

Bottazzi, Giulio; Giachini, Daniele

Working Paper

Wealth and price distribution by difusive approximation in a repeated prediction market

LEM Working Paper Series, No. 2016/13

Provided in Cooperation with:

Laboratory of Economics and Management (LEM), Sant'Anna School of Advanced Studies

Suggested Citation: Bottazzi, Giulio; Giachini, Daniele (2016) : Wealth and price distribution by difusive approximation in a repeated prediction market, LEM Working Paper Series, No. 2016/13, Scuola Superiore Sant'Anna, Laboratory of Economics and Management (LEM), Pisa

This Version is available at:

<https://hdl.handle.net/10419/174522>

Standard-Nutzungsbedingungen:

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

Terms of use:

Documents in EconStor may be saved and copied for your personal and scholarly purposes.

You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.

If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.

INSTITUTE
OF ECONOMICS



Scuola Superiore
Sant'Anna

LEM | Laboratory of Economics and Management

Institute of Economics
Scuola Superiore Sant'Anna

Piazza Martiri della Libertà, 33 - 56127 Pisa, Italy
ph. +39 050 88.33.43
institute.economics@sssup.it

LEM

WORKING PAPER SERIES

Wealth and Price Distribution by Diffusive Approximation in a Repeated Prediction Market

Giulio Bottazzi [°]
Daniele Giachini [°]

[°] Institute of Economics, Scuola Superiore Sant'Anna, Pisa, Italy

2016/13

April 2016

ISSN(ONLINE) 2284-0400

Wealth and Price Distribution by Diffusive Approximation in a Repeated Prediction Market*

Giulio Bottazzi[†] and Daniele Giachini[‡]

Istituto di Economia, Scuola Superiore Sant'Anna
Piazza Martiri della Libertà 33, 56127 Pisa - Italy

April 3, 2016

Abstract

The approximate agents' wealth and price invariant densities of the prediction market model presented in Kets et al. (2014) is derived using the Fokker-Planck equation of the associated continuous-time jump process. We show that the approximation obtained from the evolution of log-wealth difference can be reliably exploited to compute all the quantities of interest in all the acceptable parameter space. When the risk aversion of the trader is high enough, we are able to derive an explicit closed-form solution for the price distribution which is asymptotically correct.

JEL Classification: C60, D53, G11, G12

1 Introduction

Kets et al. (2014) propose a model for repeated prediction markets where two agents at each round split their investment between a risky asset, that is a bet on a risky event, and a risk free security. Agents are fractional Kelly traders: they decide their investment by mixing individual belief with the prevailing market price. The authors derive their most relevant conclusions about agents' wealth and bet

*We gratefully acknowledge the funding from the European Union's Horizon 2020 research and innovation programme under grant agreement No 640772 - DOLFINS.

[†]bottazzi@sssup.it

[‡]d.giachini@sssup.it

price performing extensive simulations. Here we propose a different approach: instead of computing the quantities of interests averaging over long enough artificial time series, we compute both wealth and price invariant distribution via diffusive approximation. Following Bottazzi and Dindo (2014) we restate the original model in terms of a pure exchange economy, in which two agents repeatedly exchange two short-lived assets, one that pays 1 dollar if the event occurs and 0 otherwise, and one that does the opposite. In the non-trivial case in which both agents survive indefinitely, that is in which the repeated betting does not lead to the ruin of one of them, we derive the Fokker-Planck equation of their relative wealth. The continuous time version of the model is obtained assuming that a homogeneous Poisson process drives the arrival of the subsequent trading rounds. The second order truncation of the Kramer-Moyal expansion is characterized by an asymptotically vanishing probability current and allows for the simple derivation of an invariant distribution. We show that the obtained approximation is remarkably good in all the relevant regions of the parameter space. When the mixing parameter of the fractional Kelly traders is small enough, that is when they are sufficiently risk averse (Bottazzi and Giachini, 2016), we show that it is possible to apply the diffusive approximation directly to the price process obtaining a closed-form solution which is asymptotically correct. As a by product, we are able to prove that when the mixing parameter goes to zero, the price converge to the true probability of the risky event, as suggested in Kets et al. (2014).

2 The model

Time is discrete and at every date t one of the two possible states of the world $\{0, 1\}$ is realized following a Bernoulli process s_t . The probability π^* that $s_t = 1$ is supposed unknown. Two agents $i \in \{1, 2\}$ posses different individual opinions about this probability, π^i , and bet on the outcome by exchanging two assets in unit supply: the first asset pays 1 only when $s_t = 1$ and the second asset pays 1 only when $s_t = 0$. Agents behave according to a so-called fractionally Kelly rule (MacLean et al., 1992, 2005), investing a fraction of wealth

$$\alpha^i(p_t) = c\pi^i + (1 - c)p_t, \quad (1)$$

on the first asset and what remains, $1 - \alpha^i(p_t)$ on the second. The rule (1) correspond to a linear combination of the Kelly rule, shortly described by the prescription “bet your believes”, and the risk-less strategy of investing proportionally to the prevailing market price. The “mixing” coefficient c is assumed equal among the two bettors, and without loss of generality we pose $\pi^2 > \pi^1$. The system is closed: no wealth is consumed and the parimutuel betting procedure redistributes the total amount bet among the winners, proportionally to the amount they wagered.

Normalizing the total wealth to 1 and denoting with w_t the wealth of the first agent, this procedure implies to the price fixing condition (Bottazzi and Giachini, 2016)

$$p_t = \pi^1 w_{t-1} + \pi^2 (1 - w_{t-1}). \quad (2)$$

After the state s_t is realized, the new wealth level becomes

$$w_t = \begin{cases} \frac{\alpha_t^1 w_{t-1}}{p_t} & \text{if } s_t = 1, \\ \frac{(1 - \alpha_t^1) w_{t-1}}{1 - p_t} & \text{if } s_t = 0. \end{cases} \quad (3)$$

As discussed in Bottazzi and Giachini (2016), the system has two possible long-run outcomes. One possibility is that one agent dominates and ends up owning the entire wealth, $\lim_{t \rightarrow \infty} w_t = 0, 1$. In this case the market price converges to the belief of the dominating agent, $\lim_{t \rightarrow \infty} p_t = \pi^2, \pi^1$. Alternatively, both agents survive, the wealth shares persistently fluctuate and p_t keeps moving in (π^1, π^2) . In the first case, the analysis of the asymptotic distribution of wealth and price is trivial. Thus, in the rest of this paper, we focus on the region of the parameter space in which the two agents both survive. A necessary and sufficient condition for this to happen is that (Bottazzi and Giachini, 2016)

$$\begin{aligned} \pi^* \log(c\pi^j + (1 - c)\pi^i) + (1 - \pi^*) \log(1 - c\pi^j - (1 - c)\pi^i) \geq \\ \pi^* \log(\pi^i) + (1 - \pi^*) \log(1 - \pi^i), \end{aligned} \quad (4)$$

for $i, j = 1, 2$ and $i \neq j$. This condition is always satisfied for c sufficiently small.

3 Poisson arrival and relative wealth distribution

It is convenient to rewrite the evolution of the model in terms of the relative log wealth $z_t = \log w_t - \log(1 - w_t)$ so as to obtain an unbounded process on the real line. This model is discrete in time and we proceed to obtain a continuous time version using a homogeneous Poisson process. Assume that in a short period of time δt there is a probability $\lambda \delta t$ that a new trading round takes place, a new state of the world is realized and the system is updated according to $z_{t+\delta t} = g(z_t, s_t)$ with

$$g(z, s) = z + \log \frac{\alpha^1(p)(2s - 1) + 1 - s}{\alpha^2(p)(2s - 1) + 1 - s}, \quad \text{and} \quad p = \frac{\pi^1 e^z + \pi^2}{1 + e^z}.$$

At the same time, there is a probability $1 - \lambda\delta t$ that nothing happens and $z_{t+\delta t} = z_t$. In terms of the probability density of the process $f_z(x, t) = d\text{Prob}\{z_t \leq x\}/dx$, the infinitesimal Chapman-Kolmogorov equation reads

$$f_z(x, t + \delta t) = (1 - \lambda\delta t)f_z(x, t) + \lambda\delta t \int dy \int d\pi(s) f_z(y, t) \delta(x - g(y, s)) ,$$

where $\delta(\cdot)$ is the Dirac delta function and $\pi(s)$ the probability measure of the underlying state process. Re-arranging terms and taking the limit for $\delta t \rightarrow 0$ one obtains the master equation

$$\frac{\partial}{\partial t} f_z(x, t) = \lambda \int dy f_z(y, t) \mathcal{K}(x, y) , \quad \mathcal{K}(x, y) = \int d\pi(s) \delta(x - g(y, s)) - \delta(x - y) .$$

By truncating its Kramer-Moyal expansion

$$\frac{\partial}{\partial t} f(x, t) = \lambda \sum_{n=1}^{\infty} \left(-\frac{\partial}{\partial x} \right)^n D_n(x) f_z(x, t) , \quad D_n(x) = \frac{1}{n!} \int dz (z - x)^n \mathcal{K}(z, x)$$

at the second term, the Fokker-Planck equation is derived,

$$\frac{\partial}{\partial t} f_z(x, t) = -\frac{\partial}{\partial x} \left(\lambda D_1(x) f(x, t) - \frac{\partial}{\partial x} (\lambda D_2(x) f(x, t)) \right) , \quad (5)$$

with

$$D_1 = \left[\pi^* \log \frac{c\pi^1 + (1-c)p(x)}{c\pi^2 + (1-c)p(x)} + (1 - \pi^*) \log \frac{1 - c\pi^1 - (1-c)p(x)}{1 - c\pi^2 - (1-c)p(x)} \right] ,$$

$$D_2 = \frac{1}{2} \left[\pi^* \left(\log \frac{c\pi^1 + (1-c)p(x)}{c\pi^2 + (1-c)p(x)} \right)^2 + (1 - \pi^*) \left(\log \frac{1 - c\pi^1 - (1-c)p(x)}{1 - c\pi^2 - (1-c)p(x)} \right)^2 \right] .$$

Since the process is unbounded, we can assume the natural boundary conditions¹ and derive the invariant distribution of (5), formally

$$f_z(x) = \frac{f_0}{D_2(x)} \exp \left(\int_{x_0}^x dy \frac{D_1(y)}{D_2(y)} \right) , \quad (6)$$

where f_0 is a normalization constant. The integral in (6) can be easily performed numerically for any acceptable parametrization. This is much faster than performing Monte Carlo simulations² and has the advantage of directly providing an approximation of the long-run invariant distribution. As can be seen in Fig. 1, the agreement of (6) with the invariant distribution is very good, for any set of parameter values.

¹The probability current is asymptotically vanishing in all cases in which the conditions in (4) are satisfied.

²Due to the strong autocorrelation of the process, especially when c is small, one has to generate very long series before obtaining reliable Monte Carlo estimates.

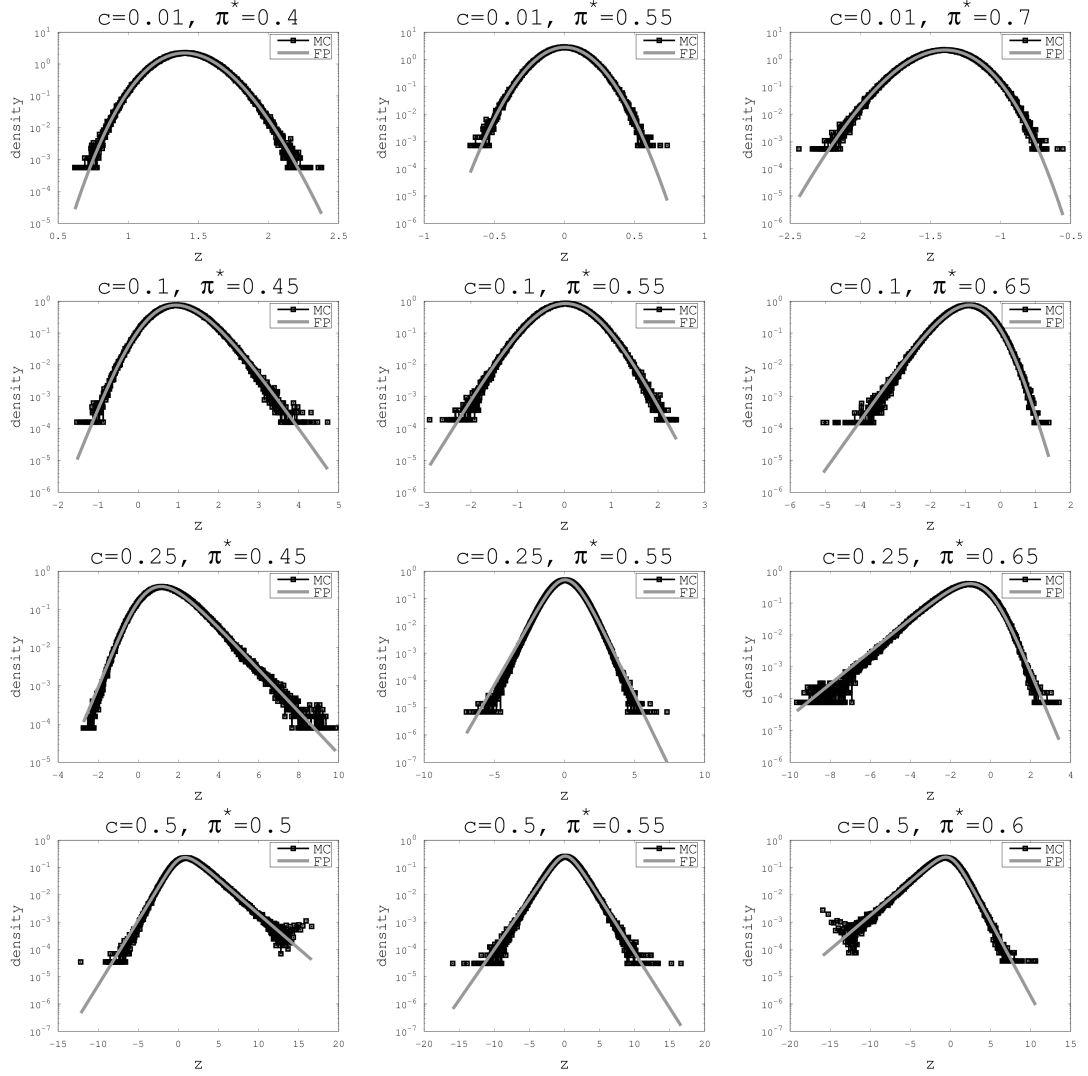


Figure 1: Approximated (FP) and simulated (MC) invariant density for $\pi^1 = 0.3$, $\pi^2 = 0.8$ and different values of c and π^* in semi-log scale. In terms of cumulated distributions, the difference between the recorded densities and the invariant distributions found via diffusive approximation is on average 0.0059 and never larger than 0.0561.

4 Small c and Price Distribution

The dynamics of the system can also be studied in terms of price. From (1), (2) and (3) it is

$$p_{t+1} = p_t + \frac{c(\pi^2 - p_t)(p_t - \pi^1)}{p_t + s_t - 1} . \quad (7)$$

Again, postulating a Poisson arrival of trading rounds we can obtain a continuous time master equation. The Kramer-Moyal expansion becomes now a power expansion in c

$$D_n^p(x) = c^n \int d\pi(s) \left(\frac{(\pi^2 - x)(x - \pi^1)}{x + s - 1} \right)^n = c^n M_n(x) . \quad (8)$$

So we expect the diffusive approximation to improve the lower the value of c . At the same time, however, the diffusive approximation based on price dynamics breaks down for higher value of c . To see it, let's first formally derive the stationary solution of the Fokker-Planck equation (5) for $f_p(x, t) = d \text{Prob} \{z_t \leq x\} / dx$ with the coefficient as in (8). It turns out that in this case the integral in the exponent of (6) can be computed explicitly to give

$$f_p(x) = \frac{2f_0}{c^2} \frac{(x - \pi^1)^{2\frac{A_1}{c}-2}}{(\pi^2 - x)^{2\frac{A_2}{c}+2}} (\pi^*(1 - \pi^*) + (\pi^* - x)^2)^{-\frac{1+A_1-A_2}{c}-1} x^2 (1 - x)^2 \exp \left\{ 2 \left(\frac{A_2}{\pi^2 - \pi^*} + \frac{A_1}{\pi^* - \pi^1} \right) \frac{\sqrt{\pi^*(1 - \pi^*)}}{c} \arctan \left(\frac{\pi^* - x}{\sqrt{\pi^*(1 - \pi^*)}} \right) \right\} , \quad (9)$$

where f_0 is a normalization constant and with

$$A_i = \frac{(\pi^* - \pi^i)(1 - \pi^i)\pi^i}{(\pi^2 - \pi^1)(\pi^*(1 - \pi^*) + (\pi^* - \pi^i)^2)} , \quad i = 1, 2 .$$

The formal solution (9) represents a good approximation of the invariant distribution of the bounded price process only if the associated potential $\Phi_p(x) = -\log f_p(x)$ diverges to plus infinity at the boundaries, that is when the reflecting barrier hypothesis is satisfied (Risken, 1984). From (9) it is immediate to see that

$$\begin{aligned} \lim_{x \rightarrow \pi^1+} \Phi_p(x) &\sim \lim_{x \rightarrow \pi^1+} \left(1 - \frac{A_1}{c} \right) \log(x - \pi^1) , \\ \lim_{x \rightarrow \pi^2-} \Phi_p(x) &\sim \lim_{x \rightarrow \pi^2-} \left(1 + \frac{A_2}{c} \right) \log(\pi^2 - x) . \end{aligned}$$

Notice that A_1 is by definition positive and A_2 negative. Thus, we can conclude that (9) represents a good approximation only when $c < A_1, -A_2$. It is important

to remark that the absorbing barrier condition, that is a potential diverging to minus infinity, which characterizes (9) for large values of c , is a mere artifact of the diffusive approximation. In fact, as long as (4) are satisfied, we know that the process is stationary and both barriers are reflecting. The diffusive approximation of the price dynamics seems to be less reliable than the diffusive approximation of the wealth ratio: when c is large, the truncation of the Kramer-Moyal expansion at the second term discards relevant information about the behavior of the price process and misinterprets the nature of the boundaries.

We conclude the paper addressing the hypothesis, advanced in Kets et al. (2014), that when the mixing parameter becomes asymptotically small, $c \rightarrow 0$, then the prevailing market price converges to the true value, $p \rightarrow \pi^*$. Formally, the invariant price density, with reference to (5) and (8), can be written as

$$f_p(x) = \frac{f_0}{c^2 M_2(x)} \exp \left(\frac{1}{c} \int_{\pi^*}^x dy \frac{M_1(y)}{M_2(y)} \right) .$$

Notice that $M_1(x) > 0$ when $x < \pi^*$ and $M_1(x) < 0$ when $x > \pi^*$, hence the expression in the exponent is never positive and has its maximum for $x = \pi^*$. We can thus apply the asymptotic expansion of the Laplace-type integral to obtain

$$\lim_{c \rightarrow 0} \int dz f_p(x) x^n \sim (\pi^*)^n f_0 \sqrt{\frac{2\pi}{c^3 M_2(\pi^*) |M_1'(\pi^*)|}}$$

which, by the normalization condition of the probability density, reduces to

$$\lim_{c \rightarrow 0} \int dz f_p(x) x^n \sim (\pi^*)^n$$

implying that $\lim_{c \rightarrow 0} f_p(x) = \delta(x - \pi^*)$, confirming the Kets et al. (2014) hypothesis.

5 Conclusion

The diffusive approximation of both the wealth and the price invariant distribution for the repeated prediction market model of Kets et al. (2014) were derived. Our method relies on embedding the discrete time model in continuous time via a Poisson process and computing the diffusive approximation of the state variable of interest via the Fokker-Planck equation. We showed that the derived approximation for the invariant distribution of the log wealth ratio is very good in the entire relevant region of the parameter space where (4) are satisfied, so that the asymptotic distribution of all other quantities can be reliably derived via a density function transformation. We also obtained a closed form expression for the

diffusive approximation of the price invariant density when the mixing parameter c is sufficiently small. This expression confirms the hypothesis, advanced in Kets et al. (2014), that the prevailing market price converges to the true probability value when the value of c goes to zero.

References

- Bottazzi, G. and P. Dindo (2014). Evolution and market behavior with endogenous investment rules. *Journal of Economic Dynamics and Control* 48, 121–146.
- Bottazzi, G. and D. Giachini (2016). Far from the madding crowd: Collective wisdom in prediction markets. LEM Papers Series 2016/14, Laboratory of Economics and Management (LEM), Sant’Anna School of Advanced Studies, Pisa, Italy.
- Kets, W., D. M. Pennock, R. Sethi, and N. Shah (2014). Betting strategies, market selection, and the wisdom of crowds. In *Twenty-Eighth AAAI Conference on Artificial Intelligence*.
- MacLean, L., W. T. Ziemba, and G. Blazenko (1992). Growth versus security in dynamic investment analysis. *Management Science* 38(11), 1562–1585.
- MacLean, L. C., W. T. Ziemba, and Y. Li (2005). Time to wealth goals in capital accumulation. *Quantitative Finance* 5(4), 343–355.
- Risken, H. (1984). *Fokker-planck equation*. Springer.