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Unit Root Testing in ARMA Models: A Likelihood Ratio Approach*

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Abstract: In this paper I propose a Likelihood Ratio test for a unit root (LR) with a local-to-unity Autoregressive parameter embedded in ARMA(1,1) models. By dealing explicitly with dependence in a time series through the Moving Average, as opposed to the long Autoregressive lag approximation, the test shows gains in power and has good small-sample properties. The asymptotic distribution of the test is shown to be independent of the short-run parameters. The Monte Carlo experiments show that the LR test has higher power than the Augmented Dickey Fuller test for several sample sizes and true values of the Moving Average parameter. The exception is the case when this parameter is very close to -1 with a considerably small sample size.

Keywords: Likelihood ratio test; ARMA model; Unit root test.

JEL Classification: C22

Resumen: En este artículo propongo una prueba de raíz unitaria de razón de verosimilitud (LR) con un parámetro Autorregresivo localmente cercano a la unidad en modelos ARMA(1,1). Al tratar explícitamente la dependencia en una serie de tiempo con una Media Móvil, y no una aproximación Autorregresiva con un gran número de rezagos, la prueba muestra ganancias en términos de su potencia y tiene propiedades deseables en pequeñas muestras. Se demuestra que la distribución asintótica de la prueba es independiente de parámetros de corto plazo. Los experimentos Monte Carlo muestran que la prueba LR tiene mayor potencia que la prueba de Dickey-Fuller Aumentada para varios tamaños de muestra y valores del verdadero parámetro de la Media Móvil. La excepción es el caso cuando este parámetro es muy cercano -1 con un tamaño de muestra considerablemente reducido.

Palabras Clave: Prueba de razón de verosimilitud; Modelo ARMA; Prueba de raíz unitaria.

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1 Introduction

The econometric analysis of an economic time series requires the econometrician to decide how to model two key properties of said series: (i) the inherent dependence between the observations, and (ii) whether the series is stationary or non-stationary by means of a unit root test. Under a small set of assumptions, the Autoregressive Moving Average (*ARMA*) set-up presents itself as a parsimonious and flexible way to address the dependence since it is parametric and, typically, easy to estimate through well known techniques.

Despite the popularity among empiricists of the Autoregressive (*AR*) model, there are a number of arguments in favour of using an *ARMA* model instead when analysing an error process that features serial correlation. First, modelling explicitly this serial correlation is of value for the econometrician. The temporal aggregation, stemming from the patterns of data collection, has been shown to hide information about the cycle of an economic time series (see [Rossana and Seater \(1995\)](#)); thus creating serial correlation. Second, the same authors conclude that a mixed *ARMA* model will be better addressing several macroeconomic time series. Third, a mixed *ARMA* model will yield a parsimonious approximation to the actual process. This in turn improves the power of the unit root test.

A great bulk of available unit root tests, however, is based on estimating “long” *AR* models, thereby only approximating any dependence of the error process.¹ Among the most popular unit root tests are the Dickey-Fuller (*DF*) test and its augmented version (*ADF*) due to [Dickey and Fuller \(1979\)](#) and [Said and Dickey \(1984\)](#), respectively.

While prominent in its insights and widespread use, these tests present some drawbacks. The *DF* test is based on the assumption of i.i.d. Gaussian disturbances, but many economic time series feature some form of dependence. The *ADF* deals with dependence, but this requires the *AR* order of the linear regression to increase with the sample size. This, however, has a cost: by increasing the *AR* order, the power of the *ADF* - that is, the probability of rejecting the null hypothesis- decreases.

The latter introduces a trade-off between testing with low power, on the one hand, and dealing appropriately with the serial correlation on the other. Up to now this issue has been circumvented by choosing the *AR* order with an information criteria. The test I propose in this paper does not possess said trade-off since dependence is modelled explicitly, and it has better power properties than the *ADF* test.

¹See [Stock \(1994\)](#) and [Haldrup and Jansson \(2007\)](#) for a thorough review of the unit root tests available. Semi-parametric tests such as that advocated by [Phillips and Perron \(1988\)](#) are not discussed since the test proposed here is fully parametric.

In their seminal work [Elliott, Rothenberg, and Stock \(1996\)](#), ERS henceforth, proposed the Likelihood Ratio test for the presence of a unit root -the Point Optimal test- and the $DF - GLS$ test. Their tests have better power properties than the ADF test, particularly dealing with deterministic components - a constant mean or a linear trend. ERS worked on a local-to-unity framework, and this allows easy comparisons across models. The framework also allows simpler ways to write both the test statistic and its asymptotic distribution. Through the Neyman-Pearson Lemma, ERS get an upper bound for the power of the test in an $AR(p)$ process, which overlaps the Gaussian Power Envelope. The ERS tests have been part of the unit root testing practice they appeared and are now included in econometrics software as a standard component.

The mechanics in the computation of the $DF - GLS$ test statistic can be summarised in two steps: (i) de-trend the economic time series through Generalised Least Squares (GLS) or Maximum Likelihood (ML), (ii) test the de-trended series for a unit root with the ADF test. This draws the econometrician to face the same trade-off as in the implementation of the ADF test. The test I propose differs from those in ERS in two respects: (i) the avoidance of the aforementioned trade-off and (ii) the computation is made in one-step.

In this paper I outline the construction of a likelihood ratio unit root test for $ARMA(1, 1)$ models with no deterministic terms (i.e. a constant and a trend). Focusing in the particular case of the $ARMA(1, 1)$ may seem restrictive, but can be justified on the following basis. If the data generating process behind an economic time series is suspected to be a continuous time process (e.g. consumption or pricing decisions), but the econometrician only has access to data collected in discrete points in time, the *exact discrete* representation of the data generating process will often be an $ARMA(1, 1)$ model as shown by [Bergstrom \(1984\)](#) and [Chambers \(2009\)](#). This model also buys a number of benefits. First, the main parameters driving the dynamics are included. Second, consistency results and asymptotic distributions are easier to present. Third, as in [Said and Dickey \(1985\)](#), the $ARMA(1, 1)$ case provides a foundation for more general cases.

This paper contains 6 additional sections. Section 2 outlines the DGP and the model to be estimated. In section 3 I present the set-up to estimate through ML and then derive the asymptotic behaviour of the estimates. The LR test is presented in section 4, followed by the empirical analysis in section 5. Section 6 has an application of the test to inflation series in several countries. Finally, section 7 presents some concluding remarks.

Notation

Throughout this paper I use the following notation: \longrightarrow denotes convergence; \longrightarrow_p convergence in probability; \longrightarrow_d convergence in distribution; \Rightarrow weak convergence. For matrix operators I use: $\|A\| = \text{tr}^{1/2}\{A'A\}$; $\|A\|_1 = \sum_t \sum_s |a_{ts}|$; $\det |A|$ for determinant of matrix A ; $\text{tr}\{A\}$ for trace of matrix A .

2 The Data Generating Process

In this section I introduce the elements on which the rest of the paper is built. In particular, the probability space, the data generating process and the error process are introduced. This will pave the way for the estimation set-up in the following sections.

The framework on which I develop the analysis requires a sample space Ω_n and \mathcal{F}_n , the σ -field of Ω_n . Moreover, Θ is the parameter space with typical element θ_n . P_{n,θ_n} is a probability measure for n observations of a time series indexed by θ_n . Thus, the triple $(\Omega_n, \mathcal{F}_n, P_{n,\theta_n})$ forms a complete probability space underlying the DGP as stated in the next assumption.

Assumption 2.A (Data Generating Process). *The observed time series, $\{y_t\}_{t=0}^n$, is generated by*

$$\Delta y_t = (\rho_0 - 1)y_{t-1} + u_{0,t}, \quad (2.1)$$

$$\rho_0 - 1 = \frac{c}{n}, \quad c < 0. \quad (2.2)$$

The local-to-unity set-up, given by (2.2), goes back at least to Phillips (1987a) and has been exploited greatly in the unit root testing literature, while the stochastic behaviour of the error process $\{u_{0,t}\}_{t=0}^n$ in (2.1) can be thought as a draw from the probability space showing linear dependence with behaviour summarised by the following assumption.

Assumption 2.B (Error Process). *The error process $\{u_{0,t}\}_{t=0}^n$, depends on a parameter vector $\theta_{20} = (\alpha_0, \sigma_0^2)'$ and satisfies: (i) $\varepsilon_t \sim i.i.d(0, \sigma_0^2)$. (ii) $\varepsilon_s = 0$ for $s \leq 0$. (iii) $u_{0,t} = \varepsilon_t - \alpha_0 \varepsilon_{t-1}$. (iv) $\sup_t E|u_{0,t}|^\delta < \infty$ for $\delta > 2$.*

Both (i) and (ii) are standard in econometric analysis, while (iii) sets up the MA and (iv) is necessary in view that a MA process is a special case of a mixing process where δ controls the dispersion as discussed in Phillips (1987a). Moreover, the assumed value of δ is needed as the ML estimators will typically involve the second moments of σ_0^2 . The error process

sequence $\{u_{0,t}\}_{t=0}^n$ can also be written in vector form by stacking its last n elements into the $n \times 1$ vector u_0 .

Write the observed time series in vector form stacking the observations so that $\Delta y = (\Delta y_1, \dots, \Delta y_n)'$ is a $n \times 1$ vector, similarly $y_{-1} = (y_0, \dots, y_{n-1})'$ and $u = (u_1, \dots, u_n)'$, to get

$$u(\theta_{1n}) = \Delta y - (\theta_{1n} - 1)y_{-1}.$$

Assumption 2.B implies that the error vector, u_0 , has a covariance matrix $\Gamma(\theta_{20})$ with typical jk element $\gamma(j - k) = Cov(u_{0,t-j}, u_{0,t-k})$. Moreover, $\Gamma(\theta_{20})$ satisfies: (i) $\gamma(0) = (1 + \alpha_0^2)\sigma_0^2$. (ii) $\gamma(1) = \gamma(-1) = -\alpha_0\sigma_0^2$. (iii) $\gamma(j - k) = 0$ for all $|j - k| > 1$. The latter implies that $\Gamma(\theta_{20})$ is a band-Toeplitz matrix containing $\gamma(j)$ in its j th-band. Finally, define the (finite) long run variance $\sigma_u^2 = \sum_{k=-\infty}^{\infty} \gamma_k = (1 - \alpha_0)^2\sigma_0^2$.

3 Maximum Likelihood Estimation and Asymptotics

In this section the estimation is presented along with a series of preliminary definitions and Lemmas that will ease the introduction of the main asymptotic results. After the set-up of the ML estimation problem, I show that the obtained estimators are consistent, a task that will require to split the objective function in two parts: one containing long-run parameters and a second part with short-run parameters driving the dependence. The section concludes with the derivation of the asymptotic distribution of the ML estimator.

ML estimation requires the properties of the parameter space to be clearly defined, which I present in the next assumption.

Assumption 3.A (Parameter Space). (a) The parameter space Θ is (i) convex and (ii) compact. (b) Write $\Theta = \Theta_1 \times \Theta_2$, where Θ_1 contains the long run parameters (i.e. $\theta_{1n} = \rho_n$) and Θ_2 contains only short-run parameters (i.e. $\theta_{2n} = (\alpha_n, \sigma_n^2)'$). (c) Θ_2 contains only elements that ensure $\Gamma^{-1}(\theta_{2n})$ exists.

To estimate the parameter vector $\theta_n = (\theta_{1n}, \theta_{2n})'$ write the log-likelihood function for $\{u_{0,t}\}_{t=0}^n$,²

$$l^n(\theta_n) = -\frac{n}{2} \ln |2\pi| - \frac{1}{2} \ln \det |\Gamma(\theta_{2n})| - \frac{1}{2} u(\theta_{1n})' \Gamma^{-1}(\theta_{2n}) u(\theta_{1n}), \quad (3.1)$$

²The likelihood function is constructed “as if” the disturbances are normally distributed, anticipating an approximation based on a limiting distribution.

or equivalently, define the loss function $Q_n(\theta_n) = -2l^n(\theta_n) - n \ln |2\pi|$ so that the objective is now to find the minimizer of

$$Q_n(\theta_n) = \ln \det |\Gamma(\theta_{2n})| + u(\theta_{1n})' \Gamma^{-1}(\theta_{2n}) u(\theta_{1n}). \quad (3.2)$$

Let $\Gamma_n = \Gamma(\theta_{2n})$ to ease notation, and following [Saikkonen \(1995\)](#) add and subtract $\theta_{10} y_{-1}$ from $u_n = u(\theta_{1n})$ to obtain

$$Q_n(\theta_n) = Q_{1n}(\theta_n) + Q_{2n}(\theta_{2n}),$$

where

$$Q_{1n}(\theta_n) = (\theta_{1n} - \theta_{10})^2 y_{-1}' \Gamma_n^{-1} y_{-1} - 2(\theta_{1n} - \theta_{10}) y_{-1}' \Gamma_n^{-1} u_0, \quad (3.3)$$

$$Q_{2n}(\theta_{2n}) = \ln \det |\Gamma_n| + u_0' \Gamma_n^{-1} u_0, \quad (3.4)$$

where by making explicit the distance $(\theta_{1n} - \theta_{10})$, consistency results can be clearly presented and additional advantages are presented in the form of two remarks for future reference.

Remark 3.B. *The loss function $Q_{1n}(\theta_n)$ depends on both long and short-run parameters. The loss function $Q_{2n}(\theta_{2n})$ as defined in (3.4) depends only on short-run parameters.*

Remark 3.C. *$Q_{1n}(\theta_0) = 0$. $Q_{2n}(\theta_{2n})$ is equivalent to the estimation problem of a stationary MA(1) process.*

3.1 Preliminaries

To present the main result on consistency of this paper, I need to introduce a number of preliminary results and notation. The Wiener Processes $\mathcal{W}(r)$, as characterised in [White \(2001\)](#), and the Ornstein-Uhlenbeck Process defined by [Phillips \(1987b\)](#).

Definition 3.D (Wiener Process). *Within $(\Omega_n, \mathcal{F}_n, P_{n, \theta_n})$. Then $\mathcal{W} : [0, \infty) \times \Omega \rightarrow \mathbb{R}$ is a Wiener Process if for $r \in [0, \infty)$, $\mathcal{W}(r, \cdot)$ is measurable- \mathcal{F}_n , and in addition:*

- i. *The process starts at zero: $P_{n, \theta_n}[\mathcal{W}(0, \cdot) = 0] = 1$.*
- ii. *The increments are independent: If $0 \leq r_0 \leq r_1 \leq \dots \leq r_k < \infty$, then $\mathcal{W}(r_i, \cdot) - \mathcal{W}(r_{i-1}, \cdot)$ is independent of $\mathcal{W}(r_j, \cdot) - \mathcal{W}(r_{j-1}, \cdot)$, $j = 1, \dots, k, j \neq i$ for all $i = 1, \dots, k$.*

iii. The increments are normally distributed: For $0 \leq a \leq b < \infty$ the increment $\mathcal{W}(b, \cdot) - \mathcal{W}(a, \cdot)$ is distributed as $N(0, b - a)$.

To ease notation write: $\mathcal{W} = \mathcal{W}(r)$ and $\int_0^\infty \mathcal{W} = \int_0^\infty \mathcal{W}(r)dr$.

Definition 3.E (Ornstein-Uhlenbeck Process). For r and s real numbers, the functional $\mathcal{J}_c(r)$ of the form

$$\begin{aligned}\mathcal{J}_c(r) &= \int_0^r \exp[(r-s)c]d\mathcal{W}(s) \\ &= \mathcal{W}(r) + c \int_0^r \exp[(r-s)c]\mathcal{W}(s)ds,\end{aligned}$$

is the Ornstein-Uhlenbeck Process associated with c . The process satisfies:

- i. Being the solution to the stochastic differential equation $d\mathcal{J}_c(r) = c\mathcal{J}_c(r)dr + d\mathcal{W}(r)$.
- ii. $\mathcal{J}_c(0) = 0$.
- iii. $\mathcal{J}_c(r)$ is distributed as $N\left(0, \frac{\exp[2rc]-1}{2c}\right)$.

To ease notation write $\mathcal{J}_c = \mathcal{J}_c(r)$ and $\int_0^1 \mathcal{J}_c = \int_0^1 \mathcal{J}_c(r)dr$.

Remark 3.F. With \mathcal{J}_c as given in Definition 3.E, in the particular case of $c = 0$, \mathcal{J}_c collapses to \mathcal{W} .

Taking advantage of Γ_0^{-1} being itself a covariance matrix (i.e. symmetric and positive definite), I introduce the following Definition and Lemma.

Definition 3.G (Spectral Density of Γ_0^{-1}). Let g_{ts} denote the ts element of Γ_0^{-1} and with spectral function $g(\omega)$ with $\omega \in (-\pi, \pi]$ so that each of its elements can be recovered using

$$\begin{aligned}g_{ts} &= \int_0^1 e^{i(t-s)\omega} g(\omega)d\omega, \\ g(\omega) &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ij\omega} g_j.\end{aligned}$$

Lemma 3.H. If $\{y_t\}_{t=0}^n$ and $\{u_{0,t}\}_{t=0}^n$ are given by assumptions 2.A and 2.B, then

$$n^{-1}y'_{-1}\Gamma_n^{-1}u_0 \Rightarrow 2\pi g(0) \left[\sigma_u^2 \int_0^1 \mathcal{J}_c d\mathcal{W} + \frac{\sigma_u^2 - \gamma_0}{2} \right] + \sum_{j=-\infty}^{\infty} g_j \Phi_j, \quad (3.5)$$

$$n^{-2}y'_{-1}\Gamma_n^{-1}y_{-1} \Rightarrow 2\pi g(0)\sigma_u^2 \int_0^1 \mathcal{J}_c^2, \quad (3.6)$$

where $\Phi_j = 0$ if $j = 0$; $\Phi_j = \sum_{k=0}^{j-1} \gamma_k$ if $j > 0$; and $\Phi_j = -\sum_{k=1}^{|j|} \gamma_k$ if $j < 0$.

Proof. I prove first (3.5). Start by writing

$$\begin{aligned}
n^{-1}y'_{-1}\Gamma_n^{-1}u_0 &= n^{-1}\sum_{t=1}^n\sum_{s=1}^ny_{t-1}u_{0,s}g_{ts} \\
&= n^{-1}\sum_{t=1}^n\sum_{s=1}^ny_{t-1}u_{0,s}\int_0^1e^{i(t-s)\omega}g(\omega)d\omega \\
&= n^{-1}\sum_{t=1}^n\sum_{s=1}^ny_{t-1}u_{0,s}\sum_{j=-\infty}^{\infty}g_j\int_0^1e^{i(t-s-j)\omega}d\omega \\
&= \sum_{j=-\infty}^{\infty}g_j\left[n^{-1}\sum_{t=1}^ny_{t-1}u_{0,t-j}\right], \tag{3.7}
\end{aligned}$$

where the second and third lines are obtained by direct substitution, and the fourth line follows from orthogonality of the integral for $s = t - j$. Now I take cases, if $j = 0$ in (3.7)

$$n^{-1}\sum_{t=1}^ny_{t-1}u_{0,t-j}\Rightarrow\sigma_u^2\int_0^1\mathcal{J}_c d\mathcal{W}+\frac{\sigma_u^2-\gamma_0}{2}, \tag{3.8}$$

from Lemma 1 in Phillips (1987b). If $j > 0$ I need to use the backward solution of the stochastic difference equation (2.1) at $t - 1$,

$$y_{t-1}=\theta_{10}^jy_{t-j-1}+\sum_{k=0}^{j-1}\theta_{10}^k u_{0,t-1-k},$$

which together with the bracket expression in (3.7) yields

$$n^{-1}\sum_{t=1}^ny_{t-1}u_{0,t-j}=\theta_{10}^jn^{-1}\sum_{t=1}^ny_{t-j-1}u_{0,t-j}+n^{-1}\sum_{t=1}^n\sum_{k=0}^{j-1}\theta_{10}^k u_{0,t-1-k}u_{0,t-j},$$

the first term has as asymptotic limit (3.8) since $\lim_{n\rightarrow\infty}\theta_{10}^k=1$ for all k , and the second term converges to

$$n^{-1}\sum_{t=1}^n\sum_{k=0}^{j-1}\theta_{10}^k u_{0,t-1-k}u_{0,t-j}\xrightarrow{p}\sum_{k=0}^{j-1}\gamma_k.$$

Finally, if $j < 0$ I need the forward solution of the stochastic difference equation, this is

$$y_{t-j} = \theta_{10}^{-j+1} y_{t-1} + \sum_{k=0}^{|j|} \theta_{10}^k u_{0,t+k},$$

which makes

$$n^{-1} \sum_{t=1}^n y_{t-1} u_{0,t-j} = \theta_{10}^{-(-j+1)} n^{-1} \sum_{t=1}^n y_{t-j-1} u_{0,t-j} - \theta_{10}^{-(-j+1)} n^{-1} \sum_{t=1}^n \sum_{k=0}^{|j|} \theta_{10}^k u_{0,t+k} u_{0,t-j},$$

where, again, the first term converges to (3.8) and the second has limit

$$-\theta_{10}^{-(-j+1)} n^{-1} \sum_{t=1}^n \sum_{k=0}^{|j|} \theta_{10}^k u_{0,t+k} u_{0,t-j} \longrightarrow_p - \sum_{k=1}^{|j|} \gamma_k,$$

adding across values of j completes the proof of (3.5).

To prove (3.6) start by noting that

$$\begin{aligned} n^{-2} y'_{-1} \Gamma_0^{-1} y_{-1} &= n^{-2} \sum_{t=1}^n \sum_{s=1}^n y_{t-1} y_{s-1} g_{ts} \\ &= \sum_{j=-\infty}^{\infty} g_j \left[n^{-2} \sum_{t=1}^n y_{t-1} y_{t-j-1} \right], \end{aligned}$$

if I follow the same steps used to obtain (3.7). If $j = 0$ I use again Lemma 1 from Phillips (1987b) to obtain

$$n^{-2} y'_{-1} \Gamma_0^{-1} y_{-1} \Rightarrow \sigma_u^2 \int_0^1 \mathcal{J}_c^2. \quad (3.9)$$

When $j > 0$, using the backward solution note that I can write

$$n^{-2} \sum_{t=1}^n y_{t-1} y_{t-j-1} = \theta_{10}^{-j} n^{-2} \sum_{t=1}^n y_{t-1}^2 - n^{-2} \theta_{10}^{-j} \sum_{t=1}^n y_{t-1} \sum_{k=0}^{j-1} \theta_{10}^k u_{0,t-1-k},$$

the first term converges to (3.9) while the second is $o_p(1)$ in view of (3.5). The same argument shows that if $j < 0$ the limit I get is (3.9), adding across j completes the proof. ■

Now I define a few more objects, following Saikkonen (1995) let $\eta > 0$ (not necessarily

the same across results), $\nu > 0$, $\delta_1, \delta_2 > 0$. Define the closed balls

$$B_{1,\nu} = \left\{ \theta_{1n} \in \Theta_1 : |\theta_{1n} - \theta_{10}| \leq \frac{\delta_1}{n^\nu} \right\}, \quad (3.10)$$

$$\bar{B}_{1,\nu} = \left\{ \theta_{1n} \in \Theta_1 : |\theta_{1n} - \theta_{10}| \geq \frac{\delta_1}{n^\nu} \right\}, \quad (3.11)$$

$$\bar{B}_2 = \{ \theta_{2n} \in \Theta_2 : \|\theta_{2n} - \theta_{20}\| \geq \delta_2 \}. \quad (3.12)$$

Also let

$$\begin{aligned} M_1 &= |y'_{-1} \Gamma_n^{-1} u_0| = O_p(n), \\ M_2 &= |y'_{-1} \Gamma_n^{-1} y_{-1}| = O_p(n^2), \end{aligned}$$

as implied by Lemma 3.H, and

$$\bar{M}_i = \sup_{\theta_{2n}} M_i; \underline{M}_i = \inf_{\theta_{2n}} M_i.$$

3.2 Consistency of $\hat{\theta}_n$

Proving consistency of the ML estimator $\hat{\theta}_{1n}$ in the framework outlined above is a task considerably different from that in the textbook case where $\{y_t\}_{t=0}^n$ is stationary. The main reason, as explained by Chambers and McCrorie (2007) who found themselves in the same conundrum, is that $Q_n(\theta_n)$ does not satisfy weak uniform convergence. Specifically, $Q_n(\cdot)$ converges at different rates in different directions of the parameter space. To overcome this difficulty they rely on the results developed by Saikkonen (1995), and I will use them here extensively.

Proposition 3.I (Consistency of $\hat{\theta}_n$). *Let $\nu \in [0, 1)$. Given assumptions 2.A and 2.B, ML estimate $\hat{\theta}_n$ satisfies $n^\nu(\hat{\theta}_{1n} - \theta_{10}) \rightarrow_p 0$ and $\hat{\theta}_{2n} - \theta_{20} \rightarrow_p 0$.*

The proof of Proposition 3.I is based on Saikkonen (1995) and is divided in several Lemmas for ease of presentation. Lemma 3.J establishes consistency of the long run parameter when $\nu = 0$, Lemma 3.K does so for $\nu \in (0, 1)$, and Lemmas 3.L and 3.M prove the consistency of the short run parameter vector.

Lemma 3.J. *If $Q_n(\theta_n)$ is given by (3.2) and $\nu = 0$, then*

$$\lim_{n \rightarrow \infty} P_{n, \theta_0} \left\{ \inf_{\theta_n \in \bar{B}_{1,0} \times \Theta_2} Q_n(\theta_n) - Q_n(\theta_0) > 0 \right\} = 1.$$

Equivalently,

$$\inf_{\theta_n \in \bar{B}_{1,0} \times \Theta_2} n^{-1} Q_{1n}(\theta_n) + \inf_{\theta_n \in \bar{B}_{1,0} \times \Theta_2} n^{-1} Q_{2n}(\theta_{2n}) - n^{-1} Q_{2n}(\theta_{20}) > \eta. \quad (3.13)$$

Proof. I prove (3.13) in two parts. First I claim $\inf_{\theta_n \in \bar{B}_{1,0} \times \Theta_2} n^{-1} Q_{1n}(\theta_n) > 0$. To see this note

$$n^{-1} Q_{1n}(\theta_n) \geq n^{-1} |\theta_{1n} - \theta_{10}|^2 \underline{M}_2 - 2n^{-1} |\theta_{1n} - \theta_{10}| \bar{M}_1,$$

or

$$\inf_{\theta_n \in \bar{B}_{1,0} \times \Theta_2} n^{-1} Q_{1n}(\theta_n) \geq n^{-1} \delta_1^2 \underline{M}_2 - 2n^{-1} \delta_1 \bar{M}_1,$$

where considering elements of Θ_1 on the set defined in (3.11) the first element is diverging to $+\infty$ despite $|\theta_{1n} - \theta_{10}|^2$ being bounded, since $\underline{M}_2 = O_p(n^2)$. The second element is $O_p(1)$ since Θ_1 is bounded and it can be shown that $\bar{M}_1 = O_p(n)$ from (3.5); thus $n^{-1} Q_{1n}(\theta_n) > 0$.

For the second part, let $\tilde{\theta}_{2n}$ be the (infeasible) ML estimate for the stationary MA(1) process in $Q_{2n}(\theta_{2n})$. Note that standard theory on ML estimation now applies and $\tilde{\theta}_{2n} - \theta_{20} = O_p(n^{-1/2})$. I claim that $\inf_{\theta_n \in \bar{B}_{1,0} \times \Theta_2} n^{-1} Q_{2n}(\tilde{\theta}_{2n}) - n^{-1} Q_{2n}(\theta_{20}) = o_p(1)$. To see this, write

$$\begin{aligned} \inf_{\theta_n \in \bar{B}_{1,0} \times \Theta_2} n^{-1} Q_{2n}(\tilde{\theta}_{2n}) - n^{-1} Q_{2n}(\theta_{20}) &\geq \inf_{\theta_n \in \bar{B}_{1,0} \times \Theta_2} n^{-1} \ln \det |\tilde{\Gamma}_n| - n^{-1} \ln \det |\Gamma_0| \\ &+ \inf_{\theta_n \in \bar{B}_{1,0} \times \Theta_2} n^{-1} u_0' \tilde{\Gamma}_n^{-1} u_0 - n^{-1} u_0' \Gamma_0^{-1} u_0. \end{aligned}$$

The first term in the right of the last expression is $o_p(1)$ as proven in Lemma 1 of Yao and Brockwell (2006). The same authors show that the second term is $o_p(1)$ in their Theorem 1. ■

Lemma 3.K. If $Q_n(\theta_n)$ is given by (3.2) and $\nu \in (0, 1)$, then

$$\lim_{n \rightarrow \infty} P_{n, \theta_0} \left\{ \inf_{\theta_n \in \bar{B}_{1,\nu} \times \Theta_2} Q_n(\theta_n) - Q_n(\theta_0) > 0 \right\} = 1.$$

Equivalently,

$$\inf_{\theta_n \in \bar{B}_{1,\nu} \times \Theta_2} Q_{1n}(\theta_n) + \inf_{\theta_n \in \bar{B}_{1,\nu} \times \Theta_2} Q_{2n}(\theta_{2n}) - Q_{2n}(\theta_{20}) > \eta. \quad (3.14)$$

Proof. Here I prove (3.14), starting with the claim: $\inf_{\theta_n \in \bar{B}_{1,\nu} \times \Theta_2} Q_{1n}(\theta_n) > 0$. To see this

write

$$\begin{aligned} Q_{1n}(\theta_n) &\geq |\theta_{1n} - \theta_{10}|^2 \underline{M}_2 - 2|\theta_{1n} - \theta_{10}| \overline{M}_1 \\ &= |\theta_{1n} - \theta_{10}|^2 \underline{M}_2 \left[1 - \frac{2\overline{M}_1}{|\theta_n - \theta_{10}| \underline{M}_2} \right]. \end{aligned}$$

Considering $\theta_n \in \overline{B}_{1,\nu}$ as defined in (3.11), I can now write

$$\inf_{\theta_n \in \overline{B}_{1,\nu} \times \Theta_2} Q_{1n}(\theta_n) \geq \delta_1^2 n^{-2\nu} \underline{M}_2 \left[1 - \frac{2n^{-1} \overline{M}_1}{n^{-(1+\nu)} \delta_1 \underline{M}_2} \right].$$

The term within brackets in the last expression converges to 1 since $\nu \in (0, 1)$, while $\delta_1^2 n^{-2\nu} \underline{M}_2$ is diverging to $+\infty$ proving the claim.

Regarding the short-term part of (3.14), again, let $\tilde{\theta}_{2n}$ be the (infeasible) ML estimate for the stationary $MA(1)$ process, so that standard results on ML estimation of stationary process apply (e.g. from Hannan (1973)) and I have $\tilde{\theta}_{2n} - \theta_{20} = O_p(n^{-1/2})$ and $\inf_{\theta_n \in \overline{B}_{1,\nu} \times \Theta_2} Q_{2n}(\tilde{\theta}_{2n}) - Q_{2n}(\theta_{20}) = O_p(1)$. ■

Lemma 3.L. *If $Q_{1n}(\theta_n)$ is given by (3.3) and $\nu \in (1/2, 1)$, then*

$$\lim_{n \rightarrow \infty} P_{n,\theta_0} \left\{ \sup_{\theta_n \in B_{1,\nu} \times \overline{B}_2} |n^{-1} Q_{1n}(\theta_n)| \leq \eta \right\} = 1.$$

Proof. Write directly

$$\sup_{\theta_n \in B_{1,\nu} \times \overline{B}_2} |n^{-1} Q_{1n}(\theta_n)| \leq \sup_{\theta_{1n} \in B_{1,\nu} \times \overline{B}_2} |\theta_n - \theta_{10}|^2 n^{-1} \overline{M}_2 + 2 \sup_{\theta_{1n} \in B_{1,\nu} \times \overline{B}_2} |\theta_{1n} - \theta_{10}| n^{-1} \overline{M}_1.$$

Recall $\nu \in (1/2, 1)$ and, from Lemma 3.H, $M_1 = O_p(n)$ and $M_2 = O_p(n^2)$. Since $\theta_n \in B_{1,\nu} \times \overline{B}_2$, with these sets defined in (3.10) and (3.12), note that both terms on the right are $o_p(1)$. Start with the first term,

$$\sup_{\theta_{1n} \in B_{1,\nu} \times \overline{B}_2} |\theta_n - \theta_{10}|^2 n^{-1} \overline{M}_2 \leq \delta_1^2 n^{-(1+2\nu)} \overline{M}_2 = o_p(1).$$

Finally,

$$\sup_{\theta_{1n} \in B_{1,\nu} \times \overline{B}_2} |\theta_{1n} - \theta_{10}| n^{-1} \overline{M}_1 \leq \delta_1 n^{-(1+\nu)} \overline{M}_1 = o_p(1).$$

Lemma 3.M. *If $Q_{2n}(\theta_{2n})$ is given by (3.4), then*

$$\lim_{n \rightarrow \infty} P_{n, \theta_0} \left\{ \inf_{\theta_{2n} \in \overline{B}_2} n^{-1} Q_{2n}(\theta_{2n}) - n^{-1} Q_{2n}(\theta_0) > \eta \right\} = 1.$$

Proof. Write

$$\begin{aligned} \inf_{\theta_{2n} \in \overline{B}_2} n^{-1} Q_{2n}(\theta_{2n}) - n^{-1} Q_{2n}(\theta_0) &\geq \inf_{\theta_{2n} \in \overline{B}_2} n^{-1} \ln \det |\Gamma_n| - n^{-1} \ln \det |\Gamma_0| \\ &+ \inf_{\theta_{2n} \in \overline{B}_2} n^{-1} u_0' \Gamma_n^{-1} u_0 - n^{-1} u_0' \Gamma_0^{-1} u_0. \end{aligned}$$

The first term on the right is $o_p(1)$ from Lemma 1 of [Yao and Brockwell \(2006\)](#). The second term satisfies standard ML theory arguments for stationary time series. In particular, for $\theta_{2n} \in \overline{B}_2$ defined in (3.12), I have $n^{-1} u_0' \Gamma_n^{-1} u_0 > n^{-1} u_0' \Gamma_0^{-1} u_0$ as proved by the same authors in their Theorem 1. This proves the Lemma. ■

Proof of Proposition 3.I. From Lemmas 3.J and 3.K,

$$\lim_{n \rightarrow \infty} P_{n, \theta_0} \left\{ \inf_{\theta_n \in \overline{B}_{1, \nu} \times \Theta_2} Q_n(\theta_n) - Q_n(\theta_0) > 0 \right\} = 1,$$

which together with a standard convergence result such as Theorem 21.6 from [Davidson \(1994\)](#) implies $n^\nu (\widehat{\theta}_{1n} - \theta_{10}) \rightarrow_p 0$.

Also, from Lemmas 3.L and 3.M it follows that

$$\lim_{n \rightarrow \infty} P_{n, \theta_0} \left\{ \inf_{\theta_n \in \Theta_1 \times \overline{B}_2} Q_n(\theta_n) - Q_n(\theta_0) > 0 \right\} = 1,$$

and again with the aid of Theorem 21.6 from [Davidson \(1994\)](#) I conclude $\widehat{\theta}_{2n} - \theta_{20} \rightarrow_p 0$. ■

3.3 Asymptotic Distribution of $\widehat{\theta}_n$

After I have proved consistency of $\widehat{\theta}_n$, in this section I present its asymptotic distribution. The textbook way to obtain the asymptotic distribution of the ML estimate is through a mean value expansion and the asymptotics of the Score vector and the sample information matrix. To follow this road, I would need to prove that the sample information matrix is stochastic equicontinuous as explained by [Saikkonen \(1995\)](#). The Score vector of the ML problem in

this paper yields closed forms for $\widehat{\theta}_{1n} - \theta_{10}$ and $\widehat{\theta}_{2n} - \theta_{20}$. Thus, instead of proving stochastic equicontinuity, I apply the expansions proposed by [Phillips \(1991\)](#) and [Saikkonen \(2001\)](#) and used by [Chambers and McCrorie \(2007\)](#).

The *normalised* Score vector $s_n(\theta_n)$, corresponding to the optimization problem (3.2) is partitioned accordingly to the previous section into $\theta_n = (\theta_{1n}, \theta'_{2n})'$,

$$s_{1n}(\theta_n) = 2n^{-1} [(\theta_{1n} - \theta_{10})y'_{-1}\Gamma_n^{-1}y_{-1} - y'_{-1}\Gamma_n^{-1}u_0], \quad (3.15)$$

$$s_{2n,j}(\theta_n) = n^{-1/2} \frac{\partial Q_{1n}(\theta_n)}{\partial \theta_{2n,j}} + n^{-1/2} \frac{\partial Q_{2n}(\theta_n)}{\partial \theta_{2n,j}}, \quad j = 1, 2, \quad (3.16)$$

where

$$\begin{aligned} \frac{\partial Q_{1n}(\theta_n)}{\partial \theta_{2n,j}} &= (\theta_{1n} - \theta_{10})^2 y'_{-1} \frac{\partial \Gamma_n^{-1}}{\partial \theta_{2n,j}} y_{-1} - 2(\theta_{1n} - \theta_{10}) y'_{-1} \frac{\partial \Gamma_n^{-1}}{\partial \theta_{2n,j}} u_0, \\ \frac{\partial Q_{2n}(\theta_n)}{\partial \theta_{2n,j}} &= \text{tr} \left\{ \Gamma_n^{-1} \frac{\partial \Gamma_n}{\partial \theta_{2n,j}} \right\} + u'_0 \frac{\partial \Gamma_n^{-1}}{\partial \theta_{2n,j}} u_0, \end{aligned}$$

$j = 1, 2$ and, as usual, solving the system $s_n(\widehat{\theta}_n) = 0$ yields the ML estimate for a given n , $\widehat{\theta}_n$. I present the closed form of $\widehat{\theta}_1$ for future reference, and then its asymptotic distribution. I write $\widehat{\Gamma}_n = \Gamma(\widehat{\theta}_{2n})$ in the rest of the paper.

Remark 3.N. $s_{1n}(\theta_n)$ as given in (3.15) implies a closed form for $\widehat{\theta}_{1n}$,

$$\widehat{\theta}_{1n} - \theta_{10} = \frac{y'_{-1} \widehat{\Gamma}_n^{-1} u_0}{y'_{-1} \widehat{\Gamma}_n^{-1} y_{-1}}.$$

Proposition 3.O (Asymptotic Distribution of $\widehat{\theta}_{1n}$). *Given $s_{1n}(\theta_n)$ from (3.15), then*

$$n(\widehat{\theta}_{1n} - \theta_{10}) \Rightarrow \frac{\int_0^1 \mathcal{J}_c d\mathcal{W}}{\int_0^1 \mathcal{J}_c^2}.$$

An important feature of the result presented in this Proposition is the absence of short-term parameters, θ_{20} . The following Lemmas and Definitions shed light on why this is so, and will ease the presentation of the proof of the Proposition.

Lemma 3.P. *If $\widehat{\theta}_{2n} - \theta_{20} \rightarrow_p 0$, then*

$$n^{-2}y'_{-1}\widehat{\Gamma}_n^{-1}y_{-1} - n^{-2}y'_{-1}\Gamma_0^{-1}y_{-1} \rightarrow_p 0,$$

$$n^{-1}y'_{-1}\widehat{\Gamma}_n^{-1}u_0 - n^{-1}y'_{-1}\Gamma_0^{-1}u_0 \rightarrow_p 0.$$

Proof. The proof is contained in [Chambers and Hernandez \(2015\)](#) Lemma 4. ■

Definition 3.Q (Spectral Density of Γ_0). *Denote the spectral density of $\{u_{0,t}\}_{t=0}^n$ by $f(\omega; \theta_{20})$, with $\omega \in (-\pi, \pi]$, the ts element of Γ_0 , γ_{t-s} , can be recovered by*

$$\begin{aligned}\gamma_{t-s} &= \int_{-\pi}^{\pi} e^{i(t-s)\omega} f(\omega; \theta_{20}) d\omega, \\ f(\omega; \theta_{20}) &= \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} e^{-ij\omega} \gamma_{t-j},\end{aligned}$$

consequently, $2\pi f(0; \theta_{20}) = \sigma_u^2$.

Definition 3.R (Spectral Density of Ψ). *Let Ψ be a $n \times n$ matrix with ts element, ψ_{t-s} , satisfying*

$$\begin{aligned}\psi_{t-s} &= \int_{-\pi}^{\pi} e^{i(t-s)\omega} [4\pi^2 f(\omega; \theta_{20})]^{-1} d\omega, \\ [4\pi^2 f(\omega; \theta_{20})]^{-1} &= \sum_{j=-\infty}^{\infty} e^{-ij\omega} \psi_{t-j}.\end{aligned}$$

Lemma 3.S. *If Γ_0 is the true covariance matrix of the error process $\{u_{0,t}\}_{t=0}^n$ and if Ψ is given by Definition 3.R, then*

$$n^{-2}y'_{-1}\Gamma_0^{-1}y_{-1} - n^{-2}y'_{-1}\Psi y_{-1} \rightarrow_p 0,$$

$$n^{-1}y'_{-1}\Gamma_0^{-1}u_0 - n^{-1}y'_{-1}\Psi u_0 \rightarrow_p 0.$$

Proof. See Lemma 3 of [Chambers and Hernandez \(2015\)](#). ■

Lemma 3.T. *If $\widehat{\theta}_{2n} - \theta_{20} \rightarrow_p 0$ and if $\{y_t\}_{t=0}^n$ and $\{u_{0,t}\}_{t=0}^n$ are given by assumptions 2.A and 2.B, then*

$$\begin{aligned} n^{-2}y'_{-1}\widehat{\Gamma}_n^{-1}y_{-1} &\Rightarrow \int_0^1 \mathcal{J}_c^2, \\ n^{-1}y'_{-1}\widehat{\Gamma}_n^{-1}u_0 &\Rightarrow \int_0^1 \mathcal{J}_c d\mathcal{W}. \end{aligned}$$

The following proof is based on the proof of Lemma 2 of [Chambers and Hernandez \(2015\)](#) and is pivotal for the main result.

Proof. To prove the first item, note

$$\begin{aligned} n^{-2}y'_{-1}\widehat{\Gamma}_n^{-1}y_{-1} &= n^{-2}y'_{-1}(\widehat{\Gamma}_n^{-1} - \Gamma_0^{-1})y_{-1} + n^{-2}y'_{-1}(\Gamma_0^{-1} - \Psi)y_{-1} + n^{-2}y'_{-1}\Psi y_{-1} \\ &= n^{-2}y'_{-1}\Psi y_{-1} + o_p(1), \end{aligned}$$

where the first line is direct and the second follows from Lemmas 3.P and 3.S. Now, I claim that $n^{-2}y'_{-1}\Psi y_{-1} \Rightarrow \int_0^1 \mathcal{J}_c^2$. To see this, note that the same steps of the proof of Lemma 3.H yield

$$n^{-2}y'_{-1}\Psi y_{-1} \Rightarrow 2\pi \sum_{j=-\infty}^{\infty} \psi_j \left[\sigma_u^2 \int_0^1 \mathcal{J}_c^2 \right],$$

but $\sigma_u^2 = 2\pi f(0; \theta_{20})$ from Definition 3.Q and $\sum_{j=-\infty}^{\infty} \psi_j = [4\pi^2 f(0; \theta_{20})]^{-1}$ from Definition 3.R; thus the claim follows.

Addressing the second item, I use again Lemmas 3.P and 3.S so that

$$n^{-1}y'_{-1}\widehat{\Gamma}_n^{-1}u_0 = n^{-1}y'_{-1}\Psi u_0 + o_p(1),$$

and claim $n^{-1}y'_{-1}\Psi u_0 \Rightarrow \int_0^1 \mathcal{J}_c d\mathcal{W}$. I prove this claim in two steps. First, note how the logic of proof of Lemma 3.H implies

$$n^{-1}y'_{-1}\Psi u_0 \Rightarrow 2\pi \sum_{j=-\infty}^{\infty} \psi_j \left[\sigma_u^2 \int_0^1 \mathcal{J}_c d\mathcal{W} + \frac{\sigma_u^2 - \gamma_0}{2} + \Phi_j \right].$$

The second step requires me to prove that $2\pi \sum_{j=-\infty}^{\infty} \psi_j \left[\frac{\sigma_u^2 - \gamma_0}{2} + \Phi_j \right] = 0$. Recall the

definition of Φ_j from Lemma 3.H and write

$$2\pi \sum_{j=-\infty}^{\infty} \psi_j \left[\frac{\sigma_u^2 - \gamma_0}{2} + \Phi_j \right] = \pi(\sigma_u^2 - \gamma_0) \sum_{j=-\infty}^{\infty} \psi_j + 2\pi\gamma_0 \sum_{j=1}^{\infty} \psi_j - 2\pi \sum_{j=1}^{\infty} \psi_j \gamma_j, \quad (3.17)$$

where the latter expression follows from symmetry of Γ_0 . From the symmetry of the Fourier coefficients of Ψ note that

$$\sum_{j=1}^{\infty} \psi_j = \frac{1}{2} \left[\sum_{j=-\infty}^{\infty} \psi_j - \psi_0 \right] = \frac{1}{2} [4\pi^2 f(0; \theta_{20})]^{-1} - \frac{1}{2} \psi_0,$$

which together with Definition 3.Q allows me to write the right side of (3.17) as

$$\pi(\sigma_u^2 - \gamma_0) [4\pi^2 f(0; \theta_{20})]^{-1} + \pi\gamma_0 \left[[4\pi^2 f(0; \theta_{20})]^{-1} - \psi_0 \right] - 2\pi \sum_{j=1}^{\infty} \psi_j \gamma_j,$$

or

$$\frac{1}{2} - \pi\psi_0\gamma_0 - 2\pi \sum_{j=1}^{\infty} \psi_j \gamma_j = 0.$$

To see the last equality, from Zygmund (1959) I know that for the Fourier series, $2\pi \sum_{j=-\infty}^{\infty} \psi_j \gamma_j e^{ijx}$, there exist a function

$$h(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [2\pi f(x-s; \theta_{20})]^{-1} 2\pi f(s; \theta_{20}) ds,$$

implying $h(0) = 1$ and thus $2\pi \sum_{j=-\infty}^{\infty} \psi_j \gamma_j = 1$, or

$$2\pi \sum_{j=1}^{\infty} \psi_j \gamma_j = \frac{1}{2}(1 - 2\pi\psi_0\gamma_0),$$

where the equality is explained by the symmetry of the Fourier coefficients. ■

Proof of Proposition 3.O. Following [Chambers and McCrorie \(2007\)](#), I start by expanding $s_{1n}(\widehat{\theta}_n)$

$$\begin{aligned} s_{1n}(\widehat{\theta}_n) &= n^{-1}(\widehat{\theta}_{1n} - \theta_{10})y'_{-1}(\widehat{\Gamma}_n^{-1} - \Gamma_0^{-1})y_{-1} - n^{-1}y'_{-1}(\widehat{\Gamma}_n^{-1} - \Gamma_0^{-1})u_0 \\ &+ n^{-1}(\widehat{\theta}_{1n} - \theta_{10})y'_{-1}(\Gamma_0^{-1} - \Psi)y_{-1} - n^{-1}y'_{-1}(\Gamma_0^{-1} - \Psi)u_0 \\ &+ n^{-1}(\widehat{\theta}_{1n} - \theta_{10})y'_{-1}\Psi y_{-1} - n^{-1}y'_{-1}\Psi u_0. \end{aligned}$$

The first line of the right side is $o_p(1)$ from [Lemma 3.P](#) and $n(\widehat{\theta}_{1n} - \theta_{10}) = O_p(1)$. The second line is also $o_p(1)$ by direct application of [Lemma 3.S](#). This allows me to write $s_{1n}(\widehat{\theta}_n) = 0$ as

$$n(\widehat{\theta}_{1n} - \theta_{10}) = \frac{n^{-1}y'_{-1}\Psi u_0}{n^{-2}y'_{-1}\Psi y_{-1}} + o_p(1),$$

finally, [Lemma 3.T](#) together with the Continuous Mapping Theorem yield the result. ■

Corollary 3.U. From $s_{1n}(\widehat{\theta}_n)$, I can also obtain the closed form,

$$\widehat{\theta}_{1n} - 1 = \frac{y'_{-1}\Psi \Delta y}{y'_{-1}\Psi y_{-1}} + o_p(1),$$

with corresponding asymptotic behaviour,

$$n(\widehat{\theta}_{1n} - 1) \Rightarrow \frac{\int_0^1 \mathcal{J}_c d\mathcal{J}_c}{\int_0^1 \mathcal{J}_c^2}.$$

Proof. The closed form for $\widehat{\theta}_{1n} - 1$ is obtained by subtracting one on both sides of the expression in [Remark 3.N](#) and using the definition of u_0 with [Lemmas 3.P](#) and [3.S](#). The asymptotic behaviour is analogous to that in [Proposition 3.O](#). ■

Now I present the following Lemma that allows me to ignore $Q_{1n}(\theta_n)$ in derivations of the asymptotic distribution of $\widehat{\theta}_{2n}$.

Lemma 3.V. $Q_{1n}(\theta_n)$ as defined by [\(3.3\)](#) is not relevant to derive the asymptotic distribution of $\widehat{\theta}_{2n}$. Equivalently,

$$n^{-1/2}Q_{1n}(\theta_n) \xrightarrow{p} 0.$$

Proof. From (3.3) note that

$$\begin{aligned} n^{-1/2}Q_{1n}(\theta) &= n^{2\nu}(\theta_{1n} - \theta_{10})^2 n^{-(2\nu+1/2)} y'_{-1} \Gamma_n^{-1} y_{-1} \\ &\quad - 2n^\nu(\theta_{1n} - \theta_{10}) n^{-(\nu+1/2)} y'_{-1} \Gamma_n^{-1} u_0. \end{aligned}$$

The first term on the right is $o_p(1)$, $y'_{-1} \Gamma_n^{-1} y_{-1}$ is only bounded in probability for $\nu = 3/4$ while $n^{3/2}(\theta_{1n} - \theta_{10})^2 = o_p(1)$. The second term is also $o_p(1)$, $y'_{-1} \Gamma_n^{-1} u_0$ is bounded in probability only if $\nu = 1/2$, but $n^{1/2}(\theta_{1n} - \theta_{10}) = o_p(1)$ from Proposition 3.I. ■

Proposition 3.W (Asymptotic Distribution of $\widehat{\theta}_{2n}$). *Given Lemma 3.V, and (3.16) the asymptotic distribution of $\widehat{\theta}_{2n}$ is that of a ML estimate for a stationary MA(1) process. This is,*

$$n^{1/2} \left(\widehat{\theta}_{2n} - \theta_{20} \right) \rightarrow_d N \left(0, V^{-1}(\theta_{20}) \right),$$

where the kl element of $V(\theta_{20})$ is given by

$$V_{kl}(\theta_{20}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^{-1}(\omega; \theta_{20}) \frac{\partial f(\omega; \theta_{20})}{\partial \theta_{20,k}} f^{-1}(\omega; \theta_{20}) \frac{\partial f(\omega; \theta_{20})}{\partial \theta_{20,l}},$$

and $f(\omega; \theta_{20})$ is the spectral density of $\{u_{0,t}\}_{t=0}^n$.

Proof. See Hannan (1973). ■

4 Unit Root Test

In this section I introduce the likelihood ratio unit root test and its asymptotic properties. With the ML framework presented in the previous section and the asymptotic behaviour of $\widehat{\theta}_n$ at hand, I can now present the LR test statistic and set up the hypothesis testing problem in the next Definition and Proposition.

Definition 4.A (LR Test Statistic). *For a given sample size, n , and the log-likelihood function defined by (3.1), the Likelihood Ratio Statistic for testing the Null Hypothesis $\theta_{10} = 1$, against the Alternative Hypothesis $\theta_{10} < 1$, is given by*

$$LR_c = 2 \left[l^n(\widehat{\theta}_n) - l^n(\widetilde{\theta}_n) \right],$$

where $\widehat{\theta}_n$ and $\widetilde{\theta}_n$ are the unrestricted and restricted ML estimates, respectively.

Proposition 4.B (Asymptotic Distribution of LR_c). *If $n^\nu(\hat{\theta}_{1n} - \theta_{10}) \rightarrow_p 0$ for $\nu \in [0, 1)$, and $\hat{\theta}_{2n} - \theta_{20} \rightarrow_p 0$, and the Null Hypothesis to test is that of Definition 4.A, then*

$$LR_c \Rightarrow \frac{\left[\int_0^1 \mathcal{J}_c d\mathcal{J}_c \right]^2}{\int_0^1 \mathcal{J}_c^2}.$$

Proof of Proposition 4.B. The proof is based on one fact, that LR_c can be written as a second order mean value expansion of the log-likelihood function $l^n(\tilde{\theta})$ about $\hat{\theta}_n$ (see Davidson (2000) p.290)

$$LR_c = -(\hat{\theta}_n - \tilde{\theta}_n)' \bar{\mathbb{H}}_n (\hat{\theta}_n - \tilde{\theta}_n),$$

where $\bar{\mathbb{H}}_n$ is the Hessian of $l^n(\theta_n)$ evaluated at $\bar{\theta}_n$, a convex combination of $\hat{\theta}_n$ and $\tilde{\theta}_n$. A convenient feature of $\bar{\mathbb{H}}_n$ is that it can be divided in blocks according to the rate of convergence of its elements. In particular

$$\bar{\mathbb{H}}_n = - \begin{bmatrix} y'_{-1} \bar{\Gamma}_n^{-1} y_{-1} & \bar{A}'_1 \\ \bar{A}_1 & \bar{H} \end{bmatrix}$$

where \bar{A}_1 is a 2×1 vector with elements

$$\bar{A}_{1,[i,1]} = \frac{\partial^2 l^n(\theta_n)}{\partial \theta_{1n} \partial \theta_{2n,i}} = y'_{-1} \frac{\partial \Gamma_n^{-1}}{\partial \theta_{2n,i}} u_0, \quad i = 1, 2. \quad (4.1)$$

and \bar{H} is the 2×2 Hessian of the short-run parameters.

$$\bar{H}_{[i,j]} = -\frac{1}{2} \frac{\partial}{\partial \theta_{2n,i}} \text{tr} \left\{ \Gamma_n^{-1} \frac{\partial \Gamma_n}{\partial \theta_{2n,j}} \right\} - u'_n \frac{\partial^2 \Gamma_n^{-1}}{\partial \theta_{2n,i} \partial \theta_{2n,j}} u_0 \quad i, j = 1, 2. \quad (4.2)$$

Thus, I get

$$\begin{aligned} LR_c &= \left[n(\hat{\theta}_{1n} - 1) \right]^2 \frac{1}{n^2} y'_{-1} \bar{\Gamma}_n^{-1} y_{-1} \\ &\quad - 2n(\hat{\theta}_{1n} - 1) \left(n^{1/2}(\hat{\theta}_{2n} - \theta_{20}) - n^{1/2}(\tilde{\theta}_{2n} - \theta_{20}) \right)' \frac{1}{n^{3/2}} \bar{A}_1 \\ &\quad + n^{1/2} \left((\hat{\theta}_{2n} - \theta_{20}) - (\tilde{\theta}_{2n} - \theta_{20}) \right)' \frac{1}{n} \bar{H} n^{1/2} \left((\hat{\theta}_{2n} - \theta_{20}) - (\tilde{\theta}_{2n} - \theta_{20}) \right), \end{aligned}$$

where the second and third terms on the right are $o_p(1)$ since the elements of \bar{A}_1 are $O_p(n)$,

and \bar{H} has only $O_p(n^{1/2})$ elements. After some algebra and using $n^{-2}y'_{-1}(\bar{\Gamma}_n^{-1} - \Gamma_0^{-1})y_{-1} = o_p(1)$, I have the expression

$$LR_c = \frac{1}{n} \Delta y' \bar{\Gamma}_n^{-1} y_{-1} n(\hat{\theta}_{1n} - 1) + o_p(1),$$

thus, I only need to invoke Lemma 3.T, Proposition 3.O, and the Continuous Mapping Theorem to obtain the result. ■

Remark 4.C. *Stock (1994) lists the characteristics that good unit root tests have: (i) the test is independent of the parameters for the constant, the trend or serial correlation; (ii) the test has good power in large samples; and (iii) the test has both good power and small size distortions when computed over different models and samples.*³

Remark 4.D. *The asymptotic distribution of LR_c presented in Proposition 4.B coincides with that found in Johansen (1988), Larsson (1998) and Rothenberg and Stock (1997). Moreover it is independent of serial correlation parameters. Thus, it satisfies the first of Stock's requirements for a good test of Remark 4.C (i).*

Corollary 4.E (Asymptotic Distribution of LR_c under the Null Hypothesis). *If $c = 0$, then I get the squared of the Dickey-Fuller t -statistic's asymptotic distribution*

$$LR_0 \Rightarrow \frac{\left[\int_0^1 \mathcal{W} d\mathcal{W} \right]^2}{\int_0^1 \mathcal{W}^2}.$$

Proof. The proof is straight forward from Remark 3.F. ■

5 Empirical Analysis

Once I have the LR_c test, the next step entails the evaluation of its empirical properties. The analytical tool to do so is the Power Envelope, and its derivation is presented in this section. It includes a subsection with preliminary definitions that should help the exposition of the power envelope, both in finite samples and its definition for the large sample case. The empirical properties of the LR_c are then presented along with the algorithm to obtain them. The section concludes with a comparison of the power properties of LR_c with the Augmented Dickey Fuller test.

³Stock (1994) pp 2764. The Null and alternative hypothesis he considers are standard.

5.1 Preliminaries

A few preliminary definitions and notation are necessary here, in particular, recall I make a *Type I* Error if I incorrectly reject the Null Hypothesis. Additionally, I make a *Type II* Error if I fail to reject the Null Hypothesis when the true parameter is not satisfying it. Thus, I can introduce a straightforward definition of the power the test and Power Function.

Definition 5.A (Power Function). *The power of the test is defined as the probability of rejecting the Null Hypothesis, and $\phi(\theta_n)$ is the Power Function satisfying*

(i) $\phi(\theta_n) = Pr(\text{Type I error})$, or

(ii) $\phi(\theta_n) = 1 - Pr(\text{Type II error})$.

Remark 5.B. *Given definitions 4.A and 5.A, a unit root test that has low power will under-reject the Null Hypothesis of a unit root.*

5.2 Power Envelope: Finite Samples

I will derive the Power Envelope following the exposition of [Jansson and Nielsen \(2012\)](#). Define the rejection region for the test statistic, \mathcal{R} , and the size of the test, $\bar{\alpha}$. Recall the LR_c statistic is given by Definition (4.A) and define the Power Envelope for each \mathcal{R} , $\bar{\alpha}$, n , and c as

$$\Pi_n^{\bar{\alpha}}(\theta_{1n}(c)) = \max_{Pr(LR_c \in \mathcal{R}): \phi(\theta_{1n}(0)) = \bar{\alpha}} \phi(\theta_{1n}(c)).$$

An application of the Neyman-Pearson Lemma yields,

$$\Pi_n^{\bar{\alpha}}(\theta_{1n}(c)) = P_{\theta_{1n}(c)}(LR_c > k_n^{\bar{\alpha}}), \quad (5.1)$$

where $P_{\theta_{1n}(c)}(\cdot)$ is the probability measure underlying $\theta_{1n}(c)$, and $k_n^{\bar{\alpha}}$ satisfies

$$P_{\theta_{1n}(0)}(LR_c > k_n^{\bar{\alpha}}) = \bar{\alpha}. \quad (5.2)$$

This is, $k_n^{\bar{\alpha}}$ is the critical value obtained from the frequency distribution of LR_0 as given by Proposition 4.B at the $(1 - \bar{\alpha})$ quantile and sample size n .

Remark 5.C. (i) $k_n^{\bar{\alpha}}$ does not depend on c . (ii) By construction, the Power Envelope as described by (5.1) corresponds to the family of most powerful unit root tests.⁴ (iii) This family of tests has an Asymptotic Power Envelope.

⁴Stock (1994) p. 2771.

5.3 Asymptotic Power Envelope

An additional tool for the evaluation of the test is the Asymptotic Power Envelope, which provides a benchmark for the asymptotic behaviour of the family of tests and is given by

$$\lim_{n \rightarrow \infty} \Pi_n^{\bar{\alpha}}(\theta_{1n}(c)) = \max_{Pr(LR_c \in \mathcal{R}) : \lim_{n \rightarrow \infty} \phi(\theta_{1n}(0)) = \bar{\alpha}} \lim_{n \rightarrow \infty} \phi(\theta_{1n}(c)).$$

Let $\lim_{n \rightarrow \infty} k_n^{\bar{\alpha}} = k^{\bar{\alpha}}$, taking limits to (5.1) and using the Neyman-Pearson Lemma

$$\lim_{n \rightarrow \infty} \Pi_n^{\bar{\alpha}}(\theta_{1n}(c)) = \lim_{n \rightarrow \infty} P_{\theta_{1n}(c)} [LR_c > k^{\bar{\alpha}}] = P_1 \left[\frac{\left[\int_0^1 \mathcal{J}_c d\mathcal{J}_c \right]^2}{\int_0^1 \mathcal{J}_c^2} > k^{\bar{\alpha}} \right], \quad (5.3)$$

where, $k^{\bar{\alpha}}$ satisfies $P_1 \left[\frac{\left[\int_0^1 \mathcal{W} d\mathcal{W} \right]^2}{\int_0^1 \mathcal{W}^2} > k^{\bar{\alpha}} \right] = \bar{\alpha}$.

Remark 5.D. *The Asymptotic Power Envelope as defined by equation (5.3) does not depend on the short run parameters θ_{2n} .*

5.4 Empirical Properties

I conducted Monte Carlo experiments to obtain the empirical properties of the test statistic in Definition 4.A. Recall the asymptotic expression for LR_c in Proposition 4.B corresponds to that found in previous work, in particular in Johansen (1988). This means there are critical values that can be used as a benchmark, and they are a useful guide. Regarding the discrete approximations to the Wiener processes and the Ornstein-Uhlenbeck processes, I have followed the methods outlined in Johansen (1995) chapter 15 and Saikkonen and Lutkepohl (1999) p. 61, respectively. Finally, selection of the sample size, n , and true values for the short run parameters related to the error process correspond to that of the recent literature (c.f. Jansson and Nielsen (2012)).

Asymptotic Size

To obtain the asymptotic distribution of LR_c under the Null Hypothesis, I simulate the asymptotic expression in Proposition 4.B through discrete approximations to Wiener Processes. Results are contained in Table 1 for 10^6 replications and the distribution is shown in Figure 1.

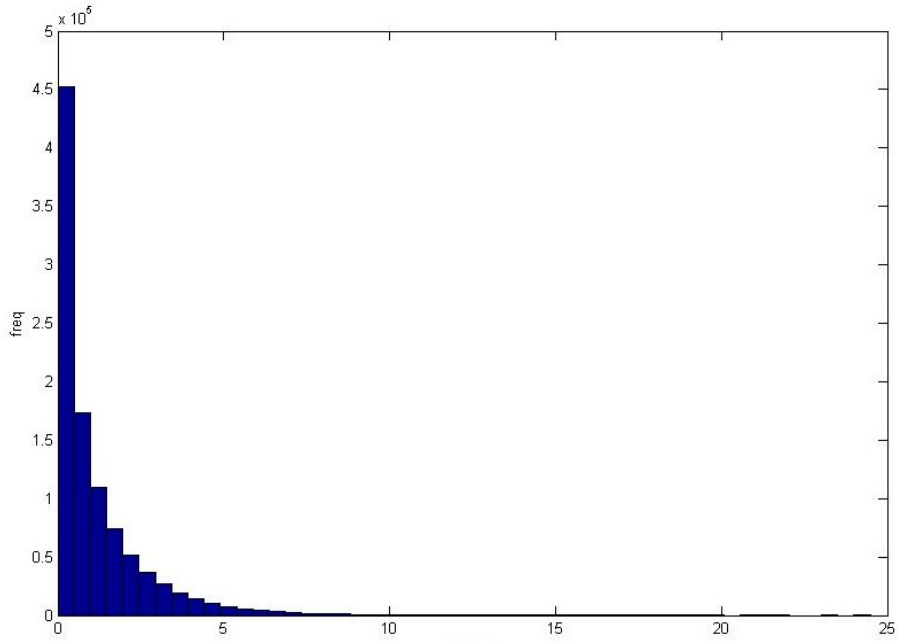


Figure 1: Asymptotic Distribution of LR_0 .

Sample/ $(1 - \bar{\alpha})$	85%	90%	95%	97.5%	99%	99.5%
$n = 10^4$	2.3345	2.9857	4.1332	5.3210	6.9288	8.1821

Note: Wiener Processes were simulated in Matlab[®] with variance $1/n$ in 10^6 Monte Carlo replications.

Table 1: Critical Values from Simulations of Wiener Process

Sample/ $(1 - \bar{\alpha})$	50%	75%	80%	85%	90%	95%	97.5%	99%
$n = 400$	0.60	1.56	1.89	2.32	2.98	4.14	5.30	7.02

Table 2: Critical Values from Johansen’s Test out of 5,000 Monte Carlo replications.

Given that the asymptotic distribution of LR_c coincides with that of [Johansen \(1988\)](#) rank test as mentioned, I can use it as a benchmark to check its validity. In [Table 2](#) I reproduce the critical values given in [Johansen \(1995\)](#) Table 15.1. The outcome of the comparison is contained in the following Remark for future reference.

Remark 5.E. *The similarity between the values for $(1 - \bar{\alpha}) = 95\%$ in [Table 1](#) and [Table 2](#) guarantees the experiments are designed correctly.*

Asymptotic Power Envelope

To simulate the Asymptotic Power Envelope (5.3), I could have used Johansen's, but the critical values presented in Table 1 enjoy a computational advantage as they were obtained by a considerably larger number of repetitions, 10^6 against 5000. I use the following algorithm to get the simulations.

1. Set initial parameters: $n = 10^4$ for the sample size as an approximation to infinite sample size and choose $k^{\bar{\alpha}}$ for $\bar{\alpha} = 5\%$, this is the 4th column of Table 1 and the true $\sigma_0^2 = 1$. Moreover $maxit$ is the total number of repetitions and $maxj$ as the number of values c can take.
2. Define a count M of dimensions $maxit, maxj$.
3. Define a sensible grid for the local-to-unity parameter c . I use $c_j = -j$ with $j \in \{0, 1, 2, \dots, 30\}$ which is the same as in Jansson and Nielsen (2012).
4. Set $it = 1$ for the first repetition.
5. Obtain an i.i.d sequence of standard normal errors $\{\epsilon\}_{t=0}^n$ with the random number generator.
6. Set $\epsilon_1 = 0$, and construct a *discretized* Wiener Process defined as: $W_t = \sum_{s=1}^t \epsilon_s$.
7. For $j = 0$, construct a *discretized* Ornstein-Uhlenbeck Process: $J_{c_j,t} = (1+c_j/n)J_{c_j,t-1} + \epsilon_t$ where $J_{c_j,0} = 0$ for all j .⁵
8. Compute $dJ_{c_j,t} = J_{c_j,t} - J_{c_j,t-1}$.
9. Compute and save the asymptotic test statistic from Proposition 4.B

$$LR_{c_j} = \left[n^{-2} \sum_{s=1}^n J_{c_j,s}^2 \right]^{-1} \left[n^{-1} \sum_{s=1}^n J_{c_j,s} dJ_{c_j,s} \right]^2.$$
10. Change the value of j to $j + 1$ and go back to Step 7. Repeat the loop until $j = maxj$.
11. Once the loop on j is finished, change it to $it + 1$ and go back to Step 4. Repeat the loop until $it = maxit$. I repeated the experiment 10^6 times; thus $maxit = 10^6$.

⁵The discrete approximations to the Ornstein-Uhlenbeck Processes are obtained following Saikkonen and Lutkepohl (1999) pp 61.

Finally, compute the Empirical Asymptotic Power

$$\widehat{\Pi}_{\infty}^{\alpha}(\theta_{1n}(c_j)) = \frac{1}{10^6} \sum_{s=1}^{10^6} \mathbb{I}\{LR_{c_j,s} > k_n^{\alpha}\},$$

for each j where $\mathbb{I}\{\cdot\}$ is the indicator function. The outcome of the simulations is presented in Figure 2.

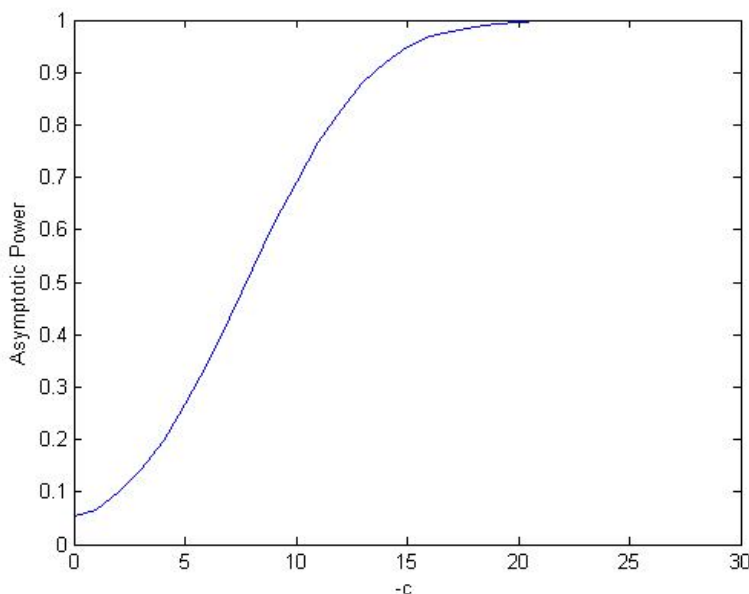


Figure 2: Asymptotic Power Envelope for LR_c .

Power Function

In this subsection I outline the algorithm to construct the Power Function, $\phi(\theta_{1n}(c))$, and present the results from the Monte Carlo experiments. These include results for (small) sample sizes $n \in \{50, 100, 250, 500\}$ and a set of Moving Average (MA) parameters $\alpha_0 \in \{\pm 0.8, \pm 0.5, 0\}$ with a grid for c as previously defined.

Algorithm

To get the Power Function for each value α_0 and each n , I use the following algorithm.

1. Set initial parameters: $n \in \{50, 100, 250, 500\}$ for the sample size, and the “true” $a \in \{\pm 0.8, \pm 0.5, 0\}$. Set the total number of repetitions $maxit$, and the number of

values c can take, $maxj = 30$.

2. Define the grid for the local-to-unity parameter c . I use $c_j = -j$ with $j \in \{0, 2, 4, \dots, 28, 30\}$.
3. Define a vector of “true” values of r with element j : $r_j = 1 - c_j/n$.
4. Set $it = 1$ for the first repetition.
5. Obtain an i.i.d. sequence of standard normal errors $\{\epsilon_t\}_{t=0}^n$.
6. Create the “true” DGP for c_j given by $\{y_t\}_{t=0}^n$, where $y_t = r_j y_{t-1} + \epsilon_t + a\epsilon_{t-1}$ and $\epsilon_0 = y_0$.
7. Estimate the unrestricted and the restricted ($c = 0$) models with the following sub-algorithm:
 - (a) For $j = 1$, write the model as: $\epsilon_t = y_t - r_j y_{t-1} - a\epsilon_{t-1}$, where r_j and a are the parameters to be estimated.
 - (b) Since in practice I only observe the sequence $\{y_t\}_{t=0}^n$ I need to get a sequence of estimated errors, $\{\hat{\epsilon}_t\}_{t=0}^n$, to estimate r_j and a . Note that, for given $y_0 = \hat{\epsilon}_0 = 0$ the process can be written in iterative form:

$$\begin{aligned}\hat{\epsilon}_1 &= y_1 \\ \hat{\epsilon}_2 &= y_2 - r_j y_1 - a\hat{\epsilon}_1 \\ &\vdots \\ \hat{\epsilon}_n &= y_n - r_j y_{n-1} - a\hat{\epsilon}_{n-1}\end{aligned}$$

to use Maximum Likelihood on

$$-\frac{n}{2} \ln |s^2| - \frac{1}{2s^2} \sum_{t=1}^n \hat{\epsilon}_t^2.$$

- (c) Obtain $\hat{\theta} = (\hat{r}_j, \hat{a}, \hat{s}^2)'$ and $\tilde{\theta} = (1, \tilde{a}, \tilde{s}^2)'$.
8. Compute and save LR_c from Definition 4.A. Compute and save the *Modified Akaike Criteria Augmented Dickey Fuller*, ADF^* , test statistic as detailed in [Ng and Perron \(2001\)](#).

9. Change $j = 1$ to $j = j + 1$ and go back to step 6. When the case $j = \max j$ is finish, change $it = 1$ to $it = it + 1$ and go back to step 5. Repeat until $it = \max it$.

This allows me to obtain sequences $\{LR_{c_{j,s}}\}_{s=1}^{10^4}$ and $\{ADF_{c_{j,s}}^*\}_{s=1}^{10^4}$. Finally, I compute the (empirical) power for each c_j as:

$$\widehat{\Pi}_n^{\bar{\alpha}}(c_j) = \frac{1}{10^4} \sum_{s=1}^{10^4} \mathbb{I}\{LR_{c_{j,s}} > k_n^{\bar{\alpha}}\}, \quad (5.4)$$

$$\widehat{\Pi}_{ADF^*,n}^{\bar{\alpha}}(c_j) = \frac{1}{10^4} \sum_{s=1}^{10^4} \mathbb{I}\{ADF_{c_{j,s}}^* > k_{ADF^*,n}^{\bar{\alpha}}\}, \quad (5.5)$$

where $\mathbb{I}\{\cdot\}$ in (5.4) is the indicator function that is equal to 1 if $LR_{c_{j,s}} > k_n^{\bar{\alpha}}$ is true, and 0 otherwise. Likewise, $\mathbb{I}\{\cdot\}$ is 1 if $ADF_{c_{j,s}}^* > k_{ADF^*,n}^{\bar{\alpha}}$ and 0 otherwise in (5.5), $k_{ADF^*,n}^{\bar{\alpha}}$ is the critical value for the ADF at k^* lags obtained with the Modified Akaike Criteria.

Monte Carlo Results

Figure 3 shows the Asymptotic Power Envelope from Figure 2 and the Power Function for each value of α_0 and n . Several general conclusions emerge from the inspection of the Monte Carlo experiments. First, for all values of n , the simulations show negligible size-bias. This is a desirable property in view of Remark 4.C. Second, for a large sample, $n = 500$, all power functions are considerably close to the asymptotic power envelope. Third, in line with the derived asymptotic behaviour in this paper, as n increases, the Power Functions for all values of α_0 converge towards the Asymptotic Power Envelope (c.f. lower-right plot in Figure 3).

Particular conclusions, can also be drawn from the empirical exercise. First, for each value of n , if $\alpha_0 > 0$, the test shows power levels close to the asymptotic power envelope. When $n \geq 100$, the power is close to optimal for $\alpha_0 = 0$ as well. I conclude that when $\alpha_0 \geq 0$ I can test for a unit root with LR_c and expect to have close to optimal power properties and negligible size-bias.

For the case of $\alpha_0 < 0$ I need to be more careful. There is a caveat when analysing an $ARMA(1, 1)$ model. It is important to keep track of the values of the roots for the lag polynomials, this is, if $\theta_0 = -\alpha_0$, then the stochastic process $\{y_t\}_{t=0}^n$ degenerates into a

white noise process. To see why, write the model as

$$(1 - \theta_{10}L)y_t = (1 + \alpha_0L)\varepsilon_t.$$

$$\theta_{10} = 1 + \frac{c}{n}, c < 0.$$

With $\theta_{10} = -\alpha_0$, note how the previous expression reduces to $y_t = \varepsilon_t$. In view of the definition of θ_{10} , and given the choice of the values for the parameter c , I should expect that the roots *nearly* cancel depending on the value of n .

In particular, I should observe the following “slumped” power functions. First, the combination $n = 50$, $c \in \{-9, -10, -11\}$, and $\alpha_0 = -0.8$. Second, the combination $n = 50$, $c \in \{-24, -26, -27\}$, and $\alpha_0 = -0.5$. Third, the case of $n = 100$, $c \in \{-19, -20, -21\}$, and $\alpha_0 = -0.8$. Fourth, if $n = 250$, $c \in \{-29, -30\}$, and $\alpha_0 = -0.8$. Finally, no loss of power is expected for $n = 500$.

Figure 3 corroborate the expectations on the cancelling roots where is clear that as n increases the “slumps” correct gradually. The experiments with $n = 50$ have three “slumps” in $\alpha_0 \in \{-0.8, -0.5, 0\}$, whereas that for $n = 100$ only has two in $\alpha_0 \in \{-0.8, -0.5\}$ and for $n = 250$ only one in $\alpha_0 = -0.8$. Finally, at $n = 500$ there is no loss of power derived by cancelling roots.

I conclude from the analysis that for a large sample, $n = 500$, all desirable properties proposed by [Stock \(1994\)](#) given in Remark 4.C are satisfied. For smaller sample sizes the negative values of the *MA* parameter may fail to have a good power while retaining negligible size-bias.

Comparison with the Benchmark

Having established the empirical properties of the power of the LR_c , I compare its performance to that of the ADF^* test. This test is a good benchmark since it is still popular among practitioners in cases with no deterministic terms. Moreover, the work from [Ng and Perron \(2001\)](#) has reduced considerably the size distortion caused by the serial correlation.

Table 3 displays the rejection rates for both the LR_c and the ADF^* tests for all values of α_0 and a set of values of the local-to-unity parameter, including the Null Hypothesis. In particular, the table contains $\rho_0 \in \{1, 0.99, 0.98, 0.96, 0.94\}$. The size bias for LR_c is smaller in the majority of cases than that of the ADF^* , as shown in the column corresponding to $\rho_0 = 1$. The rest of the Table shows that, except for the case of $\alpha_0 = -0.8$, LR_c has higher power, even for small sample sizes.

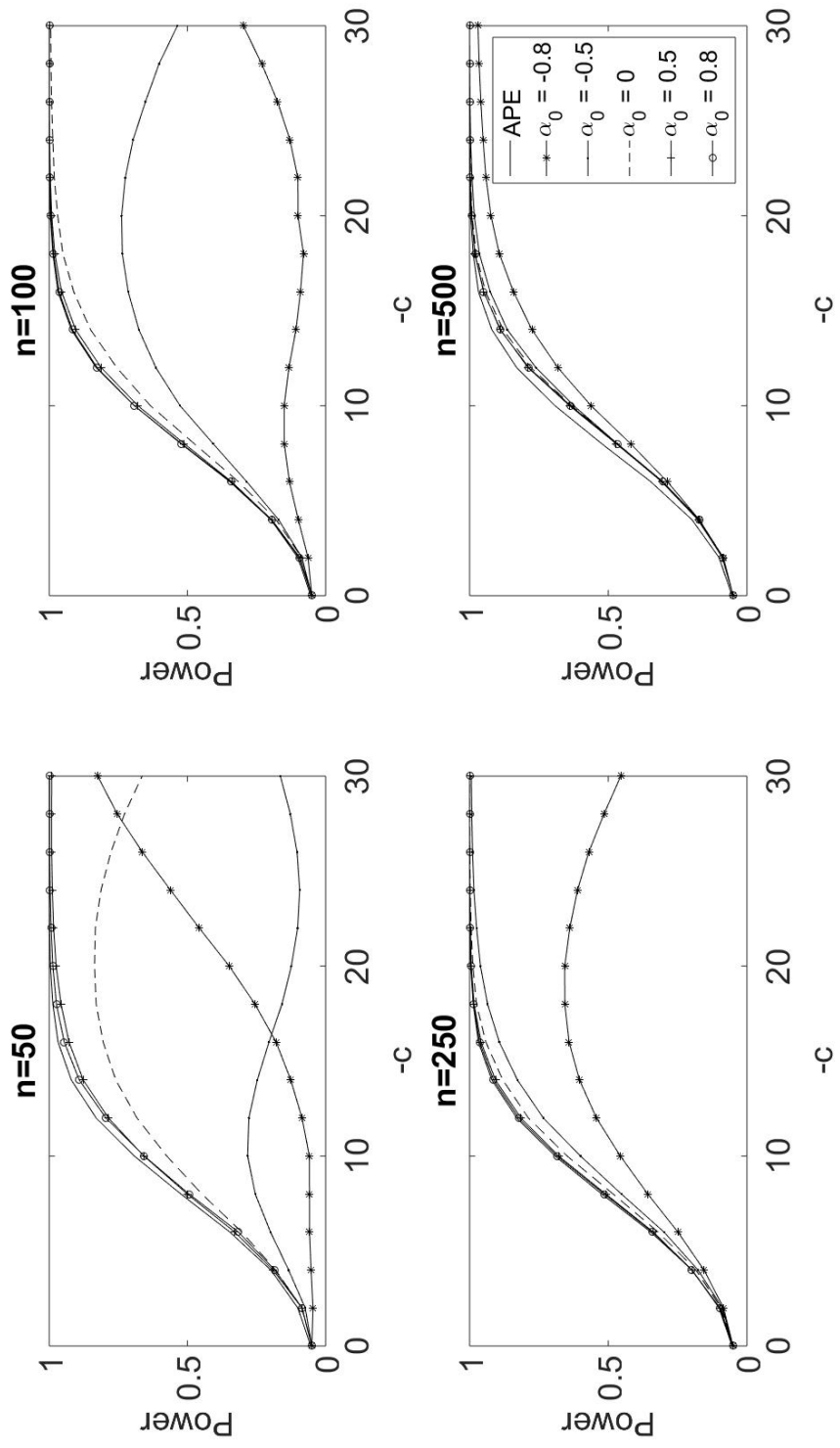


Figure 3: Asymptotic Power Envelope and Power for LR_c .

For several combinations of parameters the rejection rate of the LR_c test is more than two times that of the ADF^* , this is true for example when $\rho_0 = 0.96$ and $n \leq 100$. Another feature displayed in the Table corresponds to the stark differences in rejection rates between the tests when $\rho_0 = 0.94$ and $n = 250$. The latter is significant as the true local-to-unity parameter is well off from the Null Hypothesis value and the ADF^* test is still failing to reject in more than 25 per cent of the cases. For a large sample size, in this case $n = 500$, the differences in power between the tests for $\alpha_0 = -0.8$ decrease considerably.

Figure 4 shows the Power Function of the LR_c and ADF^* with k^* lags determined by the Modified Akaike Criterion. Each sub-plot corresponds to a value of α_0 and each Figure to a sample size. There are features that can be observed across the figure. First, for all combinations of n and $\alpha_0 \geq 0$, LR_c has better power properties than ADF^* . Second, as c moves away from zero, the differences in power between the tests grow whenever $\alpha_0 > 0$.

Particular conclusions are, first, gains in power are largest for small sample sizes $n \in \{50, 100\}$ and $\alpha_0 \geq 0$, as shown in the first and second rows of Figure 4. Second, there is no misspecification to worry about when the true process resembles an $ARMA(1, 1)$, even in small samples. This conclusion does not hold, however, for the case of $\alpha_0 = -0.8$. When the true value of $\theta_{1n}(c)$ approximates -0.8 the test has lower power than the ADF^* . Finally, if $n = 500$, the LR_c test will perform better for each value of α_0 . Note that even though the ADF^* test converges to the Asymptotic Power Envelope as n increases, the LR_c has higher power in the majority of the cases.

6 Empirical Application

For the empirical application I borrow that from [Ng and Perron \(2001\)](#), and test the performance of LR_c against ADF^* on inflation from the GDP Deflator for the G7 countries -U.S., Canada(CAN), Japan (JPN), Great Britain (GBR), Germany (GER), Italy (ITA) and France (FRA). The term structure literature provides a good motivation for testing inflation for a unit root when parameters are estimated with a VAR model. In particular, [Ang and Piazzesi \(2003\)](#) and [Kim \(2009\)](#) use inflation as a factor in their models for the term structure of the interest rate.

I obtained the data from the FRED database. The largest data span starts in the first quarter of 1955 through the fourth quarter of 2014 for the U.S. and GBR, while the shortest, that of Japan, starts in the second quarter of 1994. As Ng and Perron, I compute the inflation rates as 400 times the log-differences of successive quarters. In order to have mean-zero processes, I

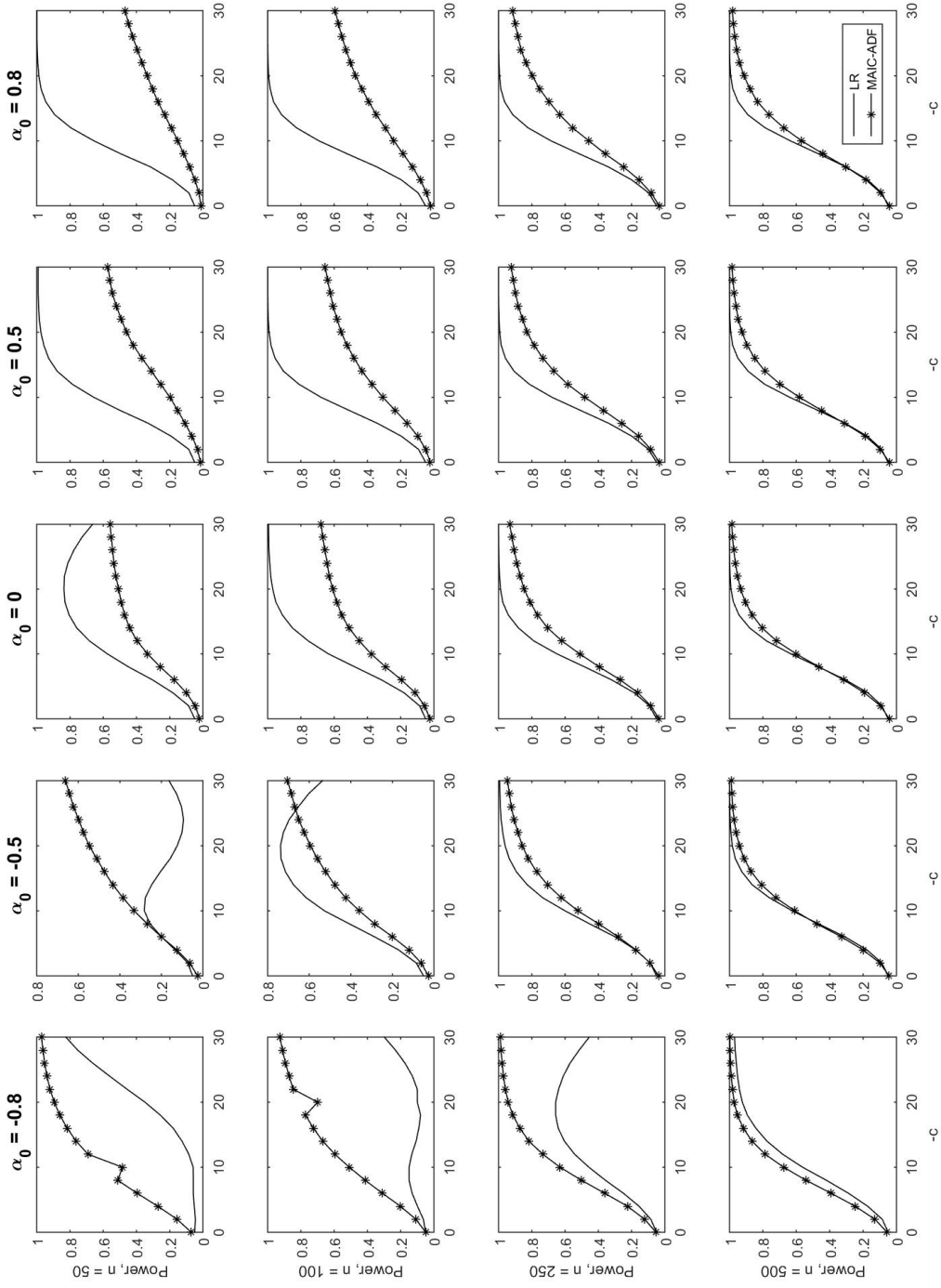


Figure 4: Power Comparison of LR_c with ADF^* .

n	α_0	$\rho_0 = 1$		$\rho_0 = 0.99$		$\rho_0 = 0.98$		$\rho_0 = 0.96$		$\rho_0 = 0.94$	
		LR_c	ADF^*	LR_c	ADF^*	LR_c	ADF^*	LR_c	ADF^*	LR_c	ADF^*
50	-0.8	0.065	0.070					0.061	0.156		
	-0.5	0.050	0.026					0.073	0.064		
	0	0.050	0.021					0.086	0.046		
	0.5	0.051	0.015					0.087	0.033		
	0.8	0.055	0.012					0.095	0.025		
100	-0.8	0.056	0.050			0.071	0.108	0.109	0.201	0.147	0.309
	-0.5	0.050	0.027			0.085	0.062	0.168	0.120	0.283	0.199
	0	0.051	0.024			0.090	0.056	0.187	0.112	0.325	0.196
	0.5	0.051	0.022			0.094	0.049	0.197	0.097	0.344	0.160
	0.8	0.051	0.020			0.096	0.045	0.198	0.082	0.348	0.129
250	-0.8	0.055	0.053	0.090	0.122	0.169	0.224	0.480	0.630	0.625	0.817
	-0.5	0.055	0.038	0.097	0.090	0.192	0.174	0.631	0.523	0.846	0.706
	0	0.052	0.035	0.101	0.085	0.199	0.161	0.678	0.512	0.896	0.705
	0.5	0.052	0.034	0.101	0.084	0.206	0.157	0.694	0.481	0.913	0.666
	0.8	0.052	0.034	0.101	0.083	0.207	0.156	0.697	0.457	0.920	0.633
500	-0.8	0.055	0.056	0.185	0.247	0.588	0.675	0.932	0.972	0.974	0.998
	-0.5	0.057	0.044	0.194	0.200	0.660	0.609	0.987	0.941	1.000	0.990
	0	0.058	0.043	0.199	0.192	0.682	0.600	0.993	0.934	1.000	0.987
	0.5	0.058	0.042	0.199	0.185	0.690	0.581	0.996	0.924	1.000	0.984
	0.8	0.058	0.042	0.200	0.183	0.693	0.568	0.996	0.914	1.000	0.983

Table 3: Rejection rates for a set of values for ρ_0 . The rejection rates for the parameter combinations in blank were not computed as these values of ρ_0 were not feasible given the set of values for n and c .

subtract the mean of each inflation series.

Following Ng and Perron and [Kim \(2009\)](#), I estimate an $ARMA(1, 1)$ model for each country and obtain values of the estimate of the AR ranging from -0.6767 (JAP) to 0.9786 (FRA). The estimates for the MA parameter are negative except for JPN and ITA. The results are summarized in [Table 4](#). Several interesting features emerge from the application. First, the range of values for $\hat{\alpha}_n$ covers that of the simulation. Second, the cancelling root issue discussed above is present for the cases of JPN and ITA. Third, $\tilde{\alpha}_n$ is negative in all cases and close to -1 for GER, an issue that has been widely discussed in the literature as in [Ng and Perron \(2001\)](#).

	U.S.	CAN	JPN	GBR	GER	ITA	FRA
Start	1955Q1	1961Q2	1994Q2	1955Q2	1970Q2	1991Q2	1960Q2
(Obs)	(239)	(215)	(83)	(239)	(179)	(95)	(219)
$AR(\hat{\theta}_{1n})$	0.9622	0.9213	-0.6767	0.9584	0.9702	-0.8749	0.9786
(S.E.)	(0.0188)	(0.0427)	(0.2631)	(0.0235)	(0.0206)	(0.0674)	(0.0210)
$MA(\hat{\alpha}_n)$	-0.4292	-0.5474	0.6495	-0.6908	-0.8217	0.7331	-0.7078
(S.E.)	(0.0428)	(0.1110)	(0.2664)	(0.0377)	(0.0507)	(0.0744)	(0.0483)
$MA(\tilde{\alpha}_n)$	-0.4540	-0.6541	-0.7848	-0.7310	-0.8439	-0.8115	-0.7269
(S.E.)	(0.0543)	(0.0032)	(0.0653)	(0.0492)	(0.0754)	(0.0516)	(0.0443)

Note: De-mean Quarterly Inflation: 400 times the log-differences of GDP Deflator about the mean. All series end in 2014Q4.

Table 4: ML estimates for the De-mean GDP Deflator Inflation in G7 countries. $\hat{\theta}_n$ and $\tilde{\theta}_n$ are the unrestricted and restricted parameter vector estimates, respectively.

	U.S.	CAN	JPN	GBR	GER	ITA	FRA
k^{MAIC}	4	4	5	3	4	4	5
Implicit c	8.9890	16.8322	137.4927	9.8991	5.3106	176.2438	4.6607
LR_c	4.0937	6.6095	10.7454	4.2118	2.7690	31.8083	1.9978
(p-value)	(0.0550)	(0.0150)	(0.0050)	(0.0350)	(0.1450)	(0.0000)	(0.1850)
ADF^*	-2.2467	-2.3678	-1.7026	-2.6946	-2.8920	-1.6835	-1.6428
(p-value)	(0.0242)	(0.0178)	(0.0836)	(0.0075)	(0.0044)	(0.0871)	(0.0944)

Table 5: De-mean GDP Deflator Inflation in G7 countries unit root tests. ADF^* is the Augmented Dickey-Fuller test with lag structure chosen from MAIC.

Table 5 contains the outcome of the unit root tests. In particular, the first row displays k^{MAIC} , the number of autoregressive parameters in addition to the unit root parameter and the variance estimate, that the $MAIC$ suggests (i.e. the ADF^* test for the U.S. requires to estimate 6 parameters). This confirms that for all countries the computation of the LR_c test requires a more parsimonious model. The second row computes the implicit value of the local-to-unity parameter, given the number of observations and the estimate for $\hat{\theta}_{1n}$, this should provide a basis for comparison of the power of each test, in view of the Monte Carlo experiments above. Both the LR_c and the ADF^* tests reject the null hypothesis for the U.S., CAN and GBR, but the former was found to have higher power (see the third row in Figure 4).

The tests yield different results for the rest of the countries. For JPN and ITA the LR_c rejects the null hypothesis whereas the ADF^* fails to reject. As shown in the second row of Figure 4, the LR_c has a higher power (i.e. a lower probability of making a Type II Error).

Finally, the tests for GER and FRA, which need to deal with $\hat{\alpha}_n$ close to -1, are computed with a sample size that is not large enough for LR_c to have higher power than ADF^* , as shown in the third row of Figure 4.

7 Concluding Remarks

Testing the nonstationarity of a particular time series with AR based tools has been the norm in parametric unit root testing. But this is not the only way to model dependence. The LR_c I propose here provides an alternative to the AR paradigm, and has more flexibility in dealing with dependence. This is true, in particular, when the error process shows dependence in the form of a MA process since the asymptotic distribution of LR_c is independent of the short run parameters. The LR_c test proves to be just as good in power terms as the ADF^* test for true values of the MA parameter $\alpha_0 \in \{\pm 0.8, \pm 0.5, 0\}$. For a small sample size of $n \leq 100$, the test has close to optimal power only if $\alpha_0 > 0$, but as the sample size increases optimality is gained for the cases $\alpha_0 \in \{-0.8, -0.5, 0\}$.

Comparing the LR_c test with the ADF^* for sample size $n = 250$, Monte Carlo simulations show that LR_c has higher power than ADF^* , except for the case of $\alpha_0 = -0.8$. The low power is explained by the caveat detailed in Section 5.4, when θ_{10} is close to $\alpha_0 = -0.8$ the test can not distinguish between the time series and a pure white noise process. For a sample size of $n = 500$ LR_c has close to optimal power properties for all values of α_0 . The empirical application shows both the advantages and shortcomings of the LR_c test. The model analysed in this paper can be extended to account for deterministic components, in particular a polynomial trend of order one or zero. I leave this task for a follow-up paper. The model can also be extended to the more general $ARMA(p, q)$, a work which is already in progress in [Chambers and Hernandez \(2015\)](#). The latter in turn will possibly require an information criterion to choose the lag lengths, and will be pursued in future research.

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