

Zantek, Paul F.; Hanson, Timothy; Damien, Paul; Popova, Elmira

## Article

# A decision dependent stochastic process model for repairable systems with applications

Operations Research Perspectives

## Provided in Cooperation with:

Elsevier

*Suggested Citation:* Zantek, Paul F.; Hanson, Timothy; Damien, Paul; Popova, Elmira (2015) : A decision dependent stochastic process model for repairable systems with applications, Operations Research Perspectives, ISSN 2214-7160, Elsevier, Amsterdam, Vol. 2, pp. 73-80, <https://doi.org/10.1016/j.orp.2015.03.002>

This Version is available at:

<https://hdl.handle.net/10419/178251>

### Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

### Terms of use:

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*



<http://creativecommons.org/licenses/by-nc-nd/4.0/>



# A decision dependent stochastic process model for repairable systems with applications



Paul F. Zantek<sup>a</sup>, Timothy Hanson<sup>b,\*</sup>, Paul Damien<sup>c</sup>, Elmira Popova<sup>d</sup>

<sup>a</sup> American Medical Systems, 10700 Bren Rd. W., Minnetonka, MN 55343, United States

<sup>b</sup> University of South Carolina, 1523 Greene St., Columbia, SC 29208, United States

<sup>c</sup> University of Texas at Austin, 1 University Station, B6000, Austin, TX 78712, United States

<sup>d</sup> University of Texas at Austin, 301 E. Dean Keeton St., Austin, TX 78705, United States

## ARTICLE INFO

### Article history:

Available online 30 March 2015

### Keywords:

Kijima model  
Decision dependence  
Maintenance schedule  
Repairable system  
Spiral down effect

## ABSTRACT

Management decisions regarding maintenance protocols critically hinge on the underlying probability distribution of the time between failures in most repairable systems. Replacement of the system with a new one resets the system age to zero, whereas a repair does not alter the system age but may shift the *parameters* of the failure-time distribution. Additionally, maintenance decisions lead to left-truncated observations, and right-censored observations. Thus, the underlying stochastic process governing a repairable system evolves based on the management decision taken.

This paper mathematically formalizes the notion of how management actions impact the functioning of a repairable system over time by developing a new stochastic process model for such systems. The proposed model is illustrated using both simulated and real data. The proposed model compares favorably to other models for well-known data on Boeing airplanes. The model is further illustrated and compared to other models on failure time and maintenance data stemming from the South Texas Project nuclear power plant.

© 2015 The Authors. Published by Elsevier Ltd.  
This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

## 1. Introduction

Many repairable processes have finite lifetimes that may require *corrective maintenance* (CM) during their lifetimes, such as adjustment, restoration, or lubrication. The process owner could instead opt for replacement with a new process, referred to as *preventive maintenance* (PM); an example of this kind of repair would be a complete overhaul of the system. An important application, considered later in this article, is nuclear-power generation. A nuclear power plant is comprised of numerous systems that fail at random times, thus requiring frequent maintenance. At each maintenance time, management decides whether the maintenance should be corrective or preventive, and this potentially influences the length of time until the next system failure. Failures may have significant implications for safety and operating costs, as well as the ability to satisfy customer demand for electricity.

A widespread assumption in the reliability literature is that the parameters of the *failure-time intensity* are unchanged after CM,

commonly termed the *minimal repair assumption* or *minimal repair hypothesis*. In order to test and, if necessary, relax the minimal repair assumption, a new stochastic process is introduced in which the failure intensity following a CM is allowed to be distinctly different than that following a PM; the failure intensity can reflect repairs that improve reliability or make it worse. Relevant properties of this stochastic process are characterized, and a two-stage procedure is proposed for maximum likelihood estimation of its parameters. As a byproduct of the maximum likelihood estimation, Wald confidence intervals for the parameters of the failure-time distribution are constructed. In addition, a likelihood ratio test (LRT) of the minimal repair hypothesis is developed.

A crucial, practical feature in any repairable system is the presence of right censored failure times. Our stochastic process model allows for such events, namely when maintenance is performed prior to a failure occurring, common in maintenance schedules. The properties of our methods, including coverage probabilities of the Wald confidence intervals and the sensitivity of the LRT, are studied using simulation.

To exemplify the methodological advances, we analyze two datasets: (a) a classic repairable systems dataset on Boeing air conditioners, and (b) the maintenance history and failure times for

\* Corresponding author.

E-mail addresses: [Paul.Zantek@ammd.com](mailto:Paul.Zantek@ammd.com) (P.F. Zantek), [hansont@stat.sc.edu](mailto:hansont@stat.sc.edu) (T. Hanson), [paul.damien@mcombs.utexas.edu](mailto:paul.damien@mcombs.utexas.edu) (P. Damien).

<http://dx.doi.org/10.1016/j.orp.2015.03.002>

2214-7160/© 2015 The Authors. Published by Elsevier Ltd. This is an open access article under the CC BY-NC-ND license (<http://creativecommons.org/licenses/by-nc-nd/4.0/>).

a water chilling system in the South Texas Project Nuclear Power Plant.

The stochastic process model considered in this article is related to models of repairable systems in the reliability literature; for comprehensive reviews of this literature, see [1,2], and [3]. Renewal processes are commonly used if all the maintenance repairs are PM, bringing the system to a “good-as-new” state each time (known as *perfect repair*). This assumption simply restarts a common failure intensity to its value at zero after every repair. Note then that the term *good-as-new* is misleading in the presence of decreasing intensity (and inter-failure hazard), as systems that have not been repaired recently are actually more reliable than those that have; this phenomenon is seen in the power plant data in Section 5.

Non-homogeneous Poisson processes (NHPP) are used if all repairs are CM, i.e. bring the system to a “good-as-old” state (known as *minimal repair*), leaving the failure intensity unchanged; this can happen, for example, by replacing a failed sub-component of a system. The NHPP is formally nested in the model proposed in Section 2. Although common, the basic assumption of a consistently “minimal” CM repair is questionable; usually several types of maintenance, with varying degrees of effectiveness, are undertaken throughout the lifetime of the system. For a recent example assuming minimal repair, see [4].

Brown and Proschan [5] assume that repairs are either good-as-new (PM) or bad-as-old (minimal repair) with probabilities  $p$  and  $1 - p$ , respectively. Block, Borges and Savits [6] allow these probabilities  $p(t)$  to vary with system age; Whitaker and Samaniego [7] assume the type of maintenance is known. Doyen [8] presents a nonparametric estimation approach to this model for unknown but fixed  $p$  along with a review of recent literature on imperfect repairs and maintenance scheduling. Presnell, Hollander and Sethuraman [9] develop a test for the minimal repair assumption in a particular model that Block, Borges and Savits [6] proposed; however, if minimal repair is rejected, the question remains as to whether CM makes the system better or worse than in the case of minimal repair. In many applications this distinction is crucial. If one ignores maintenance decisions, Cooper, de Mello and Kleywegt [10] point out that decisions based on the incorrect assumption of sufficient minimal repairs could lead to a “spiral down” effect, where system reliability gets worse after repair cycles, i.e., more failures than expected; this happens because the assumed minimal repairs are actually worse than ‘good-as-old’. The model we propose in Section 2 allows for a follow-up analysis of whether CM makes system reliability better or worse than it was right before failure.

Kijima [11] proposed a model that includes perfect, minimal, and in-between repairs by introducing the *effective age* of the system after each repair, essentially providing a quantitative measure of whether the repair was successful. A particular case of Kijima’s model allowing imperfect repair is considered by Mettas and Zhao [12], who proposed a method to find the maximum likelihood estimates of the model’s parameters. Following Kijima [11], Dorado, Hollander, and Sethuraman [13] allow for repairs of varying degree by including so-called known *life supplements*, numbers between zero and one indicating the degree of repair between perfect and minimal. Veber, Nagode and Fajdiga [14] assume one overall life supplement that is unknown, i.e. each repair reduces the effective age of the system by the same fraction  $q$ . As an extension to a common  $q$ , Pan and Rigdon [15] allow the repair effectiveness parameter to vary from system to system. Gasmi [16] considers the Weibull distribution in an alternating imperfect repair scheme, i.e. PM followed by CM repeatedly, with common life supplement  $q$ . Recently Li and Hanson [17] propose to regress the life-supplement of each repair on covariates such as repair type, materials used, et cetera using a Bayesian nonparametric model. Tanwar, Rai, and Bolia [18] review much of the related literature on Kijima-type models.

Our model joins a growing body of literature allowing for differing types of departure from minimal repair, including Kijima [11]. Doyen and Gaudoin [19] consider several classes of imperfect repair models for increasing failure intensities, including models where (a) failure intensity is reduced by a constant factor relative to the current intensity; (b) failure intensity is reduced by a constant factor, but only relative to the most recent repair; and (c) several models based on the virtual age of the system, akin to Kijima’s [11] models.

We note that the present article does not provide a method to determine which of CM or PM is optimal for a system at a given point in time. Such decisions critically depend on the context; for instance, maintenance decisions in the context of a nuclear power plant process versus a medical billing records process would be substantially different. Second, the mathematical framework needed to handle such context-specific decisions requires stochastic optimization routines that are outside the scope of the intended aims of this research. However, such routines require information about the underlying probability distribution of the *time until failure* following each of PM or CM. That is, a decision-maker must have sound knowledge of how the system’s reliability is affected by maintenance decisions at any given point in time. Dimitrov, Chukova, and Khalil [20] consider the related problem of maintenance costs with imperfect repair, namely warranty costs within a Kijima Type I model. Garg, Rani, and Sharma [21,22] consider maintenance scheduling for a paper mill assuming a Weibull distribution. Doyen [8] also reviews recent literature on imperfect repair and maintenance scheduling.

## 2. Methodology

The aim of Section 2.1 is to develop a new mathematical framework that encapsulates the impact of management’s maintenance decisions on the parameters of the failure-time distribution of repairable systems. In Section 2.2, the failure-time distribution is modeled as a Weibull since, in addition to its wide-spread use in reliability applications, it has desirable theoretical and practical properties that will be highlighted.

### 2.1. A general decision-dependent stochastic process model for repairable systems

Consider a system that is put into operation at time  $t_0 = 0$ . The time until this system fails has probability density function (pdf)  $f(y|\theta)$ , where  $\theta$  is a vector of parameters indexing the density from a class such as the Weibull family of distributions. At any time, the system’s owner is allowed to perform maintenance of one of two types. In the first type, the system’s components are replaced or some other major restoration is performed such that the system’s age is reset to zero. This is *preventive maintenance* (PM). The second type of maintenance involves partial repairs or upgrades that do *not* (necessarily) restore the process to an “as good as new” state. This is *corrective maintenance* (CM).

There exists a set of increasing times  $\{t_1, \dots, t_n | t_i < t_{i+1} \forall i = 0, 1, \dots, n-1\}$  where, at each time, a decision is made to perform either a PM or a CM. The *time series of decisions* is denoted  $\{d_i\}_{i=0}^n$ , where  $d_i = 0$  if a PM is performed at time  $t_i$  and  $d_i = 1$  if a CM is performed; for example,  $d_0 = 0$  because we start with a newly restored system. In addition, the time of the most recent PM is denoted  $t_i^* = \max\{t_j | j < i, d_j = 0\}$ ; for example,  $t_1^* = 0$  always because  $d_0 = 0$ . A PM decision at time  $t_{i-1}$ , i.e.,  $d_{i-1} = 0$ , resets the age of the system to zero. The length of time until the next failure (at time  $t_i$ ) then has pdf  $f(t_i - t_i^* | \theta) = f(t_i - t_{i-1} | \theta)$ , where  $\theta$  is the parameter vector indexing the density associated with a newly restored system.

Now consider a CM decision at time  $t_{i-1}$ , i.e.,  $d_{i-1} = 1$ . Since CM is not a complete restoration of the system, the system's age is not reset to zero but instead remains at its value at the time of the CM, which is given by  $t_{i-1} - t_i^*$ : this is just the age that the system has accumulated since the last PM, which occurred at time  $t_i^*$ . The time from the CM until the next failure is thus truncated at  $t_{i-1} - t_i^*$ , and hence has pdf

$$f(t_i - t_i^* | t_i > t_{i-1}, \theta + \beta) = f(t_i - t_i^* | \theta + \beta) / S(t_{i-1} - t_i^* | \theta + \beta), \quad t_i > t_{i-1},$$

where  $\theta + \beta$  is the vector of parameters for the failure-time distribution following a CM and  $S(y|\cdot) = \int_y^\infty f(u|\cdot)du$  is the reliability (or survival) function. Including  $\beta$  in the model allows the failure-time distribution following a CM to have different parameters than that following a PM, since each component of  $\beta$  is free to assume positive or negative values. The evolutionary nature of the stochastic process model is illustrated in Fig. 1 for the first three failure times in a new system.

Suppose the distributional parameters following a CM are the same as those following a PM. This means  $\beta = \mathbf{0}$  and the pdf of the time until the next failure becomes  $f(t_i - t_i^* | \theta) / \Pr\{t_i > t_{i-1}\}$ . This is just the pdf of the time until failure for a newly restored system, conditional on the failure occurring after the CM; in other words, the system is restored to its condition at the instant before the CM. This is known as *minimal repair* in the literature on repairable systems.

However, if at any point in time the parameters are *decision dependent* ( $\beta \neq \mathbf{0}$ ), the reliability function associated with a CM could be more (or less) favorable than in the case of minimal repair, and this may lead to different maintenance decisions.

In the above formulation, no assumption was made about the form of the probability density function  $f(y|\theta)$ . A practitioner could use one of many well-known parametric forms for  $f$ . But as noted in the beginning of this section it is common practice to let  $f$  follow a Weibull distribution in reliability applications—a repairable system is one such application. The Weibull distribution is sufficiently flexible to allow increasing, decreasing, or constant hazard rates. Moreover, this distribution asymptotically approximates the reliability of any system comprised of numerous similar components in parallel, and as such is ideal for modeling system lifetimes. See [21,22] for a recent application of the Weibull distribution to the fuzzy reliability of a paper pulping mill.

### 2.2. Testing minimal repair hypothesis under a Weibull model for $f$

An excellent reference for Weibull distributions is Rinne [23]. Suppose that the failure time for a newly restored system has a Weibull  $(\alpha, \lambda)$  distribution with probability density function parameterized as

$$f(y|\alpha, \lambda) = \alpha \lambda y^{\alpha-1} \exp(-\lambda y^\alpha),$$

where  $\alpha$  is the shape parameter and  $\lambda$  is the scale parameter. The Weibull reliability and hazard rate functions, respectively, are

$$S(y|\alpha, \lambda) = \exp(-\lambda y^\alpha)$$

and

$$h(y) = \alpha \lambda y^{\alpha-1}.$$

For the case of minimal repair  $\beta = \mathbf{0}$ , the reliability function following a CM is given by

$$\begin{aligned} \tilde{S}(t_i - t_i^* | \theta) &= S(t_i - t_i^* | \theta) / S(t_{i-1} - t_i^* | \theta) \\ &= \exp[-\lambda(t_i - t_i^*)^\alpha] / \exp[-\lambda(t_{i-1} - t_i^*)^\alpha] \\ &= \exp\{-\lambda[(t_i - t_i^*)^\alpha - (t_{i-1} - t_i^*)^\alpha]\} \quad t_i > t_{i-1}. \end{aligned}$$

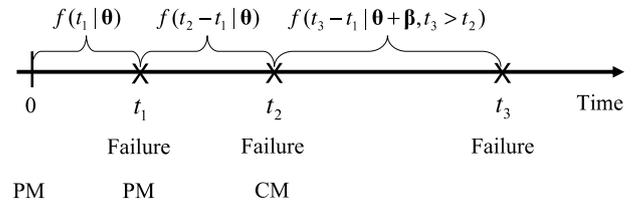


Fig. 1. Illustration of model for the first three failure times.

If instead the Weibull parameters are *decision dependent*, we have  $\beta = (\beta_1 \ \beta_2)'$  and the reliability function is given by

$$\begin{aligned} \tilde{S}(t_i - t_i^* | \theta + \beta) &= S(t_i - t_i^* | \theta + \beta) / S(t_{i-1} - t_i^* | \theta + \beta) \\ &= \exp[-(\lambda + \beta_2)(t_i - t_i^*)^{\alpha+\beta_1}] / \exp[-(\lambda + \beta_2) \\ &\quad \times (t_{i-1} - t_i^*)^{\alpha+\beta_1}] \\ &= \exp\{-(\lambda + \beta_2)[(t_i - t_i^*)^{\alpha+\beta_1} \\ &\quad - (t_{i-1} - t_i^*)^{\alpha+\beta_1}]\} \quad t_i > t_{i-1}. \end{aligned}$$

Suppose that the scale (but not shape) parameter *depends* on the decision. In this case, we have  $\beta = (0 \ \beta_2)'$  and the reliability function following CM  $\tilde{S}(t_i - t_i^* | \theta + \beta)$  reduces to

$$\tilde{S}(t_i - t_i^* | \theta + \beta) = \exp\{-(\lambda + \beta_2)[(t_i - t_i^*)^\alpha - (t_{i-1} - t_i^*)^\alpha]\} \quad t_i > t_{i-1}.$$

For  $\beta_2 < 0$ , it can be seen that  $\tilde{S}(t_i - t_i^* | \theta + \beta) \geq \tilde{S}(t_i - t_i^* | \theta)$  for all  $t_i > t_{i-1}$ ; in other words, at any time following a CM the reliability is at least as large as that under minimal repair  $\beta_1 = \beta_2 = 0$ . Alternatively, for  $\beta_2 > 0$ , the reliability is *at most* as large as that under minimal repair. Hence, it is critical to formally test whether  $\beta = \mathbf{0}$ .

**Managerial impact of testing  $\beta_2 > 0$  and  $\beta_2 < 0$ :** From a management perspective, if the scale  $\lambda$  increases after CM, i.e. we conclude  $\beta_2 > 0$ , CM does not bring the system to good-as-old but actually to a worse condition than it was before the CM. Thus, the system receiving successive CMs will fail more rapidly than what management expects under minimal repair. Under this scenario, it may be prudent for management to set the system clock back to zero with a PM more quickly (assuming increasing inter-failure hazard), or else implement measures to improve CM. If the scale  $\lambda$  decreases after CM, i.e. we conclude  $\beta_2 < 0$ , CM brings the system to better-than-old and the system is actually younger than what management perceives.

Given data, the minimal repair hypothesis  $H_0: \beta = \mathbf{0}$  may be tested via a standard likelihood ratio test (LRT), whose test statistic is  $2[\ln L(\hat{\theta}, \hat{\beta}) - \ln L(\hat{\theta}, \mathbf{0})]$ , where  $\ln L(\hat{\theta}, \hat{\beta})$  is the log likelihood evaluated at the maximum likelihood estimates of  $\theta$  and  $\beta$  and  $\ln L(\hat{\theta}, \mathbf{0})$  is the log likelihood evaluated at the maximum likelihood estimate of  $\theta$  under the constraint  $\beta = \mathbf{0}$ . The critical value of the test is the  $1 - \alpha$  quantile of the chi-squared distribution with two degrees of freedom, where  $\alpha$  is the significance level.

### 3. Maximum likelihood estimation

To describe the estimation of the Weibull model parameters  $\theta$  and  $\beta$ , first note that if PM or CM is performed at time  $t_i$  on a working system (no accompanying failure), the failure time stemming from the previous maintenance decision,  $d_{i-1}$ , is right censored, indicated by  $\delta_i = 0$ ; otherwise, for uncensored failure times,  $\delta_i = 1$ . Together, these variables form the vector process  $\{(t_i, d_i, \delta_i)\}_{i=1}^n$  with initial conditions  $t_0 = 0$  and  $d_0 = 0$ .

The likelihood function may be constructed by accumulating the contributions from the distributions associated with each maintenance decision. Following a PM decision at time  $t_{i-1}$ , i.e.,

$d_{i-1} = 0$ , a failure at time  $t_i$  has likelihood contribution  $f(t_i - t_i^* | \theta)$ . If the failure time is censored at time  $t_i$ , however, the contribution to the likelihood is  $S(t_i - t_i^* | \theta)$ . Now, for a CM decision  $d_{i-1} = 1$  at time  $t_{i-1}$ , the discussion in Section 2 implies that the likelihood contribution is  $f(t_i - t_i^* | \theta + \beta) / S(t_{i-1} - t_i^* | \theta + \beta)$ , but if the failure time is censored the contribution would be  $S(t_i - t_i^* | \theta + \beta) / S(t_{i-1} - t_i^* | \theta + \beta)$ . The likelihood function is then given by

$$L(\theta, \beta) = \prod_{i=1}^n [f(t_i - t_i^* | \theta)^{\delta_i} S(t_i - t_i^* | \theta)^{1-\delta_i}]^{1-d_{i-1}} \times \left[ \frac{f(t_i - t_i^* | \theta + \beta)^{\delta_i} S(t_i - t_i^* | \theta + \beta)^{1-\delta_i}}{S(t_{i-1} - t_i^* | \theta + \beta)} \right]^{d_{i-1}}$$

Note that for  $d_{i-1} = 0$ , the contribution is from observed or right-censored data; for  $d_{i-1} = 1$ , the contribution is from left-truncated data that are also possibly right censored.

Maximization of the likelihood function above is a complex problem for which there is no known closed-form solution. Therefore, the maximum likelihood estimates were found using a numerical optimization method. A two-stage estimation procedure was implemented in a SAS macro named DDSME (Decision Dependent Survival Model Estimation), as well as in an R script. DDSME is available upon request, whereas the R script is given in the Appendix.

#### 4. Illustration on Boeing air conditioner systems

A classic dataset introduced by Proschan [24] is considered for illustration. Time between failures (and associated repairs) for air conditioners in 13 planes were recorded. The first time recorded is assumed to be time from a PM. Additionally, four of the planes underwent one major overhaul during the study period; we assume these also are PM giving a total of 17 PM repairs. Upon consideration of each plane separately, plane 7908, with 21 CM times, showed the largest departure from the minimal repair (i.e. non-homogeneous Poisson process) assumption with a  $p$ -value of 0.009 under the likelihood ratio test. A follow-up Wald test shows that the scale term significantly increases ( $p = 0.007$ ) from PM to CM, indicating worse-than-old condition of the air conditioners after a CM repair.

The proposed model was compared to three others, (a) a Weibull renewal process; (b) a Weibull non-homogeneous Poisson process which obtains when  $\beta = \mathbf{0}$ ; and (c) a Kijima Type II [11] model with common life-supplement  $q > 0$  associated with each CM. Under the Kijima model,  $q = 1$  also implies the non-homogeneous Poisson process; when  $q = 0$  a renewal process is obtained. Models (a) and (b) are described in [25]. The Akaike Information Criterion (AIC) for these three fitted models are 437.8, 437.6, 439.6, respectively; the AIC for our proposed model is 432.2. For these data, our model is preferred. Under the Kijima Type II model, the estimate of  $q$  is 2.6 and the inter-failure hazard is increasing; this confirms, via a different imperfect repair model, that CM makes reliability worse than it was just before the repair, at least for plane 7908.

#### 5. A nuclear power generation application

The model and proposed estimation procedure are now illustrated using a history of maintenance decisions and failure-times for one system from the South Texas Project Nuclear Operating Company located in Bay City, Texas. The system of interest is the essential chillers, a group of six 300-ton air conditioners, three for each nuclear reactor unit. These chillers provide cold water for the cooling coils of various air handling units to provide a suitable environment for personnel and equipment during normal, faulted and

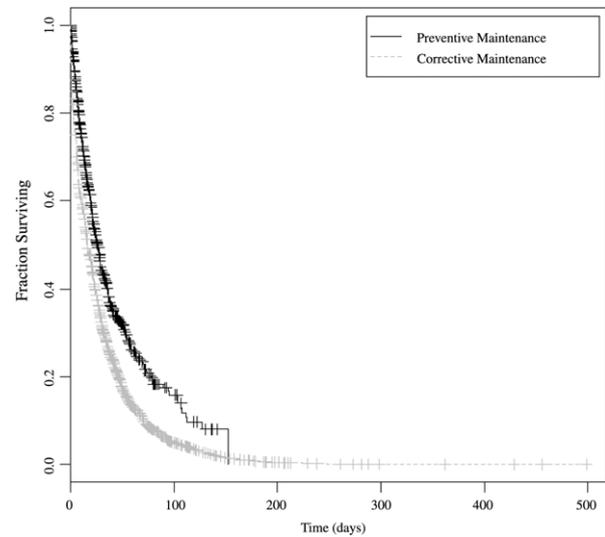


Fig. 2. Survival curves.

upset conditions. All three chillers in a reactor are automatically started if either a safety injection signal or loss of offsite power from the switchyard is detected. The dataset consists of daily times of maintenance interventions and type of maintenance performed; in addition, for each maintenance time, there is a record that indicates whether it corresponds to a failure time. We have a total of 2572 observations that combines the records of all six chillers from 1987 until 2009.

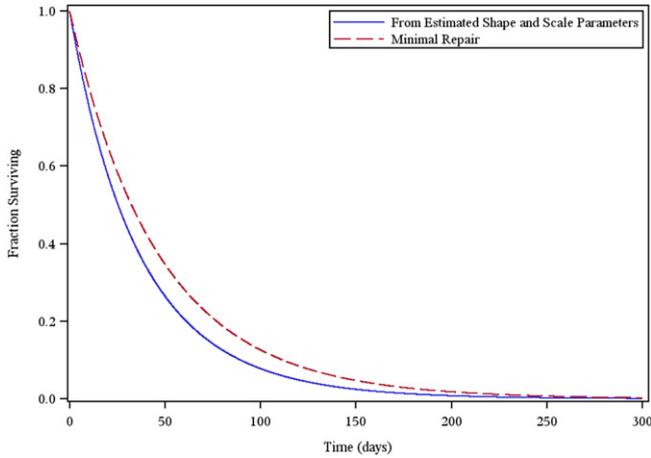
Fits to the dataset of a Weibull renewal process, a Weibull non-homogeneous Poisson process, and a Weibull Kijima Type II models yield AIC values of 11 467.4, 11 535.9, and 11 469.4; the proposed model has 11 518.9. The renewal process and the Kijima model are predictively better for these data and have essentially the same maximized log-likelihoods. The estimated  $q$  parameter is 0.0001 in the Kijima model, confirming the renewal process. The rather striking differences in AIC among the models stems from the many inter-failure times being quite small following both PM and CM—241 are less than an hour and 90 are about a minute. These are likely due to maintenance records that record every instance of a unit being switched on and off as an inter-failure time, even when this happens several times during the course of one overall repair. If we remove all inter-failure times of less than one hour, the AIC values are 11 337.0, 11 354.4, 11 338.7, and 11 346.4, respectively for the four models under consideration. The renewal process wins again, but this time the AIC values are more similar. A renewal process assumes all inter-failure times are identically and independently distributed from a Weibull distribution and is commonly referred to as *good-as-new*. For these data, the inter-failure hazard estimate from a renewal fit decreases sharply as time increases. Both PM and CM bring the system time back to zero when it is most likely to fail. This confirms that CM brings the system into a state much worse than it was right before failure. For these data, every repair allows the system to run, but restores it to its most vulnerable state. System reliability improves the longer it runs without any repairs.

We now discuss the implications to system reliability via an analysis of the survival (or reliability) curves stemming from the model proposed in this paper. The empirical reliability curves for PM and CM, accounting for right censoring and left truncation (for CM), are shown in Fig. 2. The results from applying our estimation procedure to the data are reported in Table 1.

The LRT developed in Section 2.2 rejects the null hypothesis of minimal repair (chi-squared statistic = 20.98,  $df = 2$ ,  $p$ -value < 0.0001), implying that the maintenance decision does

**Table 1**  
Estimation results from history of water-chiller unit at South Texas Power.

Weibull parameter	ML estimate	95% confidence interval	
		Lower endpoint	Upper endpoint
Shape PM $\alpha$	0.9193	0.8626	0.9785
Shape CM $\alpha + \beta_1$	0.8513	0.7764	0.9335
Scale PM $\lambda$	0.0324	0.0261	0.0401
Scale CM $\lambda + \beta_2$	0.0584	0.0394	0.0866



**Fig. 3.** Survival curves for corrective maintenance.

affect the parameters of the failure-time distribution of the chilling system at the power plant. One can further ascertain if the shape, scale or both parameters are different for a PM as opposed to a CM failure-time distribution. The null hypothesis that the shape parameter is the same for the PM and CM failure-time distributions is not rejected by a Wald test ( $t$  statistic =  $-1.35$ ,  $df = 2572$ ,  $p$ -value =  $0.1765$ ). However, the Wald test rejects that the scale parameter is the same for the PM and CM failure-time distributions ( $t$  statistic =  $2.58$ ,  $df = 2572$ ,  $p$ -value =  $0.0099$ ).

It was noted in Section 2 that the survival or reliability function following a CM is

$$\tilde{S}(t_i - t_i^* | \theta + \beta) = \exp\{-\lambda + \beta_2\} [(t_i - t_i^*)^{\alpha + \beta_1} - (t_{i-1} - t_i^*)^{\alpha + \beta_1}].$$

A graph of this function with the parameters replaced by their maximum likelihood estimates (i.e.,  $\alpha = 0.9193$ ,  $\lambda = 0.0324$ ,  $\beta_1 = 0.0680$ ,  $\beta_2 = 0.0260$ ) is presented in Fig. 3. For comparison, also presented in Fig. 3 is the reliability function assuming minimal repair:

$$\tilde{S}(t_i - t_i^* | \theta) = \exp\{-\lambda[(t_i - t_i^*)^\alpha - (t_{i-1} - t_i^*)^\alpha]\},$$

where, again, the parameters are replaced by their maximum likelihood estimates ( $\alpha = 0.9193$ ,  $\lambda = 0.0324$ ).

Both reliability functions in Fig. 3 are for a CM performed when the system age equals the mean of the PM distribution  $\Gamma(1 + 1/\alpha)/\lambda^{1/\alpha}$ , where  $\Gamma(b) = \int_0^\infty r^{b-1} \exp(-r) dr$  is the gamma function. It is seen from Fig. 3 that, on any given day, the estimated reliability probability is at most that of a minimal repair; in other words, following a CM, the estimated reliability function of the system is less favorable than that of a system of the same age that has not undergone a CM.

For these water chiller data, now consider the impact of assuming minimal repair. Under this assumption, we can compute the Wald 95% confidence interval for the Weibull shape parameter, which is  $[0.9105, 0.9902]$ , and which includes only cases where the hazard function is monotonically decreasing. Thus, the plant manager would conclude that a CM would always restore the

system to ‘better-than-new’, and hence would always choose CM unless its cost exceeds that of PM by some threshold value. Moreover, future data might seem to reinforce the decision to always perform CM because, from Table 1, the CM shape parameter appears to be significantly less than one: its 95% confidence interval is  $[0.7764, 0.9335]$ ; this once again implies a decreasing hazard rate and preference for CM. Thus, the decision to always perform CM would likely not improve even if the parameter estimates are updated. This is analogous to the ‘spiral-down’ effect noted by Cooper, de Mello and Kleywegt [10], wherein decisions could become progressively worse after updating parameter estimates of the underlying probability model.

### 6. Evaluation of estimation procedure and LRT

Since the nuclear power plant in Texas intends to implement our model in real time, it is imperative to evaluate the accuracy of the estimation procedure (Section 3) and the LRT (Section 2).

#### 6.1. Simulating a Weibull-based decision dependent stochastic process

Using the model detailed in Section 2, and random number generation, we first construct a simulated history of maintenance and failure times  $t_0, t_1, \dots, t_n$  and decisions  $d_0, d_1, \dots, d_n$ ; these are used to assess whether the estimation procedure described in Section 3 ‘recovers’ the Weibull shape and scale values used in simulating the data.

To simulate the history of maintenance times, we first specify the values of the Weibull shape and scale parameters that govern the distribution of failure times following each CM and PM decision. Then, an inversion-transform method is used to sample from each of the corresponding failure-time distributions as follows.

Given a PM at time  $t_{i-1}$  and the Weibull assumption, the length of time until the next failure  $t_i - t_{i-1}$  may be generated from the inverse Weibull cdf evaluated at a uniform  $(0, 1)$  random number  $u$ , i.e.,  $t_i - t_{i-1} = [-\ln(1 - u)/\lambda]^{1/\alpha}$ . If instead a CM is performed at time  $t_{i-1}$ , then from the assumptions in Section 2, the length of time  $t_i - t_i^*$  (given  $t_i > t_{i-1}$ ) has cdf

$$\tilde{F}(t_i - t_i^* | \theta + \beta) = 1 - \exp\{-\lambda + \beta_2\} [(t_i - t_i^*)^{\alpha + \beta_1} - (t_{i-1} - t_i^*)^{\alpha + \beta_1}],$$

which has inverse

$$\tilde{F}^{-1}(u | \theta + \beta) = \inf\{t_i - t_{i-1} : \tilde{F}(t_i - t_i^* | \theta + \beta) \geq u\} = [(t_{i-1} - t_i^*)^{\alpha + \beta_1} - \ln(1 - u)/(\lambda + \beta_2)]^{1/(\alpha + \beta_1)}.$$

Hence, the failure time  $t_i$  may be generated as

$$t_i = t_i^* + [(t_{i-1} - t_i^*)^{\alpha + \beta_1} - \ln(1 - u)/(\lambda + \beta_2)]^{1/(\alpha + \beta_1)} \quad i = 1, 2, 3, \dots$$

Right censoring of failure times is simulated as follows. For each failure time  $t_i$ , an independent and identically distributed time  $c_i$ , called the censoring time, is simulated. If  $c_i < t_i$ , then the maintenance is performed at time  $c_i$ , and the failure time  $t_i$  is right censored at that time; otherwise, if  $c_i \geq t_i$  the maintenance is performed at time  $t_i$  and there is no censoring. At each maintenance time, a fair coin is tossed to determine whether PM or CM is performed.

The values of the shape and scale parameters considered in the simulations are given in Table 2; these ranges were selected based on input from the nuclear plant engineers. The shape parameter values for PM are chosen to give decreasing ( $\alpha = 0.5$ ), constant ( $\alpha = 1$ ), and increasing ( $\alpha = 3$ ) hazard functions. The value

**Table 2**  
Estimates of power for test of minimal repair.

Case	Shape parameter		Scale parameter		Mean $\mu$		Power ( $n = 200$ )	Power ( $n = 2000$ )
	PM $\alpha$	CM $\alpha + \beta_1$	PM $\lambda$	CM $\lambda + \beta_2$	PM	CM		
1	0.5	0.5	0.44721	0.44721	10	10	0.05	0.05
2	0.5	0.53510	0.44721	0.44721	10	8	0.07	0.38
3	0.5	0.47522	0.44721	0.44721	10	12	0.07	0.19
4	0.5	0.5	0.44721	0.5	10	8	0.06	0.25
5	0.5	0.5	0.44721	0.40825	10	12	0.07	0.18
6	1	1	0.1	0.1	10	10	0.06	0.04
7	1	1.09016	0.1	0.1	10	8	0.17	0.94
8	1	0.93775	0.1	0.1	10	12	0.12	0.68
9	1	1	0.1	0.125	10	8	0.12	0.82
10	1	1	0.1	0.0833	10	12	0.11	0.58
11	3	3	0.0007121	0.0007121	10	10	0.06	0.05
12	3	3.31241	0.0007121	0.0007121	10	8	0.90	1.00
13	3	2.78623	0.0007121	0.0007121	10	12	0.54	1.00
14	3	3	0.0007121	0.001391	10	8	0.746	1.00
15	3	3	0.0007121	0.0004121	10	12	0.502	1.00

of the scale parameter  $\lambda$  for PM is chosen such that the mean time until failure is  $\mu = 10$ . Now, given  $\mu$  and  $\alpha$ , the scale parameter is found from  $\lambda = [\Gamma(1 + 1/\alpha)/\mu]^\alpha$  because the mean of the Weibull distribution is given by  $\mu = \Gamma(1 + 1/\alpha)/\lambda^{1/\alpha}$ . For each pair of PM shape and scale values, five pairs of CM shape and scale values are considered. The first CM pair corresponds to minimal repair, i.e., the pair is the same as that for PM. The remaining four CM pairs have the shape or scale value shifted such that the mean time until failure is 20% greater (or less) than the PM mean of 10.

As in the case of PM, the scale parameter for CM,  $\lambda + \beta_2$ , that gives the desired mean value may be found directly from the formula for the mean of the Weibull distribution. The shape parameter  $\alpha + \beta_1$ , however, is found numerically. From the formula for the mean of the Weibull distribution, it follows that  $(\alpha + \beta_1)\{\ln[\Gamma(1 + 1/(\alpha + \beta_1))] - \ln \mu\} - \ln(\lambda + \beta_2) = 0$ , from which the shape parameter  $\alpha + \beta_1$  is found via a binary search.

Note that the above approach to choosing the shape and scale values results in a 20% difference between CM and PM occurring for maintenance performed when the system age is zero. However, it would seem that maintenance would typically *not* be performed when the system age is zero. It is perhaps ideal if the shape and scale values are instead chosen such that the 20% difference occurs at the *mean* system age at maintenance. Implementing this is both difficult and infeasible, because this mean is not known analytically and, in fact, may depend on the CM shape and scale parameters. To circumvent this difficulty, since real data are available at the nuclear power plant, a pragmatic approach that the nuclear power plant engineers are experimenting with is to assume a value for the system age  $t_{i-1} - t_i^*$  at the time of maintenance; for example, the mean of the distribution stemming from a PM, as in Section 4. Nonetheless, using any system age greater than zero is complicated because the mean time until failure following a CM on a system of age  $t_{i-1} - t_i^*$  is

$$\int_{t_{i-1}-t_i^*}^{\infty} \tilde{S}(r|\theta + \beta) dr = \exp[(\lambda + \beta_2)(t_{i-1} - t_i^*)^{\alpha+\beta_1}] \times \int_{t_{i-1}-t_i^*}^{\infty} \exp[-(\lambda + \beta_2)r^{\alpha+\beta_1}] dr,$$

which does not have a closed-form expression. For the sake of illustration, in this simulation study a system age of zero is used and the CM shape and scale values are found as described above.

For each combination of parameter values given in Table 2, 10,000 datasets of  $n = 200$  maintenance times, and 200 datasets of  $n = 2000$  each are simulated. Then, the proposed estimation procedure is applied to each of these datasets. For each parameter, the performance of the estimation procedure is measured by

the estimated coverage probability—the fraction of datasets for which the 95% Wald confidence interval includes the value of the parameter. The coverage calculations are performed using a SAS macro DDSMSE (Decision Dependent Survival Model Simulated Estimation) available on request.

A key takeaway from the simulation results for the 15 cases in Table 2 is this: the estimated coverage probabilities of each Wald confidence interval for the shape and scale parameters of the different Weibull distributions range from 94% to 96%, and are all approximately equal to the nominal 95% value, as desired. That is, the estimation procedure described in Section 3 is robust.

## 6.2. Evaluating the power of the LRT used in testing hypotheses 1 and 2

The simulated datasets described above are now used to study the power of the LRT with respect to the null hypothesis of minimal repair defined previously. Recall that the null hypothesis is that the shape and scale parameters are invariant to the maintenance decision ( $H_0 : \beta = \mathbf{0}$ ). The power of the LRT is estimated for all 15 cases of parameter values considered in Section 6.1, assuming a significance level of 5%.

The estimated power is reported in the last column of Table 2. Note that for Cases 1, 6, and 11 in Table 2, the data were simulated under the null hypothesis of minimal repair; hence, the estimated power is an estimate of the LRT's significance level. In each of these three cases, the estimate of the significance level approximately equals the nominal value of 5%, as desired.

For the remaining twelve cases in Table 2, the alternative hypothesis of *decision dependent parameters* holds; hence, a power of one is desired. The estimated power in the presence of *decision dependence* ranges from 0.06 to 0.90 for  $n = 200$  and from 0.18 to 1.00 for  $n = 2000$ . Moreover, the estimated power seems to be strongly affected by the PM shape parameter  $\alpha$ . For  $\alpha = 0.5$  (decreasing hazard), the estimated power is only slightly greater than the estimate of the significance level (0.05) for  $n = 200$ , whereas for  $\alpha = 3$  and  $n = 200$  the estimated power ranges from 0.50 to 0.90. These numbers improve substantially by increasing the sample size tenfold to  $n = 2000$ —in fact the power increases to one when  $\alpha = 3$  and  $n = 2000$ . Finally, the estimated power for a 20% decrease in the mean time until failure is usually larger than that for a 20% increase.

Acknowledging that no statistical test is perfect, nonetheless, as sample sizes increase, *ceteris paribus*, the proposed methodology performs as it should. It is important to note that in nuclear power plant maintenance, collecting and updating data are given considerable attention since reliability cannot be overvalued; hence, engineers and managers who implement statistical and



```

par=maxk$par # MLE's from full model
se=sqrt(diag(solve(maxk$hessian))) # standard errors
#####
# renewal process
#####
rp=survreg(Surv(t,c)~1,dist="weibull")
#####
# AIC: RP, NHPP, KM, ours respectively
#####
cat("Weibull renewal process AIC='',2*2-2*rp$loglik[1],
"/n")
cat("Weibull NHPP AIC='',2*2+2*maxr$value,"/n")
cat("Weibull Kijima Type II AIC='',2*3+2*maxk$value,"/n")
cat("Weibull ZHDP AIC='',2*4+2*maxf$value,"/n")

```

## References

- [1] Finkelstein M. Failure rate modelling for reliability and risk. London: Springer-Verlag; 2008.
- [2] Lindqvist BH. On the statistical modeling and analysis of repairable systems. *Statist Sci* 2006;21:532–51.
- [3] Rausand M, Høyland A. System reliability theory: models, statistical methods, and applications. 2nd ed. Hoboken, New Jersey: John Wiley & Sons; 2004.
- [4] Balakrishnan N, Kamps U, Kateri M. Minimal repair under a step-stress test. *Statist Probab Lett* 2009;79:1548–58.
- [5] Brown M, Proschan F. Imperfect repair. *J Appl Probab* 1983;20:851–9.
- [6] Block HW, Borges WS, Savits TH. Age-dependent minimal repair. *J Appl Probab* 1985;22:370–85.
- [7] Whitaker LR, Samaniego FJ. Estimating the reliability of systems subject to imperfect repair. *J Amer Statist Assoc* 1989;84:301–9.
- [8] Doyen L. Semi-parametric estimation of Brown–Proschan preventive maintenance effects and intrinsic wear-out. *Comput Statist Data Anal* 2014;77:206–22.
- [9] Presnell B, Hollander M, Sethuraman J. Testing the minimal repair assumption in an imperfect repair model. *J Amer Statist Assoc* 1994;89:289–97.
- [10] Cooper WL, de Mello TH, Kleywegt AJ. Models for the spiral-down effect in revenue management. *Oper Res* 2006;54:968–97.
- [11] Kijima M. Some results for repairable systems with general repair. *J Appl Probab* 1989;26:89–102.
- [12] Mettas A., Zhao W.. Modeling and analysis of repairable systems with general repair. In: 2005 Proceedings annual reliability and maintainability symposium, Alexandria, Virginia. 2005.
- [13] Dorado C, Hollander M, Sethuraman J. Nonparametric estimation for a general repair model. *Ann Statist* 1997;25:1140–60.
- [14] Veber B, Nagode M, Fajdiga M. Generalized renewal process for repairable systems based on finite Weibull mixture. *Reliab Eng Syst Saf* 2008;93:1461–72.
- [15] Pan R, Rigdon S. Bayes inference for general repairable systems. *J Qual Technol* 2009;41(1):82–94.
- [16] Gasmi S. Parameter estimation in an alternating repair model. *J Statist Plann Inference* 2011;141:3605–16.
- [17] Li L, Hanson T. A Bayesian semiparametric regression model for reliability data using effective age. *Comput Statist Data Anal* 2014;73:177–88.
- [18] Tanwar M, Rai RN, Bolia N. Imperfect repair modeling using Kijima type generalized renewal process. *Reliab Eng Syst Saf* 2014;124:24–31.
- [19] Doyen L, Gaudoin O. Classes of imperfect repair models based on a reduction of failure intensity or virtual age. *Reliab Eng Syst Saf* 2004;84:45–56.
- [20] Dimitrov B, Chukova S, Khalil Z. Warrantee costs: An age-dependent failure/repair model. *Naval Res Logist* 2004;51:959–76.
- [21] Garg H, Rani M, Sharma SP. Preventive maintenance scheduling of the pulping unit in a paper mill. *Japan J Indust Appl Math* 2013;30:397–414.
- [22] Garg H, Sharma SP, Rani M. Weibull fuzzy probability distribution for analyzing the behavior of pulping unit in a paper industry. *Int J Ind Syst Eng* 2013;14:395–413.
- [23] Rinne H. The Weibull distribution: a handbook. Florida: Chapman & Hall/ CRC; 2008.
- [24] Proschan F. Theoretical explanation of observed decreasing failure rate. *Technometrics* 1963;5:375–83.
- [25] Yañez M, Joglar F, Modarres M. Generalized renewal process for analysis of repairable systems with limited failure experience. *Reliab Eng Syst Saf* 2002;77:167–80.