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Risk Contracts with Private Information and One-Sided Commitment

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TEXTO PARA DISCUSSÃO

No. 635

Risk Sharing Contracts with Private
Information and
One-Sided Commitment

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Risk Sharing Contracts with Private Information and One-Sided Commitment*

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Abstract

In a repeated unobserved endowment economy in which agents negotiate long-term contracts with a financial intermediary, we study the implications of the interaction between incentive compatibility and participation constraints for risk sharing. In particular, we assume that after a default episode, agents consume their endowment and remain in autarky forever (one-sided commitment). We find that, once away from autarky today, if the probability of drawing the highest possible endowment shock is small enough, the optimal contract prevents agents from reaching autarky tomorrow and, thus, from being “impoverished”. Moreover, an invariant cross-sectional distribution of life-time utilities (or values) exists. Our numerical simulations suggest that the mass of agents living in autarky is zero in the limit.

Keywords: risk sharing contracts; private information; one-sided commitment.

JEL Classification: D31, D82, D86.

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1 Introduction

This paper studies optimal risk sharing contracts in an endowment economy with private information and one-sided commitment. Each risk-averse agent is endowed with a sequence of perishable goods, distributed identically and independently over time and agents. A long-term contract can be signed with a risk-neutral financial intermediary (or principal). Private information imposes incentive compatibility constraints as in Thomas and Worrall [1990]. One-sided commitment introduces a set of ex-post participation constraints. As in Thomas and Worrall [1988] and Kocherlakota [1996], we assume that after repudiating the contract, agents consume their endowment and, then, remain in autarky forever. In contrast with those papers, we assume that the principal can credibly commit to the long-term contract.

Focusing on both private information and one-sided commitment allows this paper to encompass two important contributions of the existing literature. On the one hand, Thomas and Worrall [1990] study the implications of private information for risk sharing in an endowment economy assuming full commitment. On the other hand, Ljungqvist and Sargent [2012]’s version of Thomas and Worrall [1988] only consider one-sided commitment in the context of complete information. When private information is the sole contracting friction, the principal spreads continuation values in order to provide cheaper incentives. The so-called immiseration result, in which the continuation values become arbitrarily negative with probability one, arises. Under complete information, when commitment is instead one-sided, and in sharp contrast, continuation values increase over time reaching a finite limit in finite time, when full-insurance is achieved.¹

Our model also echoes some earlier work by Phelan [1995], who considers both incentive and participation constraints although with a different modeling of participation. In particular, ex-ante participation constraints simply impose a lower bound on the set of possible continuation values. Hence, they do not depend on current realizations of the endowment. In contrast, we consider ex-post participation constraints that allow

¹Thomas and Worrall [1988] informally mention this result, although they carry their analysis assuming that the principal can also renege on the contract.

agents to consume their endowment after a default episode.² Despite some similarities with previous contributions, the interaction between ex-post participation and incentive compatibility constraints, which is the focus of this paper, has novel implications.

Our main result (Proposition 1 in Section 4) states that, once away from the autarky state today (i.e. once agents were promised a value higher than the value of autarky), if the probability of drawing the highest possible endowment shock is small enough, the optimal contract prevents agents from reaching the autarky state tomorrow. In other words, the optimal contract prevents agents from being “immiserated” (or “impoverished”), in the sense that the optimal contract does not deliver the value of autarky tomorrow, which is the greatest lower bound on the set of feasible continuation values. Our numerical simulations suggest that the mass of agents living in autarky can be zero in the limit. This is in sharp contrast with the immiseration result.³

In order to prove Proposition 1, we state two intermediate lemmas that characterize the optimal contract at the autarky state. These lemmas are also useful to develop some intuition behind the main result. First, we find that, if an agent is at the autarky state and has access to financial markets, financiers cannot spread continuation values to provide cheaper incentives unless the highest realization of the endowment is drawn.⁴ In other words, if the agent draws any realization of the endowment other than the highest possible one, then he remains stuck in autarky. Second, we find that, at the autarky state, some intertemporal trade occurs between the financial intermediary and agents who draw the highest endowment shock. Hence, the autarky state is not absorbing.⁵

The intuition behind our main result is as follows. In a problem with asymmetric information, the possibility to spread continuation values is a profitable tool to provide incentives. But at the autarky state, the principal cannot spread continuation values

²In related contexts, Hertel [2004] and Broer et al. [2017] also consider both private information and limited commitment (or enforcement). Broer et al. [2017] study consumption risk sharing in a similar environment with persistent shocks, public insurance and ex-ante participation constraints, whereas Hertel [2004] studies risk sharing contracts between two risk-averse agents as in Kocherlakota [1996].

³When private information is the sole friction in the model, Phelan [1998] argues that the crucial assumption to generate the immiseration result rests on preferences.

⁴This result follows directly from the restrictions on continuation values and transfers that arise from the constraints in the recursive problem.

⁵This result is reminiscent of the literature on dynamic risk sharing contracts with private information. In general, whenever a lower bound on continuation values is present, it is not an absorbing state. See, for example, Atkeson and Lucas [1995] and Wang [1995], among others.

for types other than the highest one. If the probability of getting the highest possible endowment shock remains small enough, then autarky is a persistent state. In this case, the impossibility to spread continuation values for lower types becomes excessively costly. Hence, the inclination of the value function of the principal becomes positive in the neighborhood of the autarky state, which prevents him from promising the value of autarky.

Moreover, we show that the constraint that can prevent agents from being “immiserated” is the participation constraint that makes the agent hit by the lowest income shock indifferent between live in autarky and honor the contract. Moreover, we also show that none of the states reached is absorbing. In particular, Proposition 2 shows that a non-degenerate invariant cross-sectional distribution of life-time utilities (or values) exists. These results contrast with Ljungqvist and Sargent [2012]’s version of Thomas and Worrall [1988], who show that continuation values converge to an absorbing state, pinned down by the participation constraint of the highest type.

Finally, our results also differ from Phelan [1995], who shows that the lower bound on the set of continuation values is a recurrent state and, thus, a positive mass of agents lives at this state in the limit.⁶ Phelan [1995] also shows that a non-degenerate invariant distribution exists. As the main difference between this paper and Phelan [1995] is the presence of ex-post participation constraints, we solve numerically for the optimal contract with and without these constraints, so to highlight the role of them in the model.

The remainder of this paper is organized as follows. Section 2 presents the model. Section 3 states some intermediate results. Section 4 studies the dynamics of the model. Section 5 discusses numerically some properties of the model. Finally, Section 6 concludes.

2 The Model

We consider an economy in which many infinitely lived ex ante identical agents can sign a single long-term contract with a financial intermediary (or principal). In each period, an agent is endowed with θ units of a perishable consumption good. We assume that θ is

⁶The numerical simulations in Broer et al. [2017] highlight such bunching at the lower bound.

private information, drawn from the set $\{\theta_1 < \theta_2 < \dots < \theta_n\}$, with $n \geq 2$. In particular, we assume that endowment shocks are independently and identically distributed over agents and time, with $\pi_j = \text{prob}(\theta = \theta_j) > 0$, $j = 1, \dots, n$, such that $\sum_{j=1}^n \pi_j = 1$.

Each agent derives utility from a consumption stream $\{c_t\}_{t=0}^{\infty}$. Preferences are separable over time, such that the discounted instantaneous utility at t is denoted by $\delta^t u(c_t)$, where $\delta \in (0, 1)$ is the discount factor. We assume that u is strictly increasing, strictly concave, twice continuously differentiable and bounded above, i.e. $\sup u(c) < \infty$. Finally, we normalize life-time utility by the factor $(1 - \delta)$:

$$(1 - \delta) \sum_{t=0}^{\infty} \delta^t u(c_t).$$

This normalization is important solely to prove that the value function of the principal is strictly concave for δ high enough, a result we use in Section 4 to derive a Lagrange functional for the problem. Results reported in Section 3 do not depend on this normalization.

Financial intermediaries are risk neutral with free access to credit markets, where they can borrow and lend at a constant risk-free interest rate given by $\frac{1}{\delta} - 1$. Hence, agents and financial intermediaries discount the future at the same rate. Financial intermediaries can credibly commit to a long-term loan contract designed to maximize their life-time profits, which are also normalized by the factor $(1 - \delta)$. In particular, at $t = 0$, they offer a long-term contract to agents that promises a given normalized life-time utility (or value) v_0 . In contrast, agents can walk away from the contract at any time at a cost of living in autarky forever.

The optimal contracting problem can be written recursively. Suppose an agent enters a period with a given promised value v . Hence, for each θ_j , the contract assigns a transfer b_j to the agent, which can be negative, and a promised continuation value w_j that the contract must honor in the beginning of the next period.

For a given value v promised to the agent at the end of the previous period, the contract $\{b_j, w_j\}_{j=1}^n$ offered by financial intermediaries must respect four restrictions. First, financial intermediaries must honor the last period promised value. To do so, the expected

value of the contract must be equal to v . Hence, the promise-keeping constraint reads as follows:

$$(PK) \quad \sum_{j=1}^n \pi_j [(1 - \delta)u(\theta_j + b_j) + \delta w_j] = v.$$

Second, since θ is private information, agents can misreport their endowment shocks. Incentive compatibility requires that:

$$(IC) \quad (1 - \delta)u(\theta_j + b_j) + \delta w_j \geq (1 - \delta)u(\theta_j + b_k) + \delta w_k, \text{ for all } j, k.$$

Third, we assume that agents cannot commit to honor the contract. Once an agent reneges on the contract, he is excluded from the financial market, and forced to remain in autarky forever. Hence, the contract must respect the following participation constraints:

$$(PC) \quad (1 - \delta)u(\theta_j + b_j) + \delta w_j \geq (1 - \delta)u(\theta_j) + \delta w_{aut}, \text{ for all } j,$$

where $w_{aut} = \sum_{j=1}^n \pi_j u(\theta_j)$ is the normalized expected life-time utility of living in autarky forever.

Many contributions in the literature also assume this specific form of ex-post participation constraints.⁷ In particular, the nature of the punishment, i.e. living in autarky forever, is crucial to derive our analytical results. This outside option entails a lower bound w_{aut} on the set of continuation values, a constraint that will be used to derive some results in the next sections. Indeed, both (PK) and (PC) imply that:

$$v = \sum_{j=1}^n \pi_j [(1 - \delta)u(\theta_j + b_j) + \delta w_j] \geq \sum_{j=1}^n \pi_j [(1 - \delta)u(\theta_j) + \delta w_{aut}] = w_{aut}.$$

Hence, $w_j < w_{aut}$ for some j violates the next period promise keeping constraint. Finally, notice that no contract can provide incentives by promising values higher than $w_{max} = \sup u(c_t) < \infty$.

⁷A non-exhaustive list includes Thomas and Worrall [1988], Kocherlakota [1996], Attanasio and Ríos-Rull [2000], Kehoe and Levine [2001], Ligon et al. [2002], Krueger and Perri [2006], Krueger and Perri [2011], Tian and Zhang [2013] and Laczó [2014].

Therefore, given a promised utility v , the principal solves

$$W(v) = \max_{\{b_j, w_j \in [w_{aut}, w_{max}]\}_{j=1}^n} \sum_{j=1}^n \pi_j [-(1 - \delta)b_j + \delta W(w_j)]$$

subject to (PK), (IC) and (PC),

where W is the value function of the principal.

This problem encompasses two important contributions from the literature. Ljungqvist and Sargent [2012]’s version of Thomas and Worrall [1988] ignores (IC) from the problem. Similarly, under additional assumptions on u , Thomas and Worrall [1990] solve the case in which (PC) is absent, which implies that the set of constraints $w_j \geq w_{aut}$, for all j , is also absent from the problem.⁸

Phelan [1995] also considers participation constraints in a model of risk sharing with private information, although with a different modeling of participation. In particular, he assumes that the decision to repudiate the contract is taken in the beginning of the period, or equivalently, one-period contracts are enforceable. In addition, the principal can also renege on the contract at a fixed cost. As a result, participation constraints translate into a lower and an upper bound on the set of continuation values.⁹ If he assumed that the outside option is to live in autarky forever, then $w_j \geq w_{aut}$ for all j would follow. Hence, the program above without (PC) but with the set of constraints $w_j \geq w_{aut}$, for all j , is akin to the setup in Phelan [1995].¹⁰

3 Characterization at $v = w_{aut}$

In the next subsections, we state some intermediate lemmas used to prove our main result (Proposition 1 in Section 4). These lemmas might be of interest by themselves.

⁸To be precise, both contributions do not normalize life-time utility and profits by $(1 - \delta)$. It is easy to verify that this normalization would not alter the results in Ljungqvist and Sargent [2012] and Thomas and Worrall [1990].

⁹In his model, this lower bound is endogenized by assuming that the outside option is the value of signing a long-term contract with another financial intermediary, which is determined in equilibrium. The main messages of the paper, i.e. some risk-sharing occurs and a non-degenerate distribution of values exists in the limit, would follow if this lower bound were treated exogeneously.

¹⁰In contrast with this paper, Phelan [1995] simplifies his framework along two dimensions. First, he considers an economy with $n = 2$. Second, he assumes constant absolute risk aversion preferences.

The first subsection explores the interaction between (PK), (IC) and (PC) to show that, at $v = w_{aut}$, unless the highest endowment shock realizes, the principal cannot spread continuation values to provide incentives. The second shows that, at $v = w_{aut}$, some intertemporal trade occurs between the financial intermediary and agents who draw the highest shock. Hence, although autarky impairs the amount of risk sharing that can be achieved, it is not an absorbing state. Some risk sharing occurs at $v = w_{aut}$.

3.1 The Interaction Between (PK), (IC) and (PC)

In this subsection we argue that the interaction between promise keeping (PK), incentive compatibility (IC) and participation (PC) constraints has the potential to substantially limit the amount of risk sharing in this economy.

A standard result in the literature states that (IC) and concavity of u impose restrictions on the transfers that can be made by the principal. In particular, $b_{j-1} \geq b_j$ and $w_{j-1} \leq w_j$ for $j \geq 2$. In words, in order to provide incentives, transfers must decrease with income whereas continuation values must increase. The following auxiliary lemma states that the interaction between (IC), (PC) and concavity of u imposes further restrictions on the transfers that can be made by the principal.

The lemma considers the case in which (PC) is binding at one type, which must eventually happen along the optimal contract path. Indeed, in the absence of (PC) from the problem, Thomas and Worrall [1990] show that v becomes arbitrarily negative (immiseration result). Hence, (PC) must eventually bind at some j .

Lemma 1. *If $i < k$ ($i > k$) and (PC) is binding at k , then $b_i \geq 0$ ($b_i \leq 0$). In addition, if (PC) is not binding at i , then $b_i > 0$ ($b_i < 0$).*

Proof. (PC) and (IC) imply that

$$\begin{aligned} (1 - \delta)u(\theta_k) + \delta w_{aut} &= (1 - \delta)u(\theta_k + b_k) + \delta w_k \geq (1 - \delta)u(\theta_k + b_i) + \delta w_i \geq \\ &\geq (1 - \delta)[u(\theta_k + b_i) - u(\theta_i + b_i)] + (1 - \delta)u(\theta_i) + \delta w_{aut}. \end{aligned}$$

Concavity of u and $i < k$ ($i > k$) imply that $b_i \geq 0$ ($b_i \leq 0$). Finally, if (PC) is not

binding at i , the last inequality is strict, which completes the proof. \square

In words, if (PC) is binding at a given type, say k , then every type below (above) it must receive positive (negative) transfers. The relevant part of the lemma for the rest of the analysis is that $b_i \geq 0$ if $i < k$ and (PC) is binding at k . Notice that Lemma 1 does not make use of (PK). If we consider the role of (PK), even further restrictions on the contract offered by the principal apply. In fact, recall from previous section that (PC) and (PK) imply that $w_j \geq w_{aut}$.

The rest of this subsection and the next one characterize properties of the optimal contract at $v = w_{aut}$. This is useful for two reasons. First, it allows us to show that a variant of the immiseration result, in which agents get stuck in autarky forever, does not follow (Lemma 3 in Section 3.2). In other words, some risk sharing occurs at $v = w_{aut}$. Second, it allows us to derive conditions under which the optimal contract does not assign the value of autarky as a continuation value (Proposition 1 in Section 4). In this case, our numerical simulations suggest that the mass of agents living in autarky in the limit can be zero.

The next lemma states that if an agent was promised the value of autarky in the previous period, then he remains in autarky unless hit by the highest realization of the endowment, θ_n . Moreover, transfers to the highest type b_n are negative.

Lemma 2. *If $v = w_{aut}$, then $w_j = w_{aut}$ and $b_j = 0$ for $j = 1, \dots, n - 1$. Moreover, (PC) also binds at $j = n$ and $b_n \leq 0$.*

Proof. Note that (PC) is binding for all j . Otherwise, (PK) would imply that

$$w_{aut} = \sum_{j=1}^n \pi_j [(1 - \delta)u(\theta_j + b_j) + \delta w_j] > \sum_{j=1}^n \pi_j [(1 - \delta)u(\theta_j) + \delta w_{aut}] = w_{aut},$$

which yields a contradiction. Since (PC) is binding at $j = 1, \dots, n$, then $w_j \geq w_{aut}$ implies $b_j \leq 0$. In addition, (PC) binding at $j = n$ and Lemma 1 imply that $b_j \geq 0$ for $j = 1, \dots, n - 1$. \square

Lemma 2 shows that the interaction between (PK), (IC) and (PC) has a severe implication

for risk sharing at $v = w_{aut}$. Unless the highest type realizes, the principal cannot spread continuation values to provide incentives. Although the proof is straightforward, the intuition rests on understanding how different combinations of the constraints in the problem restrict transfers and continuation values. At autarky, (PK) forces (PC) to be binding for all types. But if (PC) is binding for all types, given that $w_j \geq w_{aut}$, transfers must be weakly negative for all types. What makes the highest type special is the absence of another type with positive mass that has incentive to mimic him downward. Lemma 1 shows that the incentive of a binding type ($j = n$ in this case) to mimic smaller types forces transfers to be weakly positive for these smaller types. Hence, the only choice variables that are not pinned down by the set of constraints at $v = w_{aut}$ are $b_n \leq 0$ and $w_n \geq w_{aut}$.¹¹

An immediate implication of Lemma 2 is that autarky is a highly persistent state if π_n is small, as expected in unequal societies. In particular, one may conjecture that $\pi_n = 0$ makes autarky an absorbing state. This conjecture is wrong. A close inspection of the proof of Lemma 2 reveals that $\pi_n > 0$ is crucial to conclude that (PC) is binding at n , and thus, apply Lemma 1. In fact, by assuming that $\pi_n = 0$, there would be redundancy in the analysis as $j = n - 1$ would play the role of the highest type in practice.^{12,13}

3.2 Risk Sharing

Despite the severe implication of Lemma 2, the following proposition shows that some risk sharing occurs when $v = w_{aut}$. In fact, some intertemporal trade occurs between the financial intermediary and agents who draw θ_n . In particular, agents transfer part

¹¹One can show that the optimal contract at $v = w_{aut}$ maximizes the principal's profits from transacting with $j = n$, $-(1 - \delta)b_n + \delta W(w_n)$, subject to (PC) binding at $j = n$, $(1 - \delta)u(\theta_n + b_n) + \delta w_n = (1 - \delta)u(\theta_n) + \delta w_{aut}$.

¹²This claim is not straightforward. Since $\pi_n = 0$, the financial intermediary's profits and (PK) do not depend on w_n and b_n . Hence, one needs to show that exist values for w_n and b_n that preserve (IC) and (PC). Given that $b_{n-1} \leq 0$ (Lemma 2), concavity of u implies that this is accomplished by setting $w_n = w_{n-1}$ and $b_n = b_{n-1}$.

¹³In Hemsley and Zilberman [2017], two of us consider a small departure of this environment that makes autarky an absorbing state. In particular, we drop the assumption that the endowment θ takes value in a finite space, and assume, instead, that θ takes value in a compact set, say $[\underline{\theta}, \bar{\theta}]$. In principle, one may also allow for both discrete and continuous types. As the arguments in this paper and Hemsley and Zilberman [2017] suggest, in order to autarky be an absorbing state, the type space must be connected near its upper bound, say $\bar{\theta}$. Indeed, when $v = w_{aut}$, the interaction between (PC), (IC) and (PK) forces all discrete and almost all continuous types to remain in autarky, including those slightly below $\bar{\theta}$.

of their endowment to the financial intermediary, $b_n < 0$, in exchange for a promise of life-time utility above the autarky value, $w_n > w_{aut}$.

Lemma 3. *If $v = w_{aut}$, then $w_n > w_{aut}$ and $b_n < 0$.*

Proof. Suppose $v = w_{aut}$ and consider the following contract, slightly different from autarky. At some t , the principal receives $\varepsilon > 0$ if the highest endowment realizes. At $t + 1$, upon the realization of the highest endowment in the previous period, the agent receives $\xi > 0$ in all possible states. In the remaining periods and contingencies, no transfers occur. Set ε and ξ , such that:

$$(1 - \delta)u(\theta_n - \varepsilon) + \delta \sum_{j=1}^n \pi_j u(\theta_j + \xi) = (1 - \delta)u(\theta_n) + \delta \sum_{j=1}^n \pi_j u(\theta_j).$$

Clearly, this contract satisfies incentive compatibility and participation constraints. Moreover, the agent is indifferent between this contract and autarky forever. Take a first order Taylor approximation at the equation above around $\varepsilon = 0$ and $\xi = 0$. Thus,

$$-(1 - \delta)u'(\theta_n)\varepsilon + \delta \sum_{j=1}^n \pi_j u'(\theta_j)\xi = 0 \iff \varepsilon = \frac{\delta}{1 - \delta} \frac{\sum_{j=1}^n \pi_j u'(\theta_j)}{u'(\theta_n)} \xi > \frac{\delta}{1 - \delta} \xi.$$

The inequality follows from $u'' < 0$ and, thus, $\frac{\sum_{j=1}^n \pi_j u'(\theta_j)}{u'(\theta_n)} > 1$.

The principal's net revenue obtained in this contract, $\pi_n[(1 - \delta)\varepsilon - \delta\xi]$, is positive for ε and ξ small enough. Hence, autarky is not optimal. Lemma 2 implies that $b_n < 0$ and $w_n > w_{aut}$. □

In words, despite the severe implication of Lemma 2, some risk sharing occurs in this economy. A variant of this result is also present in Thomas and Worrall [1990], who consider an extension in which both incentive compatibility and participation constraints interact. In particular, they assume that both principal and agents may renege on the contract and, thus, are subject to participation constraints. They show that, under high enough discount factors, some risk sharing always occurs. Lemma 3 considers one-sided commitment instead and is valid for all values of δ .

This result also echoes Phelan [1995], who considers an economy with $n = 2$, constant

absolute risk aversion (CARA) preferences, and another modeling of participation constraints described above. In this case, he shows that some intertemporal trade occurs when the promised value is at its lower bound. In contrast, Lemma 3 holds for generic preferences as long as u satisfies strict concavity.

In related contexts with asymmetric information, whenever a lower bound on continuation values is present, other papers derive similar implications. Wang [1995], Atkeson and Lucas [1995] and Hertel [2004], for example, show that this lower bound is not an absorbing state, and a non-degenerate invariant distribution exists (something we discuss in the next section).

Despite similarities with previous contributions, Lemmas 2 and 3 say something novel. Once in autarky, except for the highest type, the principal cannot spread continuation values to provide incentives. Therefore, the agent only leaves autarky if the highest possible realization of the endowment θ_n is drawn, which happens with probability π_n . If π_n is small as expected in many unequal economies, then autarky is a highly persistent state. These two reasons, impossibility to properly provide incentives and high persistence, imply that autarky is a costly state. In the next section, we explore this implication for the dynamics of the optimal contract.

4 Dynamics

In this section we argue that if $v > w_{aut}$, there is a small enough probability of drawing the highest realization of the endowment such that the optimal contract prevents agents from reaching autarky tomorrow. This is a direct implication from the fact that the autarky state becomes costlier as π_n gets smaller.

We also assume that δ is high enough. This assumption, jointly with the normalization of life-time utilities and profits by the factor $1 - \delta$, are used to show that the value function of the principal, W , is strictly concave (see Appendix A).¹⁴ Except through the strict concavity of W , the results in this paper do not rely on this assumption or normalization.

¹⁴In practice, this normalization works as if the discount factor could take values above one were life-time utilities and profits not normalized.

Hence, any other set of assumptions, with or without this normalization, that guarantees strict concavity of W would be enough.

Appendix A derives a Lagrange functional for the principal's problem. We show that one can attach Lagrange multipliers to the constraints (PK), (IC), (PC) and $w_j \geq w_{aut}$, and derive the set of optimality conditions.¹⁵ Let μ , $\lambda_{j,j-1}$, $\lambda_{j,j+1}$, ς_j and $\delta\xi_j$ be the Lagrange multipliers associated with (PK), (IC) that prevents j from mimicking $j-1$, (IC) that prevents j from mimicking $j+1$,¹⁶ (PC) and $w_j \geq w_{aut}$, respectively. The Lagrangian reads:

$$\begin{aligned} \mathcal{L} = & \sum_{j=1}^n \pi_j (-(1-\delta)b_j + \delta W(w_j)) + \mu \left[\sum_{j=1}^n \pi_j ((1-\delta)u(\theta_j + b_j) + \delta w_j) - v \right] + \\ & + \sum_{j=2}^n \lambda_{j,j-1} [(1-\delta)u(\theta_j + b_j) + \delta w_j - ((1-\delta)u(\theta_j + b_{j-1}) + \delta w_{j-1})] + \\ & + \sum_{j=1}^{n-1} \lambda_{j,j+1} [(1-\delta)u(\theta_j + b_j) + \delta w_j - ((1-\delta)u(\theta_j + b_{j+1}) + \delta w_{j+1})] + \\ & + \sum_{j=1}^n \varsigma_j [(1-\delta)u(\theta_j + b_j) + \delta w_j - (1-\delta)u(\theta_j) - \delta w_{aut}] + \sum_{j=1}^n \delta\xi_j (w_j - w_{aut}), \end{aligned}$$

with $\lambda_{1,0} = \lambda_{n,n+1} = \lambda_{0,1} = \lambda_{n+1,n} = 0$.

Notice that we do not account for the constraints $w_j \leq w_{max}$ in the Lagrangian. A simple argument shows that these constraints are never binding along the optimal contract path. Indeed, the value function $W(v)$ is bounded below by the normalized life-time profits when the principal pays a constant amount for all types in all periods, i.e. $-\bar{b}(v) \leq W(v)$, where $\bar{b}(v)$ solves $\sum_{j=1}^n \pi_j u(\theta_j + \bar{b}) = v$. Analogously, $W(v)$ is bounded above by the normalized life-time profits generated by the first-best unconstrained contract, which guarantees full-insurance by assuring constant consumption for all types in all periods, i.e. $W(v) \leq \sum_{j=1}^n \pi_j [\theta_j - \bar{c}(v)]$, where $\bar{c}(v) = u^{-1}(v)$. Since $w_{max} = \sup u(c) < \infty$, these

¹⁵To do so, we define a relaxed version of the program by allowing the principal to choose a joint distribution probability over transfers and continuation values. In other words, we convexify the program, so that the maximization problem is well-defined and can be casted as a Lagrange functional. Then, we show that W is differentiable and, if δ is high enough, strictly concave. Hence, the solution of the relaxed version must be deterministic and, thus, feasible within the original program.

¹⁶A standard result states that concavity of u implies that it is sufficient to account for local upward and downward (IC) constraints.

bounds above imply that

$$\lim_{v \rightarrow w_{max}} W(v) = \lim_{v \rightarrow w_{max}} W'(v) = -\infty.$$

Consequently, if $v < w_{max}$, it is never optimal for the principal to set $w_j = w_{max}$ for some j .

This result is used in the next proposition to show that $W'(w_{aut})$ becomes arbitrarily large as π_n gets arbitrarily small. This guarantees that for each $v > w_{aut}$, there exists π_n small enough such that the optimal continuation value is interior, i.e. $w_j \in (w_{aut}, w_{max})$. In other words, once away from autarky today, the optimal contract prevents the agent from reaching the autarky state tomorrow. We are ready to state our main result.

Proposition 1. *For each $v > w_{aut}$, there is $\underline{\pi}(v)$ such that $w_j > w_{aut}$ for $j = 1, \dots, n$ and for all $\pi_n < \underline{\pi}(v)$.*

We sketch the proof below, and fill the details in the Appendix B. After manipulating the first order conditions of the Lagrangian above with respect to w_j , substituting $\mu = -W'(v)$ (envelope theorem, see Milgrom and Segal [2002]), and using Lemmas 2 and 3 to evaluate the resulting equation at $v = w_{aut}$, one obtains:

$$W'(w_{aut}) = W'(w_n) + \frac{1}{\pi_n} \sum_{j=1}^n (\varsigma_j + \xi_j),$$

where w_n , with a slight abuse of notation, is the optimal continuation value for type- n at $v = w_{aut}$. Also, the multipliers are evaluated at $v = w_{aut}$.

At $v = w_{aut}$, we show that the optimality conditions imply that $\lim_{\pi_n \rightarrow 0} \sum_{j=1}^n (\varsigma_j + \xi_j) > 0$. Intuitively, in the absence of binding participation constraints, a well-known result states that the principal spreads continuation values in order to provide incentives. Since $v = w_{aut}$, then $w_1 < w_{aut}$ would violate one of the participation constraints of the problem. Moreover, strict concavity of W , $w_n \in (w_{aut}, w_{max})$ and $\lim_{v \rightarrow w_{max}} W'(v) = -\infty$ imply that $W'(w_n) > -\infty$. Since this result is valid for all distributions of $\{\pi_j\}_{j=1}^n$,

including those with $\pi_n \rightarrow 0$, then $\lim_{\pi_n \rightarrow 0} W'(w_n) > -\infty$. Hence,

$$\lim_{\pi_n \rightarrow 0} W'(w_{aut}) = \infty.$$

This kind of “Inada condition” guarantees that for each $v > w_{aut}$, in a neighborhood of $\pi_n = 0$ (i.e. for all $\pi_n < \underline{\pi}(v)$), the optimal contract prevents agents from being “immiserated” (or “impoverished”) tomorrow, in the sense that the optimal contract does not deliver the value of autarky, which is the greatest lower bound on the set of feasible continuation values.

Intuitively, in a problem with asymmetric information, the possibility to spread continuation values is a profitable tool to provide incentives. At $v = w_{aut}$, the financial intermediary cannot vary continuation values unless the agent draws the highest endowment θ_n . If the probability of such event, π_n , is small enough, autarky is a persistent state, which makes the impossibility to spread continuation values for lower types markedly costly. In this case, the inclination of the value function of the principal W' becomes positive in the neighborhood of $v = w_{aut}$, as v slightly above w_{aut} allows the principal to vary continuation values for lower types. Hence, the principal chooses $w_j > w_{aut}$ for all j whenever this choice is feasible, which it is for $v > w_{aut}$ but not for $v = w_{aut}$.

Due to the immiseration result in the absence of (PC) from the problem, (PC) must eventually bind at least at one j . An immediate implication of Lemma 1 and Proposition 1 is that, as long as π_n small enough implies $w_j > w_{aut}$ for all j , (PC) cannot bind at $j > 1$.¹⁷ Therefore, the constraint that can prevent agents from being “immiserated” is the one that makes the lowest type, $j = 1$, indifferent between live in autarky forever and honor the contract.

Another immediate implication of the steps in the proof of Proposition 1 is that, due to the need of spreading continuation values given the presence of (IC) in the problem, none

¹⁷Indeed, suppose that (PC) binds at some $k > 1$, then Proposition 1 and (IC) imply that

$$(1 - \delta)u(\theta_k) + \delta w_{aut} = (1 - \delta)u(\theta_k + b_k) + \delta w_k > (1 - \delta)u(\theta_k + b_j) + \delta w_{aut}, \text{ for all } j \neq k.$$

Hence, $b_j < 0$ for $j \neq k$. But Lemma 1 implies that $b_j \geq 0$ for $j = 1, \dots, k - 1$, a contradiction.

of the states reached is absorbing.¹⁸ In particular, $w_n > v$ for all $v \in [w_{aut}, w_{max}]$.¹⁹ Importantly, this result does not rely on π_n being small enough.

These two implications contrast with Ljungqvist and Sargent [2012]’s version of Thomas and Worrall [1988], who ignore (IC) from the problem. In this case, v converges to a finite value, an absorbing state that makes the highest type $j = n$, rather than the lowest type $j = 1$, indifferent between live in autarky forever and honor the contract.

Notice that $\underline{\pi}$ depends on v . Our proof is silent on whether or not exists a fixed $\underline{\pi}$ (independent of v) such that for all $v > w_{aut}$ and for all $\pi_n < \underline{\pi}$, then $w_j > w_{aut}$ for all j . If this stronger result is valid, in the limit, the mass of agents living in autarky is zero for all $\pi_n \in (0, \underline{\pi})$. Indeed, if $v_0 > w_{aut}$, the mass of agents living in the autarky state is always zero along the optimal contract path. If $v_0 = w_{aut}$ instead, Lemmas 2 and 3 imply that the mass of agents living in the autarky state at period t is $(1 - \pi_n)^t$, which converges to zero in the long run.²⁰ Nonetheless, a weaker implication of Proposition 1, Lemmas 2 and 3 applies. As π_n gets arbitrarily close to zero (i.e. $\pi_n \rightarrow 0$), the mass of agents living in autarky along the optimal contract path gets arbitrarily close to zero (one) when $v_0 > w_{aut}$ ($v_0 = w_{aut}$).

Importantly, at least for a specific parametrization of the model, our numerical simulations below suggest the aforementioned stronger result holds (i.e., zero mass of agents at the lower bound on the set of feasible continuation values in the limit). This is in sharp contrast with Thomas and Worrall [1990], who consider the problem without (PC) and, thus, also without the set of constrains $w_j \geq w_{aut}$, for all j . In this case, v converges to its lower bound, $-\infty$, almost surely (immiseration result). It also differs from Phelan [1995], who considers a problem without (PC) but with a lower bound and an upper bound (smaller than w_{max}) on the set of continuation values. In his model, these states are recurrent along the optimal contract path, and the limit distribution has a positive

¹⁸Notice that w_{max} is an absorbing state, but never reached along the optimal contract path.

¹⁹Indeed, suppose that $w_n \leq v$. Equation (3) in the Appendix B, used to prove Proposition 1, and strict concavity of W imply that $w_j = v$ and $\varsigma_j = \xi_j = 0$ for all j . In this case, a well-known result states that the principal would like to spread continuation values, $w_1 < v < w_n$, yielding a contradiction.

²⁰Notice that, since $W'(w_{aut}) > 0$ for π_n small enough, both the financial intermediary and agents are better off by signing an optimal contract with $v_0 > w_{aut}$ rather than $v_0 = w_{aut}$. Hence, none of the agents would ever be in autarky.

mass of agents, smaller than one, at its lower bound.²¹

The next proposition shows that, for every $\pi_n > 0$, a non-degenerate invariant cross-sectional distribution exists.

Proposition 2. *The Markov process $\{v_t\}$ implied by the optimal choices of $\{w_j\}_{j=1}^n$ has a non-degenerate invariant distribution.*

Proof. Let $w_j(v)$ be the optimal continuation value when the promised value was v and shock j was drawn. Let P be a transition function that maps elements of the compact set $[w_{aut}, w_{max}]$ into Borel sets of $[w_{aut}, w_{max}]$. In particular,

$$P(v, A) = \sum_{j=1}^n \pi_j \mathbf{1}_{\{w_j(v) \in A\}}, \quad (1)$$

where A is a Borel set of $[w_{aut}, w_{max}]$. Notice that $\mathbf{1}$ is the indicator function.

By the Theorem of Maximum, w_j is continuous in v and, thus, P has the Feller property. Theorem 12.10 in Stokey and Lucas [1989] implies that an invariant distribution exists. Non-degeneracy follows from the fact that none of the states reached is absorbing. □

Notice that this proposition is silent on whether the invariant distribution is unique and stable. Hence, convergence from any initial value, v_0 , towards this invariant distribution is not guaranteed. This proposition could be strengthened if one shows that the transition function, P , defined in (1), satisfies monotonicity. Hence, if w_j is bounded above by some $\bar{w} < w_{max}$, Theorem 12.12 in Stokey and Lucas [1989] guarantees convergence from any initial value towards the unique invariant distribution associated with (1).

For instance, if $w_j(v)$ is non-decreasing in v , which is satisfied in the numerical example below, then P satisfies monotonicity. Although we could not prove it generally, the result that $w_j(v)$ is non-decreasing in v seems plausible in other contexts.²²

Moreover, all results reported and discussed in this paper would follow if we impose an

²¹See also the numerical simulations in Broer et al. [2017], who also consider ex-ante participation constraints.

²²See, for instance, the discussion in Farhi and Werning [2007], page 383.

upper bound smaller than w_{max} instead. This can be motivated by allowing limited commitment on the side of the principal. In particular, the financial intermediary can renege the contract at a given (normalized) fixed cost $C \in (0, \infty)$ and, then, be excluded from the financial market forever. As in Phelan [1995], this assumption of limited commitment generates an upper bound $\bar{w} < w_{max}$ on the space of continuation values such that $W(\bar{w}) = -C$.

The main difference between this paper and Phelan [1995] is the presence of (PC) in a context with private information.²³ In order to further understand the role of (PC), next section solves numerically for the optimal contract when preferences are CARA and continuation values are bounded below by w_{aut} and above by $\bar{w} < w_{max}$. The aim is to contrast economies with and without (PC).

5 Numerical Solution

In this section, we solve the model with CARA preferences, i.e. $u(c) = -\exp(-\gamma c)$ with $\gamma > 0$, numerically. Appendix C describes the solution method. In order to highlight the importance of the participation constraints, we also consider the case in which (PC) is ignored from the financial intermediary problem, but the set of constraints $w_j \geq w_{aut}$ for all j is kept in the problem. As explained above, this case is akin to Phelan [1995], who assumes that the possibility to walk away from the contract occurs at the beginning of a period. Hence, this numerical exercise also highlights the importance of the timing assumed in this paper for the results.

Keeping in mind that is impossible to exhaust all possible parametrization of the model, we focus the discussion on the differences between the optimal contract with and without (PC). We consider the following parametrization: $\gamma = 1$, $\frac{\delta}{1-\delta} = 0.8$, $n = 3$, $\pi_1 = \pi_2 = \pi_3 = 1/3$, $\{\theta_1, \theta_2, \theta_3\} = \{0, 1, 2\}$. As in Phelan [1995], we assume an upper bound $\bar{w} < w_{max}$ on the space of continuation values such that $W(\bar{w}) = -C$. Otherwise, the numerical solution would be imprecise in the neighborhood of w_{max} . We choose C such

²³There are, of course, other differences. For example, regarding preferences, Phelan [1995] assumes CARA preferences, whereas we assume that δ is high enough in this section.

that $\bar{w} = -(1 - \delta)0.2$.

Figure 1 plots the optimal contract (continuation values and transfers) for each type, along with the forty-five degree line (dashed-line). The left graphs account for participation constraints (PC) in the problem, whereas the right ones ignore them. Figure 2 reproduces in larger scale the behavior of continuation values near the lower bound.

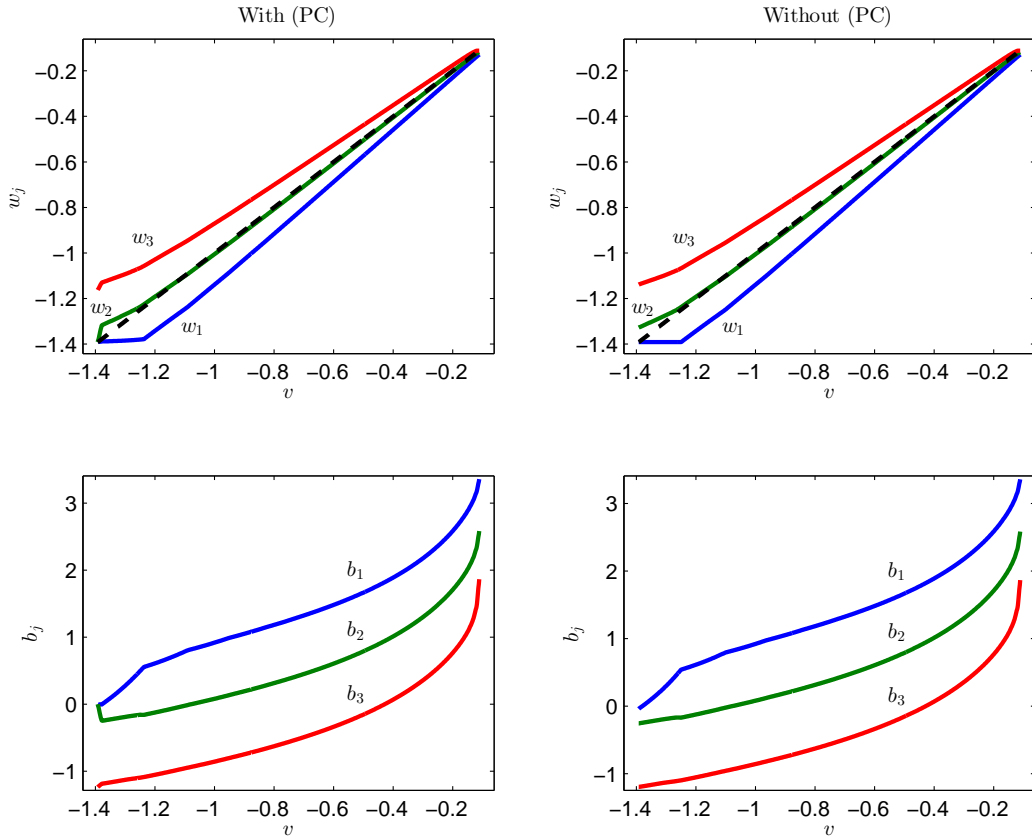


Figure 1: Optimal contract.

Consider the optimal contract in the right plots without (PC). The numerical solution shows that for the lowest type, below a certain threshold value, w_{aut} is the optimal continuation value. Hence, $w_j \geq w_{aut}$ binds at $j = 1$ in the neighborhood of $v = w_{aut}$. Given that w_1 as a function of v is bounded above by the forty-five degree line, the autarky state is reached with positive probability (e.g. after realizing a finite sequence of the lowest endowment θ_1). Similarly, once in autarky, agents leave it with positive probability (e.g. after drawing θ_2 or θ_3). In the long run, a positive mass of agents lives in autarky.

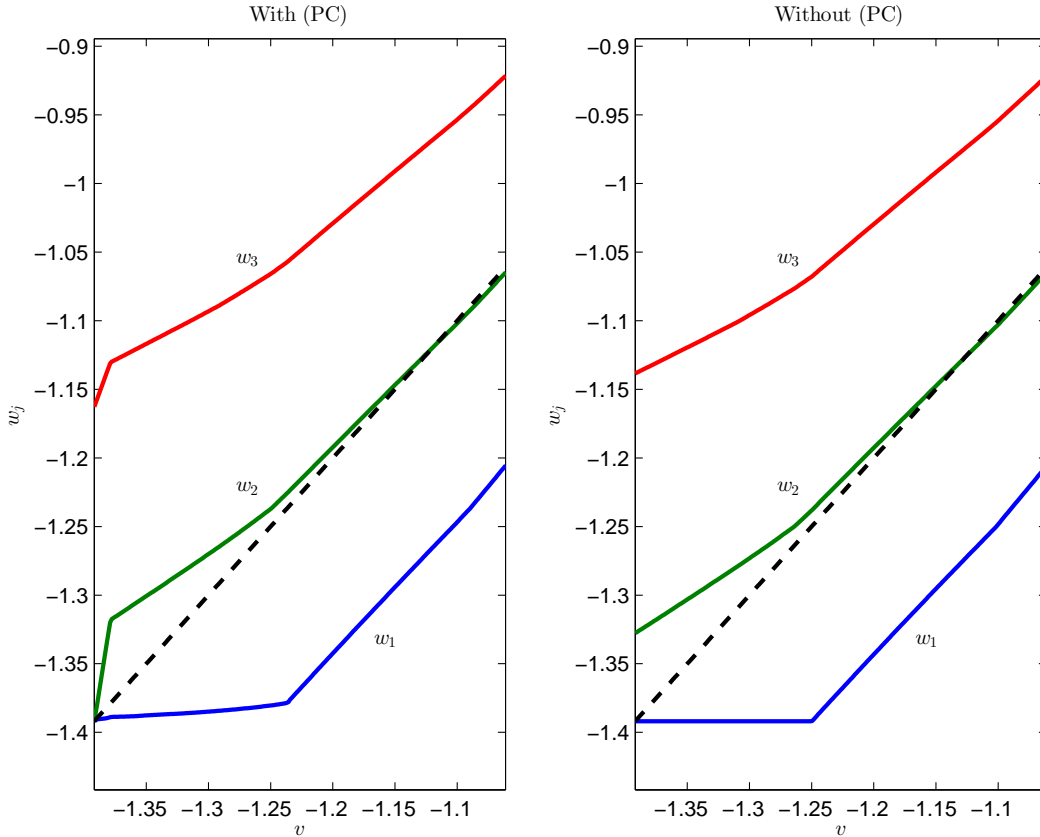


Figure 2: Optimal contract (large scale).

In contrast, once (PC) is accounted for in the left plots, our numerical solution suggests that autarky cannot be reached with positive probability. Indeed, w_1 is a strictly increasing function of v . Moreover, (PC) binds at $j = 1$ slightly above $v = w_{aut}$ but $w_j \geq w_{aut}$ does not bind. Interestingly, notice that as an agent gets closer to autarky, not only w_2 and w_3 as functions of v are above the forty-five degree line, but they also become steeper. Thus, in case θ_2 or θ_3 is draw, this agent gets farther away from the autarky state, meaning that even more consumption is postponed. In this case, our numerical simulations suggest that the mass of agents living in autarky in the limit is zero.

Given the same parametrization, we argue that these differences between contracts arise from the fact that autarky is a costlier state once (PC) is accounted for. First, the principal cannot spread continuation values for types other than $j = n$ in order to provide incentives in autarky. Second, autarky is a more persistent state as the agent leaves it whenever θ_n is drawn, which occurs with probability π_n . In contrast, once (PC) is ignored

from the problem, the principal can promise values higher than w_{aut} for all types other than $j = 1$ and, thus, agents leave autarky whenever any endowment other than θ_1 is drawn, which happens with probability $1 - \pi_1 > \pi_n$.

The left graph of Figure 3 plots the value functions of the principal with and without (PC), represented by the full- and dashed-line, respectively. As differences between them are not visible, the right graph reproduces in larger scale their shape near the lower bound w_{aut} . Notice that slightly above w_{aut} , the inclination of the value function is positive in the model with (PC), but negative in the model without (PC). Given that spreading continuation values becomes markedly costly near w_{aut} once (PC) is accounted for, the principal could increase his profits by promising more consumption in the future to the agent.

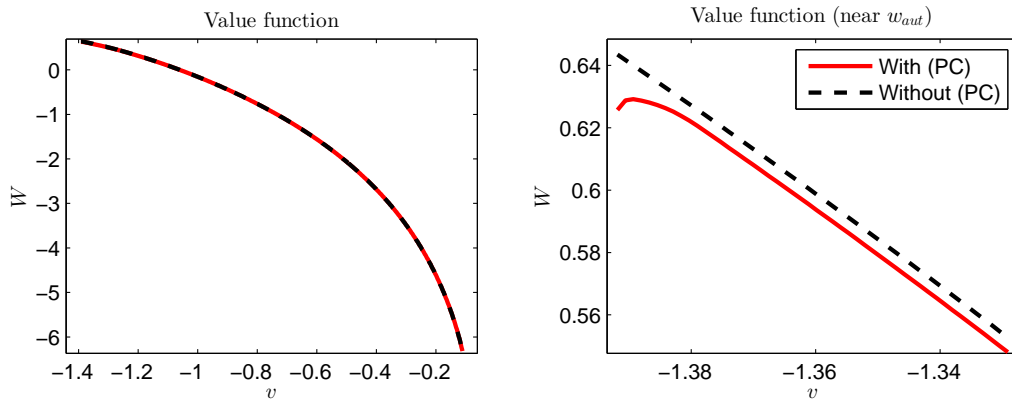


Figure 3: Value function.

In order to inspect how the optimal contract changes with different values of π_n , we allow π_3 to vary from 0.2 to 0.8, and let $\pi_1 = \pi_2 = (1 - \pi_3)/2$. Figure 4 plots the continuation values (near the lower bound w_{aut}) prescribed by the optimal contract for $\pi_3 \in \{0.20, 0.44, 0.68\}$ with (PC) present in the model. Figure 5 plots the numerical right-derivatives of w_1 , w_2 and w_3 at $v = w_{aut}$ as functions of π_n . The left panel accounts for (PC), whereas the right panel does not.

These figures summarize three possible cases within the model with (PC).

First, π_n is small enough ($\pi_3 = 0.2$ in Figure 4 and $\pi_3 \in [0.2, 0.4)$ in Figure 5), which is the case discussed above. Again, the cost of being in autarky is relatively large and, thus,

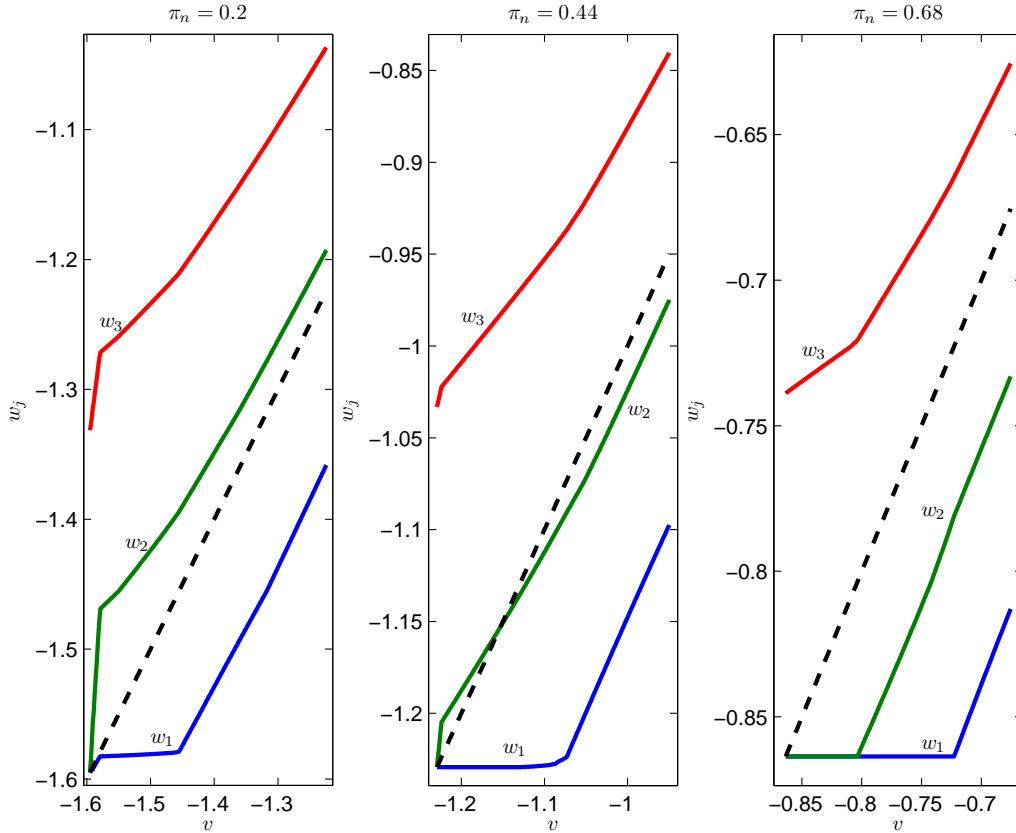


Figure 4: Optimal contract (continuation values) for different values of π_n (large scale).

w_j strictly increases with v for all j . Notice that, for lower values of π_3 , when autarky is costlier, w_j as function of v tends to be steeper at $v = w_{aut}$ for all j , hindering agents from getting closer to the autarky state as they approach it. Notice that the opposite happens in the model without (PC).

Second, if π_n takes intermediate values ($\pi_3 = 0.44$ in Figure 4 and $\pi_3 \in [0.4, 0.5)$ in Figure 5), then w_2 and w_3 are strictly increasing functions of v , but w_1 becomes flat below a certain threshold promised value. Hence, w_{aut} is the optimal continuation value for the lowest type $j = 1$ below this threshold.

Third, if π_n is high enough ($\pi_3 = 0.68$ in Figure 4 and $\pi_3 \in [0.5, 0.8]$ in Figure 5), the cost of being in autarky is relatively small and, thus, w_{aut} is the optimal continuation value for both the lowest and intermediate types ($j = 1, 2$) below certain threshold promised values. In this case, and for this parametrization of the model, the optimal contracts with and without (PC) are identical in our numerical experiments. In other words, (PC)

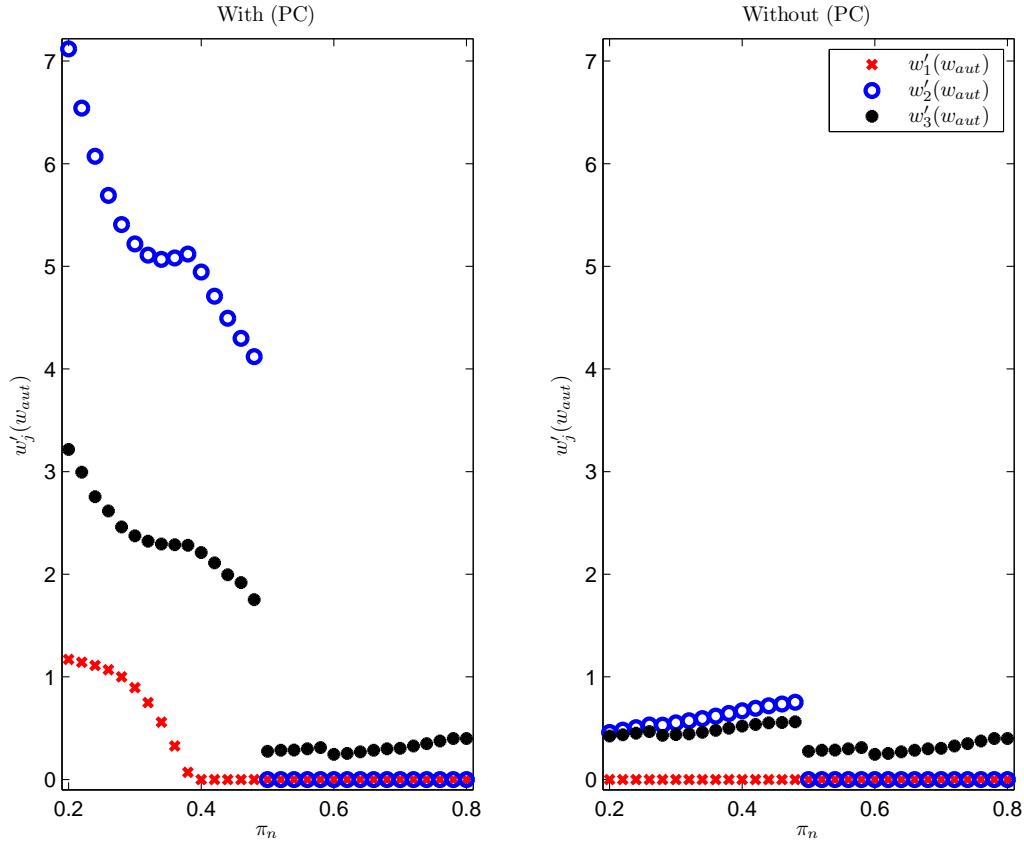


Figure 5: Numerical right-derivatives of w_1 , w_2 and w_3 .

never binds along the optimal contract path.

In the second and third cases, the autarky state can be reached in finite time with positive probability, which implies a positive mass of agents living in autarky in the long run. Thus, the model with (PC) may generate a persistent fraction of agents that is hand-to-mouth in the sense that, absent financial instruments to borrow and save, they consume their own endowment.

Finally, Figure 6 plots the numerical right-derivative of W at $v = w_{aut}$ as a function of π_n . We consider both problems with and without (PC). If (PC) is absent, for this parametrization of the model, the inclination of $W(w_{aut})$ is negative for all values of π_n considered. In contrast, if the optimal contract must respect (PC), the inclination of $W(w_{aut})$ grows without bound as π_n gets smaller. This difference is driven by the impossibility, once (PC) is present, to provide cheaper incentives as π_n gets smaller and v is slightly above w_{aut} .

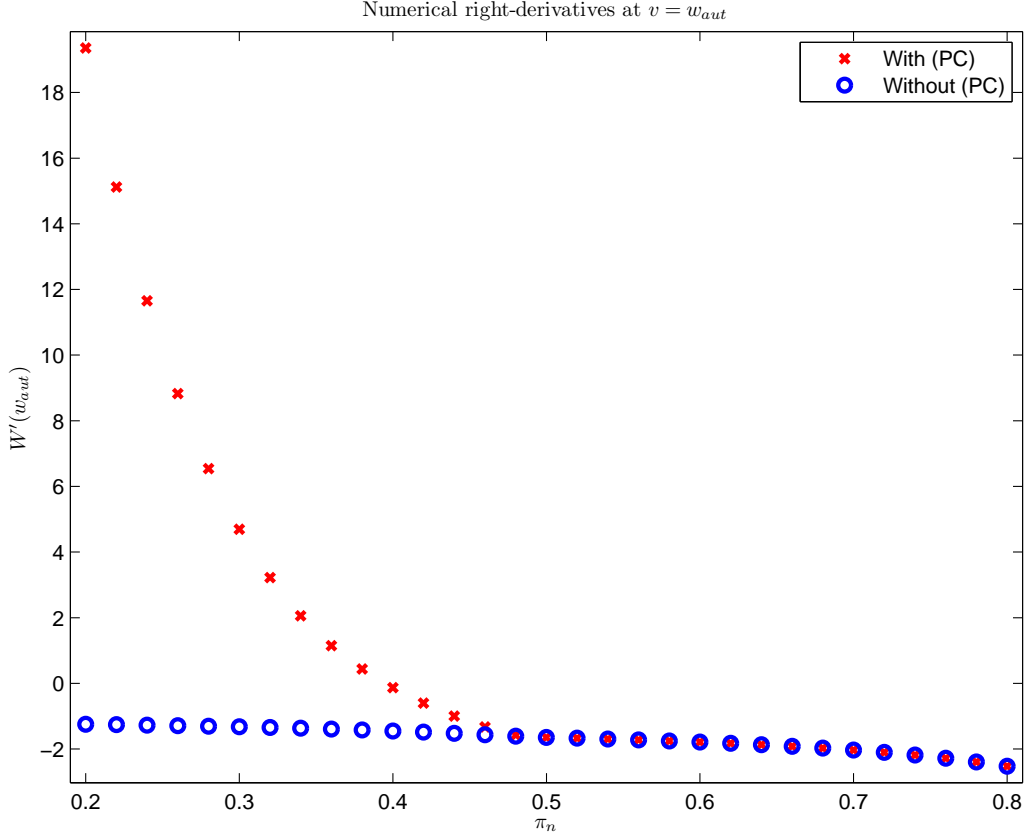


Figure 6: Numerical right-derivatives of $W(w_{aut})$.

6 Conclusion

This paper studies the implications of the interaction between incentive compatibility and participation constraints for risk sharing contracts. In particular, we assume that after a default episode, agents consume their endowment and live in autarky forever. We show that once in the autarky state, financial intermediaries cannot spread continuation values to provide incentives for agents unless they have been hit by the highest possible endowment shock. Hence, the only possibility to leave autarky is to be hit by such highly favorable shock. In unequal societies, where the probability of such event is arguably small, autarky remains a persistent state. These reasons make autarky a relatively costly state, and thus, the optimal contract prevents agents from being “impoverished”. In particular, the value function of the principal becomes positive in the neighborhood of the autarky state. Moreover, we show that an invariant cross-sectional distribution exists. We also discuss throughout the paper how these results compare with key contributions in

the literature. Finally, in the context of a numerical example, we highlight the importance of the modeling of participation constraints to generate these results.

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Appendix A: Derivation of a Lagrange Functional

In order to derive the Lagrange functional in Section 4, we define a relaxed version of the program by allowing the principal to choose a joint distribution probability over transfers and continuation values. First, we show that this relaxed version can be casted as a Lagrange functional. Second, we show that, if δ is high enough, then W is strictly concave. Hence, the solution of the relaxed version must be deterministic and, thus, feasible within the original program.

Let \mathcal{B} and \mathcal{W} be, respectively, the set from which transfers b and continuation values w are drawn. These sets can, of course, be taken to be bounded and closed. Let $\Delta(\mathcal{B}, \mathcal{W})$ be the set of all probability distributions on the Borel sets of $\mathcal{B} \times \mathcal{W}$. A stochastic recursive mechanism is a mapping that takes announcements θ_j , $j \in \{1, \dots, n\}$, into a joint probability distribution over actions and continuation values (i.e., take announcements into $\Delta(\mathcal{B}, \mathcal{W})$). Let $\mathcal{P} \equiv (\Delta(\mathcal{B}, \mathcal{W}))^{\{\theta_1, \dots, \theta_n\}}$.

Let:

$$W(v) = \max_{\{P(b, w|\theta_j)\}_j \in \mathcal{P}} \sum_{j=1}^N \pi_j \int [-(1-\delta)b + \delta W(w)] dP(b, w|\theta_j) \quad (\text{P1})$$

subject to

$$(\text{PK}) \quad \sum_{j=1}^N \pi_j \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) = v;$$

$$(\text{IC}) \quad \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) \geq \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_k), \quad \forall j, k;$$

$$(\text{PC}) \quad \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) \geq (1-\delta)u(\theta_j) + \delta w_{aut}, \quad \forall j.$$

The first equality refers to the promise keeping constraint (PK). The first set of inequalities is the incentive compatibility constraints (IC) that a stochastic recursive mechanism has to satisfy, whereas the second set of inequalities is the participation constraints (PC) that a stochastic recursive mechanism has to satisfy as well.

Notice that (PC) and (PK) imply that $v \geq w_{aut}$. Hence, the infimum of \mathcal{W} is w_{aut} .

B.1 Existence of Lagrange multipliers for P1

We now establish that a solution to such a constrained optimization program (P1) can be found by maximization of a suitable Lagrange functional.

Lemma 4. *If $\{P^*(b, w|\theta_j)\}_j \in \mathcal{P}$ solves (P1), then there exists Lagrange multipliers μ , $\lambda_{j,j-1}$, $\lambda_{j,j+1}$ and ς_j associated with (PK), (IC) that prevents j from mimicking $j-1$, (IC) that prevents j from mimicking $j+1$ and (PC), respectively, so that $\{P^*(b, w|\theta_j)\}_j \in \mathcal{P}$ also maximizes the appropriate Lagrangian associated with program (P1).*

In particular, the Lagrange functional is given by:

$$\begin{aligned} \mathcal{L} \left(\{P(b, w|\theta_j)\}_j \right) &= \sum_{j=1}^N \pi_j \int [-(1-\delta)b + \delta W(w)] dP(b, w|\theta_j) + \\ &+ \mu \left[\sum_{j=1}^N \pi_j \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) - v \right] + \\ &+ \sum_{j=2}^n \lambda_{j,j-1} \left[\int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) - \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_{j-1}) \right] + \\ &+ \sum_{j=1}^{n-1} \lambda_{j,j+1} \left[\int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) - \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_{j+1}) \right] + \\ &+ \sum_{j=1}^n \varsigma_j \left[\int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) - (1-\delta)u(\theta_j) - \delta w_{aut} \right]. \end{aligned}$$

We prove the Lemma above in a couple of steps.

Step 1. *The maximization program (P1) is convex.*

Proof. It follows by noticing that: i) the objective functional is linear in $\{P(b, w|\theta_j)\}_j$, ii) the set \mathcal{P} is convex; iii) all payoffs in the constraints are linear in $\{P(b, w|\theta_j)\}_j$; and iv) the constraints are either weak inequality constraints, (IC) and (PC), or equality constraints, (PK). □

Step 2. *The maximization problem (P1) is well-defined.*

Proof. The set $\mathcal{B} \times \mathcal{W}$ is compact (in the standard Euclidean Topology in \mathfrak{R}^2); hence, the set $\Delta(\mathcal{B}, \mathcal{W})$ of all distributions over $\mathcal{B} \times \mathcal{W}$ is compact in the weak-* topology (see, for

example, Aliprantis and Border [2006], Theorem 15.11). Since the set of constraints are either weak inequality or equality constraints, the set of feasible stochastic mechanisms is a closed subset of a weak-* compact set. Hence, it is compact itself. By Tychonoff's Theorem (see, again, Aliprantis and Border [2006]), \mathcal{P} is weak-* compact in the product topology. Last, notice that (since W is continuous) the objective is a bounded linear functional; hence, continuous. \square

Step 3. *At an optimal, only the local IC constraints might bind.*

Proof. The proof is standard, given that ex-post payoffs are linear in w and (due to the concavity of u) has decreasing differences in $\theta_j + b$. \square

Notice that within the set of constraints of (P1), an interior point does not exist, but its existence is one of the conditions we need to apply the Lagrange multiplier theorem in Section 8.3 of Luenberger [1969]. In order to proceed, we define the following sequence of optimization programs indexed by $\epsilon \in [0, \bar{\epsilon}]$, for a strictly positive and finite $\bar{\epsilon}$. Let:

$$V_\epsilon(v) = \max_{\{P(b, w|\theta_j)\}_{j \in \mathcal{P}}} \sum_{j=1}^n \pi_j \int [-(1-\delta)b + \delta V_\epsilon(w)] dP(b, w|\theta_j) \quad (\text{P2})$$

subject to

$$(\text{PK}+) \quad \sum_{j=1}^n \pi_j \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) \geq v - \epsilon;$$

$$(\text{PK}-) \quad -\sum_{j=1}^n \pi_j \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) \geq -(v + \epsilon);$$

$$(\text{IC}) \quad \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) \geq \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_k), \quad \forall j, k;$$

$$(\text{PC}) \quad \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) \geq (1-\delta)u(\theta_j) + \delta w_{aut}, \quad \forall j.$$

Clearly, within the set of constraints of (P2), an interior point exists. Moreover, similar arguments to ones used in the proofs of Steps 1, 2 and 3 apply to the program above (P2). Finally, when $\epsilon = 0$, the inequalities (PK+) and (PK-) are equivalent to (PK) and, thus, the program (P2) coincides with (P1); hence, $V_0(v) = W(v)$ for all v .

The next steps prove the existence of Lagrange multipliers for program (P2) and, then, for program (P1) by considering the limit case of program (P2) as $\epsilon \rightarrow 0$.

Step 4. Fix $\epsilon > 0$. If $\{P_\epsilon^*(b, w|\theta_j)\}_j \in \mathcal{P}$ solves (P2), then there exists Lagrange multipliers $\mu_+^\epsilon, \mu_-^\epsilon, \lambda_{j,j-1}^\epsilon, \lambda_{j,j+1}^\epsilon$ and ς_j^ϵ associated with (PK+), (PK-), (IC) that prevents j from mimicking $j-1$, (IC) that prevents j from mimicking $j+1$ and (PC), respectively, so that $\{P_\epsilon^*(b, w|\theta_j)\}_j \in \mathcal{P}$ also maximizes the appropriate Lagrangian associated with program (P2).

Proof. All the conditions of Theorem 1, in Section 8.3, of Luenberger [1969] are satisfied. The result then follows because only local IC constraints might bind. \square

The Lagrange functional, associated with program (P2), is given by:

$$\begin{aligned} \mathcal{L}_\epsilon \left(\{P(b, w|\theta_j)\}_j \right) &= \sum_{j=1}^n \pi_j \int [-(1-\delta)b + \delta W(w)] dP(b, w|\theta_j) + \\ &+ \mu_+^\epsilon \left[\sum_{j=1}^n \pi_j \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) - (v - \epsilon) \right] = \\ &- \mu_-^\epsilon \left[\sum_{j=1}^n \pi_j \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) - (v + \epsilon) \right] + \\ &+ \sum_{j=2}^n \lambda_{j,j-1}^\epsilon \left[\int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) - \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_{j-1}) \right] + \\ &+ \sum_{j=1}^{n-1} \lambda_{j,j+1}^\epsilon \left[\int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) - \int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_{j+1}) \right] + \\ &+ \sum_{j=1}^n \varsigma_j^\epsilon \left[\int [(1-\delta)u(\theta_j + b) + \delta w] dP(b, w|\theta_j) - (1-\delta)u(\theta_j) - \delta w_{aut} \right]. \end{aligned}$$

Step 5. If $\{P_\epsilon^*(b, w|\theta_j)\}_j \in \mathcal{P}$ solves (P2), then there exists Lagrange multipliers $\mu, \lambda_{j,j-1}, \lambda_{j,j+1}$ and ς_j associated with (PK), (IC) that prevents j from mimicking $j-1$, (IC) that prevents j from mimicking $j+1$ and (PC), respectively, so that $\{P_\epsilon^*(b, w|\theta_j)\}_j \in \mathcal{P}$ when $\epsilon \rightarrow 0$ also maximizes the appropriate Lagrangian associated with program (P1).

Proof. Pick an arbitrary $\{P(b, w|\theta_j)\}_j$ that satisfies all the constraints of (P1). Due to the lower hemi-continuity of the constrained set of (P2), for any sequence of ϵ going to zero,

there is a sequence of $\{P_\epsilon(b, w|\theta_j)\}_j$ that satisfies all the constraints of (P2) and converges to $\{P(b, w|\theta_j)\}_j$.

Let $\{P_\epsilon^*(b, w|\theta_j)\}_j$ be the solution to (P2). Given that the constrained set of (P2) is continuous (i.e., upper hemi-continuous and lower hemi-continuous) in ϵ , Ausubel and Deneckere's generalized Theorem of the Maximum implies that $\{P_\epsilon^*(b, w|\theta_j)\}_j$ is upper hemi-continuous and $V_\epsilon(v)$ is continuous in ϵ . Therefore, the limit of $\{P_\epsilon^*(b, w|\theta_j)\}_j$ when $\epsilon \rightarrow 0$ is a solution of (P1), i.e. $\{P^*(b, w|\theta_j)\}_j$.

By Step 4, there are Lagrange multipliers $\mu_+^\epsilon, \mu_-^\epsilon, \lambda_{j,j-1}^\epsilon, \lambda_{j,j+1}^\epsilon$ and ζ_j^ϵ , so that $\{P_\epsilon^*(b, w|\theta_j)\}_j \in \mathcal{P}$ solves the appropriate Lagrangian associated with (P2). That is,

$$\begin{aligned} \sum_{j=1}^n \pi_j \int [-(1-\delta)b + \delta V_\epsilon(w)] dP_\epsilon^*(b, w|\theta_j) &= \mathcal{L}_\epsilon \left(\{P_\epsilon^*(b, w|\theta_j)\}_j \right) \geq \\ &\geq \mathcal{L}_\epsilon \left(\{P_\epsilon(b, w|\theta_j)\}_j \right) \geq \sum_{j=1}^n \pi_j \int [-(1-\delta)b + \delta V_\epsilon(w)] dP_\epsilon(b, w|\theta_j). \end{aligned}$$

Let $\mu_+, \mu_-, \lambda_{j,j-1}, \lambda_{j,j+1}$ and ζ_j be the limsup as $\epsilon \rightarrow 0$ of $\mu_+^\epsilon, \mu_-^\epsilon, \lambda_{j,j-1}^\epsilon, \lambda_{j,j+1}^\epsilon$ and ζ_j^ϵ , respectively. Notice that only one of μ_- and μ_+ can be strictly positive. If μ_- is strictly positive, let $\mu = -\mu_-$. Otherwise, let $\mu = \mu_+$. By taking limits as $\epsilon \rightarrow 0$ (along the subsequence that defines the limsup) on both sides of the above expression (and using continuity of \mathcal{L}_ϵ), one obtains

$$\begin{aligned} \sum_{j=1}^n \pi_j \int [-(1-\delta)b + \delta W(w)] dP^*(b, w|\theta_j) &= \mathcal{L} \left(\{P^*(b, w|\theta_j)\}_j \right) \geq \\ &\geq \mathcal{L} \left(\{P(b, w|\theta_j)\}_j \right) \geq \sum_{j=1}^n \pi_j \int [-(1-\delta)b + \delta W(w)] dP(b, w|\theta_j). \end{aligned}$$

Since $\{P(b, w|\theta_j)\}_j$ is arbitrary, the result follows. \square

B.2 Differentiability and strict concavity of W

Finally, we establish that W is differentiable and strictly concave for δ large enough. Hence, the solution to program (P1) is deterministic. Indeed, the financial intermediary must be indifferent between all (b, w) in the support of $P(b, w|\theta_j)$ for some j . Due to

strict concavity of W , this cannot hold for non-degenerate $\{P(b, w|\theta_j)\}_j$, $j = 1, \dots, n$.

Lemma 5. *W is differentiable at the interior of \mathcal{W} . Moreover, $W'(v) = -\mu$.*

Proof. Notice that the objective is linear in $\{P(b, w|\theta_j)\}_j$ and the constraint set is convex in $\{P(b, w|\theta_j)\}_j$. Hence, by standard arguments, W is concave. It then follows that all conditions of the Envelope Theorem for saddle point problems with parameterized constraints of Milgrom and Segal [2002], Corollary 5, are satisfied. Such result establishes that W is directionally differentiable, with $W'(v^+) \geq W'(v^-)$. But since W is concave, $W'(v^+) \leq W'(v^-)$. Therefore, $W'(v^+) = W'(v^-) = W'(v)$. Finally, by Milgrom and Segal [2002], Theorem 1, $W'(v) = -\mu$. \square

Lemma 6. *For δ large, W is strictly concave.*

We prove the lemma above in a series of steps.

Step 1. *If W is linear, then full efficiency is attained.*

Proof. Assume that W is linear. Then, from Lemma 4, for all $v \in \mathcal{W}$,

$$W(v) = a - \mu v,$$

for some a . We construct a contract $\{b_j, w_j\}_j$ that delivers v to the agents and first best payoffs to the principal.

Notice, first, that in search for an optimum, it is without loss of generality to restrict attention to expected continuation values \tilde{w}_j , $j = 1, \dots, n$, given by

$$\tilde{w}_j = \int w dP(b, w|\theta_j),$$

where $\{P(b, w|j)\}_j \in \mathcal{P}$ is an arbitrary stochastic recursive mechanism. Indeed, the

Lagrangian then reads

$$\begin{aligned}
& \sum_{j=1}^n \pi_j \left(\int [-(1-\delta)b] dP(b, w|\theta_j) + \delta [a - \mu \tilde{w}_j] \right) \\
& + \mu \left[\sum_{j=1}^n \pi_j \left(\int (1-\delta) u(\theta_j + b) dP(b, w|\theta_j) + \delta \tilde{w}_j \right) - v \right] \\
& + \sum_{j=2}^n \lambda_{j,j-1} \left[\int (1-\delta) u(\theta_j + b) dP(b, w|\theta_j) + \delta \tilde{w}_j - \left(\int (1-\delta) u(\theta_j + b) dP(b, w|\theta_{j-1}) + \delta \tilde{w}_{j-1} \right) \right] \\
& + \sum_{j=1}^{n-1} \lambda_{j,j+1} \left[\int (1-\delta) u(\theta_j + b) dP(b, w|\theta_j) + \delta \tilde{w}_j - \left(\int (1-\delta) u(\theta_j + b) dP(b, w|\theta_{j+1}) + \delta \tilde{w}_{j+1} \right) \right] \\
& + \sum_{j=1}^n \varsigma_j \left[\int (1-\delta) u(\theta_j + b) dP(b, w|\theta_j) + \delta \tilde{w}_j - (1-\delta) u(\theta_j) - \delta w_{aut} \right].
\end{aligned}$$

The first order condition (FOC) with respect to \tilde{w}_j yields:

$$\lambda_{j,j-1} - \lambda_{j+1,j} + \lambda_{j,j+1} - \lambda_{j-1,j} + \varsigma_j = 0,$$

with $\lambda_{1,0} = \lambda_{0,1} = \lambda_{n+1,n} = \lambda_{n,n+1} = 0$. Summing over j yields:

$$\sum_{j=1}^n \varsigma_j = 0, \text{ which implies } \varsigma_j = 0, \text{ for all } j.$$

Evaluating the FOC at $j = n$ implies $\lambda_{n,n-1} = \lambda_{n-1,n}$. Since both constraints cannot bind at the same time, then $\lambda_{n,n-1} = \lambda_{n-1,n} = 0$. Proceeding inductively: $\lambda_{j,j-1} = \lambda_{j,j+1} = 0$ for all j . Substituting into the Lagrangian functional:

$$\sum_{j=1}^n \pi_j (1-\delta) \left(\int [-b + \mu u(\theta_j + b)] dP(b, w|\theta_j) \right) + \delta a - \mu v,$$

which calls for a pointwise maximization. The FOC with respect to b is given by:

$$u'(\theta_j + b_j) = \frac{1}{\mu}, \text{ for all } j.$$

So full risk-sharing will be optimal. At such allocation, incentives to misreport are necessarily downward. Therefore, it suffices to guarantee that high types do not want to

mimic low types:

$$u(c) + \frac{\delta}{(1-\delta)}w_j = u(\theta_j + b_{j-1}) + \frac{\delta}{(1-\delta)}w_{j-1} = u(c + \theta_j - \theta_{j-1}) + \frac{\delta}{(1-\delta)}w_{j-1}.$$

where c is the amount of constant consumption that prevails in the optimal mechanism.

Hence:

$$w_j = w_{j-1} + \frac{1-\delta}{\delta} [u(c + \theta_j - \theta_{j-1}) - u(c)], \text{ for all } j. \quad (2)$$

Now, pick $w_1 \geq w_{aut}$ and c that satisfy (PK). Since full risk-sharing is attained with satisfaction of (PK), the result follows. \square

Step 2. *If W is linear over an open set A , there exists $\widehat{\delta} < 1$, so that, if $\delta > \widehat{\delta}$, efficient payoffs are attainable.*

Proof. Without loss of generality, pick $v \in A$. Clearly, there exists $\widehat{\delta} < 1$, such that both w_1 and

$$w_n = w_1 + \frac{1-\delta}{\delta} \left[\sum_{s=2}^n [u(c + \theta_s - \theta_{s-1}) - u(c)] \right],$$

obtained from the recursion in equation (2), are in A for $\delta \geq \widehat{\delta}$, and (PK) is satisfied. Step 1 completes the proof. \square

Step 3. *Full efficiency cannot be attained regardless of $\delta \in (0, 1)$.*

Proof. Our setting is exactly as Thomas and Worrall [1990] with an additional constraint: the (PC) constraint. For any given $v \in \mathcal{W}$, the value $W^{TW}(v)$ attained by Thomas and Worrall [1990] is an upper bound to the values attainable in this paper. The result then follows from the fact that efficient payoffs cannot be attained in their setting regardless of $\delta \in (0, 1)$. \square

Since W is concave, for δ large, Steps 2 and 3 imply that W must be strictly concave, which completes the proof of Lemma 5.

Appendix B: Proof of Proposition 1

In this section, we prove Proposition 1. The first order conditions (FOCs) of the Lagrangian in Section 4 with respect to w_j and b_j are:

$$\begin{aligned}\pi_j[W'(w_j) + \mu] &= \lambda_{j+1,j} - \lambda_{j,j-1} + \lambda_{j-1,j} - \lambda_{j,j+1} - \varsigma_j - \xi_j, \quad \text{and} \\ \pi_j[1 - \mu u'(\theta_j + b_j)] &= (\lambda_{j,j-1} + \lambda_{j,j+1} + \varsigma_j)u'(\theta_j + b_j) - \lambda_{j+1,j}u'(\theta_{j+1} + b_j) - \lambda_{j-1,j}u'(\theta_{j-1} + b_j),\end{aligned}$$

respectively. Moreover, $\mu = -W'(v)$ (envelope theorem, see Milgrom and Segal [2002]).

By summing the FOCs with respect to w_j in j , and substituting $\mu = -W'(v)$, one obtains:

$$\sum_{j=1}^n \pi_j W'(w_j) + \sum_{j=1}^n (\varsigma_j + \xi_j) = W'(v). \quad (3)$$

By using Lemmas 2 and 3, evaluate the equation above at $v = w_{aut}$. After rearranging the terms:

$$W'(w_{aut}) = W'(w_n) + \frac{1}{\pi_n} \sum_{j=1}^n (\varsigma_j + \xi_j),$$

where w_n , with a slight abuse of notation, is the optimal continuation value for type- n at $v = w_{aut}$. Also, the multipliers are evaluated at $v = w_{aut}$.

We complete the proof in two steps. First, we show that, at $v = w_{aut}$, the optimality conditions imply that $\lim_{\pi_n \rightarrow 0} \sum_{j=1}^n (\varsigma_j + \xi_j) > 0$, such that²⁴

$$\lim_{\pi_n \rightarrow 0} W'(w_{aut}) = \infty.$$

Second, we show that the condition above implies that if $v > w_{aut}$ then w_{aut} is not reachable within a neighborhood of $\pi_n = 0$, i.e. there is $\underline{\pi}(v) > 0$ such that $w_j > w_{aut}$ for all j and for all $\pi_n < \underline{\pi}(v)$.

²⁴Recall that strict concavity of W , $w_n \in (w_{aut}, w_{max})$ and $\lim_{v \rightarrow w_{max}} W'(v) = -\infty$ imply that $W'(w_n) > -\infty$. Since this result is valid for all distributions of $\{\pi_j\}_{j=1}^n$, including those with $\pi_n \rightarrow 0$, then $\lim_{\pi_n \rightarrow 0} W'(w_n) > -\infty$.

Step 1. At $v = w_{aut}$, $\lim_{\pi_n \rightarrow 0} \sum_{j=1}^n (\varsigma_j + \xi_j) > 0$.

Proof. We prove that $\sum_{j=1}^n (\varsigma_j + \xi_j) > 0$ for all distributions $\{\pi_j\}_{j=1}^n$, including those with $\pi_n \rightarrow 0$. Suppose by contradiction that, at $v = w_{aut}$, $\sum_{j=1}^n (\varsigma_j + \xi_j) = 0$. Hence, $\varsigma_j = \xi_j = 0$ for all j . Notice that the FOCs with respect to w_j and b_j (including $j = n$) become:

$$\begin{aligned} \pi_j[W'(w_j) + \mu] &= \lambda_{j+1,j} - \lambda_{j,j-1} + \lambda_{j-1,j} - \lambda_{j,j+1}, \quad \text{and} \\ \pi_j[1 - \mu u'(\theta_j + b_j)] &= (\lambda_{j,j-1} + \lambda_{j,j+1})u'(\theta_j + b_j) - \lambda_{j+1,j}u'(\theta_{j+1} + b_j) - \lambda_{j-1,j}u'(\theta_{j-1} + b_j), \end{aligned}$$

respectively. By summing the FOCs with respect to w_j in j and substituting $\mu = -W'(w_{aut})$, one obtains:

$$\sum_{j=1}^{n-1} \pi_j W'(w_j) + \pi_n W'(w_n) = W'(w_{aut}). \quad (4)$$

Whether $\pi_n > 0$ or $\pi_n \rightarrow 0$ does not matter for the arguments below. We break the analysis into two cases: $W'(w_{aut}) < 0$ and $W'(w_{aut}) \geq 0$.

Case 1: $W'(w_{aut}) < 0$. Since W is strictly concave, $w_{aut} \leq w_1$ and $w_j \leq w_{j+1}$, equation (4) implies that $w_j = w_{j+1} = w_{aut}$ for $j = 1, \dots, n-2$. The FOCs with respect to w_j (excluding $j = n$) become:

$$\lambda_{j+1,j} - \lambda_{j,j-1} + \lambda_{j-1,j} - \lambda_{j,j+1} = 0.$$

Given that $\lambda_{1,0} = \lambda_{0,1} = 0$, a simple iterative argument implies that $\lambda_{j+1,j} = \lambda_{j,j+1}$, for all j . Moreover, $\lambda_{j+1,j} = \lambda_{j,j+1} = 0$.²⁵ Hence, the FOCs with respect to b_j (excluding

²⁵Suppose not, then strict concavity of u , $b_{j+1} \leq b_j$ and $\lambda_{j+1,j} > 0$ imply that

$$(1 - \delta)[u(\theta_j + b_j) - u(\theta_j + b_{j+1})] > (1 - \delta)[u(\theta_{j+1} + b_j) - u(\theta_{j+1} + b_{j+1})] = \delta(w_{j+1} - w_j).$$

Hence,

$$(1 - \delta)u(\theta_j + b_j) + \delta w_j > (1 - \delta)u(\theta_j + b_{j+1}) + \delta w_{j+1},$$

and thus, $\lambda_{j,j+1} = 0$, a contradiction.

$j = n$) become:

$$u'(\theta_j + b_j) = -\frac{1}{W'(w_{aut})} > 0.$$

Hence, since $u'' < 0$, $b_j > b_{j+1}$, for $j = 1, \dots, n - 2$. But from (IC), $w_j = w_{j+1}$ implies $b_j = b_{j+1}$, a contradiction.

Case 2: $W'(w_{aut}) \geq 0$. Since W is strictly concave, $w_{aut} \leq w_1$ and $w_j \leq w_{j+1}$, equation (4) implies that $w_1 = w_{aut}$. Given that $\lambda_{1,0} = \lambda_{0,1} = 0$, the FOC with respect to w_1 and the arguments in footnote 24 imply that $\lambda_{2,1} = \lambda_{1,2} = 0$. Hence, the FOC with respect to b_1 becomes:

$$u'(\theta_1 + b_1) = -\frac{1}{W'(w_{aut})} \leq 0,$$

a contradiction with $u' > 0$. □

Step 2. For each $v > w_{aut}$, $\lim_{\pi_n \rightarrow 0} W'(w_{aut}) = \infty$ implies that there is $\underline{\pi}(v)$ such that $w_j > w_{aut}$ for $j = 1, \dots, n$ and for all $\pi_n < \underline{\pi}(v)$.

Proof. Suppose by contradiction that at $v > w_{aut}$, $w_j = w_{aut}$ is an optimal choice for some j . Since $w_{aut} \leq w_1$ and $w_j \leq w_{j+1}$, it must be the case that $w_1 = w_{aut}$. Consider the FOC with respect to w_1 after plugging $\mu = -W'(v)$ and evaluating at $w_1 = w_{aut}$:

$$\xi_1 + \varsigma_1 = \pi_1[W'(v) - W'(w_{aut})] + \lambda_{2,1} - \lambda_{1,2}.$$

Given that the maximization problem is well-defined, $\lambda_{2,1} < \infty$ for all distributions $\{\pi_j\}_{j=1}^n$, including those with $\pi_n \rightarrow 0$. Hence, $\lim_{\pi_n \rightarrow 0} W'(w_{aut}) = \infty$ and $\lim_{\pi_n \rightarrow 0} W'(v) < \infty$ (recall that $v > w_{aut}$ and W is strictly concave) imply that $\xi_1 + \varsigma_1 < 0$ in a neighborhood of $\pi_n = 0$, a contradiction. □

Appendix C: Numerical Solution Method

In order to solve numerically the problem, we use the value function iteration method with tolerance of 10^{-6} for convergence. In all experiments, we consider an equally spaced grid between w_{aut} and \bar{w} with 1001 gridpoints. In each iteration, we use the sequential quadratic programming algorithm embedded in the *fmincon* command in MATLAB. This method allows us to compute the policy and value functions reported in Figures 1, 2 and 3. As we vary π_n between 0.2 and 0.8 with step size of 0.02 to generate Figures 4, 5 and 6, we apply the aforementioned solution method for each value of π_n . Notice that w_{aut} varies with π_n , thus the step size between adjacent gridpoints also varies with π_n .