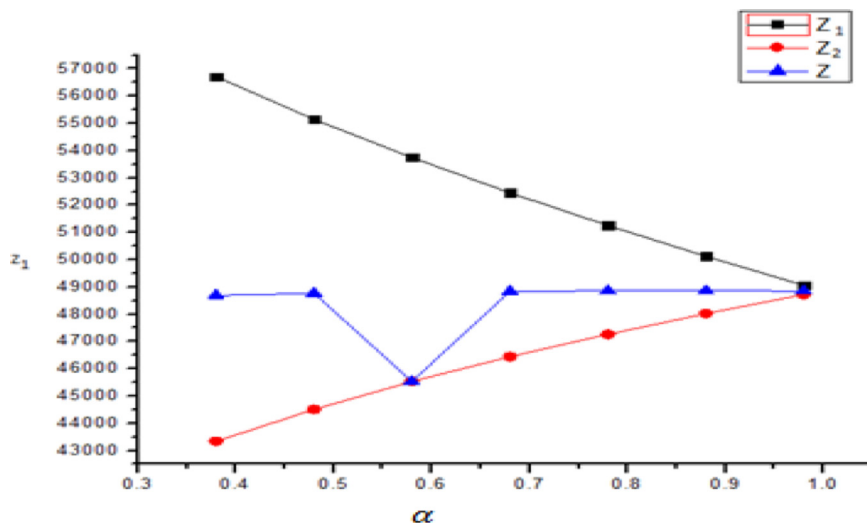


Fig. 4. Effect of α on Z_1 , Z_2 and Z of sub case 4.2 of Case 4.

Table 2
Effect of b on Z_1 and Z_2 of different sub cases of Case 1.

b	sub case 1.1		sub case 1.2		sub case 1.3	
	Z_1	Z_2	Z_1	Z_2	Z_1	Z_2
0.2	26,476.29	25,343.17	23,458.16	22,403.27	23,222.14	22,178.02
0.4	26,498.76	25,365.64	23,480.31	22,425.42	23,244.83	22,200.71
0.6	26,521.22	25,388.10	23,502.46	22,447.57	23,267.52	22,223.40
0.8	26,543.69	25,410.50	23,524.60	22,469.71	23,290.21	22,246.08

Table 3
Effect of θ on Z_1 and Z_2 of different sub cases of Case 2.

θ	sub case 2.1		sub case 2.2		sub case 2.3	
	Z_1	Z_2	Z_1	Z_2	Z_1	Z_2
0.3	51,841.01	35,060.10	48,778.64	31,971.38	49,134.94	33,179.13
0.35	55,802.05	41,859.19	52,711.38	38,733.06	53,905.84	40,722.27
0.4	60,519.31	48,679.43	57,395.37	45,517.62	59,706.28	48,565.98
0.45	66,156.29	55,929.00	62,993.43	52,730.42	66,778.85	57,197.11
0.5	72,913.80	63,956.08	69,705.27	60,717.57	75,426.78	67,065.95

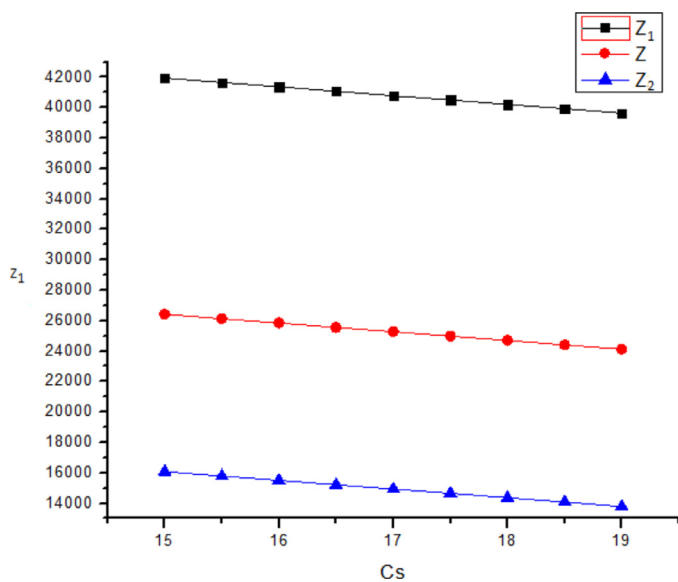


Fig. 5. Effect of C_s on Z_1 , Z_2 and Z of sub case 2.3 of Case 2.

with the increase of interest payable (i_c), the upper and lower α -cuts of inventory cost of retailer increase for sub 1.3 of case 1. It is due to the fact that if the rate of interest payable increases, then the retailer has to pay more amounts to the supplier. So the inventory cost with respect to the retailer increases. Fig. 7 shows that with the increase of purchasing price (c_p), the upper and lower α -cuts of inventory cost of retailer increases for sub 2.2 of case 2. In real life situation, if the purchasing cost is high then the retailer has to pay more amount for relevant inventory cost. In this case the annual inventory cost of retailer increases. Fig. 8 shows that with the increase of production rate (K), the upper and lower α -cuts of inventory cost of retailer increases for sub case 1.3 of case 1. Generally, if the production rate is high, then the retailer has the amount for the production. So in this case the annual inventory cost is more.

7. Managerial insights

In this EPQ model of deteriorating items, with partial trade credit financing via fuzzy differential approach, the managerial insights are:

- (i) Among all the cases, case 1 is more accepted in real life situation. Since in Sub case 1.3 of case 1 the retailer's relevant cost is minimum.
- (ii) Now a day, the retailers do not agree to offer full credit period to his/her customers. They accept full credit period from the supplier and offer partial trade credit period to his customers. In this situation, our proposed model is helpful for a decision maker to make a decision to optimize the total relevant cost of retailer.
- (iii) Solving of fuzzy differential equation with the help of gH derivative approach is more acceptable and logical with respect to other methods.
- (iv) We are living in a world full of uncertainty and ambiguity. We usually ask ourselves questions that we are uncertain about our answers. Is it going to rain tomorrow? What will be the exchange rate of Indian currency next month? Why, where and how should I invest? "Fuzzy"-perhaps is a most appropriate tool to deal uncertainty and ambiguity. During the inventory problems, it has often observed that, some of the inventory parameters are to be treated as uncertain variables. There are so many practical situations in real life where this phenomena happen. For instance, suppose a decision maker (DM) wants to make a business plan

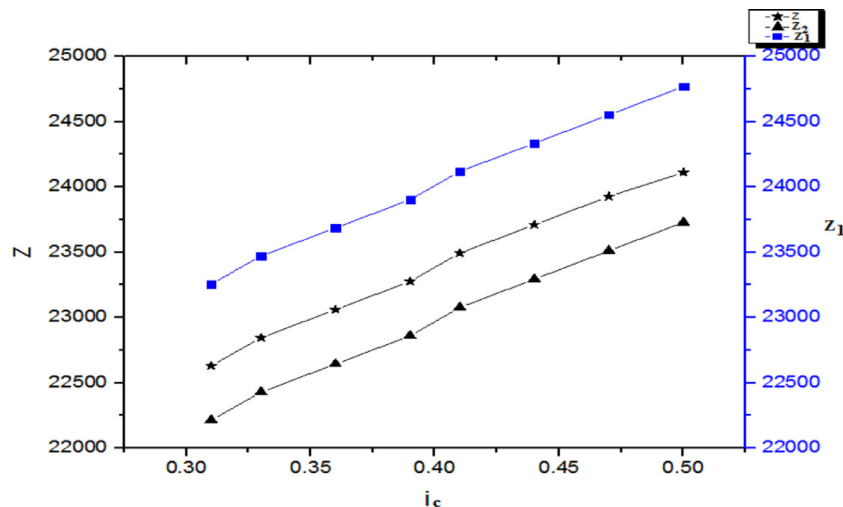


Fig. 6. Effect of i_c on Z_1 , Z_2 and Z of sub case 1.3 of Case 1.

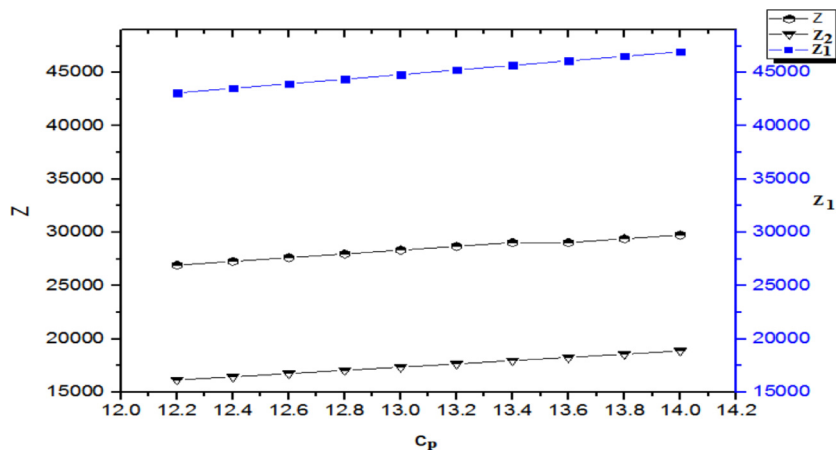


Fig. 7. Effect of c_p on Z_1 , Z_2 and Z of sub case 2.2 of Case 1.

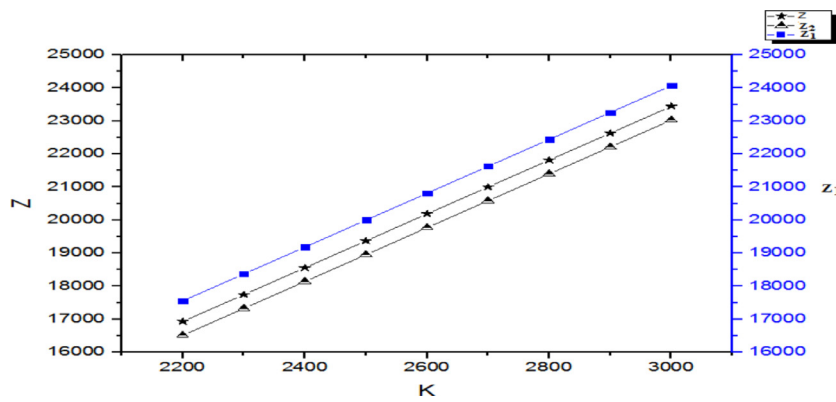


Fig. 8. Effect of K on Z_1 , Z_2 and Z of sub case 1.3 of Case 1.

for the upcoming month in advance. For that the DM needs the information about the capacity of plant/production capacity, an idea about the demand that required for smooth business, all of which are not precisely known to the DM. In such case, DM is often rely on the expert's opinion or probabilistic statistics. In this type of situation, use of fuzziness or stochastic or some other like that is more suitable.

8. Conclusion and future research scope

In this paper, some useful ideas were presented to deal with inventory control problem with left right type fuzzy demand. The main contributions can be summarized as the following six aspects:

- (i) In this volatile situation demands are taken as fuzzy in nature. Sometimes demands are represented as triangular and trapezoidal fuzzy number. Left right type fuzzy demand which was not yet applied in any fuzzy inventory problem was applied in this paper.
- (ii) gH derivative approach proposed by Bede and Stefanini [6] were discussed and successfully applied to the proposed model to find the total variable cost.
- (iii) Partial trade credit financing which is one of the central features in supply chain management was applied for the current model using gH derivative approach.
- (iv) According to literature survey for the first time in a single mathematical formulation, we introduced an economic production quantity model with time dependent demand in fuzzy environment where demand is taken as left right type fuzzy number. With the use of gH derivative approach the

proposed model is solved for the minimum cost of the retailer.

- (v) In the current investigation, several sub cases of different cases with two level credit periods are formulated. A decision maker can take the managerial decisions depending upon the existing situation for minimum cost. A decision maker choose sub case 1.3 of case 1 for minimum cost of the retailer, it is due to the fact that in this sub case retailers cost is minimum. This is justified to our real life situation that in sub case 1.3 of case 3 the retailer's trade credit period is greater than the total cycle time, so in this case the retailer has to pay less interest to the supplier and consequently inventory cost will be minimum.
- (vi) Some new real life based important facts are provided as managerial decisions in this paper, which will help in developing the business management.

The proposed model can be extended in several ways. For future research work, we may extend this model with shortages, quantity discounts and taking selling price, ordering cost as trapezoidal fuzzy number, Intuitionistic fuzzy number etc. As it is assumed that the unit selling price is greater than the unit purchasing price, the retailer must have sufficient amounts before the end of business period and to pay the dues to the wholesaler some time before the end of the total cycle and in this situation, he will have to pay less interest to the wholesaler. Moreover, the retailer can earn more interest after that time up to the end of the business period. This new approach can be applied to our current model following Majumder et al. [48]. We can also extend this model by taking type two fuzzy number, Intuitionistic fuzzy number etc. The concept of immediate part payment and the delay-

payment for the rest can also allowed by the wholesaler for an item over a finite planning horizon or random planning horizon In addition, against an immediate part payment (variable) to the wholesaler, there is a provision for (i) borrowing money from a money lending source and (ii) earning some relaxation on credit period from the wholesaler. The models can also be developed with respect to the retailer for maximum profit. We can also extend the current model by considering both partial trade credit for supplier and retailer.

Appendix

Definition (Fuzzy Set [36]). A fuzzy set \tilde{A} is defined by $\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) : x \in A, \mu_{\tilde{A}}(x) \in [0, 1]\}$. In the pair $(x, \mu_{\tilde{A}}(x))$ the first element x belong to the classical set A , the second element $\mu_{\tilde{A}}(x)$, belong to the interval $[0, 1]$, called Membership function.

Definition (α -cut of a fuzzy set). The α -level set (or interval of confidence at level α or α -cut) of the fuzzy set \tilde{A} of X is a crisp set A_α that contains all the elements of X that have membership values in \tilde{A} greater than or equal to α i.e. $\tilde{A} = \{x : \mu_{\tilde{A}}(x) \geq \alpha, x \in X, \alpha \in [0, 1]\}$.

Definition (Fuzzy Number [37]). A fuzzy number is fuzzy set like $u : R \rightarrow I = [0, 1]$ which satisfies

- (1) u is upper semi-continuous.
- (2) $u(x) = 0$ outside the interval $[c, d]$
- (3) There are real numbers a, b such $c \leq a \leq b \leq d$
 - (3.1) $u(x)$ is monotonic increasing on $[c, a]$,
 - (3.2) $u(x)$ is monotonic decreasing on $[b, d]$,
 - (3.3) $u(x) = 1, a \leq x \leq b$

Let E^1 be the set of all real fuzzy numbers which are normal, upper semi-continuous, convex and compactly supported fuzzy sets.

Definition (Fuzzy Number (Parametric form) [12,13]). A fuzzy number u in a parametric form is a pair (u_1, u_2) of function $u_1(r), u_2(r), 0 \leq r \leq 1$, which satisfies the following requirments:

- (1) $u_1(r)$ is a bounded monotonic increasing left continuous function,
- (2) $u_2(r)$ is a bounded monotonic decreasing left continuous function,
- (3) $u_1(r) \leq u_2(r), 0 \leq r \leq 1$.

A crisp number x is simply represented by $(u_1(r), u_2(r)) = (x, x), 0 \leq r \leq 1$. By appropriate definitions, the fuzzy number space $\{(u_1(r), u_2(r))\}$ becomes a convex cone E^1 which could be embedded isomorphically and isometrically into a Banach space.

Properties. Let $x = (x_1(r), x_2(r)), y = (y_1(r), y_2(r)) \in E^1, 0 \leq r \leq 1$ and arbitrary $k \in R$.

Then

- (1) $x = y$ iff $x_1(r) = y_1(r)$ and $x_2(r) = y_2(r)$,
- (2) $x + y = (x_1(r) + y_1(r), x_2(r) + y_2(r))$,
- (3) $x - y = (x_1(r) - y_2(r), x_2(r) - y_1(r))$,
- (4) $kx = \begin{cases} (kx_1(r), kx_2(r)), & k \geq 0 \\ (kx_2(r), kx_1(r)), & k < 0 \end{cases}$

Definition (L-R Fuzzy Number [30]). A function usually denoted as L or R , is a reference function of fuzzy number iff

- (1) $L(x) = L(-x)$,
- (2) $L(0) = 1$,
- (3) L is non increasing on $[0, +\infty)$.

A Fuzzy Number $\tilde{A} = (a_1, a_2, a_3, a_4)$, is called a L-R type fuzzy number iff

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_1 \\ L\left(\frac{x-a_1}{a_2-a_1}\right), & a_1 \leq x \leq a_2 \\ 1, & a_2 \leq x \leq a_3 \\ R\left(\frac{a_4-x}{a_4-a_3}\right), & a_3 \leq x \leq a_4 \\ 0, & x \geq a_4 \end{cases}$$

and it is denoted by $\tilde{A} = (a_1, a_2, a_3, a_4)_{LR}$

Definition (Generalized L-R Fuzzy Number [30]). A Generalized Fuzzy Number

$\tilde{A} = (a_1, a_2, a_3, a_4; w)$, is called a Generalized L-R Fuzzy Number if its membership function is given by

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a_1 \\ wL\left(\frac{x-a_1}{a_2-a_1}\right), & a_1 \leq x \leq a_2 \\ w, & a_2 \leq x \leq a_3 \\ wR\left(\frac{a_4-x}{a_4-a_3}\right), & a_3 \leq x \leq a_4 \\ 0, & x \geq a_4 \end{cases}$$

and it is denoted by $\tilde{A} = (a_1, a_2, a_3, a_4; w)_{LR}$

Definition (α -cut of of L-R fuzzy number [51]). The α -cut set of a L-R fuzzy number \tilde{A} is a crisp subset of R and is defined by $(\tilde{A})_\alpha = \{x | \mu_{\tilde{A}}(x) \geq \alpha \text{ and } x \in R\}$. Hence α -cut set of a L-R fuzzy number \tilde{A} is denoted by $[a_\alpha^L, a_\alpha^R]$, where $a_\alpha^L = \inf (\tilde{A})_\alpha$ and $a_\alpha^R = \sup (\tilde{A})_\alpha$. i.e., if $\tilde{A} = (a_1, a_2, a_3, a_4)$ is a L-R type fuzzy number, then its α -cut is

$$[a_1 + L^{-1}(\alpha)(a_2 - a_1), a_4 - R^{-1}(\alpha)(a_4 - a_3)]$$

Definition [4]. Let $x, y \in E^1$. If there exists $z \in E^1$ such that $x = y + z$, then z is called the Hukuhara-difference of fuzzy numbers x and y , and it denoted by $z = x \ominus y$.

Lemma. The point should be notated that $x \ominus y \neq x + (-1)y$.

Definition [4]. Let $f : [a, b] \rightarrow E^1$ and $t_0 \in [a, b]$. We say that f is Hukuhara differential at t_0 , if there exist an element $f'(t_0) \in E^1$ such that for all $h > 0$ sufficiently small, there exists $f(t_0 + h) \ominus f(t_0), f(t_0) \ominus f(t_0 - h)$ and the limits exists in metric D .

$$\lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus f(t_0)}{h} = \lim_{h \rightarrow 0} \frac{f(t_0) \ominus f(t_0 - h)}{h} = f'(t_0)$$

Definition [3]. Let $f : (a, b) \rightarrow E$ and $x_0 \in (a, b)$. We say that f is strongly generalized differential at x_0 (Bede-Gal differential) if there exists an element $f'(x_0) \in E$, such that

- (i) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) \ominus f(x_0)$ and $f(x_0) \ominus f(x_0 - h)$ and the limits exist in the metric D

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 - h)}{h} = f'(x_0)$$

Or

- (ii) for all $h > 0$ sufficiently small, there exist $f(x_0) \ominus f(x_0 + h)$ and $f(x_0 - h) \ominus f(x_0)$ and the limits exist in the metric D

$$\lim_{h \searrow 0} \frac{f(x_0) \ominus f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

Or

- (iii) for all $h > 0$ sufficiently small, there exist $f(x_0 + h) \ominus f(x_0)$, and $f(x_0 - h) \ominus f(x_0)$ and the limits exist in the metric D

$$\lim_{h \searrow 0} \frac{f(x_0 + h) \ominus f(x_0)}{h} = \lim_{h \searrow 0} \frac{f(x_0 - h) \ominus f(x_0)}{-h} = f'(x_0)$$

Or

(iv) for all $h > 0$ sufficiently small, there exist $f(x_0) - {}^h f(x_0 + h)$ and $f(x_0) - {}^h f(x_0 - h)$ and the limits exists in the metric D

$$\lim_{h \searrow 0} \frac{f(x_0) - {}^h f(x_0 + h)}{-h} = \lim_{h \searrow 0} \frac{f(x_0) - {}^h f(x_0 - h)}{h} = f'(x_0)$$

(h and $-h$ at denominators mean $\frac{1}{h}$ and $-\frac{1}{h}$, respectively).

Remarks [5]. In, the authors consider four cases for derivatives. Here we only consider the two first cases. In the other cases, the derivative is trivial because it is reduced to a crisp element. We say that f is (i)-differentiable on (a, b) if f is differentiable with the meaning (i) and also (ii)-differentiable that f satisfies case (ii).

Definition [65]. Let $f : R \rightarrow E$ be a function and denote $f(t) = (f_L(t, r), f_R(t, r))$, for each $r \in [0, 1]$. Then

- (1) If f is (i)-differentiable, then $f_L(t, r)$ and $f_R(t, r)$ are also differentiable function and $f'(t) = (f'_L(t, r), f'_R(t, r))$
- (2) If f is (ii)-differentiable, then $f_L(t, r)$ and $f_R(t, r)$ are also differentiable function and $f'(t) = (f'_R(t, r), f'_L(t, r))$

Definition Generalized Hukuhara difference [6]. The generalized Hukuhara difference of two fuzzy numbers $u, v \in \mathfrak{R}_{\mathcal{F}}$ is defined as follows

$$u \ominus_g v = w \Leftrightarrow \begin{cases} \text{(i) } u = v \oplus w \\ \text{or (ii) } v = u \oplus (-1)w \end{cases}$$

Consider $[w]_{\alpha} = [w_L(\alpha), w_R(\alpha)]$, then $w_L(\alpha) = \min\{u_L(\alpha) - v_L(\alpha), u_R(\alpha) - v_R(\alpha)\}$ and $w_R(\alpha) = \max\{u_L(\alpha) - v_L(\alpha), u_R(\alpha) - v_R(\alpha)\}$.

Here the parametric representation of a fuzzy valued function $f : [a, b] \rightarrow \mathfrak{R}_{\mathcal{F}}$ is expressed by

$$[f(t)]_{\alpha} = [f_L(t, \alpha), f_R(t, \alpha)], \quad t \in [a, b], \alpha \in [0, 1].$$

Definition (Generalized Hukuhara derivative [6]). The generalized Hukuhara derivative of a fuzzy valued function $f : (a, b) \rightarrow \mathfrak{R}_{\mathcal{F}}$ at t_0 is defined as

$$f'(t_0) = \lim_{h \rightarrow 0} \frac{f(t_0 + h) \ominus_g f(t_0)}{h}$$

In parametric form we say that

$$f(t) \text{ is (i)-gH differentiable at } t_0 \text{ if } [f'(t_0)]_{\alpha} = [f'_L(t_0, \alpha), f'_R(t_0, \alpha)]$$

and

$$f(t) \text{ is (ii)-gH differentiable at } t_0 \text{ if } [f'(t_0)]_{\alpha} = [f'_L(t_0, \alpha), f'_R(t_0, \alpha)].$$

Theorem: Existence condition for solution of fuzzy differential equation: Let us consider the fuzzy initial value problem (FIVP)

$$x'(t) = f(t, x(t)), \quad \tilde{x}(t_0) = \tilde{x}_0, \quad a \leq t \leq b$$

Now let, $[x(t)]^{\alpha} = [x_L^{\alpha}(t), x_R^{\alpha}(t)]$, $[x_0]^{\alpha} = [x_{0,L}^{\alpha}, x_{0,R}^{\alpha}]$ and

$$[f(t, x(t))]^{\alpha} = [f_L^{\alpha}(t, x_L^{\alpha}(t), x_R^{\alpha}(t)), f_R^{\alpha}(t, x_L^{\alpha}(t), x_R^{\alpha}(t))]$$

By using the extension principle we have the membership function as

$$f(t, x(t))(s) = \sup\{x(t)(\tau) | s = f(t, \tau)\}, \quad s \in R$$

Then the result $f(t, x(t))$ is also a fuzzy function.

Where,

$$f_L^{\alpha}(t, x_L^{\alpha}(t), x_R^{\alpha}(t)) = \min\{f(t, u) | u \in [x_L^{\alpha}(t), x_R^{\alpha}(t)]\}$$

and

$$f_R^{\alpha}(t, x_L^{\alpha}(t), x_R^{\alpha}(t)) = \max\{f(t, u) | u \in [x_L^{\alpha}(t), x_R^{\alpha}(t)]\}$$

Lemma. If the solution $x_L^{\alpha}(t)$ is an increasing function, where as $x_R^{\alpha}(t)$ is a decreasing function then the solution is called strong solution. Otherwise it is a weak solution.

i.e., $\frac{\partial x_L^{\alpha}(t)}{\partial \alpha} \geq 0$, $\frac{\partial x_R^{\alpha}(t)}{\partial \alpha} \leq 0$ and $x_L^{\alpha}(t, \alpha = 1) = x_R^{\alpha}(t, \alpha = 1)$ then it is strong solution, otherwise weak solution.

Lemma. If $[x_L(t, \alpha), x_R(t, \alpha)]$ be the solution of a FDE then the more general way to find the strong solution of a fuzzy differential equation is

$$x_L^*(t, \alpha) = \min\{x_L^{\alpha}(t), x_R^{\alpha}(t)\} \text{ and } x_R^*(t, \alpha) = \max\{x_L(t, \alpha), x_R(t, \alpha)\}$$

Where $[x_L^*(t, \alpha), x_R^*(t, \alpha)]$ is a strong solution.

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