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## Article

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# A fast and simple branching algorithm for solving small scale fixed-charge transportation problem



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## ABSTRACT

In this paper, we develop a simple algorithm for obtaining the global solution to a small scale fixed-charge transportation problem (FCTP). The procedure itself is very quick. The proposed method solves FCTP by decomposing the problem into series of smaller sub-problems, which is novel and can be useful to researchers solving any size of the problem.

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## 1. Introduction

In a transportation problem (TP), products need to be transported from a number of sources to a number of destinations. Decisions need to be made on the amount of products transported between each two locations to minimize total transportation cost. Typically, only a variable cost proportional to the number of products transported is incurred. However, in many real-world problems, a fixed/setup cost is also incurred when the transportation amount is positive. This problem is referred to as fixed charge transportation problem (FCTP) [1,2]. The fixed cost has made the problem difficult to solve. Various solution methods have been proposed in the optimization literature, and different claims have been made on their computational success [3–10]. Among these methods, branch-and-bound methods are one of the most effective ones and have had many applications [11–18]. For example, Adlakha, Kowalski and Lev [19] developed an effective branching method for FCTP. Their method starts with a linear formulation, and then sequentially separates the fixed cost and finds a direction to improve the value. The algorithm converges to an optimal solution as the lower and upper bounds are continually tightened. In this paper, we present improvements to that method applicable to small problems, which can accelerate the solution. The proposed method solves FCTP by decomposing the problem into series

of smaller sub-problems, which is novel and can be useful to researchers solving any size of the problem.

We present FCTP formulation in Section 2 and reiterate the linear approximation developed by Balinski [20]. We show computational examples for non-degenerate problems in Section 3, and for degenerate problems in Section 4. We conclude the paper in Section 5.

## 2. Fixed charge transportation problem

Assume that there are  $m$  ( $i = 1, 2, \dots, m$ ) suppliers and  $n$  ( $j = 1, 2, \dots, n$ ) customers in a transportation problem. Supply at each supplier  $i$  is  $a_i$  units, and demand at each customer  $j$  is  $b_j$  units. Let  $x_{ij}$  be the number of units shipped by supplier  $i$  to customer  $j$ . The cost of shipping  $x_{ij}$  units is composed of two parts—a variable cost per unit  $c_{ij}$ , and a fixed cost  $f_{ij}$  for opening this route. The objective is to minimize the total cost of meeting all demands, given all supply constraints. The fixed charge transportation problem (FCTP) is formulated as follows:

$$\text{P: Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n (c_{ij}x_{ij} + f_{ij}y_{ij}) \quad (1)$$

$$\text{Subject to } \sum_{j=1}^n x_{ij} = a_i, \quad i = 1, 2, \dots, m, \quad (2)$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad j = 1, 2, \dots, n, \quad (3)$$

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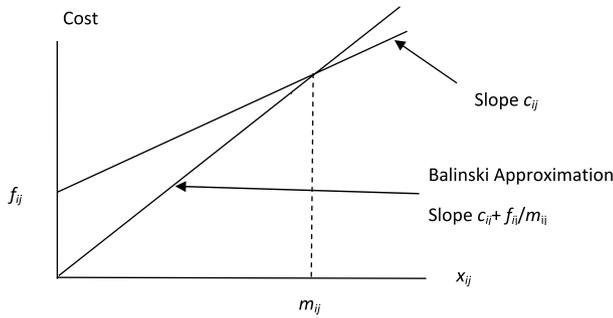


Fig. 1. Balinski linear approximation.

$$x_{ij} \leq m_{ij}y_{ij}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \quad (4)$$

$$\text{where } m_{ij} = \min(a_i, b_j), \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \quad (5)$$

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j, \quad (6)$$

$$y_{ij} = \begin{cases} 1, & \text{if } x_{ij} > 0, \\ 0, & \text{otherwise} \end{cases} \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n \quad (7)$$

$$a_i, b_j, c_{ij}, f_{ij} \geq 0; \quad x_{ij} \geq 0, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n. \quad (8)$$

Balinski [20] provided a linear approximation of FCTP. He relaxed the integer constraint on  $y_{ij}$ , with the property that

$$y_{ij} = x_{ij}/m_{ij} \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, n. \quad (9)$$

Under this relaxation, an FCTP problem is transformed into a classical TP without fixed charge, where the unit transportation costs  $C_{ij} = c_{ij} + f_{ij}/m_{ij}$ . We refer to this problem as **PB**. Let the optimal solution to problem **PB** be  $\{x_{ij}^B\}$ , then we can easily obtain a feasible solution of  $\{x_{ij}^B, y_{ij}^B\}$  of **P** by setting  $y_{ij}^B = 1$  if  $x_{ij}^B$  is positive and  $y_{ij}^B = 0$  otherwise. Balinski shows that the optimal value,  $Z(\mathbf{PB})$ , provides a lower bound on the optimal value  $Z^*(\mathbf{P})$  of FCTP, i.e.,  $Z(\mathbf{PB}) = C_{ij}x_{ij}^B \leq Z^*(\mathbf{P})$ . Geometrically, the Balinski linear approximation can be represented as in Fig. 1.

Adlakha et al. [19] presented a branching algorithm for solving the FCTP problem. Building upon this algorithm, we now propose a simple and fast branching algorithm for small problems. Several criteria were provided by Kowalski [21] for determining if a problem is degenerate. Using these criteria, our proposed model takes different procedures for non-degenerate and degenerate problems, which will be demonstrated separately in the following sections.

### 3. Computational experiment for a problem with non-degenerate solution

First, we present the algorithm when a problem is non-degenerate. Using the criteria from Kowalski [21] we can see that the example in Table 1 can have only non-degenerate distributions. For such problems, our proposed algorithm uses the branching algorithm presented by Adlakha et al. [19], but branches the most “heavy loaded” locations (those with the highest fixed costs) to achieve immediate convergence. The first step is to subtract the smallest fixed charge from every fixed charge. Then we select one of the most heavily loaded locations to branch. The first branch assumes that no shipping will be made at the selected location and eliminates the fixed costs in that given row or column. The second branch assumes no fixed cost at that location. These steps are demonstrated below.

For Example 1, the first step is to reduce the fixed charge by the smallest fixed charge, which is 10 in this case. Details of this operation can be found in Kowalski and Lev [22]. The corresponding coefficients of the problem are given in Table 2.

Table 1  
Cost matrix  $(f_{ij}, c_{ij})$  for Example 1.

	$b_1$	$b_2$	$b_3$	Supply
$a_1$	(20, 1)	(10, 2)	(10, 3)	15
$a_2$	(20, 1)	(20, 2)	(20, 1)	30
$a_3$	(10, 3)	(20, 1)	(10, 2)	15
Demand	20	20	20	

Table 2  
Cost matrix  $(f_{ij}, c_{ij})$  for reduced fixed charge.

	$b_1$	$b_2$	$b_3$	Supply
$a_1$	(10, 1)	(0, 2)	(0, 3)	15
$a_2$	(10, 1)	(10, 2)	(10, 1)	30
$a_3$	(0, 3)	(10, 1)	(0, 2)	15
Demand	20	20	20	

Table 3  
Cost matrix  $(f_{ij}, c_{ij})$  of modified FCTP.

	$b_1$	$b_2$	$b_3$	Supply
$a_1$	(0, 1)	(0, 2)	(0, 3)	15
$a_2$	X	(0, 2)	(0, 1)	30
$a_3$	(0, 3)	(10, 1)	(0, 2)	15
Demand	20	20	20	

Table 4  
Corresponding Balinski matrix.

	$b_1$	$b_2$	$b_3$
$a_1$	1	2	3
$a_2$	X	2	1
$a_3$	3	1.66	2

The next step is to find a location to branch. The criterion for branching is to find a position that can eliminate as many fixed charges as possible. For Example 1 the best will be to remove a location in the second row. We can start with elimination of location (2, 1). Then using the principle from Adlakha and Kowalski [23] Tables 3 and 4 are obtained.

The optimal solution to this problem is:

	$b_1$	$b_2$	$b_3$
$a_1$	15	0	0
$a_2$	0	10	20
$a_3$	5	10	0

Let  $Z(\mathbf{MPB})$  denote the optimal value of the Balinski version of the modified FCTP problem (with extracted fixed charges), and let  $Z(\mathbf{MP})$  denote the matching value of the modified FCTP problem with extracted fixed charges. Lev & Kowalski [24] introduced the principle of a minimum interval  $\Delta$ , which they used for modeling fixed charge problems with polynomials. The minimum interval is the largest common factor of supply and demand which is 5 for Example 1. The same value is also the minimum interval for the objective function. That means that the objective function can only take values with intervals of 5. Therefore, for a given feasible solution, if the difference between a lower bound and an upper bound is smaller than the minimum interval, then this solution is the optimal one. In the above example  $Z(\mathbf{MPB}) = 86.66$  and  $Z(\mathbf{MP}) = 90$ . Since the results are within the minimum interval 5 this is the optimal solution. The corresponding  $Z_{\text{FCTP}} = 20 + 20 + 20 + 10 + 20 + 20 + 15 + 20 + 20 + 15 + 10 = 170$ .

Next we branch the problem with load allocation at (2, 1) by removing the fixed charge at this location (see Tables 5 and 6).

**Table 5**  
Cost matrix  $(f_{ij}, c_{ij})$  for reduced fixed charge.

	$b_1$	$b_2$	$b_3$	Supply
$a_1$	(10, 1)	(0, 2)	(0, 3)	15
$a_2$	(0, 1)	(10, 2)	(10, 1)	30
$a_3$	(0, 3)	(10, 1)	(0, 2)	15
Demand	20	20	20	

**Table 6**  
Corresponding Balinski matrix.

	$b_1$	$b_2$	$b_3$
$a_1$	1.66	2	3
$a_2$	1	2.5	1.5
$a_3$	3	1.66	2

The optimal solution to this problem is:

	$b_1$	$b_2$	$b_3$
$a_1$	10	5	0
$a_2$	10	0	20
$a_3$	0	15	0

The Balinski optimal cost  $Z(\mathbf{MPB}) = 91.66$  and the corresponding  $Z(\mathbf{MP}) = 95$ . Again since the results are within the minimum interval 5 this is the optimal solution. The corresponding  $Z_{\text{FCPT}} = 20 + 10 + 20 + 20 + 20 + 10 + 10 + 10 + 20 + 15 = 155$ , which is the global solution. In this example, we obtained the solution by solving two  $3 \times 3$  regular transportation problems. Clearly branching on the most “heavy loaded” locations ensures a fast convergence for this problem.

**4. Computational experiments for a problem with possible degenerate solution**

The next example is a degenerate one. After altering some of the supplies and demands in Example 1 we create Example 2 shown in Table 7, which can be shown to have degenerate distributions using the criteria from Kowalski [21]. Following we check if one of the values  $a_i$  or  $b_j$  is equal to the minimum interval  $\Delta$  (which is also 5). We identify  $a_1 = 5$ . In such a case, the following analysis can be performed. Since we know that  $a_1$  is equal to the minimum interval then there can be only one  $x_{1,j}$  loading in row 1. This allows us to do one of two operations. The first operation would be reducing each fixed charge in row 1 by the minimum fixed charge (equal to 10) in that row. The second operation could be assigning one of the locations in row 1 to ship  $a_1$ , then the original  $3 \times 3$  FCTP can be broken into three  $2 \times 3$  sub problems, which as will be shown may be further broken into even smaller sub problems. In the following analysis we will focus only on the second principle/operation.

Based on the principle above the optimization of the above problem is equivalent to an optimization of three smaller sub problems. In Sub problem 1, location (1, 1) is assigned to ship 5 and thus the first row can be removed from further consideration. Similarly, in Sub problem 2 location (1, 2) is removed and in Sub problem 3 location (1, 3) is removed. In each case this results in the removal of the first row and in an adjustment of the corresponding demand value.

*Sub problem 1*

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	(20, 1)	(20, 2)	(20, 1)	30
$a_3$	(10, 3)	(20, 1)	(10, 2)	15
Demand	5	20	20	

**Table 7**  
Cost matrix  $(f_{ij}, c_{ij})$  for Example 2.

	$b_1$	$b_2$	$b_3$	Supply
$a_1$	(20, 1)	(10, 2)	(10, 3)	5
$a_2$	(20, 1)	(20, 2)	(20, 1)	30
$a_3$	(10, 3)	(20, 1)	(10, 2)	15
Demand	10	20	20	

*Sub problem 2*

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	(20, 1)	(20, 2)	(20, 1)	30
$a_3$	(10, 3)	(20, 1)	(10, 2)	15
Demand	10	15	20	

*Sub problem 3*

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	(20, 1)	(20, 2)	(20, 1)	30
$a_3$	(10, 3)	(20, 1)	(10, 2)	15
Demand	10	20	15	

Since one of the adjusted demands is now equal to the minimum interval, the breaking can be continued further. For example Sub problem 1 can be broken into two smaller sub problems:

*Sub problem 1A*

	$b_2$	$b_3$	Supply
$a_2$	(20, 2)	(20, 1)	25
$a_3$	(20, 1)	(10, 2)	15
Demand	20	20	

*Sub problem 1B*

	$b_2$	$b_3$	Supply
$a_2$	(20, 2)	(20, 1)	30
$a_3$	(20, 1)	(10, 2)	10
Demand	20	20	

And Sub problem 2 can be broken into 3 sub problems with supply  $b_1$  broken into three possible combinations  $x_{11} = 5$  and  $x_{21} = 5, x_{12} = 10$  and  $x_{21} = 10$ .

*Sub problem 2A*

	$b_2$	$b_3$	Supply
$a_2$	(20, 2)	(20, 1)	25
$a_3$	(20, 1)	(10, 2)	10
Demand	15	20	

*Sub problem 2B*

	$b_2$	$b_3$	Supply
$a_2$	(20, 2)	(20, 1)	20
$a_3$	(20, 1)	(10, 2)	15
Demand	15	20	

Sub problem 2C

	$b_2$	$b_3$	Supply
$a_2$	(20, 2)	(20, 1)	30
$a_3$	(20, 1)	(10, 2)	5
Demand	15	20	

We will skip Sub problem 3 as its breaking is similar to that of Sub problem 2. Following this series it is possible to break the problem even further. This way it is possible to optimize the FCTP only using a simple Excel sheet and comparing the results. For illustration purposes we stop the breaking process here and continue with the example by solving Sub problems 1–3 the same way as Example 1. Using the principle from Adlakha & Kowalski [23] the following series of reduced matrices are obtained.

Sub problem 1 with reduced fixed charges

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	(20, 1)	(0, 2)	(0, 1)	30
$a_3$	(10, 3)	(20, 1)	(10, 2)	15
Demand	5	20	20	

Sub problem 2 with reduced fixed charges

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	(20, 1)	(20, 2)	(0, 1)	30
$a_3$	(10, 3)	(20, 1)	(10, 2)	15
Demand	10	15	20	

Sub problem 3 with reduced fixed charges

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	(20, 1)	(0, 2)	(20, 1)	30
$a_3$	(10, 3)	(20, 1)	(10, 2)	15
Demand	10	20	15	

Balinski coefficients for Sub problem 1 with reduced fixed charges

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	5	2	1	30
$a_3$	5	2.33	2.66	15
Demand	5	20	20	

Balinski coefficients for Sub problem 2 with reduced fixed charges

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	3	3.33	1	30
$a_3$	4	2.33	2.66	15
Demand	10	15	20	

Balinski coefficients for Sub problem 3 with reduced fixed charges

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	3	2	2.33	30
$a_3$	4	2.33	2.66	15
Demand	10	20	15	

Sub problem 1, 2, 3A and 3B optimal distributions

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	-	10	20	30
$a_3$	5	10	-	15
Demand	5	20	20	

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	10	-	20	30
$a_3$	-	15	-	15
Demand	10	15	20	

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	10	20	-	30
$a_3$	-	-	15	15
Demand	10	20	15	

	$b_1$	$b_2$	$b_3$	Supply
$a_2$	10	5	15	30
$a_3$	-	15	-	15
Demand	10	20	15	

The values for **MPB**, **MP** and the original problems are:

Sub problem 1:  $Z(\mathbf{MPB}) = 88.33, Z(\mathbf{MP}) = 95, Z = 160$ ;

Sub problem 2:  $Z(\mathbf{MPB}) = 85, Z(\mathbf{MP}) = 85, Z = 125$ ;

Sub problem 3A:  $Z(\mathbf{MPB}) = 110, Z(\mathbf{MP}) = 110, Z = 155$ ;

Sub problem 3B:  $Z(\mathbf{MPB}) = 110, Z(\mathbf{MP}) = 110, Z = 155$ .

Note that in Sub problems 2, 3A and 3B,  $Z(\mathbf{MPB}) = Z(\mathbf{MP})$  so the solutions found are optimal for those sub problems. However, the difference between  $Z(\mathbf{MP})$  and  $Z(\mathbf{MPB})$  in Sub problem 1 is equal to 6.66, which is greater than the minimum interval equal to 5, which means that there is a possibility to lower the value of the FCTP solution in this branch of sub problems. While it would be possible to run another iteration to merge the lower bound with the matching FCTP distribution, in our case it is not necessary, due to the principle of truncating parallel branches shown by Adlakha, Kowalski and Lev [19]. After adjustment of the lower bound by adding the “extracted” fixed and variable values, it is equal to 153.33 with matching FCTP value of 160. Since the lower bound for the Sub problem 1 is greater than the  $Z = 125$  obtained for Sub problem 2, this branch can be truncated and discounted from further analysis. Therefore the optimal distribution is given by Sub problem 2, which was obtained by solving four regular  $2 \times 3$  transportation problems.

5. Conclusions

In this paper, we propose a simple and fast branching method for solving small scale fixed-charge transportation problem. The algorithm has two novelties. First, branching can start from the “heavy loaded” locations therefore convergence may be accelerated. In our example, more than one fixed charge can be extracted at the same time, which allows for solving a  $3 \times 3$  problem in one iteration. If the first iteration does not converge to an optimal solution, another “heavy loaded” location can be selected and the resulting calculation is still faster than a regular branching method. Secondly, if one of the demands or supplies is equal to the minimum load, then we can subtract the minimum element from given row or column, or break the problem into a series of reduced problems. The sub problems can be further broken into smaller sub problems until an optimal solution is obtained.

In our example, the proposed method solves FCTP very efficiently. This method provides an alternative to solving small scale problems in a fast way without using computer software. Although it cannot be considered superior to the existing methods, our main goal is to present a simple method that yields the optimal solution and is easily understood even by students with weak mathematical backgrounds. Therefore, it can be used for education purposes as a pedagogical tool in a classroom environment.

One of our future research directions is to generalize the method to more general settings, such as FCTP with uncertainties, or with nonlinear cost. In addition, although the proposed method is used for small scale problems, the idea of decomposing the problem into series of smaller problems can be useful to researchers solving any size of the problem. It is our hope that this may shed some light on developing new algorithms for large scale problems in the future.

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