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Nikolay Gospodinov and Bin Wei

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JEL classification: G12, E43, E44, C32

Key words: bond prices, TIPS, inflation derivatives, oil prices, no-arbitrage, affine models, out-of-sample forecasting

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Forecasts of Inflation and Interest Rates in No-Arbitrage Affine Models*

Nikolay Gospodinov[†] and Bin Wei[‡]

February 9, 2016

Abstract

In this paper, we examine the forecasting ability of an affine term structure framework that jointly models the markets for Treasuries, inflation-protected securities, inflation derivatives, and oil future prices based on no-arbitrage restrictions across these markets. On methodological side, we propose a novel way of incorporating information from these markets into an affine model. On empirical side, two main findings emerge from our analysis. First, incorporating information from inflation options can often produce more accurate inflation forecasts than those based on the Survey of Professional Forecasters. Second, incorporating oil futures tends to improve short-term inflation and longer-term nominal yield forecasts.

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1 Introduction

Affine term structure models can produce better forecasts of nominal Treasury yields than those obtained by simply assuming that yields follow a random walk process (Duffee, 2002). They are capable of simultaneously fitting the behavior of expected excess returns over time and term structure shapes in the cross section. More recently, researchers have extended the affine term structure framework to jointly model the markets for nominal Treasury securities and Treasury Inflation Protected Securities (TIPS). The no-arbitrage restriction across these markets for nominal and real Treasury securities sheds light on the dynamics of the inflation process. Another potentially useful source of forward-looking information about inflation are inflation derivatives (e.g., inflation swaps, caps, and floors) whose market has exhibited a rapid growth in recent years. Finally, given the increased correlation between oil prices and market-based measures of inflation compensation, commodity futures may also provide differential information about the short- and medium-term movements of inflation and inflation expectations.

In this paper, we study the following questions: How well can the joint affine term structure models forecast simultaneously inflation, nominal and real interest rates? How can we incorporate inflation derivatives and commodity futures into the affine term structure framework and do they lead to forecasting improvements?

To address these questions, we develop a unified affine term structure framework that links the markets for nominal and real Treasury securities, inflation derivatives, and oil futures assuming no-arbitrage across these markets.¹ Our model builds on the growing literature on decomposing the break-even inflation into inflation expectations and risk premium that includes Abrahams, Adrian, Crump and Moench (2015), Ang, Bekaert and Wei (2008), Chen, Liu and Cheng (2010), Christensen, Lopez and Rudebusch (2010), D’Amico, Kim, and Wei (2014), Grishchenko and Huang (2013), Haubrich, Pennacchi and Ritchken (2012), Hördahl and Tristani (2012), Joyce, Lildholdt and Sorensen (2010), Kitsul and Wright (2013), among others. It is well known that nominal and real Treasury yields take an affine functional form (D’Amico, Kim, and Wei, 2014) and it is relatively straightforward to include inflation swaps into an affine term structure model (Haubrich, Pennacchi, and Ritchken, 2012). However, it is not trivial to incorporate the markets for inflation

¹Incorporating additional independent information from surveys or other asset markets also help the potential identification of hidden or unspanned factors (Fisher and Gilles, 2000; Duffee, 2011; Chernov and Mueller, 2012; Joslin, Priebsch and Singleton, 2014) that pass undetected through the term structure of interest rates. For example, as Chernov and Mueller (2012) point out, a factor can remain hidden from the nominal and real term structure of interest rates if it has an equal but opposite effect on inflation expectations and inflation risk premium.

options and oil futures into an affine model and preserve the affine structure.

For inflation options, the difficulty arises from the fact that their prices are highly nonlinear functions of the underlying parameters and state variables. The novelty of our paper is to show that (1) under the put-call parity, we can extract option-implied inflation expectation based on the values of an inflation cap and an inflation floor with the same tenor and strike price, and (2) importantly, the option-implied inflation expectations have an affine structure in Gaussian no-arbitrage term structure models. It is worth pointing out that the extraction of option-implied inflation expectation is model-free, but the resulting inflation expectations are obtained under the forward measure. Furthermore, under no-arbitrage, the forward, risk-neutral, and physical measures are all connected through the prices of risk, and this relation also has an affine structure that can be readily incorporated in an affine term structure model.

To extract information from oil futures, we follow Casassus and Collin-Dufresne (2005) by embedding unobservable oil factors, i.e., the spot oil price and the convenience yield, into an affine term structure model. Instead of one single yield curve factor assumed in their paper, we allow for three latent yield curve factors in addition to the two oil factors, resulting in a five-factor affine model. It is important to have at least three latent yield curve factors because they are needed to capture the level-, slope-, and curvature-movements in the nominal yield curve (Litterman and Scheinkman, 1991) and additional information from the real yield curve. To the best of our knowledge, the proposed five-factor affine model is the first comprehensive model that allows for incorporating information from the markets for nominal and real Treasury securities, inflation swaps and options, and oil futures.

We estimate our joint model by maximum likelihood via Kalman filter and evaluate its forecasting performance. First, we evaluate how the model performs in forecasting inflation against the SPF (Survey of Professional Forecasters) benchmark. We show that using nominal and real Treasury yields only, the inflation forecasting performance of the model is worse than the survey benchmark, with root-mean squared forecast errors about 1-2% larger than the survey-based forecasts. However, once inflation derivatives and/or oil futures are included, our model's inflation forecasting performance is generally at par with the survey benchmark, and exhibits a substantially better performance during the 2011-2015 period when inflation option prices became available. Also, while the accuracy of SPF inflation forecasts has been documented elsewhere (Ang, Bekaert and Wei, 2007; Faust and Wright, 2013), the source of the SPF empirical success for forecasting inflation has not been thoroughly investigated. Interestingly, our results suggest that the SPF forecasts at short

horizons are almost identical to a stylized affine model that uses information only from the yield curve of nominal and real yields.

Second, we also assess how the model performs in forecasting interest rates against the random walk benchmark. As shown in Duffee (2002), essentially affine term structure models can produce better forecasts of nominal Treasury yields than those produced by simply assuming yields follow a random walk. The success of essentially affine models stems from breaking up the tight link between risk compensation and interest rate volatility prevalent in completely affine models. In this paper, we show that when we extend the essentially affine models to model jointly nominal, real and inflation derivative markets, it is more challenging to produce interest rate forecasts that outperform the random walk model. This is perhaps largely due to the fact that the flexibility that allows the essentially affine term structure models to accurately forecast Treasury yields is compromised in the multi-market context. The reason is that to rule out arbitrage, the cross-sectional and time-series characteristics of the term structure in both Treasury and TIPS markets are inherently linked. Focusing on one single market can produce better forecasts for that market only. Once multiple markets are included in an essentially affine term structure model, no-arbitrage imposes a tight link between risk compensations demanded in these markets.² Despite the increased difficulty to outperform the interest rate forecasts by the random walk model, we find that incorporating oil futures help to improve the interest rate forecasts at a longer-term horizon and for longer-maturity bonds.

Finally, while our focus in this paper is on forecasting, the proposed model provides a useful framework for policy analysis or decision-making by businesses and households (e.g., mortgage choices). In particular, it allows us to decompose market-based measures of inflation compensation into several components and monitor the evolution of short-term and long-term inflation expectations. For example, from the middle of 2014 until the beginning of 2016, the five-year, five-year forward breakeven inflation based on TIPS declined by over 100 basis points and many observers interpreted this as a downward drift in inflation expectations. Using a similar version of the model proposed in this paper, Gospodinov and Wei (2015) showed that most of the decline in the TIPS breakeven inflation was due to technical factors and risk premia while the long-run inflation expectations remained stable. Furthermore, even though the performance of our inflation forecasts is similar to that of SPF, our model builds the whole term structure of inflation expectations (at

²Part of the underperformance of the model in forecasting interest rates can be attributed to the specificity of the out-sample forecasting period that is characterized by nominal yields that are near the lower zero bound.

any desired horizon) which is available at much higher frequency. This provides policy makers with timely and valuable information in forming their decisions.

The rest of the paper is organized as follows. Section 2 introduces our new affine model of nominal yields, TIPS and inflation option prices and derives its implications for inflation expectations, uncertainty, and risk premium. Section 3 describes the data and the estimation strategy. Section 4 contains the main empirical results. Section 4 concludes. The proofs of the main results are relegated to Appendix A. The state-space representation of the model and its estimation are discussed in the Appendix B.

2 A Joint Model of Nominal Yields, Real Yields, Oil Prices and Inflation

In this section, we describe in detail the main no-arbitrage framework that we use to jointly model nominal yields, real yields and inflation. We extend the framework in D'Amico, Kim, and Wei (2014) by incorporating information from the derivatives markets and oil futures.

The main risk factors that drive nominal and real yields are the latent variables $x_t = (x_{1t}, x_{2t}, x_{3t})'$ which are associated with the level, slope and curvature of the yield curve. The log spot oil price $s_t \equiv \ln S_t$ and oil convenience yield δ_t are introduced to capture the possible effect of oil prices on inflation expectations. Since s_t and δ_t are found to be highly positively correlated (Casassus and Collin-Dufresne, 2005), we assume that they are driven by the same shock although extending this setup to separate correlated shocks is straightforward. The dynamics the state variables $(x_t', \delta_t, s_t)'$ under the physical measure \mathbb{P} is

$$dx_t = \mathcal{K}(\mu - x_t) dt + \Sigma dW_{x,t}, \quad (1)$$

$$d\delta_t = \kappa_\delta(\mu_\delta - \delta_t) dt + \sigma_\delta dW_{\delta,t}, \quad (2)$$

$$ds_t = \left(\mu_s - \delta_t - \frac{1}{2}\sigma_s^2 \right) dt + \sigma_s dW_{\delta,t}, \quad (3)$$

where $W_{x,t} = (W_{1,t}, W_{2,t}, W_{3,t})'$, and $W_{\delta,t}$ are independent standard Brownian motions. Note that as we will show shortly, the drift of the spot price s_t can be shown to have an affine functional form under no-arbitrage.

The logarithm of the price level Q_t , denoted by $q_t \equiv \ln Q_t$, follows the process

$$dq_t = \pi_t dt + \sigma_q' dW_{x,t} + \sigma_q^\perp dW_{\perp,t}, \quad (4)$$

where $W_{\perp,t}$ is an independent standard Brownian motion and π_t is the instantaneous expected inflation rate, which is assumed to be affine in the latent variables

$$\pi_t = \rho_0^\pi + \rho_x^\pi x_t + \rho_\delta^\pi \delta_t + \rho_s^\pi s_t. \quad (5)$$

The inflation process is allowed to be affected by shocks to these state variables, $dW_{x,t} = (dW_{1,t}, dW_{2,t}, dW_{3,t})'$, as well as the inflation-specific shock $dW_{\perp,t}$ with constant volatility σ_q^\perp .³

2.1 Nominal and Real Bond Prices and Spot Rates

The nominal bond prices are determined by the following nominal pricing kernel

$$dM_t^N / M_t^N = -r_t^N dt - \Lambda_{x,t}^N dW_{x,t} - \Lambda_{\delta,t}^N dW_{\delta,t}, \quad (6)$$

where r_t^N is the nominal short rate, specified as an affine function of the latent variables

$$r_t^N = \rho_0^N + \rho_x^N x_t + \rho_\delta^N \delta_t + \rho_s^N s_t, \quad (7)$$

and the vector of prices of risk is given by

$$\Lambda_{x,t}^N = \lambda_{0,x}^N + \lambda_{1,x}^N x_t, \quad (8)$$

$$\Lambda_{\delta,t}^N = \lambda_{0,\delta}^N + \lambda_{1,\delta}^N \delta_t, \quad (9)$$

where $\lambda_{0,x}^N$ is a 3×1 vector, $\lambda_{1,x}^N$ is a 3×3 matrix, and $\lambda_{0,\delta}^N$ and $\lambda_{1,\delta}^N$ are scalar parameters.

The real and the nominal pricing kernels are linked by the no-arbitrage condition $M_t^R = M_t^N Q_t$. By Ito's lemma, it is straightforward to show that the real pricing kernel M_t^R follows (see Appendix A.1 for detailed derivation)

$$\begin{aligned} dM_t^R / M_t^R &= dM_t^N / M_t^N + dQ_t / Q_t + (dM_t^N / M_t^N) \cdot (dQ_t / Q_t) \\ &= -r_t^R dt - \Lambda_{x,t}^R dW_{x,t} - \Lambda_{\delta,t}^R dW_{\delta,t} + \sigma_q^\perp dW_{\perp,t}, \end{aligned} \quad (10)$$

where the real short rate r_t^R is given by

$$\begin{aligned} r_t^R &= r_t^N - \pi_t + \sigma_q' \Lambda_{x,t}^N - \frac{1}{2} \left(\sigma_q' \sigma_q + \left(\sigma_q^\perp \right)^2 \right) \\ &\equiv \rho_0^R + \rho_x^R x_t + \rho_\delta^R \delta_t + \rho_s^R s_t, \end{aligned} \quad (11)$$

³In an earlier version of the paper, we also allowed for time-varying volatility in the inflation process. While incorporating this inflation uncertainty factor may be theoretically appealing (see Wright, 2011), it also leads to overparameterization and additional technical problems arising from the non-Gaussianity of the model. Results for the model with stochastic volatility in inflation are available from the authors upon request.

and $\Lambda_{x,t}^R = \Lambda_{x,t}^N - \sigma_q \equiv \lambda_{0,x}^R + \lambda_{1,x}^R x_t$ and $\Lambda_{\delta,t}^R = \Lambda_{\delta,t}^N \equiv \lambda_{0,\delta}^R + \lambda_{1,\delta}^R \delta_t$ with $\lambda_{0,x}^R$, $\lambda_{1,x}^R$, $\lambda_{0,\delta}^R$ and $\lambda_{1,\delta}^R$ being of the same dimension as their nominal counterparts. Eq. (11) is the generalized Fisher equation in which the nominal short rate is decomposed into the real short rate, expected inflation rate, instantaneous inflation risk premium, and a convexity term due to Jensen's inequality.

Let $P_{t,\tau}^N$ (or $P_{t,\tau}^R$) denote the time- t price of a nominal (or real) τ -year zero-coupon bond that pays one dollar at maturity. Under the model, nominal and real bond prices can then be determined under the risk neutral measure \mathbb{P}^* :

$$P_{t,\tau}^i = E_t^{\mathbb{P}^*} \left[\exp \left(- \int_t^{t+\tau} r_s^i ds \right) \right], \text{ for } i = N, R,$$

where $E_t^{\mathbb{Q}}[\cdot]$ (respectively, $Var_t^{\mathbb{Q}}[\cdot]$) is generic notation for the expectation (respectively, variance) operator under a particular \mathbb{Q} measure, conditional on information at time t . As is standard for affine term structure models, bond prices can be shown to be exponential affine in the state variables.

2.2 Spot Oil Price

Absence of arbitrage in the oil futures market implies that

$$E_t^{\mathbb{P}^*} [dS_t] = (r_t^N - \delta_t) S_t dt. \quad (12)$$

The convenience yield δ_t can be considered as a “dividend flow”, net of storage costs, to the holder of the commodity. Using Ito's lemma and the change of probability measure, we can derive the dynamics of log spot price under the physical measure as

$$\begin{aligned} ds_t &= \left(r_t^N - \delta_t - \frac{1}{2} \sigma_s^2 + \sigma_s \Lambda_{\delta,t}^N \right) dt + \sigma_s dW_{\delta,t} \\ &\equiv (\rho_0^s + \rho_x^s x_t + \rho_\delta^s \delta_t + \rho_s^s s_t) dt + \sigma_s dW_{\delta,t}, \end{aligned} \quad (13)$$

where $\rho_0^s = \rho_0^N - \frac{1}{2} \sigma_s^2 + \sigma_s \lambda_{0,\delta}^N$, $\rho_x^s = \rho_x^N$, $\rho_\delta^s = \rho_\delta^N - 1 + \sigma_s \lambda_{1,\delta}^N$, and $\rho_s^s = \rho_s^N$.

2.3 An Affine Model of Nominal Yields, Real Yields and Oil Futures Prices

In Proposition 1 below, we derive the closed-form expressions of bond prices and yields in term of the underlying parameters.

Proposition 1 *Under this model, τ -year nominal and real bond prices take the exponential-affine form*

$$P_{t,\tau}^i = \exp \left(A_\tau^i + B_\tau^i x_t + C_\tau^i \delta_t + D_\tau^i s_t \right), \quad i = N, R \quad (14)$$

and τ -year nominal and real yields take the affine form

$$y_{t,\tau}^i = a_\tau^i + b_\tau^i x_t + c_\tau^i \delta_t + d_\tau^i s_t, \quad i = N, R, \quad (15)$$

where $a_\tau^i \equiv -A_\tau^i/\tau$, $b_\tau^i \equiv -B_\tau^i/\tau$, $c_\tau^i \equiv -C_\tau^i/\tau$, and $d_\tau^i \equiv -D_\tau^i/\tau$, and A_τ^i , B_τ^i , C_τ^i , D_τ^i ($i = N, R$) satisfy the following system of ordinary differential equations:

$$\begin{aligned} \frac{dA_\tau^i}{d\tau} &= -\rho_0^i + (\mathcal{K}\mu - \Sigma\lambda_{0,x}^i)' B_\tau^i + (\kappa_\delta\mu_\delta - \sigma_\delta\lambda_{0,\delta}^i) C_\tau^i + \left(\rho_0^N - \frac{1}{2}\sigma_s^2\right) D_\tau^i \\ &\quad + \frac{1}{2}B_\tau^{i'}\Sigma\Sigma' B_\tau^i + \frac{1}{2}\sigma_\delta^2 (C_\tau^i)^2 + \frac{1}{2}\sigma_s^2 (D_\tau^i)^2 \\ \frac{dB_\tau^i}{d\tau} &= -\rho_x^i - (\mathcal{K} + \Sigma\lambda_{1,x}^i)' B_\tau^i + \rho_x^N D_\tau^i \\ \frac{dC_\tau^i}{d\tau} &= -\rho_\delta^i - (\kappa_\delta + \sigma_\delta\lambda_{1,\delta}^i) C_\tau^i + (\rho_\delta^N - 1) D_\tau^i \\ \frac{dD_\tau^i}{d\tau} &= -\rho_s^i + \rho_s^N D_\tau^i \end{aligned}$$

with $A_0^i = B_0^i = C_0^i = D_0^i = 0$.

Proof of Proposition 1. See Appendix A. ■

It follows from Proposition 1 that (for $i = N, R$)

$$\frac{dP_{t,\tau}^i}{P_{t,\tau}^i} = r_t^i dt + B_\tau^i \Sigma dW_{x,t} + (C_\tau^i \sigma_\delta + D_\tau^i \sigma_s) dW_{\delta,t},$$

i.e., we allow the dynamics of nominal and real bond prices to be affected by $dW_{x,t}$ and $dW_{\delta,t}$.

Next, Proposition 2 shows that the oil futures prices are an exponentially affine function of the state variables and the underlying parameters.

Proposition 2 *Under this model, the oil futures price with τ -year maturity takes the form*

$$P_{t,\tau}^{oil} = E_t^{\mathbb{P}^*} [\exp(s_{t+\tau})] = \exp\left(A_\tau^{oil} + B_\tau^{oil} x_t + C_\tau^{oil} \delta_t + D_\tau^{oil} s_t\right),$$

where A_τ^{oil} , B_τ^{oil} , C_τ^{oil} , D_τ^{oil} satisfy the following system of ordinary differential equations:

$$\begin{aligned} \frac{dA_\tau^{oil}}{d\tau} &= (\mathcal{K}\mu - \Sigma\lambda_{0,x}^N)' B_\tau^{oil} + (\kappa_\delta\mu_\delta - \sigma_\delta\lambda_{0,\delta}^N) C_\tau^{oil} + \left(\rho_0^N - \frac{1}{2}\sigma_s^2\right) D_\tau^{oil} \\ &\quad + \frac{1}{2}B_\tau^{oil'}\Sigma\Sigma' B_\tau^{oil} + \frac{1}{2}\sigma_\delta^2 (C_\tau^{oil})^2 + \frac{1}{2}\sigma_s^2 (D_\tau^{oil})^2, \\ \frac{dB_\tau^{oil}}{d\tau} &= -(\mathcal{K} + \Sigma\lambda_{1,x}^N)' B_\tau^{oil} + \rho_x^N D_\tau^{oil}, \\ \frac{dC_\tau^{oil}}{d\tau} &= -(\kappa_\delta + \sigma_\delta\lambda_{1,\delta}^N) C_\tau^{oil} + (\rho_\delta^N - 1) D_\tau^{oil}, \\ \frac{dD_\tau^{oil}}{d\tau} &= \rho_s^N D_\tau^{oil}, \end{aligned}$$

with initial conditions $A_0^{oil} = B_0^{oil} = C_0^{oil} = 0$ and $D_0^{oil} = 1$.

Proof of Proposition 2. See Appendix A. ■

2.4 Inflation Derivative Prices

We now turn to inflation derivatives, i.e., inflation swaps and options. A zero-coupon inflation swap is a forward contract, whereby the inflation buyer pays a predetermined fixed nominal rate and in return receives from the seller an inflation-linked payment. By a standard argument (see, e.g., Haubrich, Pennacchi, and Ritchken, 2012), the equilibrium swap rate for an inflation swap that matures in τ periods is given by

$$y_{t,\tau}^S = y_{t,\tau}^N - y_{t,\tau}^R. \quad (16)$$

Next, we turn our attention to pricing inflation options under the forward measure $\tilde{\mathbb{P}}^\tau$. The Radon-Nikodym derivative of the forward measure $\tilde{\mathbb{P}}^\tau$ with respect to the risk neutral measure \mathbb{P}^* is given by

$$\left(\frac{d\tilde{\mathbb{P}}^\tau}{d\mathbb{P}^*} \right)_{t,t+\tau} = \frac{1}{P_{t,\tau}^N} \exp \left(- \int_t^{t+\tau} r_s^N ds \right).$$

In the rest of the paper, we omit the superscript and simply denote the forward measure by $\tilde{\mathbb{P}}$. The forward measure $\tilde{\mathbb{P}}$ is used for analytical tractability. In particular, the value of an inflation option is simply the expected payoff at maturity under the forward measure discounted by the nominal yield $y_{t,\tau}^N$; that is,

$$\begin{aligned} P_{t,\tau,K}^{CAP} &= \exp(-\tau y_{t,\tau}^N) E_t^{\tilde{\mathbb{P}}} \left[\left(\frac{Q_{t+\tau}}{Q_t} - (1+K)^\tau \right)^+ \right], \\ P_{t,\tau,K}^{FLO} &= \exp(-\tau y_{t,\tau}^N) E_t^{\tilde{\mathbb{P}}} \left[\left((1+K)^\tau - \frac{Q_{t+\tau}}{Q_t} \right)^+ \right], \end{aligned}$$

where $P_{t,\tau,K}^{CAP}$ ($P_{t,\tau,K}^{FLO}$) denotes the price of an inflation cap (floor) with time to maturity τ and a strike price K . If the put-call parity holds, then one can extract option-implied inflation expectation from option prices as follows:

$$E_t^{\tilde{\mathbb{P}}} \left[\frac{Q_{t+\tau}}{Q_t} \right] = \frac{P_{t,\tau,K}^{CAP} - P_{t,\tau,K}^{FLO}}{P_{t,\tau}^N} + (1+K)^\tau. \quad (17)$$

Two observations are worth mentioning here. First, the expression for the option-implied inflation expectation in Eq.(17) holds in general and is not model-specific. The only assumption behind this

result is the put-call parity. Second, although we can extract inflation expectations from option data using Eq.(17), the expectations are taken under the forward measure. As a result, they cannot be directly compared with the breakeven inflation rate which contains inflation expectations under the physical measure. The affine term structure model in this paper allows us to further translate the option-implied inflation expectations based on Girsanov's theorem which implies the following relationship between Brownian motions under the forward and risk-neutral measures:

$$d\tilde{W}_{x,t} = dW_t^* - \Sigma' B_\tau^N dt,$$

Denote

$$\begin{aligned} \mathcal{IE}_{t,\tau} &\equiv \frac{1}{\tau} E_t^{\mathbb{P}} [q_{t+\tau} - q_t], \\ \mathcal{IU}_{t,\tau} &\equiv \frac{1}{\tau} \text{Var}_t^{\mathbb{P}} [q_{t+\tau} - q_t], \end{aligned}$$

where $\mathcal{IE}_{t,\tau}$ is an alternative definition of inflation expectations⁴ and $\mathcal{IU}_{t,\tau}$ measures inflation uncertainty. Under the assumption that the change in log price levels follows a normal distribution, i.e., $q_{t+\tau} - q_t | \mathcal{F}_t \sim N(\tau \cdot \mathcal{IE}_{t,\tau}, \tau \cdot \mathcal{IU}_{t,\tau})$, the option-implied inflation expectations in Eq.(17) have the following exponential-affine form:

$$E_t^{\mathbb{P}} \left[\frac{Q_{t+\tau}}{Q_t} \right] = \exp \left[\tau (\mathcal{IE}_{t,\tau}) + \frac{1}{2} \tau (\mathcal{IU}_{t,\tau}) \right], \quad (18)$$

where the explicit expressions of $\mathcal{IE}_{t,\tau}$ and $\mathcal{IU}_{t,\tau}$, in terms of the state variables and model parameters, are given in Lemma A.1 in Appendix A.

Moreover, we can also derive closed-form pricing formulas for inflation caps and floors. The results are presented in Proposition 3 below.

Proposition 3 *Under the model, the prices of inflation caps and floors with maturity τ and strike K are given by*

$$P_{t,\tau,K}^{CAP} = P_{t,\tau}^N \left[\begin{aligned} &e^{\tau(\mathcal{IE}_{t,\tau} + \frac{1}{2}\mathcal{IU}_{t,\tau})} \Phi \left(\frac{-\ln(1+K) + (\mathcal{IE}_{t,\tau} + \mathcal{IU}_{t,\tau})}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}} \right) \\ &- (1+K)^\tau \Phi \left(\frac{-\ln(1+K) + \mathcal{IE}_{t,\tau}}{\sqrt{\mathcal{IU}_{t,\tau}/\tau}} \right) \end{aligned} \right], \quad (19)$$

⁴We define inflation expectations as in Christensen, Lopez and Rudebusch (2010)

$$IE_{t,\tau} \equiv -\frac{1}{\tau} \ln \left(E_t \left[\frac{Q_t}{Q_{t+\tau}} \right] \right).$$

This differs from D'Amico, Kim and Wei (2014) who define it as $IE_{t,\tau} \equiv \frac{1}{\tau} E_t \left[\ln \left(\frac{Q_{t+\tau}}{Q_t} \right) \right]$. The difference between these two slightly different measures is due to the Jensen's inequality term.

$$P_{t,\tau,K}^{FLO} = P_{t,\tau}^N \left[\begin{array}{c} -e^{\tau(\mathcal{I}\mathcal{E}_{t,\tau} + \frac{1}{2}\mathcal{I}\mathcal{U}_{t,\tau})} \Phi \left(-\frac{-\ln(1+K) + (\mathcal{I}\mathcal{E}_{t,\tau} + \mathcal{I}\mathcal{U}_{t,\tau})}{\sqrt{\mathcal{I}\mathcal{U}_{t,\tau}/\tau}} \right) \\ + (1+K)^\tau \Phi \left(-\frac{-\ln(1+K) + \mathcal{I}\mathcal{E}_{t,\tau}}{\sqrt{\mathcal{I}\mathcal{U}_{t,\tau}/\tau}} \right) \end{array} \right], \quad (20)$$

where $\Phi(\cdot)$ denotes the cumulative distribution function of a standard normal random variable.

Proof of Proposition 3. See Appendix A. ■

3 Estimation Methodology and Data

3.1 Estimation Methodology

The set of variables that enter our model consists of observables ($y_{t,\tau}^N$, $y_{t,\tau}^R$, $y_{t,\tau,K}^O$, $y_{t,\tau}^S$ and $p_{t,\tau}^{oil}$), latent (x_t , s_t and δ_t) and partially observed (q_t) state variables, where $y_{t,\tau,K}^O \equiv \frac{1}{\tau} \ln E_t^{\tilde{\mathbb{P}}} \left[\frac{Q_{t+\tau}}{Q_t} \right] = \mathcal{I}\mathcal{E}_{t,\tau} + \frac{1}{2}\mathcal{I}\mathcal{U}_{t,\tau}$ (see Eqs.(17)-(18)) and $p_{t,\tau}^{oil} = \ln P_{t,\tau}^{oil}$. The superscript ‘‘O’’ refers to the use of options data in deriving the inflation expectations under the forward measure. Since the dimension of the observables is typically larger than the dimension of the state vector, the term structure models are inherently stochastically singular (Piazzesi, 2010, p.726). There are two approaches to dealing with this singularity. One approach is to ‘‘invert’’ the state variables from a small subset (of the same dimension as the state vector) of observables, add measurement errors to the rest of the observable vector and proceed with quasi-maximum likelihood estimation. The drawback of this method is that the choice of the observables from which the state variables are extracted is arbitrary and naturally affects the quality and the dynamics of the derived state variables. An additional problem that arises in our setup is the presence of inflation derivative and oil prices.

For these reasons, we pursue the second approach in which all yields, log oil prices and option-implied inflation expectations are assumed to be observed with a measurement error. This specification arises naturally in our framework since our yield data is obtained from an interpolated zero-coupon yield curve (see also Piazzesi, 2010, for the plausibility of this assumption). This approach requires the use of a filtering method and, given the assumptions and the structure of our model, we employ the Kalman filter which is discussed later. The nominal yields, real yields, oil prices and inflation option prices are at weekly frequency while the CPI inflation is based on monthly data. Let $Y_t = (q_t, y_{t,\tau}^{Nl}, y_{t,\tau}^{Rl}, y_{t,\tau,K}^{O'}, y_{t,\tau}^{Sl}, p_{t,\tau}^{oil})'$ denote the $m \times 1$ vector of observables, where $\{q_t\}$, $t \in \{1, \dots, T_q\}$, $\{y_{t,\tau}^N\}$ with $t \in \{1, \dots, T_N\}$ and $\tau \in \{\tau_1, \dots, \tau_{m^N}\}$, $\{y_{t,\tau}^R\}$ with $t \in \{1, \dots, T_R\}$ and $\tau \in \{\tau_1, \dots, \tau_{m^R}\}$, $\{y_{t,\tau,K}^O\}$ with $t \in \{1, \dots, T_O\}$, $\tau \in \{\tau_1, \dots, \tau_{m^O}\}$ and $K \in \{K_1, \dots, K_{m^K}\}$, $\{p_{t,\tau}^{oil}\}$ with $t \in \{1, \dots, T_{oil}\}$ and $\tau \in \{\tau_1, \dots, \tau_{m^{oil}}\}$ and $m = 1 + m^N + m^R + m^S + m^O \cdot m^K + m^{oil}$.

The state-space representation of the model is characterized by the measurement equation for Y_t and the transition equation for the augmented state vector $X_t = (q_t, x'_t, s_t, \delta_t)'$ whose construction is described in Appendix B. The discretized state-space system is given by

$$\begin{aligned} X_t &= \mathcal{A} + \mathcal{B}X_{t-1} + \eta_t \\ Y_t &= a + bX_t + e_t, \end{aligned}$$

where \mathcal{A} , \mathcal{B} , a , and b are functions of the underlying parameters of interest $\theta = (\text{vec}(\mathcal{K})', \text{vech}(\Sigma\Sigma')', \mu', \rho_0^{Nt}, \rho_x^{Nt}, \rho_\delta^N, \rho_s^N, \lambda_0^{Nt}, \text{vec}(\lambda_x^N)', \rho_0^\pi, \rho_x^\pi, \rho_\delta^\pi, \rho_s^\pi, \sigma'_q, \sigma_q^\perp, \kappa_\delta, \sigma_\delta, \mu_\delta, \lambda_{0,\delta}^N, \lambda_{1,\delta}^N, \sigma_s)'$. For identification purposes, we follow D'Amico, Kim and Wei (2014) and impose the restrictions that μ is a zero vector, \mathcal{K} is a diagonal matrix and Σ is a lower triangular matrix with diagonal matrix set equal to 0.01. For forecasting, as explained in Section 4, we also impose the following parameter restrictions: $\rho_s^\pi = 0$, $\lambda_{0,\delta}^N = 0$ and $\lambda_{1,\delta}^N = 0$. The parameter vector θ is then estimated by Kalman filter (see Appendix B for details).

3.2 Data

All data variables are converted to weekly frequency and end in the last week of December 2015 (although they may have different start dates). Continuously-compounded, zero-coupon yields on U.S. Treasury notes with 1-, 2-, 4-, 7- and 10-year maturities are obtained from the U.S. Treasury yield curve of Gürkaynak, Sack and Wright (2007), maintained by the Federal Reserve Board (available at <http://www.federalreserve.gov/Pubs/feds/2006/200628/200628abs.html>). The 3- and 6-month rates are obtained from the 3- and 6-month T-bill rates with constant maturity from the Federal Reserve Board's H.15 statistical release by converting them from discount basis to continuously-compounded rates. The sample period for the nominal yields starts in the first week of January 1990. For the TIPS yields, we use data for 5-, 7- and 10-year continuously-compounded, zero-coupon yields from the TIPS yield curve⁵ of Gürkaynak, Sack and Wright (2010), maintained by the Federal Reserve Board (<http://www.federalreserve.gov/pubs/feds/2008/200805/200805abs.html>). The sample period for TIPS yields starts in the first week of January 1999. As of the end of August 2015, there is \$1.1 trillion of TIPS outstanding versus \$11.4 trillion of nominal Treasuries outstanding. The principal of the TIPS is linked to the non-seasonally adjusted CPI for all urban consumers, and is accredited monthly. TIPS offer a deflation protection (floor) as the greater of the inflation-adjusted principal and the original principal is paid at maturity.

⁵For some specific aspects of the U.S. TIPS market, see Fleckenstein, Longstaff and Lustig (2014), Fleming and Krishnan (2012), Gürkaynak, Sack and Wright (2010), and Sack and Elsasser (2004).

Data for inflation cap and floor prices, starting in the middle of February 2010, with strike prices from 1% to 3% in increments of 1% with 1- and 3-year maturities, are obtained from Bloomberg. We also obtain inflation swap data (starting in July 2004) from Bloomberg and choose the same maturities as the TIPS data, i.e., 5-, 7-, and 10-year maturities.

For our oil series, we use the prices of crude oil (WTI), traded on NYMEX, for the nearest, 1-, 3- and 12-month futures contracts. To avoid problems with lack of liquidity and higher volatility near the expiration of the contract, we roll over the current contract to the following contract on the first day of the delivery month. The oil data is available from January 1990. Weekly series for nominal yields, TIPS yields, and inflation option prices, inflation swaps and oil prices are constructed by using the Wednesday observation of each week (if the market is closed on Wednesday, we take the Tuesday observation or Thursday’s observation if the Tuesday’s is not available).

We use the CPI for all urban consumers (all items, seasonally adjusted) from the U.S. Bureau of Labor Statistics, covering the period January 1990 – September 2015. The monthly CPI is assumed to be observed on the third Wednesday of each month. The remaining weeks are treated as missing observations which are filled in via the Kalman filter. Similarly, the missing weekly observations up to January 1999 for TIPS and October 2009 for inflation options are also estimated using the Kalman filter.

Figure 1 plots the 5-year Treasury and TIPS yields along with the 5-year breakeven inflation rate. For most of the period, the 5-year breakeven rate varies between 1% and 3% except for a sharp decrease in the wake of the recent financial crisis. There are some regularities in the breakeven rate that have become more pronounced after the financial crisis and may have been caused by a seasonal carry that characterizes the TIPS market. In historical context, the recent decline in the breakeven inflation is not unusual.⁶

4 Empirical Results

We estimate four versions of the model that include different input variables. The models are denoted by \mathcal{M} with superscripts NR , O , S , oil (for nominal/real yields, option-implied inflation expectations, swaps and oil prices) for the input variables. The benchmark model, \mathcal{M}^{NR} , is the model used in D’Amico, Kim and Wei (2014) without a liquidity factor. This model uses nominal (Treasury) and real (TIPS) yields (as well as inflation) as input variables.

⁶See Gospodinov and Wei (2015) for a more detailed analysis of the dynamics and decomposition of the breakeven inflation.

The other three models are new. The first one, \mathcal{M}^{NRO} , is designed to evaluate the information content of options (option-implied inflation expectations, to be more precise) for identifying and estimating the same set of model parameters as in \mathcal{M}^{NR} . The model \mathcal{M}^{NRS} uses inflation swap information and the model \mathcal{M}^{NRoil} incorporates oil futures prices in addition to nominal and real yields.

It should be noted that model \mathcal{M}^{NRoil} is estimated by imposing some parameter restrictions – in particular, $\rho_s^\pi = 0$, $\lambda_{0,\delta}^N = 0$ and $\lambda_{1,\delta}^N = 0$ – in order to avoid overparameterization and overfitting. The first restriction is imposed to prevent the near-nonstationary and trending behavior of the spot oil price to translate directly into inflation expectations. The informational content that is aggregated in the oil futures market is allowed to operate through the convenience yield which tends to reflect global demand conditions (see Gospodinov and Ng, 2013). The last two restrictions are imposed since the prices of oil risk appear to be small and relatively unimportant.

All model specifications produce similar dynamics for the state variables in x which roughly correspond to the level (x_2), slope (x_1) and curvature (x_3) of the nominal and real term structure. The loadings of the nominal spot rate on these state variables as well as the estimated prices of risk are also similar across the different models.

The models \mathcal{M}^{NR} , \mathcal{M}^{NRO} , \mathcal{M}^{NRS} and \mathcal{M}^{NRoil} , estimated at weekly frequency, are used to produce forecasts at 3-month, 6-month and 12-month horizons. The monthly forecasts for inflation, nominal and real yields are evaluated relative to the random walk (RW) model. Since the Survey of Professional Forecasters (SPF) has been documented to provide some of the best forecasts of inflation (Ang, Bekaert and Wei, 2007; Faust and Wright, 2013), we also evaluate the forecast performance of our models relative to SPF at 1- to 4-quarter ahead horizons.

4.1 Out-of-sample forecast of inflation

Table 1 presents the out-of-sample forecasting results for annual inflation at monthly forecast horizons. The forecasts are computed recursively with an initial estimation sample January 1990 - December 2003. The out-of-sample period is 2004-2014. To assess how the performance of the different forecasting models vary over time (in particular - before, during and after the recent financial crisis), we also report results for several sub-samples: 2004-2007, 2008-2010, 2011-2014. We should note that the dating of our sample periods reflects the time when the forecast is being made. This allows us to have the same number of forecasts for each forecast horizon h ($h = 3, 6, 12$). The root mean squared forecast error (RMSE) for the different forecasting models is computed as

$\sqrt{\frac{1}{n} \sum_{i=1}^n (\tilde{\pi}_{i+h} - \hat{\pi}_{i+h})^2}$, where $\hat{\pi}_{i+h} = \frac{Q_{i+h} - Q_{i+h-12}}{Q_{i+h-12}}$ is the actual annual inflation rate at horizon h (in months) and $\tilde{\pi}_{i+h}$ is the model forecast of inflation.

Table 1 presents the RMSE for the RW model and the ratios of the RMSEs of the other models relative to that of RW. Hence, numbers greater than one indicate that RW dominates the other models and numbers smaller than one suggest that RW is outperformed by the corresponding model. The models considered in this paper dominate uniformly the RW model across forecast horizons and sub-samples. The reduction of the RMSE, relative to RW, is often quite substantial. The inflation options improve the accuracy of inflation forecasts with their introduction in 2010. Oil prices also help to reduce the RMSE at short horizons (up to 3 months) in the sub-periods 2004-2007 and 2011-2014.

Since SPF has been documented (Ang, Bekaert and Wei, 2007) to produce some of the most accurate forecasts of inflation, we compare our model against this benchmark. Since SPF is available only at quarterly frequency, we conduct the forecast comparison at this frequency with 1-, 2- 3- and 4-quarter horizons. Table 2 reports the results for different models as ratios of their RMSEs relative to that of SPF.

The results in Table 2 can be summarized as follows. As argued elsewhere in the literature (Ang, Bekaert and Wei, 2007; Faust and Wright, 2013), SPF performs very well for forecasting CPI inflation 1- to 4-quarters ahead. Adding information from options and swaps helps the model forecasts at longer horizons, except for the financial crisis period. The improvements of \mathcal{M}^{NRO} over SPF in the most recent sub-sample (when inflation option data has become available and inflation options started to trade more actively) are substantial. As in Table 1, oil futures provide some forecasting improvements at short horizons, with the exception of the financial crisis period. Figure 2 plots the year-over-year inflation rate (blue solid line), four-quarter ahead SPF survey forecast (black crosses), and model forecasts from all four models (red circles). The right-upper subplot demonstrates that the forecast based on \mathcal{M}^{NRO} performs generally better than the SPF survey forecast between 2011 and 2014 when inflation option data are readily available. In this period, the realized inflation rate dropped to a low level around zero percent in 2015 while the SPF forecast was quite stable around 2 percent (recall that the timing is specified as the time when the SPF forecast was made). By contrast, unlike the other models, model \mathcal{M}^{NRO} is able to generate inflation forecast that is more in line with the actual realization.

Presumably inflation options may well contain useful forward-looking information about future

inflation and thus help improve inflation forecasts. As we argue earlier, one particular useful information we can extract from inflation options is the model-free inflation expectations under the forward measure. Figure 3 presents the 1- and 3-year option-implied inflation expectations that are computed as described in Section 2. From Figure 3, we can see that option-implied inflation expectations co-move closely with the break-even inflation rate, and are typically greater than the latter most of the time.

To the best of our knowledge, however, the source of the good forecasting performance of SPF has not been clear apart from being a combination of forecasts from experts. Our results shed light on what factors the professional forecasters might be using in forming their inflation expectations and forecasts. First, note that the forecasting performance of \mathcal{M}^{NR} , albeit slightly worse, is very close to SPF for 1- and 3-quarter horizons. To visualize this performance over time, Figure 4 plots the model forecasts, SPF forecasts and realized annualized inflation. The closeness between the forecasts of \mathcal{M}^{NR} and SPF at shorter horizons is striking. This suggests that the median forecaster uses the information (level, slope and curvature) in the yield curve as a main predictor for future inflation. At longer horizons, the discrepancy is increasing as the SPF forecasts become flatter. This is likely due to the fact that the professional forecasters impose more mean reversion (possibly based on judgemental assessment) in their longer-term inflation forecasts. By contrast, the model state variables (especially the level x_2) are highly persistent and induce a slower mean reversion in the model forecasts.

4.2 Out-of-sample forecast of interest rates

Tables 3a-3d and 4a-4d report results for out-of-sample forecasts of nominal and real yields. We consider models \mathcal{M}^{NR} , \mathcal{M}^{NRO} , \mathcal{M}^{NRS} , \mathcal{M}^{NRoil} , and RW. RW is the benchmark model and the results for the other models are presented as a ratio of their RMSE to the RMSE of RW. The RMSEs for nominal and real yields are computed as $\sqrt{\frac{1}{n} \sum_{i=1}^n \left(\tilde{y}_{i+h,\tau}^N - \hat{y}_{i+h,\tau}^N \right)^2}$ and $\sqrt{\frac{1}{n} \sum_{i=1}^n \left(\tilde{y}_{i+h,\tau}^R - \hat{y}_{i+h,\tau}^R \right)^2}$, where $\hat{y}_{i+h,\tau}^N$ is the actual nominal yield, $\tilde{y}_{i+h,\tau}^N$ is the model forecast of nominal yield, $\hat{y}_{i+h,\tau}^R$ is the actual real (TIPS) yield, and $\tilde{y}_{i+h,\tau}^R$ is the model forecast of real yield, respectively. The forecast horizon is $h = 3, 6, 12$ months. We consider bond maturities $\tau = 0.5, 1, 2, 5$ and 10 (in years) for nominal yields and $\tau = 2, 5$ and 10 (in years) for real yields. As for inflation, we report results for the whole out-of-sample period 2004-2014 as well as for sub-samples: 2004-2007, 2008-2010 and 2011-2014.

For nominal yields, \mathcal{M}^{NR} dominates RW at shorter maturities for the 2004-2007 sub-sample. Incorporating information from inflation derivatives does not help to reduce the RMSE for nominal bond forecasts. The oil state variables appear to provide an improvement for long-maturity bonds but it is only marginal. The models are more successful at forecasting real yields at 6- and 12-month horizon with a substantial reduction of the RMSE for 2- and 5-year TIPS.

The poor forecasting performance of the model for nominal yields warrants a few remarks. First, a part of the under-performance can be attributed to the specificity of the out-sample forecasting period that is characterized by nominal yields that are near the lower zero bound. Since none of the models considered have any built-in-features to handle this type of behavior, it is not surprising that they are out-performed by the RW model. Consider, for example, the sub-sample period 2011-2014 when the relative RMSE ratios are the highest. Figure 5 plots the actual 6-month Treasury yield and its 3-month forecasts by the \mathcal{M}^{NR} , \mathcal{M}^{NRO} , \mathcal{M}^{NRS} , and \mathcal{M}^{NRoil} models. While the actual 6-month yield has stayed relatively flat and close to zero since 2008, the model forecasts have been much more volatile because our specification does not impose any restrictions related to the zero lower bound. This is not a deficiency that is specific to our model but a feature of any affine term structure model that does not account explicitly for the zero lower bound. In fact, our model forecasts exhibit a fairly realistic dynamics that can be linked to some developments in monetary policy (quantitative easing programs, FOMC statements etc.) Introducing a shadow rate as in Wu and Xia (2015) may allow our model to better approximate the behavior of the short end of the nominal yield curve during the period of unconventional monetary policy.

Another important reason, however, is that once markets for both nominal and real yields are included in an essentially affine term structure model, no-arbitrage imposes a tight link between the nominal and real prices of risk. For example, since $\Lambda_{x,t}^N = \Lambda_{x,t}^R + \sigma_q$, the parameter σ_q that governs the inflation process imposes a tight link between $\Lambda_{x,t}^N$ and $\Lambda_{x,t}^R$. Therefore, it is challenging to freely break up risk compensations demanded in both markets. That is, we may be able to break up the link between risk compensation and interest rate volatility in one market, but it is quite challenging to break up the link in both markets. This is the key reason why the essentially affine term structure models that jointly model Treasury and TIPS yields fail to produce accurate forecasts for nominal yields, although they can still produce accurate forecasts for TIPS yields and inflation.

4.3 Information contained in inflation swaps, options, and oil futures

In this subsection, we try to assess and summarize the information content from incorporating inflation derivatives and oil futures. First, we focus on inflation forecasting at quarterly forecast horizons for which SPF forecasts are available. One natural way of assessing the relative contribution of the different informational sources is to regress the realized inflation rate on the model-implied forecasts based on different markets. Specifically, we run the following regression of the realized inflation rate π_{t+h} at time $t+h$ on survey and model forecasts $\widehat{\pi}_{t+h}$ made at time t :

$$\pi_{t+h} = \alpha^{SPF} \widehat{\pi}_{t+h}^{SPF} + \alpha^{NR} \widehat{\pi}_{t+h}^{NR} + \alpha^{NRO} \widehat{\pi}_{t+h}^{NRO} + \alpha^{NRS} \widehat{\pi}_{t+h}^{NRS} + \alpha^{NRoil} \widehat{\pi}_{t+h}^{NRoil},$$

subject to the constraints:

$$\alpha^{SPF} + \alpha^{NR} + \alpha^{NRO} + \alpha^{NRS} + \alpha^{NRoil} = 1,$$

and

$$\alpha^{SPF}, \alpha^{NR}, \alpha^{NRO}, \alpha^{NRS}, \alpha^{NRoil} \in [0, 1].$$

Heuristically, the regression coefficients measure the incremental information content from a specific market or survey. For example, if the survey-based inflation forecast had completely dominated all model-implied forecasts, we would expect to see α^{SPF} close to one with the other coefficients being close to zero. The regression results are reported in Table 5. From Table 5, we can see that the coefficient α^{SPF} is around 0.5, indicating that model-based forecasts are informative as well. Table 5 also shows that the forecast based on model \mathcal{M}^{NR} is no longer informative once we include forecasts based on inflation derivatives or oil futures. Among the latter, model \mathcal{M}^{NRO} contains most information content for predicting inflation, as suggested by its large regression coefficients. This is consistent with the earlier evidence shown in Table 1.

Similar exercises can be performed to evaluate information contents in forecasting interest rates. Specifically, we run the following regression

$$y_{t+h,\tau}^i = \alpha^{RW} \widehat{y}_{t+h,\tau}^{i,RW} + \alpha^{NR} \widehat{y}_{t+h,\tau}^{i,NR} + \alpha^{NRO} \widehat{y}_{t+h,\tau}^{i,NRO} + \alpha^{NRS} \widehat{y}_{t+h,\tau}^{i,NRS} + \alpha^{NRoil} \widehat{y}_{t+h,\tau}^{i,NRoil}, \text{ for } i = N, R,$$

where $\widehat{y}_{t+h,\tau}^N$ and $\widehat{y}_{t+h,\tau}^R$ denote model h -month ahead forecasts of nominal and real yields made at time t , subject to the same constraints as the regression for inflation forecasts. The results are reported in Tables 6 and 7, regarding the forecasts of nominal and real interest rates, respectively. One interesting finding is that incorporating inflation options is no longer useful in forecasting

interest rates. On the other hand, incorporating oil futures seems to matter a lot especially when forecasting longer-term interest rates. One possible explanation for this finding is that the convenience yield may contain valuable information about the economy and is thus useful in forecasting future interest rates.

5 Conclusion

In this paper we examine the forecasting ability of affine term structure framework that links the markets for Treasuries, inflation-protected securities, inflation derivatives, and oil futures, based on no-arbitrage restrictions across these markets. In addition to fitting all these asset markets simultaneously, the model provides measures of inflation expectations, risk premium and model-implied inflation distributions with wide policy implications.

We contribute to the literature along several dimensions. First, we demonstrate the importance of no-arbitrage restrictions across different markets for improved forecasting of inflation. Incorporating information from inflation options reduces substantially the forecast error for inflation. Second, we use a novel way to introduce information from the options market into an affine framework. More specifically, we establish that option-implied inflation expectations, under the forward measure, are affine in the state variables. We link the different (physical, risk-neutral and forward) measure through common prices of risk and show that inflation options help to identify the price of risk parameters. Third, we augment the no-arbitrage model for nominal and real yields with the term structure of oil future prices. Fourth, our results suggest that the empirical success of inflation forecasts from the survey of professional forecasters can be replicated with a standard affine model that exploits only information in the nominal and real yield curve. Finally, because the no-arbitrage restrictions across markets also constrain the flexibility in modeling risk compensation and interest rate volatility, combining information from these markets poses a challenge to existing affine term structures models for their ability to simultaneously forecast inflation and interest rates.

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Appendix A: Derivation and Proofs

A.1 Derivation of the Real Pricing Kernel

By Ito's lemma and the no-arbitrage condition $M_t^R = M_t^N Q_t$, we have

$$\begin{aligned}
dM_t^R/M_t^R &= dM_t^N/M_t^N + dQ_t/Q_t + (dM_t^N/M_t^N) \cdot (dQ_t/Q_t) \\
&= -r_t^N dt - \Lambda_{x,t}^N dW_{x,t} - \Lambda_{\delta,t}^N dW_{\delta,t} \\
&\quad + \left(\pi_t + \frac{1}{2} \left(\sigma'_q \sigma_q + \left(\sigma_q^\perp \right)^2 \right) \right) dt + \sigma'_q dW_{x,t} + \sigma_q^\perp dW_{\perp,t} + \sigma'_q \Lambda_{x,t}^N \\
&= -r_t^R dt - \Lambda_{x,t}^R dW_{x,t} - \Lambda_{\delta,t}^R dW_{\delta,t} + \sigma_q^\perp dW_{\perp,t},
\end{aligned}$$

where the real short rate r_t^R is given by

$$\begin{aligned}
r_t^R &= r_t^N - \pi_t + \sigma'_q \Lambda_{x,t}^N - \frac{1}{2} \left(\sigma'_q \sigma_q + \left(\sigma_q^\perp \right)^2 \right) \\
&= \left(\rho_0^N + \rho_x^N x_t + \rho_\delta^N \delta_t + \rho_s^N s_t \right) - \left(\rho_0^\pi + \rho_x^\pi x_t + \rho_\delta^\pi \delta_t + \rho_s^\pi s_t \right) \\
&\quad + \sigma'_q \left(\lambda_{0,x}^N + \lambda_{1,x}^N x_t \right) - \frac{1}{2} \left(\sigma'_q \sigma_q + \left(\sigma_q^\perp \right)^2 \right) \\
&= \left(\rho_0^N - \rho_0^\pi - \frac{1}{2} \left(\sigma'_q \sigma_q + \left(\sigma_q^\perp \right)^2 \right) + \sigma'_q \lambda_{0,x}^N \right) \\
&\quad + \left(\rho_x^N - \rho_x^\pi + \sigma'_q \lambda_{1,x}^N \right) x_t + \left(\rho_\delta^N - \rho_\delta^\pi \right) \delta_t + \left(\rho_s^N - \rho_s^\pi \right) s_t \\
&\equiv \rho_0^R + \rho_x^R x_t + \rho_\delta^R \delta_t + \rho_s^R s_t
\end{aligned}$$

and the real prices of risk are given by

$$\begin{aligned}
\Lambda_{x,t}^R &= \Lambda_{x,t}^N - \sigma_q \equiv \lambda_{0,x}^R + \lambda_{1,x}^R x_t, \\
\Lambda_{\delta,t}^R &= \Lambda_{\delta,t}^N \equiv \lambda_{0,\delta}^R + \lambda_{1,\delta}^R \delta_t.
\end{aligned}$$

The parameters in the above equations are given by:

$$\begin{aligned}
\rho_0^R &\equiv \rho_0^N - \rho_0^\pi - \frac{1}{2} \left(\sigma'_q \sigma_q + \left(\sigma_q^\perp \right)^2 \right) + \sigma'_q \lambda_{0,x}^N, \\
\rho_x^R &\equiv \rho_x^N - \rho_x^\pi + \sigma'_q \lambda_{1,x}^N, \\
\rho_\delta^R &\equiv \rho_\delta^N - \rho_\delta^\pi, \\
\rho_s^R &\equiv \rho_s^N - \rho_s^\pi,
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{0,x}^R &\equiv \lambda_{0,x}^N - \sigma_q, \\
\lambda_{1,x}^R &\equiv \lambda_{1,x}^N, \\
\lambda_{0,\delta}^R &\equiv \lambda_{0,\delta}^N, \\
\lambda_{1,\delta}^R &\equiv \lambda_{1,\delta}^N.
\end{aligned}$$

A.2 Proof of Proposition 1

We derive the pricing formula for nominal bonds. The derivation of the pricing formula for real bonds is very similar and thus omitted.

Let

$$\Lambda_t^N \equiv \begin{pmatrix} \Lambda_{x,t}^N \\ \Lambda_{\delta,t}^N \end{pmatrix}, W_t \equiv \begin{pmatrix} W_{x,t} \\ W_{\delta,t} \end{pmatrix}.$$

Then, the Radon-Nikodym derivative of the risk neutral measure \mathbb{P}^* with respect to the physical measure \mathbb{P} is given by

$$\left(\frac{d\mathbb{P}^*}{d\mathbb{P}} \right)_{t,T} = \exp \left[-\frac{1}{2} \int_t^T \Lambda_s^{N'} \Lambda_s^N ds - \int_t^T \Lambda_s^{N'} dW_s \right]$$

By the Girsanov theorem, $dW_t^* = dW_t + \Lambda_t^N dt$ is a standard Brownian motion under the risk-neutral probability measure \mathbb{P}^* . It implies that under the risk neutral measure,

$$dx_t = \mathcal{K}^* (\mu^* - x_t) + \Sigma dW_{x,t}^*, \quad (\text{A1})$$

$$d\delta_t = \kappa_\delta^* (\mu_\delta^* - \delta_t) dt + \sigma_\delta dW_{\delta,t}^*, \quad (\text{A2})$$

$$ds_t = (\phi_0^* + \phi_x^* x_t + \phi_\delta^* \delta_t + \phi_s^* s_t) dt + \sigma_s dW_{\delta,t}^*, \quad (\text{A3})$$

$$dq_t = (\rho_0^{\pi^*} + \rho_x^{\pi^*} x_t + \rho_\delta^{\pi^*} \delta_t + \rho_s^{\pi^*} s_t) dt + \sigma_q' dW_{x,t}^* + \sigma_q^\perp dW_{\perp,t}^*, \quad (\text{A4})$$

where

$$\begin{aligned} \mathcal{K}^* &= \mathcal{K} + \Sigma \lambda_{1,x}^N, \mathcal{K}^* \mu^* = \mathcal{K} \mu - \Sigma \lambda_{0,x}^N \\ \kappa_\delta^* &= \kappa_\delta + \sigma_\delta \lambda_{1,\delta}^N, \kappa_\delta^* \mu_\delta^* = \kappa_\delta \mu_\delta - \sigma_\delta \lambda_{0,\delta}^N \\ \phi_0^* &= \phi_0 - \sigma_s \lambda_{0,\delta}^N, \phi_x^* = \phi_x, \phi_\delta^* = \phi_\delta - \sigma_s \lambda_{1,\delta}^N, \phi_s^* = \rho_s^N \\ \rho_0^{\pi^*} &= \rho_0^\pi - \lambda_{0,x}^{N'} \sigma_q, \rho_x^{\pi^*} = \rho_x^\pi - \lambda_{1,x}^{N'} \sigma_q, \rho_\delta^{\pi^*} = \rho_\delta^\pi, \rho_s^{\pi^*} = \rho_s^\pi. \end{aligned}$$

From the fact that $\exp\left(-\int_0^t r_s^N ds\right) P_{t,\tau}^N$ is a martingale under the risk neutral measure, we can derive the ODE system for the nominal yields by standard argument.

A.3 Proof of Proposition 2

Given the dynamics of s_t in Eq.(13), we can show that the oil futures price has the following exponential-affine form:

$$P_{t,\tau}^{oil} = E_t^* [\exp(s_{t+\tau})] \equiv \exp\left(A_\tau^{oil} + B_\tau^{oil} x_t + C_\tau^{oil} \delta_t + D_\tau^{oil} s_t\right).$$

Applying the Girsanov's theorem and using similar arguments as in the proof of Proposition 1, we obtain the system of ordinary differential equations

$$\begin{aligned}
\frac{dA_\tau^{oil}}{d\tau} &= (\mathcal{K}\mu - \Sigma\lambda_{0,x}^N)' B_\tau^{oil} + (\kappa_\delta\mu_\delta - \sigma_\delta\lambda_{0,\delta}^N) C_\tau^{oil} + \left(\rho_0^N - \frac{1}{2}\sigma_s^2\right) D_\tau^{oil} \\
&\quad + \frac{1}{2}B_\tau^{oil'}\Sigma\Sigma' B_\tau^{oil} + \frac{1}{2}\sigma_\delta^2 (C_\tau^{oil})^2 + \frac{1}{2}\sigma_s^2 (D_\tau^{oil})^2, \\
\frac{dB_\tau^{oil}}{d\tau} &= -(\mathcal{K} + \Sigma\lambda_{1,x}^N)' B_\tau^{oil} + \rho_x^N D_\tau^{oil}, \\
\frac{dC_\tau^{oil}}{d\tau} &= -(\kappa_\delta + \sigma_\delta\lambda_{1,\delta}^N) C_\tau^{oil} + (\rho_\delta^N - 1) D_\tau^{oil}, \\
\frac{dD_\tau^{oil}}{d\tau} &= \rho_s^N D_\tau^{oil}.
\end{aligned}$$

A.4 Lemma A.1

Under the forward measure, we have

$$\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{t,T} = \frac{\exp\left(-\int_t^T r_s^N ds\right)}{P_{t,\tau}^N}$$

and

$$\Psi_t \equiv E_t^{\mathbb{P}^*} \left[\left(\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}^*}\right)_{0,T} \right] = E_t^{\mathbb{P}^*} \left[\frac{\exp\left(-\int_0^T r_s^N ds\right)}{P_{0,T}^N} \right] = \frac{P_{t,\tau}^N}{P_{0,T}^N} \exp\left(-\int_0^t r_s^N ds\right)$$

We have

$$\begin{aligned}
d\Psi_t &= \frac{\exp\left(-\int_0^t r_s^N ds\right)}{P_{0,T}^N} [dP_{t,\tau}^N - r_t^N P_{t,\tau}^N dt] \\
&= \Psi_t \left[B_\tau^{N'} \Sigma dW_{x,t} + C_\tau^N \sigma_q^\perp dW_{\perp,t} \right]
\end{aligned}$$

By Girsanov's Theorem, we have

$$d\tilde{W}_t = dW_t^* - \frac{d\Psi_t}{\Psi_t} \cdot dW_t^*$$

or

$$\begin{aligned}
d\tilde{W}_{x,t} &= dW_t^* - \Sigma' B_\tau^N dt \\
d\tilde{W}_{\perp,t} &= dW_{\perp,t}^* \\
d\tilde{W}_{\delta,t} &= dW_{\delta,t}^* - (\sigma_\delta C_\tau^N + \sigma_s D_\tau^N) dt
\end{aligned}$$

The dynamics of the state variables under the risk neutral measure is stated in Eq.(A1)-Eq.(A3). Applying the Girsanov's Theorem yields:

$$\begin{aligned}
dx_t &= \mathcal{K}^* (\mu^* - x_t) dt + \Sigma \left(d\tilde{W}_{x,t} + \Sigma' B_\tau^N dt \right) \\
&= (\mathcal{K}^* \mu^* + \Sigma \Sigma' B_\tau^N - \mathcal{K}^* x_t) dt + \Sigma d\tilde{W}_{x,t} \\
&\equiv \tilde{\mathcal{K}} (\tilde{\mu} - x_t) dt + \Sigma d\tilde{W}_{x,t},
\end{aligned}$$

$$\begin{aligned}
d\delta_t &= \kappa_\delta^* (\mu_\delta^* - \delta_t) dt + \sigma_\delta dW_{\delta,t}^* \\
&= \kappa_\delta^* (\mu_\delta^* - \delta_t) dt + \sigma_\delta \left(d\tilde{W}_{\delta,t} + (\sigma_\delta C_\tau^N + \sigma_s D_\tau^N) dt \right) \\
&= (\kappa_\delta^* \mu_\delta^* + \sigma_\delta (\sigma_\delta C_\tau^N + \sigma_s D_\tau^N) - \kappa_\delta^* \delta_t) dt + \sigma_\delta d\tilde{W}_{\delta,t} \\
&\equiv \tilde{\kappa}_\delta (\tilde{\mu}_\delta - \delta_t) dt + \sigma_\delta d\tilde{W}_{\delta,t}
\end{aligned}$$

$$\begin{aligned}
ds_t &= (\phi_0^* + \phi_x^* x_t + \phi_\delta^* \delta_t + \phi_s^* s_t) dt + \sigma_s dW_{\delta,t}^* \\
&= (\phi_0^* + \phi_x^* x_t + \phi_\delta^* \delta_t + \phi_s^* s_t) dt + \sigma_s \left(d\tilde{W}_{\delta,t} + (\sigma_\delta D_\tau^N + \sigma_s E_\tau^N) dt \right) \\
&= ([\phi_0^* + \sigma_s (\sigma_\delta C_\tau^N + \sigma_s D_\tau^N)] + \phi_x^* x_t + \phi_\delta^* \delta_t + \phi_s^* s_t) dt + \sigma_s d\tilde{W}_{\delta,t} \\
&\equiv (\tilde{\phi}_0 + \tilde{\phi}_x x_t + \tilde{\phi}_\delta \delta_t + \tilde{\phi}_s s_t) dt + \sigma_s d\tilde{W}_{\delta,t}
\end{aligned}$$

and

$$\begin{aligned}
dq_t &= (\rho_0^{\pi*} + \rho_x^{\pi*} x_t + \rho_\delta^{\pi*} \delta_t + \rho_s^{\pi*} s_t) dt + \sigma'_q dW_{x,t}^* + \sigma_q^\perp dW_{\perp,t}^* \\
&= ([\rho_0^{\pi*} + \sigma'_q \Sigma' B_\tau^N] + \rho_x^{\pi*} x_t + \rho_\delta^{\pi*} \delta_t + \rho_s^{\pi*} s_t) dt + \sigma'_q d\tilde{W}_{x,t} + \sigma_q^\perp d\tilde{W}_{\perp,t} \\
&\equiv (\tilde{\rho}_0^\pi + \tilde{\rho}_x^\pi x_t + \tilde{\rho}_\delta^\pi \delta_t + \tilde{\rho}_s^\pi s_t) dt + \sigma'_q d\tilde{W}_{x,t} + \sigma_q^\perp d\tilde{W}_{\perp,t}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{\mathcal{K}} &= \mathcal{K}^* = \mathcal{K} + \Sigma \lambda_{1,x}^N, \\
\tilde{\mathcal{K}} \tilde{\mu} &= \mathcal{K}^* \mu^* + \Sigma \Sigma' B_\tau^N = \mathcal{K} \mu - \Sigma \lambda_{0,x}^N + \Sigma \Sigma' B_\tau^N, \\
\tilde{\kappa}_\delta &= \kappa_\delta^* = \kappa_\delta + \sigma_\delta \lambda_{1,\delta}^N, \\
\tilde{\kappa}_\delta \tilde{\mu}_\delta &= \kappa_\delta^* \mu_\delta^* + \sigma_\delta (\sigma_\delta C_\tau^N + \sigma_s D_\tau^N) = \kappa_v \mu_v - \sigma_\delta \lambda_{0,\delta}^N + \sigma_\delta (\sigma_\delta C_\tau^N + \sigma_s D_\tau^N) \\
\tilde{\pi}_t &= \tilde{\rho}_0^\pi + \tilde{\rho}_x^\pi x_t + \tilde{\rho}_\delta^\pi \delta_t + \tilde{\rho}_s^\pi s_t, \\
\tilde{\rho}_0^\pi &= \rho_0^{\pi*} + \sigma'_q \Sigma' B_\tau^N = \rho_0^\pi - \lambda_{0,x}^N \sigma_q + \sigma'_q \Sigma' B_\tau^N, \\
\tilde{\rho}_x^\pi &= \rho_x^{\pi*} = \rho_x^\pi - \lambda_{1,x}^N \sigma_q, \\
\tilde{\rho}_\delta^{\pi*} &= \rho_\delta^{\pi*} \text{ and } \tilde{\rho}_s^{\pi*} = \rho_s^{\pi*}.
\end{aligned}$$

The notations such as $\tilde{\rho}_0^\pi$, etc. denote corresponding parameters under the forward measure

Next, we compute the expected values of the state variables over the period t to $t + \tau$ under the forward measure. After tedious algebra, we can show that

$$\begin{aligned}
\mathcal{IE}_{t,\tau} &\equiv \frac{1}{\tau} E_t^{\tilde{\mathbb{P}}} [q_{t+\tau} - q_t] = \tilde{a}_\tau^\pi + \tilde{b}_\tau^{\pi'} x_t + \tilde{c}_\tau^\pi \delta_t + \tilde{d}_\tau^{\pi'} s_t \\
\mathcal{IU}_{t,\tau} &\equiv \frac{1}{\tau} Var_t^{\tilde{\mathbb{P}}} [q_{t+\tau} - q_t] = \sigma'_q \sigma_q + (\sigma_q^\perp)^2
\end{aligned}$$

where

$$\begin{aligned}
\tilde{a}_\tau^\pi &= \tilde{\rho}_0^\pi + \tilde{\rho}_x^{\pi'} \tilde{a}_\tau^x + \tilde{\rho}_\delta^\pi \tilde{a}_\tau^\delta + \tilde{\rho}_s^{\pi'} \tilde{a}_\tau^s, \\
\tilde{b}_\tau^\pi &= \tilde{b}_\tau^x \tilde{\rho}_x^\pi + \tilde{b}_\tau^{s,x'} \tilde{\rho}_s^\pi, \\
\tilde{c}_\tau^\pi &= \tilde{\rho}_\delta^\pi \tilde{b}_\tau^\delta + \tilde{\rho}_s^{\pi'} \tilde{b}_\tau^{s,\delta}, \\
\tilde{d}_\tau^{\pi'} &= \tilde{\rho}_s^{\pi'} \tilde{b}_\tau^{s,s}.
\end{aligned}$$

A.5 Proof of Proposition 3

Using the results in Lemma A.1, the price of a τ -maturity inflation cap with strike K is given by

$$\begin{aligned}
P_{t,\tau,K}^{CAP} &= \exp(-\tau y_{t,\tau}^N) E_t^{\mathbb{P}} \left[\left(\frac{Q_{t+\tau}}{Q_t} - (1+K)^\tau \right)^+ \right] \\
&= \exp(-\tau y_{t,\tau}^N) E_t^{\mathbb{P}} \left[\left(\exp \left(\ln \left(\frac{Q_{t+\tau}}{Q_t} \right) \right) - (1+K)^\tau \right)^+ \right] \\
&= \exp(-\tau y_{t,\tau}^N) \left[\exp(\tau (\mathcal{IE}_{t,\tau} + 1/2\mathcal{IU}_{t,\tau})) \right. \\
&\quad \times \Phi \left(\frac{\tau}{\sigma} [-\ln(1+K) + \mathcal{IE}_{t,\tau} + \mathcal{IU}_{t,\tau}] \right) \\
&\quad \left. - (1+K)^\tau \Phi \left(\frac{\tau}{\sigma} [-\ln(1+K) + \mathcal{IE}_{t,\tau}] \right) \right],
\end{aligned}$$

and the price of a τ -maturity inflation cap with strike K is given by

$$\begin{aligned}
P_{t,\tau,K}^{FLO} &= \exp(-\tau y_{t,\tau}^N) E_t^{\mathbb{P}} \left[\left((1+K)^\tau - \frac{Q_{t+\tau}}{Q_t} \right)^+ \right] \\
&= \exp(-\tau y_{t,\tau}^N) E_t^{\mathbb{P}} \left[\left((1+K)^\tau - \exp \left(\ln \left(\frac{Q_{t+\tau}}{Q_t} \right) \right) \right)^+ \right] \\
&= \exp(-\tau y_{t,\tau}^N) \left[-\exp(\tau (\mathcal{IE}_{t,\tau} + 1/2\mathcal{IU}_{t,\tau})) \right. \\
&\quad \times \Phi \left(-\frac{\tau}{\sigma} [-\ln(1+K) + \mathcal{IE}_{t,\tau} + \mathcal{IU}_{t,\tau}] \right) \\
&\quad \left. + (1+K)^\tau \Phi \left(-\frac{\tau}{\sigma} [-\ln(1+K) + \mathcal{IE}_{t,\tau}] \right) \right],
\end{aligned}$$

where we used the fact that for a normal random variable $\tilde{z} \sim N(\mu, \sigma^2)$,

$$\begin{aligned}
E \left[\left(ae^{\tilde{z}} - b \right)^+ \right] &= a \exp \left(\mu + \frac{\sigma^2}{2} \right) \Phi \left(\frac{\ln(a/b) + (\mu + \sigma^2)}{\sigma} \right) - b \Phi \left(\frac{\ln(a/b) + \mu}{\sigma} \right), \\
E \left[\left(b - ae^{\tilde{z}} \right)^+ \right] &= -a \exp \left(\mu + \frac{\sigma^2}{2} \right) \Phi \left(-\frac{\ln(a/b) + (\mu + \sigma^2)}{\sigma} \right) + b \Phi \left(-\frac{\ln(a/b) + \mu}{\sigma} \right).
\end{aligned}$$

Appendix B: Estimation Methodology

B.1 Transition and Measurement Equations

We first consider the discrete-time dynamics of the state variable q_t between time $t - \Delta t$ and t . From $dq_t = \pi_t dt + \sigma'_q dW_{x,t} + \sigma_q^\perp dW_{\perp,t}$, we have

$$q_t = q_{t-\Delta t} + \rho_0^\pi \Delta t + (\rho_x^\pi \Delta t) x_t + (\rho_\delta^\pi \Delta t) \delta_t + (\rho_s^\pi \Delta t) s_t + \eta_t^q$$

where $\eta_t^q \equiv \int_{t-\Delta t}^t (\sigma'_q dW_{x,u} + \sigma_q^\perp dW_{\perp,u}) \sim N(0, \Omega_{t-\Delta t}^q)$ and $\Omega_{t-\Delta t}^q \equiv Var_{t-\Delta t}(\eta_t^q) = (\sigma'_q \sigma_q + (\sigma_q^\perp)^2) \Delta t$.

Similarly, for the state variable x_t we have

$$x_t = \exp(-\mathcal{K}\Delta t) x_{t-\Delta t} + (I - \exp(-\mathcal{K}\Delta t)) \mu + \eta_t^x,$$

where $\eta_t^x = \int_0^{\Delta t} \exp(-\mathcal{K}u) \Sigma dW_{x,u} \sim N(0, \Omega_{t-\Delta t}^x)$ and

$$\Omega_{t-\Delta t}^x \equiv \text{Var}_{t-\Delta t}(\eta_t^x) = \int_0^{\Delta t} \exp(-\mathcal{K}u) \Sigma \Sigma' \exp(-\mathcal{K}'u) du = N \Xi N',$$

with $\mathcal{K} = NDN^{-1}$, $D = \text{diag}([d_1, \dots, d_N])$, and $\Xi_{i,j} = [(N^{-1}\Sigma_x)(N^{-1}\Sigma_x)']_{i,j} \frac{1 - \exp(-(d_i + d_j) \cdot t)}{(d_i + d_j)}$. The covariance matrix between η_t^x and η_t^q is given by

$$\Omega_{t-\Delta t}^{xq} = \text{Cov}_{t-\Delta t}[\eta_t^x, \eta_t^q] = \int_0^{\Delta t} \exp(-\mathcal{K}u) \Sigma \sigma_q du = \mathcal{K}^{-1} (I - \exp(-\mathcal{K}\Delta t)) \Sigma \sigma_q.$$

where I signifies the identity matrix.

Next, we consider the state variable δ_t . From $d\delta_t = \kappa_\delta (\mu_\delta - \delta_t) dt + \sigma_\delta dW_{\delta,t}$, we have

$$\delta_t = \exp(-\kappa_\delta \Delta t) \delta_{t-\Delta t} + \mu_\delta (1 - \exp(-\kappa_\delta \Delta t)) + \eta_t^\delta,$$

where $\eta_t^\delta = \sigma_\delta \int_{t-\Delta t}^t \exp(\kappa_\delta (u - t)) dW_{\delta,u} \sim N(0, \Omega_{t-\Delta t}^\delta)$ and

$$\Omega_{t-\Delta t}^\delta = E_{t-\Delta t} \left[\sigma_\delta^2 \int_{t-\Delta t}^t \exp(2\kappa_\delta (u - t)) du \right] = \sigma_\delta^2 \frac{1 - \exp(-2\kappa_\delta \Delta t)}{2\kappa_\delta}$$

Finally, for the state variable s_t we have

$$s_t - s_{t-\Delta t} = (\rho_0^s \Delta t + (\rho_x^s \Delta t)' x_{t-\Delta t} + \rho_\delta^s \Delta t \delta_{t-\Delta t} + \rho_s^s \Delta t s_{t-\Delta t}) + \eta_t^s,$$

where $\eta_t^s = \sigma_s \int_{t-\Delta t}^t \exp(-\rho_s^s (u - t)) dW_{\delta,u} \sim N(0, \Omega_{t-\Delta t}^s)$,

$$\Omega_{t-\Delta t}^s = E_{t-\Delta t} \left[\sigma_s^2 \int_{t-\Delta t}^t \exp(2(-\rho_s^s)(u - t)) du \right] = \sigma_s^2 \frac{\exp(2\rho_s^s \Delta t) - 1}{2\rho_s^s}$$

and

$$\begin{aligned} \Omega_{t-\Delta t}^{\delta s} &= \sigma_\delta \sigma_s E_{t-\Delta t} \left[\int_{t-\Delta t}^t \exp(\kappa_\delta (u - t)) dW_{\delta,u} \int_{t-\Delta t}^t \exp(-\rho_s^s (u - t)) dW_{\delta,u} \right] \\ &= \sigma_\delta \sigma_s E_{t-\Delta t} \left[\int_{t-\Delta t}^t \exp((\kappa_\delta - \rho_s^s)(u - t)) du \right] \\ &= \sigma_\delta \sigma_s \frac{1 - \exp(-(\kappa_\delta - \rho_s^s) \Delta t)}{\kappa_\delta - \rho_s^s}. \end{aligned}$$

In summary, the dynamics of the augmented state vector $X_t = (q_t, x_t', \delta_t, s_t)'$ follows the VAR process

$$X_t = \mathcal{A} + \mathcal{B}X_{t-\Delta t} + \eta_t,$$

$$\text{where } \mathcal{A} = \begin{bmatrix} \rho_0^\pi \Delta t \\ (I - \exp(-\mathcal{K}_x \Delta t)) \mu_x \\ (1 - \exp(-\kappa_\delta \Delta t)) \mu_\delta \\ \rho_0^s \Delta t \end{bmatrix}, \mathcal{B} = \begin{bmatrix} 1 & \rho_x^{\pi'} \Delta t & \rho_\delta^\pi \Delta t & \rho_s^\pi \Delta t \\ 0 & \exp(-\mathcal{K}_x \Delta t) & & \\ 0 & 0_{1 \times 3} & \exp(-\kappa_\delta \Delta t) & \\ 0 & \rho_x^{s'} \Delta t & \rho_\delta^s \Delta t & 1 + \rho_s^s \Delta t \end{bmatrix}, \eta_t = \begin{pmatrix} \eta_t^q \\ \eta_t^x \\ \eta_t^\delta \\ \eta_t^s \end{pmatrix} \sim N(0, \Omega_{t-\Delta t}) \text{ and } \Omega_{t-\Delta t} = \begin{bmatrix} \Omega_{t-\Delta t}^q & \Omega_{t-\Delta t}^{xq'} \\ \Omega_{t-\Delta t}^{xq} & \Omega_{t-\Delta t}^x \\ & & \Omega_{t-\Delta t}^\delta & \Omega_{t-\Delta t}^{\delta s} \\ & & \Omega_{t-\Delta t}^{\delta s} & \Omega_{t-\Delta t}^s \end{bmatrix}.$$

Combining the expressions for nominal and real yields in Proposition 1 and for option-implied inflation expectations in Proposition 3, the measurement equation⁷ takes the form

$$Y_t = \begin{bmatrix} 0 \\ a_\tau^N \\ a_\tau^R \\ a_\tau^{oil} \\ \bar{O} \\ a_{\tau,K} \end{bmatrix} + \begin{bmatrix} 1 & 0_{1 \times 3} & 0 & 0 \\ 0 & b_\tau^{N'} & c_\tau^N & d_\tau^N \\ 0 & b_\tau^{R'} & c_\tau^R & d_\tau^R \\ 0 & b_\tau^{oil'} & c_\tau^{oil} & d_\tau^{oil} \\ 0 & b_{\tau,K}^{O'} & c_{\tau,K}^O & d_{\tau,K}^O \end{bmatrix} X_t + \begin{bmatrix} 0 \\ e_t^N \\ e_t^R \\ e_t^{oil} \\ e_t^O \\ e_t \end{bmatrix} \\ \equiv a + bX_t + e_t,$$

where e_t is an $m \times 1$ vector of *iid* measurement errors with $e_{t,\tau_i}^N \sim N(0, (\delta_{\tau_i}^N)^2)$, $e_{t,\tau_i}^R \sim N(0, (\delta_{\tau_i}^R)^2)$, $e_{t,\tau_i}^{oil} \sim N(0, (\delta_{\tau_i}^{oil})^2)$ and $e_{t,\tau_i}^O \sim N(0, (\delta_{\tau_i}^O)^2)$ with $\delta_{\tau_i}^O = \delta^O$ for all $\tau \in \{\tau_1, \dots, \tau_m\}$.

B.2 Kalman Filter

With the derived state and observation equations, the state-space system is given by

$$X_t = \mathcal{A} + \mathcal{B}X_{t-1} + \eta_t \\ Y_t = a + bX_t + e_t$$

where \mathcal{A} , \mathcal{B} , a , and b are functions of the underlying parameters of interest $\theta = (\text{vec}(\mathcal{K})', \text{vech}(\Sigma\Sigma')', \mu', \rho_0^{N'}, \rho_x^{N'}, \rho_\delta^{N'}, \rho_s^{N'}, \lambda_0^{N'}, \text{vec}(\lambda_x^N)', \rho_0^\pi, \rho_x^{\pi'}, \rho_\delta^\pi, \rho_s^\pi, \sigma_q', \sigma_q^\perp, \kappa_\delta, \sigma_\delta, \mu_\delta, \lambda_{0,\delta}^N, \lambda_{1,\delta}^N, \sigma_s)'$. The estimation procedure, based on the Kalman filter, that we adopt in this paper is the following. Denote the one-period-ahead prediction and variance of X_t (or Y_t) as $X_{t|t-1}$ and $P_{t|t-1}$ (or $Y_{t|t-1}$ and $V_{t|t-1}$), respectively. That is,

$$X_{t|t-1} \equiv E_{t-1}[X_t] \text{ and } P_{t|t-1} \equiv \text{Var}_{t-1}[X_t] \\ Y_{t|t-1} \equiv E_{t-1}[Y_t] \text{ and } V_{t|t-1} \equiv \text{Var}_{t-1}[Y_t].$$

1. Calculate the unconditional mean and variance of X_t as the prediction and variance of X_0 , denoted by $X_{0|0}$ and $P_{0|0}$.
2. Compute the one-period-ahead prediction and variance of X_t , given $X_{t-1|t-1}$ and $P_{t-1|t-1}$

$$X_{t|t-1} = \mathcal{A} + \mathcal{B}X_{t-1|t-1} \\ P_{t|t-1} = \mathcal{B}P_{t-1|t-1}\mathcal{B}' + \Omega_{t-1}$$

⁷The vector of observables can be further augmented with inflation survey expectations as well as short-term (up to a year) and long-term (5 to 10 years) forecasts of the spot rate as in D'Amico, Kim and Wei (2014).

3. Compute the one-period-ahead prediction and variance of Y_t , given $X_{t-1|t-1}$ and $P_{t-1|t-1}$

$$\begin{aligned} Y_{t|t-1} &= a + bX_{t|t-1} \\ V_{t|t-1} &= bP_{t-1|t-1}b' + \Gamma \end{aligned}$$

4. Compute the forecast error in Y_t as

$$\epsilon_{t|t-1} = Y_t - Y_{t|t-1} \sim N(0, V_{t|t-1})$$

5. Update the prediction of X_t as

$$\begin{aligned} X_{t|t} &= X_{t|t-1} + P_{t|t-1}b'V_{t|t-1}^{-1}\epsilon_{t|t-1} \\ P_{t|t} &= P_{t|t-1} - P_{t|t-1}b'V_{t|t-1}^{-1}bP_{t|t-1} \end{aligned}$$

6. Repeat steps 2-5 for $t = 1, \dots, T$, the parameter vector θ maximizes the following pseudo log-likelihood function

$$\mathcal{L}(Y_t; X_t, \theta) = \max \sum_{t=1}^T \left[-\frac{1}{2} \left(m \ln(2\pi) + \ln |V_{t|t-1}| + \epsilon'_{t|t-1} V_{t|t-1}^{-1} \epsilon_{t|t-1} \right) \right].$$

Table 1. RMSEs for Monthly Out-of-Sample Inflation Forecasts.

horizon (months)	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
Panel A: 2004-2014					
3	0.0119	0.6824	0.6817	0.6815	0.6967
6	0.0165	0.7273	0.7257	0.7294	0.8212
12	0.0219	0.7159	0.7116	0.7332	0.9018
Panel B: 2004-2007					
3	0.0098	0.6560	0.6560	0.6449	0.6358
6	0.0106	0.8167	0.8166	0.7927	0.7401
12	0.0173	0.8708	0.8709	0.8577	0.8777
Panel C: 2008-2010					
3	0.0181	0.6658	0.6686	0.6667	0.7094
6	0.0270	0.6925	0.6960	0.6962	0.8379
12	0.0341	0.6281	0.6290	0.6418	0.9008
Panel D: 2011-2014					
3	0.0070	0.8051	0.7882	0.8128	0.7455
6	0.0097	0.8058	0.7758	0.8305	0.8161
12	0.0121	0.8464	0.8088	0.9399	0.9549

Notes: The table reports the root mean squared errors (RMSE) of the h -month ($h = 3, 6, 12$) ahead forecasts of annualized CPI inflation for the random walk (RW) model. The results for all the other models are presented as ratios of their RMSEs to RW's RMSE. The initial in-sample period is January 1990 - December 2003 and the out-of-sample period is January 2004 - December 2014. Note that the samples in Panels A to D reflect the time when the forecast is made. \mathcal{M}^{NR} denotes the model that uses data from nominal and real yields; \mathcal{M}^{NRO} uses data from nominal and real yields, as well as inflation option prices; \mathcal{M}^{NRS} uses data from nominal and real yields, and inflation swaps; and model \mathcal{M}^{NRoil} uses data from nominal and real yields, and oil futures prices.

Table 2. RMSEs for Quarterly Out-of-Sample Inflation Forecasts.

horizon (quarters)	SPF	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
Panel A: 2004-2014						
1	0.0075	1.5236	1.0118	1.0101	1.0112	1.0217
2	0.0110	1.4301	1.0202	1.0179	1.0214	1.1628
3	0.0133	1.4144	1.0179	1.0154	1.0305	1.2619
4	0.0149	1.4098	1.0059	1.0025	1.0248	1.3146
Panel B: 2004-2007						
1	0.0063	1.5277	0.9875	0.9875	0.9739	0.9424
2	0.0084	1.2654	0.9967	0.9967	0.9604	0.9075
3	0.0113	1.0644	1.0645	1.0645	1.0230	0.8985
4	0.0144	1.1914	0.9698	0.9698	0.9441	1.0036
Panel C: 2008-2010						
1	0.0109	1.5733	1.0162	1.0205	1.0179	1.0887
2	0.0167	1.5041	1.0230	1.0306	1.0280	1.2745
3	0.0193	1.5863	1.0021	1.0118	1.0238	1.4493
4	0.0197	1.6428	1.0585	1.0692	1.0736	1.5863
Panel D: 2011-2014						
1	0.0051	1.3338	1.0334	1.0090	1.0443	0.8955
2	0.0073	1.3353	1.0403	0.9952	1.0738	0.9927
3	0.0090	1.2621	0.9964	0.9462	1.0649	1.0404
4	0.0107	1.0958	0.9294	0.8774	1.0376	1.0103

Notes: The table reports the root mean squared errors (RMSE) of the h -quarter ($h = 1, 2, 3, 4$) ahead forecasts of annualized CPI inflation from SPF (Survey of Professional Forecasters). The results for all the other models are presented as ratios of their RMSEs to SPF's RMSE. The initial in-sample period is January 1990 - December 2003 and the out-of-sample period is January 2004 - December 2014. Note that the samples in Panels A to D reflect the time when the forecast is made. See the notes to Table 1 for the description of the different models.

Table 3a. RMSEs for Out-Of-Sample Nominal Yields Forecasts (2004-2014).

maturity (years)	horizon (months)	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
0.5	3	0.0043	0.9344	0.9450	0.9393	0.9805
1	3	0.0044	0.9369	0.9441	0.9372	0.9681
2	3	0.0045	1.0125	1.0122	1.0058	1.0116
5	3	0.0045	1.0324	1.0288	1.0263	1.0106
10	3	0.0044	1.0694	1.0688	1.0710	1.0547
0.5	6	0.0072	0.9300	0.9394	0.9283	0.9757
1	6	0.0070	0.9502	0.9521	0.9399	0.9789
2	6	0.0066	1.0369	1.0287	1.0189	1.0312
5	6	0.0062	1.0605	1.0528	1.0534	1.0294
10	6	0.0060	1.0697	1.0688	1.0784	1.0463
0.5	12	0.0125	0.8835	0.8865	0.8700	0.8977
1	12	0.0116	0.9290	0.9228	0.9036	0.9231
2	12	0.0101	1.0497	1.0337	1.0141	1.0047
5	12	0.0077	1.1610	1.1456	1.1376	1.0647
10	12	0.0072	1.1153	1.1124	1.1248	1.0322

Table 3b. RMSEs for Out-Of-Sample Nominal Yields Forecasts (2004-2007).

maturity (years)	horizon (months)	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
0.5	3	0.0062	0.8113	0.8382	0.8278	0.8725
1	3	0.0062	0.8384	0.8635	0.8515	0.8816
2	3	0.0060	0.9255	0.9456	0.9340	0.9452
5	3	0.0048	1.0146	1.0226	1.0143	1.0088
10	3	0.0036	1.0568	1.0520	1.0516	1.0373
0.5	6	0.0108	0.7846	0.8144	0.7964	0.8481
1	6	0.0102	0.8141	0.8417	0.8221	0.8636
2	6	0.0091	0.8914	0.9131	0.8951	0.9171
5	6	0.0063	1.0160	1.0205	1.0147	1.0050
10	6	0.0043	1.1579	1.1328	1.1577	1.1220
0.5	12	0.0195	0.7632	0.7941	0.7719	0.7988
1	12	0.0179	0.7924	0.8240	0.7982	0.8155
2	12	0.0153	0.8742	0.9021	0.8760	0.8743
5	12	0.0095	1.0586	1.0674	1.0510	1.0190
10	12	0.0055	1.3351	1.2917	1.3252	1.2617

Table 3c. RMSEs for Out-Of-Sample Nominal Yields Forecasts (2008-2010).

maturity (years)	horizon (months)	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
0.5	3	0.0042	1.0973	1.0801	1.0834	1.1029
1	3	0.0044	1.0635	1.0409	1.0433	1.0584
2	3	0.0051	1.0983	1.0670	1.0692	1.0703
5	3	0.0057	1.0253	1.0016	1.0055	1.0010
10	3	0.0055	1.0150	1.0059	1.0125	1.0144
0.5	6	0.0060	1.2302	1.1965	1.2015	1.2197
1	6	0.0063	1.2041	1.1531	1.1567	1.1716
2	6	0.0069	1.2210	1.1582	1.1628	1.1680
5	6	0.0077	1.0812	1.0388	1.0494	1.0576
10	6	0.0072	1.0145	0.9961	1.0126	1.0300
0.5	12	0.0081	1.2997	1.1948	1.1997	1.2086
1	12	0.0080	1.3677	1.2217	1.2252	1.2229
2	12	0.0071	1.5889	1.4009	1.4055	1.3627
5	12	0.0068	1.4782	1.3199	1.3380	1.2382
10	12	0.0067	1.1873	1.0876	1.1178	1.0094

Table 3d. RMSEs for Out-Of-Sample Nominal Yields Forecasts (2011-2014).

maturity (years)	horizon (monthly)	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
0.5	3	0.0004	5.2621	5.1446	5.1550	5.5538
1	3	0.0005	3.8064	3.6804	3.6929	4.0939
2	3	0.0011	1.7316	1.7277	1.7165	1.6818
5	3	0.0031	1.0921	1.1110	1.1067	1.0400
10	3	0.0042	1.1452	1.1573	1.1563	1.1174
0.5	6	0.0005	6.5007	6.2639	6.2905	6.7921
1	6	0.0007	4.2245	4.0460	4.0623	4.4432
2	6	0.0016	2.0485	2.0609	2.0468	1.9846
5	6	0.0046	1.0972	1.1398	1.1314	1.0131
10	6	0.0064	1.0805	1.1062	1.1029	1.0267
0.5	12	0.0009	6.2274	5.9043	5.9259	6.4122
1	12	0.0015	3.6988	3.5366	3.5342	3.8131
2	12	0.0021	2.6079	2.6379	2.6053	2.6022
5	12	0.0061	1.0602	1.1549	1.1378	1.0111
10	12	0.0089	0.9837	1.0464	1.0409	0.9713

Notes: The table reports the out-of-sample root mean squared errors (RMSEs) of h -month ($h = 3, 6, 12$) ahead forecasts of τ -year to maturity nominal (Treasury) yields. The results for all models, except for RW, are presented as ratios of their RMSEs to RW's RMSE. The initial in-sample period is January 1990 - December 2003 and the out-of-sample period is January 2004 - December 2014. Note that the sub-sample periods reflect the time when the forecast is made. See the notes to Table 1 for the description of the different models.

Table 4a. RMSEs for Out-of-Sample Real Yields Forecasts (2004-2014).

maturity (years)	horizon (months)	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
2	3	0.0095	1.0212	1.0277	1.1367	1.0176
5	3	0.0057	1.1348	1.1427	1.1994	0.9802
10	3	0.0041	1.5780	1.5779	1.3979	1.4785
2	6	0.0123	0.7767	0.7909	0.8563	0.8528
5	6	0.0074	0.9826	0.9953	0.9982	0.9313
10	6	0.0054	1.4748	1.4736	1.3346	1.4177
2	12	0.0137	0.7037	0.7115	0.7212	0.7752
5	12	0.0092	0.9121	0.9243	0.8565	0.9222
10	12	0.0073	1.3392	1.3434	1.2143	1.3211

Table 4b. RMSEs for Out-of-Sample Real Yields Forecasts (2004-2007).

maturity (years)	horizon (months)	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
2	3	0.0069	0.9148	0.9005	0.9002	0.7571
5	3	0.0044	1.3380	1.3015	1.2174	0.9992
10	3	0.0030	2.3167	2.2497	2.0361	1.7105
2	6	0.0097	0.8233	0.8158	0.7889	0.7864
5	6	0.0062	1.2077	1.1734	1.0937	1.0350
10	6	0.0039	2.1614	2.0847	1.9534	1.7106
2	12	0.0156	0.8349	0.8343	0.8289	0.9480
5	12	0.0093	1.0537	1.0323	0.9943	1.0496
10	12	0.0055	1.8497	1.7733	1.7308	1.5806

Table 4c. RMSEs for Out-of-Sample Real Yields Forecasts (2008-2010).

maturity (years)	horizon (months)	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
2	3	0.0151	1.0756	1.0956	1.2088	1.0633
5	3	0.0086	1.0481	1.0677	1.2262	0.8881
10	3	0.0055	1.1578	1.1454	1.1790	1.0456
2	6	0.0196	0.7696	0.7938	0.8778	0.8455
5	6	0.0107	0.8135	0.8391	0.9550	0.7615
10	6	0.0065	1.1059	1.0955	1.1115	1.0621
2	12	0.0182	0.4285	0.4411	0.4805	0.4627
5	12	0.0115	0.6305	0.6300	0.5763	0.6094
10	12	0.0075	1.1770	1.1465	0.9874	1.1578

Table 4d. RMSEs for Out-of-Sample Real Yields Forecasts (2011-2014).

maturity (years)	horizon (months)	RW	\mathcal{M}^{NR}	\mathcal{M}^{NRO}	\mathcal{M}^{NRS}	\mathcal{M}^{NRoil}
2	3	0.0056	0.8557	0.8004	1.0481	1.1045
5	3	0.0038	1.1553	1.1906	1.0647	1.2506
10	3	0.0039	1.5963	1.6646	1.2348	1.8477
2	6	0.0059	0.7036	0.6930	0.8504	1.0592
5	6	0.0051	1.1106	1.1660	0.9892	1.2282
10	6	0.0058	1.3881	1.4454	1.1674	1.5554
2	12	0.0052	1.2398	1.2866	1.2926	1.2187
5	12	0.0070	1.1080	1.1892	1.0503	1.1748
10	12	0.0085	1.1606	1.2358	1.0610	1.2986

Notes: The table reports the out-of-sample root mean squared errors (RMSEs) of h -month ($h = 3, 6, 12$) ahead forecasts of τ -year to maturity real (TIPS) yields. The results for all models, except for RW, are presented as ratios of their RMSEs to RW's RMSE. The initial in-sample period is January 1990 - December 2003 and the out-of-sample period is January 2004 - December 2014. Note that the sub-sample periods reflect the time when the forecast is made. See the notes to Table 1 for the description of the different models.

Table 5. Combination of Inflation Forecasts.

horizon	α^{SPF}	α^{RW}	α^{NR}	α^{NRO}	α^{NRS}	α^{NRoil}
1-quarter	0.3656	0	0	0.2556	0	0.3788
2-quarter	0.6780	0	0	0.1965	0.1255	0
3-quarter	0.6181	0	0	0.3819	0	0
4-quarter	0.5144	0	0	0.4856	0	0

Notes: The table reports the results of regressing realized inflation on survey-based and model-based inflation forecasts, for forecast horizons of 1- to 4-quarters. Specifically, for a given forecasting horizon h , we regress the realized inflation rate $\pi_{t,h}$ at time $t+h$ on various forecasts that are made at time t : $\pi_{t,h} = \alpha^{SPF}\hat{\pi}_{t,h}^{SPF} + \alpha^{NR}\hat{\pi}_{t,h}^{NR} + \alpha^{NRO}\hat{\pi}_{t,h}^{NRO} + \alpha^{NRS}\hat{\pi}_{t,h}^{NRS} + \alpha^{NRoil}\hat{\pi}_{t,h}^{NRoil}$, subject to the constraints: $\alpha^{SPF} + \alpha^{NR} + \alpha^{NRO} + \alpha^{NRS} + \alpha^{NRoil} = 1$ and $\alpha^{SPF}, \alpha^{NR}, \alpha^{NRO}, \alpha^{NRS}, \alpha^{NRoil} \in [0, 1]$.

Table 6. Combination of Nominal Interest Rate Forecasts.

maturity (years)	horizon (months)	α^{RW}	α^{NR}	α^{NRO}	α^{NRS}	α^{NRoil}
0.5	3	0.7626	0	0.2374	0	0
1	3	0.7706	0	0.2139	0	0.0155
2	3	0.7065	0.0189	0.2745	0	0
5	3	0	0.1256	0.3391	0	0.5353
10	3	0	0.2497	0.2350	0	0.5154
0.5	6	0.5411	0	0	0.4589	0
1	6	0.5290	0	0	0.4710	0
2	6	0.3771	0.0195	0	0.6034	0
5	6	0	0.2126	0.7034	0	0.0840
10	6	0.1874	0.3577	0.4549	0	0
0.5	12	0.3165	0	0	0.6835	0
1	12	0.2679	0	0	0.7321	0
2	12	0.1970	0	0	0.8030	0
5	12	0.1030	0	0	0.6649	0.2321
10	12	0.1225	0	0.3802	0	0.4972

Notes: The table reports the results of regressing actual nominal bond yields on model-based nominal yield forecasts, for forecast horizons of 3-, 6-, and 12-months. Specifically, for a given forecasting horizon h and a given bond maturity τ , we regress the observed bond yield $y_{t+h,\tau}^N$ at time $t+h$ on various h -month ahead forecasts that are made at time t : $y_{t+h,\tau}^N = \alpha^{RW} \hat{y}_{t+h,\tau}^{N,RW} + \alpha^{NR} \hat{y}_{t+h,\tau}^{N,NR} + \alpha^{NRO} \hat{y}_{t+h,\tau}^{N,NRO} + \alpha^{NRS} \hat{y}_{t+h,\tau}^{N,NRS} + \alpha^{NRoil} \hat{y}_{t+h,\tau}^{N,NRoil}$, subject to the constraints: $\alpha^{RW} + \alpha^{NR} + \alpha^{NRO} + \alpha^{NRS} + \alpha^{NRoil} = 1$ and $\alpha^{RW}, \alpha^{NR}, \alpha^{NRO}, \alpha^{NRS}, \alpha^{NRoil} \in [0, 1]$.

Table 7. Combination of Real Interest Rate Forecasts

maturity (years)	horizon (months)	α^{RW}	α^{NR}	α^{NRO}	α^{NRS}	α^{NRoil}
2	3	0.4291	0.1613	0	0	0.4096
5	3	0.2585	0.0745	0	0	0.6669
10	3	0.3204	0.0579	0.1777	0	0.4440
2	6	0.0606	0.1830	0	0	0.7564
5	6	0.0101	0.0166	0	0	0.9733
10	6	0.2705	0.1294	0.3428	0	0.2574
2	12	0	0	0.2055	0	0.7945
5	12	0.0371	0	0	0.3445	0.6184
10	12	0.1921	0	0	0.4835	0.3245

Notes: The table reports the results of regressing actual real (TIPS) bond yields on model-based nominal yield forecasts, for forecast horizons of 3-, 6-, and 12-months. Specifically, for a given forecasting horizon h and a given bond maturity τ , we regress the observed bond yield $y_{t+h,\tau}^R$ at time $t+h$ on various h -month ahead forecasts that are made at time t : $y_{t+h,\tau}^R = \alpha^{RW} \hat{y}_{t+h,\tau}^{R,RW} + \alpha^{NR} \hat{y}_{t+h,\tau}^{R,NR} + \alpha^{NRO} \hat{y}_{t+h,\tau}^{R,NRO} + \alpha^{NRS} \hat{y}_{t+h,\tau}^{R,NRS} + \alpha^{NRoil} \hat{y}_{t+h,\tau}^{R,NRoil}$, subject to the constraints: $\alpha^{RW} + \alpha^{NR} + \alpha^{NRO} + \alpha^{NRS} + \alpha^{NRoil} = 1$ and $\alpha^{RW}, \alpha^{NR}, \alpha^{NRO}, \alpha^{NRS}, \alpha^{NRoil} \in [0, 1]$.

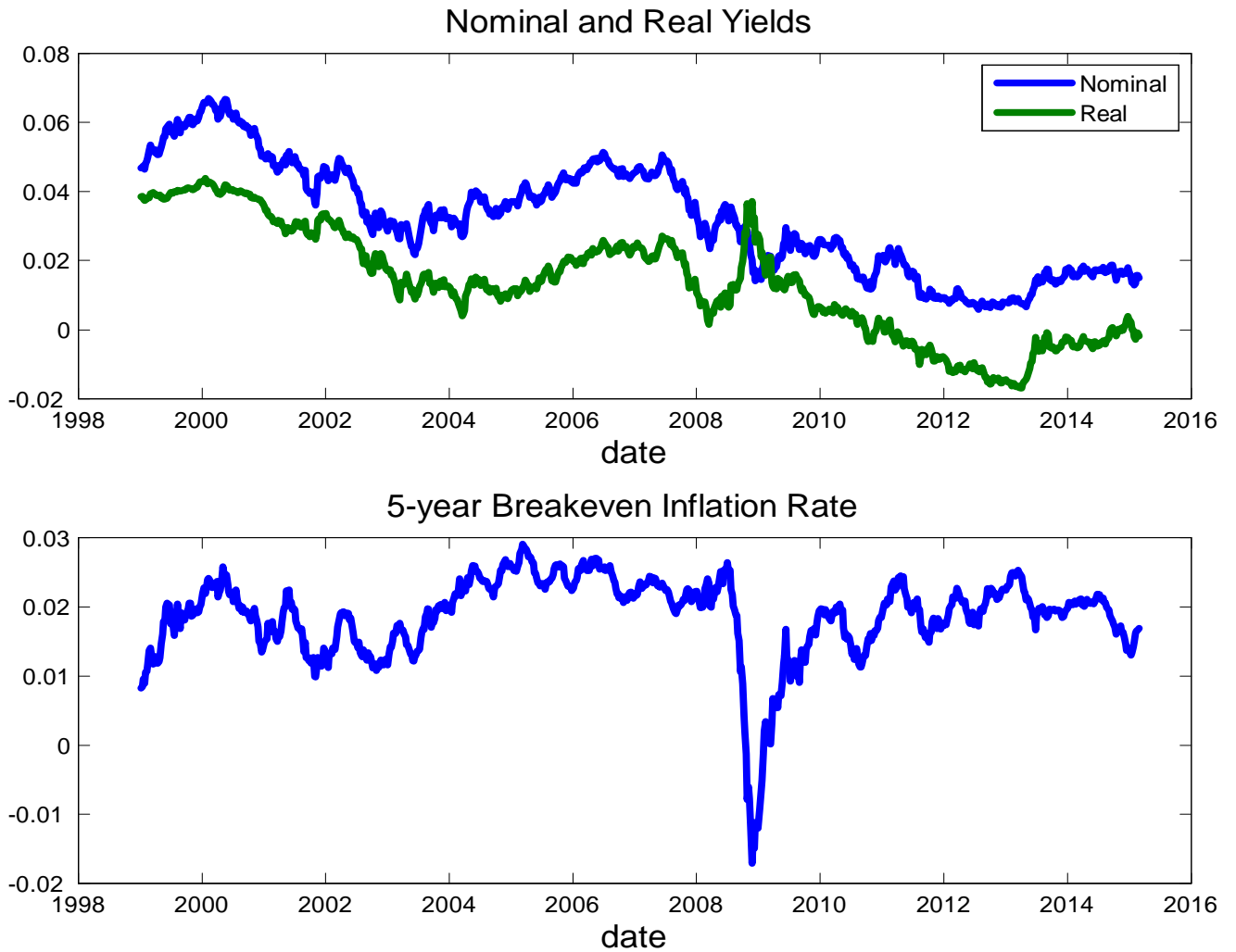


Figure 1: 5-year Treasury and TIPS yields (top graph) and 5-year breakeven inflation (bottom graph).

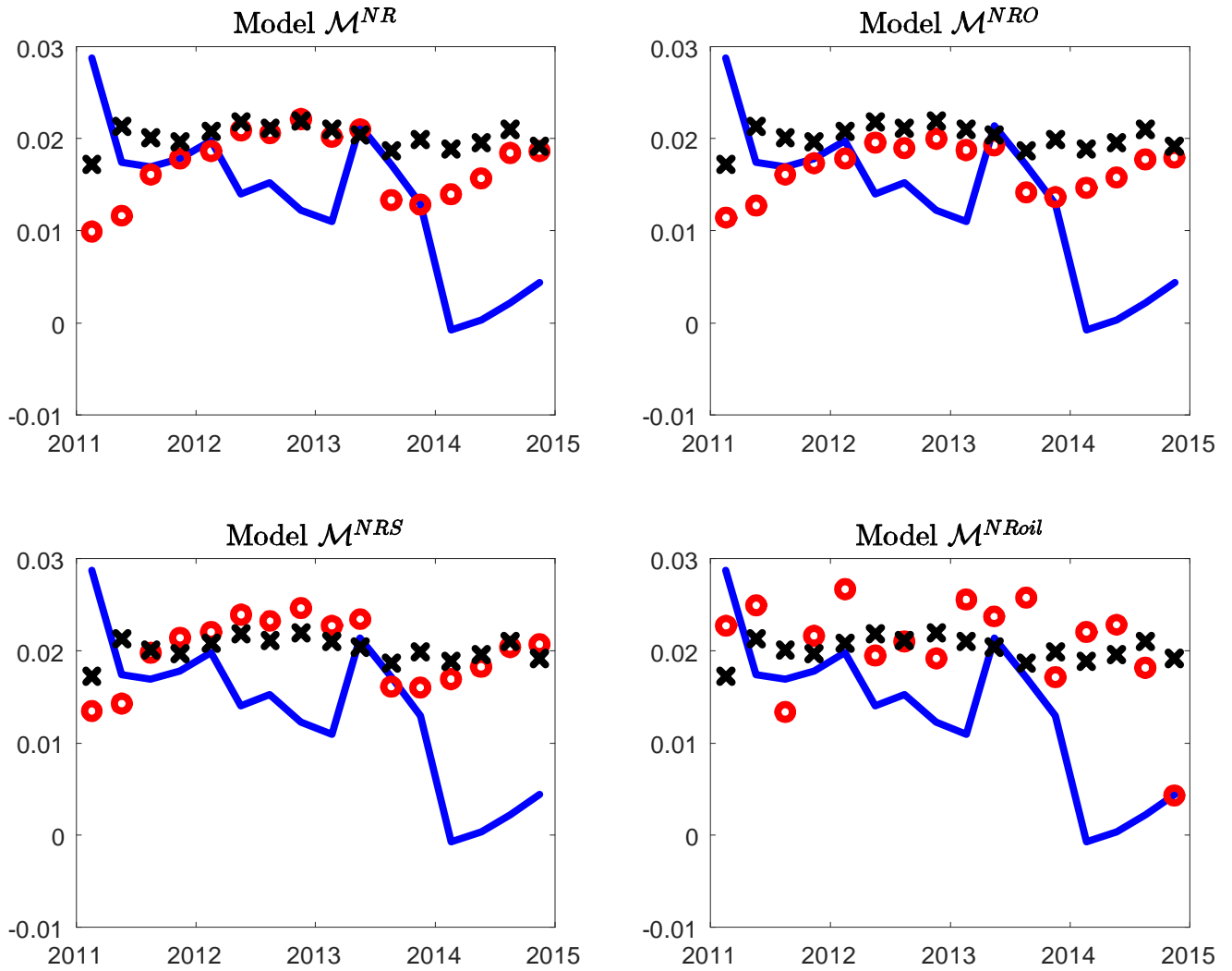


Figure 2: Actual year-over-year inflation rate (blue solid line), 4-quarter ahead SPF survey forecast (black crosses), and model forecasts (red circles) from \mathcal{M}^{NR} , \mathcal{M}^{NRO} , \mathcal{M}^{NRS} and \mathcal{M}^{NRoil} models.

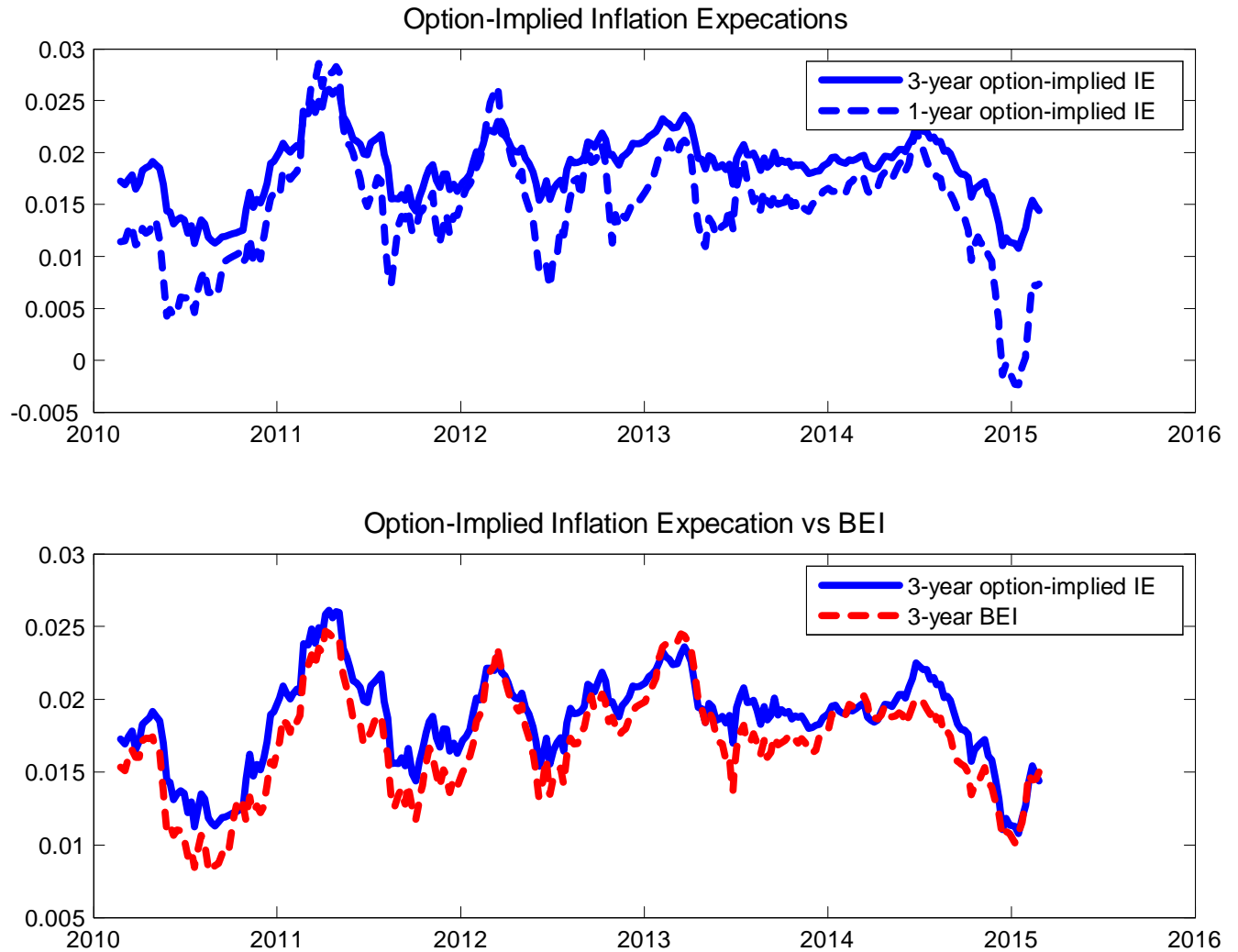


Figure 3: 1-year and 3-year option-implied inflation expectations (IE, top graph), and 3-year option-implied inflation expectations and TIPS-based breakeven inflation rate (BEI, bottom graph).

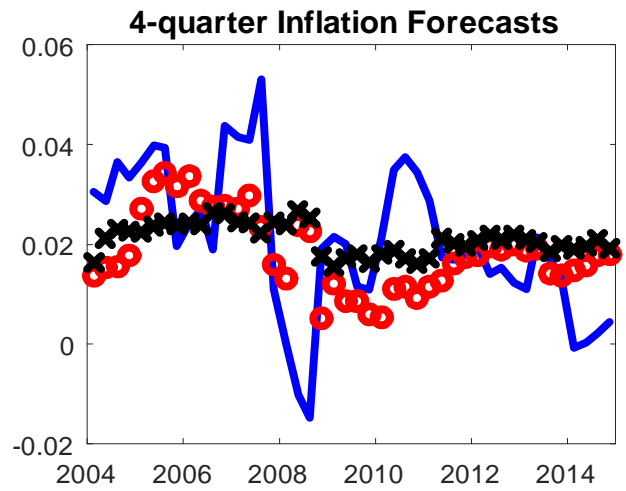
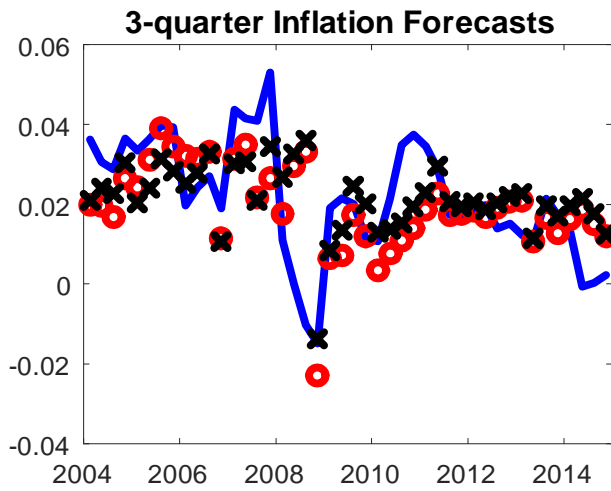
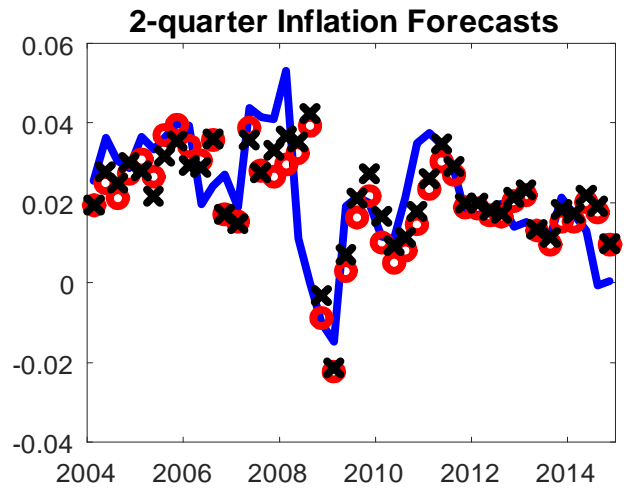
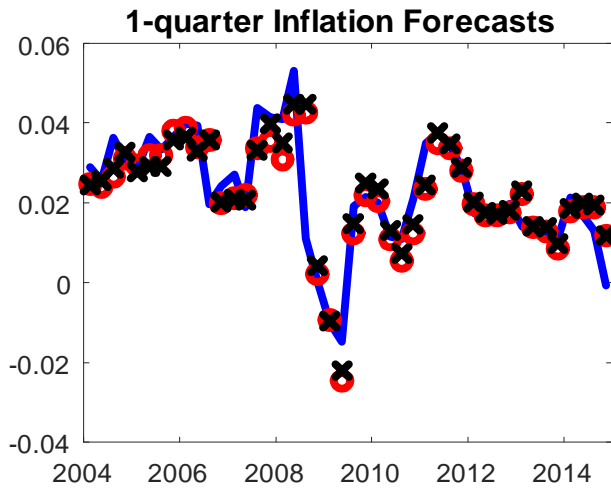


Figure 4: Actual inflation (blue solid) versus the 1-, 2-, 3-, and 4-quarter ahead forecasts based on SPF (black crosses) and \mathcal{M}^{NR} (red circles).

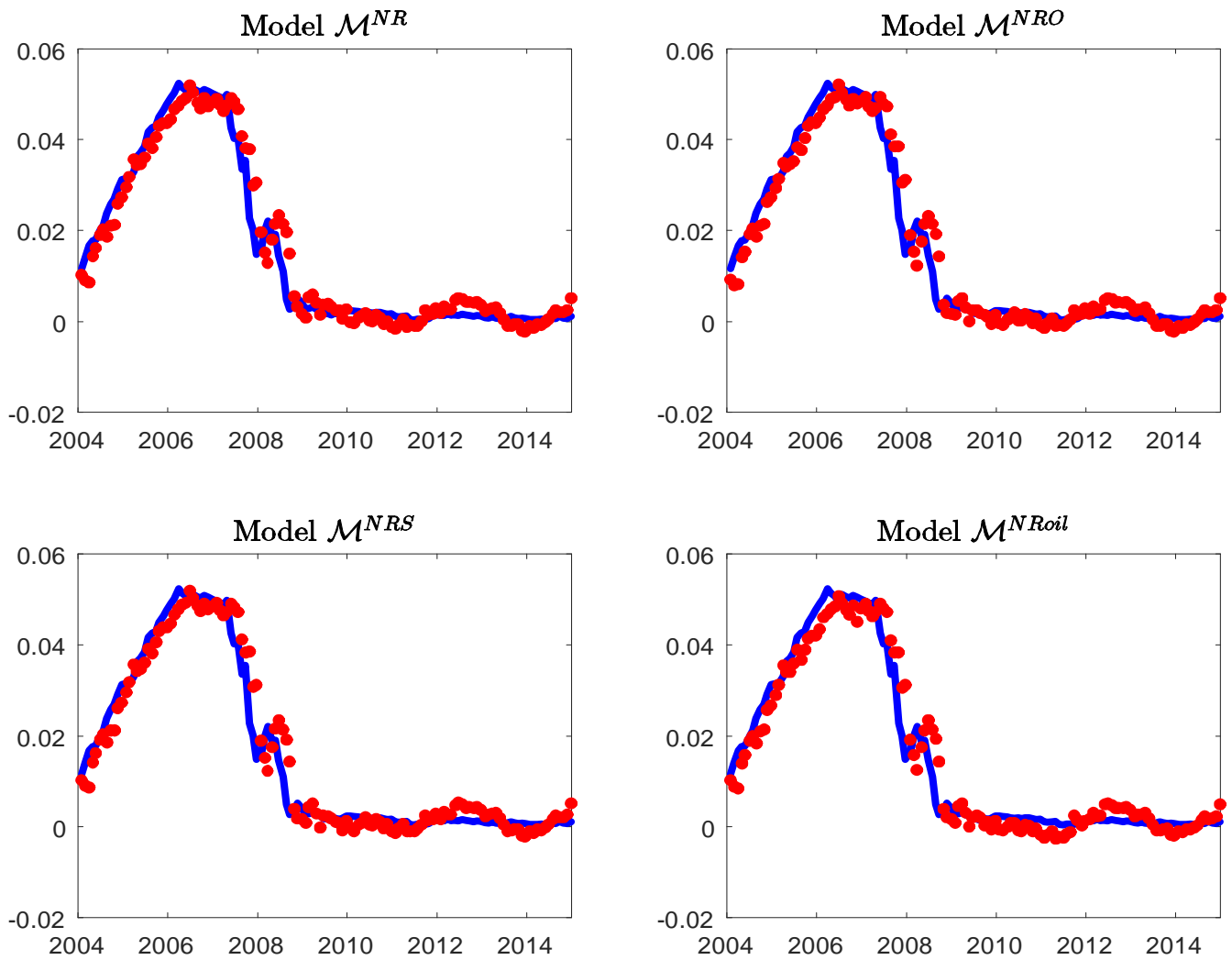


Figure 5: Actual 6-month nominal yield (blue solid line) and its 3-month ahead forecasts (red circles) from \mathcal{M}^{NR} , \mathcal{M}^{NRO} , \mathcal{M}^{NRS} and \mathcal{M}^{NRoil} models.