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Existence of Equilibria in Exhaustible Resource Markets with Economies of Scale and Inventories

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Swiss Federal Institute of Technology Zurich

Existence of Equilibria in Exhaustible Resource Markets with Economies of Scale and Inventories*

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August 22, 2014

Abstract

The paper proves the existence of equilibrium in nonrenewable resource markets when extraction costs are non-convex and resource storage is possible. Inventories flatten the consumption path and eliminate price jumps at the end of the extraction period. Market equilibrium becomes then possible, contradicting previous claims from Eswaran, Lewis and Heaps (1983). We distinguish between two types of solutions, one with immediate and one with delayed build-up of inventories. For both cases we do not only characterize potential optimal paths but also show that equilibria actually exist under fairly general conditions. It is found that optimum resource extraction involves increasing quantities over a period of time. What is generally interpreted as an indicator of increasing resource abundance is thus perfectly compatible with constant resource stocks.

Keywords : Exhaustible resources, nonconvex extraction cost, equilibrium existence, resource storage.

JEL classification : Q30, C62, D92, D41.

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1 Introduction

1.1 Non-existence results

In their seminal contribution, Eswaran, Lewis, and Heaps (1983, henceforth labelled "ELH") show that competitive equilibria generally do not exist in exhaustible resource markets, provided there are initial economies of scale in either the extraction or the utilization of the resource as an input in production. Because markets for resources like oil, coal, and gas are very important in contemporary economies and scale effects often observed on resource markets the paper received broad attention. ELH extend their result to the case where firms are realistically allowed to store extracted resources above the ground, stating that "as in the no-storage case, price must jump discontinuously at the terminal time, so that an intertemporal equilibrium does not exist" (ELH p. 162).

The present paper confirms in a very general setup that the result of ELH without storage is correct but derives in detail why the result with storage does not hold. In fact, ELH admit that costless inventories may flatten the consumption path and eliminate price jumps at the end of optimization. But, using an erroneous argument where the non convexity of the cost function is ignored, they conclude that this may not restore equilibrium existence. It is however shown in the current paper that the combination of non convex extraction costs with costless inventories makes it optimum to extract at high rates and to build up inventories in a specific extraction phase and to exclusively sell from inventories in a last phase. Prices and sales follow a smooth profile and equilibrium exists.

We derive the characteristics of extraction, sales, and price profiles of nonrenewable resources using a general framework. As a novelty we obtain that optimum resource extraction with given resource stocks involves increasing extraction quantities over some period of time. While in the standard resource economics literature this is generally seen as an indicator of increasing resource abundance, we show that increasing extraction can be perfectly compatible with constant resource stocks. The reason is that resource extractors need to increasingly exploit economies of scale to obtain constant discounted resource rents. Proofs of existence and nonexistence are fully derived which makes the point of the original contribution of ELH more rigorous. Like ELH we assume non-convex extraction costs, price-taking firms, and the use of costless resource inventories. The maximization problem of the resource-owning firm includes two stock variables, the resource stock below the ground and the inventory stock above the ground, and two control variables, the resource extraction quantity and the sell quantity. For an equilibrium solution to exist, we require the price path not to exhibit (positive) jumps but rather to be continuous in time, in order to avoid the problem that firms wish to postpone production to the time period after the

jump.

1.2 Our main findings

In this subsection, we provide an overview of our results as well as the intuition behind them. Formal statements and proofs, which involve solving complex dynamic optimization problems, are given in later sections. In all the sections we will successively consider the no-storage and the storage case. Our results for the no-storage case, formally stated in Proposition 1, confirm those of ELH: if storage is excluded, no market equilibrium (in the usual sense provided in Definition 1) exists on resource markets with non-convex extraction costs. The intuition is that, for an equilibrium to exist, prices must increase over time, the usual finding for nonrenewable resource extraction. In the absence of storage, increasing prices are only compatible with decreasing extraction. However, rather than pursuing extraction at quantities lower than those with minimum average cost, it is preferable for the firm to shut down, which generates a discontinuity in supply and an upward jump of the market price. But, such a discontinuous price path would provide incentives for the firm to postpone and stop extraction later, contradicting the existence of a market equilibrium.

When storage becomes possible, however, the previous intuition no longer applies. In fact, even if the extraction path is discontinuous, inventories can be used to generate a continuous sale path, giving rise to a continuous price path. We can show that an equilibrium exists under very general conditions. Moreover, if an equilibrium exists, it has to be of one of the two types depicted in Figure 1 (left and right). To distinguish the two types, we define a threshold resource stock, R_1^*, that depends on cost and demand functions; the formal definition will be given in Equation (37). In the first type of equilibrium, where the initial given resource stock R_0 is below the threshold ($R_0 \leq R_1^*$), firms start building inventories from period zero on. In a first period, extraction path is increasing over time. Then, at some time t_2^*, the firm stops the extraction and starts to sell out of inventories. Sales are continuous and decreasing over time, consistent with a continuously increasing price path in a general equilibrium, depicted in the upper left of the Figure. Resource stock decreases up to time t_2^* while inventories gradually build up and continuously decrease after t_2^*, which can be seen from the lower left in Figure 1. The second type of equilibrium is obtained when $R_0 > R_1^*$; then, the previous pattern is preceded by a period in which extraction is positive but storage is not yet used. In that period, the extraction and sale flows are equal and decreasing over time. Once the amount of resources reaches the threshold R_1^*, firms start building inventories, just like in the type 1 equilibrium. Sales continue to decrease while extraction increases with time. At some point in time, extraction stops and sales are pursued until stocks are completely sold.

It is worth emphasizing that, in both cases, the extraction path is non mono-

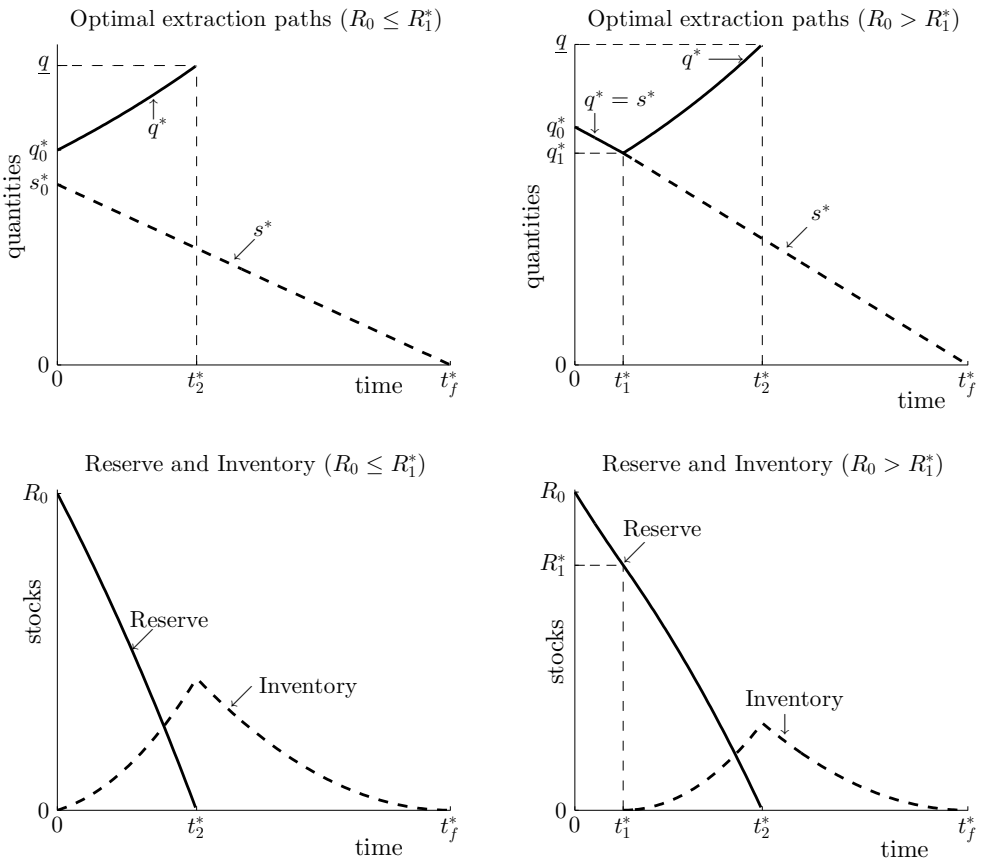

Figure 1: Extraction path. Optimal extraction (q), sales (s), reserve (R) and inventory (I) for (1) low reserves on the left and of (2) high reserves on the right.

tonic. In particular, the shutdown of the firm is always preceded by a period where extraction is rising over time. The intuition is that, instead of using low extraction levels that would generate a too large average extraction cost (because of the fixed maintenance cost), the firm prefers to maintain a relatively high flow of extraction and build inventories - before closing. During the period in which the firm is building inventories the firm's optimal strategy results from a trade off between opposing incentives. On the one hand, the firm aims at extracting resources at a level close to $\underline{q}$ which is the one that minimizes average extraction costs. On the other hand, because of the discount factor, the firm has incentives to postpone extraction as much as possible, and thus to build the inventories later in time. The simultaneous impact of both incentives leads the firm to chose an increasing extraction path ending in $\underline{q}$.

The second type equilibrium includes a period where extraction decreases with time and another where it increases. Interestingly, the period with decreasing extraction comes first. The switch to the increasing extraction period occurs only when resource stocks fall below a given threshold. This means that, here, observing an increasing extraction path is actually an indication of resource scarcity (formally an indication that the amount of resource left is below R_1^*), while a decreasing extraction path indicates that resource are still abundant, i.e. lie above R_1^*.

While it can be shown in great generality that equilibria have necessarily to look like the ones shown in Figure 1, proving that such equilibria always exist requires some additional technical assumptions. These are either related to the initial amount of resources or to the properties of the production function. More precisely, we will show that there exists another resource threshold, labeled $\overline{R}$ and greater than R_1^*, which depends on the properties of cost and demand functions as well as the discount factor, such that:

$$\text{if } R_0 < \overline{R} \text{ there exists a unique equilibrium,}$$

while we still have:

$$\text{if } \begin{cases} R_0 \leq R_1^* & \text{the equilibrium is of type 1} \\ R_0 > R_1^* & \text{the equilibrium is of type 2} \end{cases}.$$

We will also show that, with additional conditions, one gets $\overline{R} = +\infty$. Then, there always exists one and only one equilibrium.

1.3 Contribution to literature

In the aftermath of the important ELH contribution, the general aim of literature was to find ways paving the path to restore equilibrium. This seems to be imperative because in reality we find many well-functioning resource markets, despite non-convex

cost structures. Since the publication of the ELH paper, research addressed many of the ELH model assumptions, but never resource storage. Prominently, Cairns (1991) and Asheim (1992) use alternative assumptions on market behavior introducing full contestability of markets to get an equilibrium; Cairns (2008) uses an indirect approach basing competitive markets on a limit of oligopolistic equilibria. These authors show that, if there are no fixed extraction costs, the assumption of some firms entering and exiting the market with unbounded frequency is sufficient to restore equilibrium. We will motivate below why and how we think it is appropriate to exclude these "chattering controls" at the firm level for resource extraction. Fisher and Karp (1993) show that the introduction of a backstop technology restores both existence and efficiency, provided that the backstop price is sufficiently low. The realism of assuming a new technology being able to fully replace natural resources is limited; backstops will thus be disregarded in our paper. An equilibrium is also shown to exist when extraction capacity is limited, see Holland (2003); we will not use such a restriction in our analysis. Lozada (1996) compares resource depletion of the production of "normal" commodities and solves the model also in discrete time. He derives that, for U-shaped average cost curves, equilibria sometimes exist when using discrete time. We explain below why we think it is more appropriate to assume a continuous-time framework. Mason (2012) studies the smoothing effect of resource inventories in the presence of stochastic demand, which is a general feature of storage. Furthermore, several papers looking at cost and technologies of nonrenewable resource extraction are equally relevant for the present contribution. Le Van, Schubert, and Nguyen (2010) deal with a non-renewable natural resource producer using a convex–concave technology and derive the emergence of poverty traps in less developed economies. Gaudet, Moreaux, and Salant (2001) look at resource extraction when resource sites and their users are spatially distributed, which entails additional cost of resource production. Finally, Chakravorty, Moreaux, and Tidball (2008) analyze the ordering of resource extraction from different stocks depending on extraction costs. This relates to our case because sales out of (underground) stock involve different cost than those of (above-ground) inventories.

Remarkably, the most general problem of resource extraction with non-convex costs and the realistic assumption of inventories has never been addressed in literature since ELH. The present contribution fills this gap, rectifying one of the central results in resource economics. As resource markets are crucial for the broadly discussed energy and climate issues, our study on resource market equilibria also contributes to formulating adequate energy and climate policies, see e.g. Harstadt (2012). The model assumption of non-convex extraction costs has been confirmed throughout, because set-up activities like drilling wells, building pipelines, searching for deposits, etc. are still highly relevant and will even be more important in the future, when more remote deposits will be explored. At the same time, resource inventories have

played an increasingly important role in recent years, see e.g. Mason (2012); their inclusion in an extraction framework is therefore warranted.

The remainder of the paper is organized as follows. Section 2 introduces the model framework. Formal statements and derivations of the findings described in this introduction are provided in Section 3. Section 4 concludes.

2 The setting

Time is assumed to be continuous. We analyze a representative resource firm which is endowed with an initial stock of non renewable resources R_0 and aims at maximizing discounted profits. The firm can choose the flow of resource extraction q and the flow of sales s, which may differ when storage is possible. The levels of resource extraction and sales at time t are denoted q_t and s_t, respectively. The firm is price taker - like in the original contribution of ELH[1] - and the instantaneous resource price at time t is denoted by P_t. Discounted profits are computed using an exogenous discount rate δ.

Central to the analysis is the assumption that extraction costs are not convex. Moreover, we exclude infinitely rapid variations in the extraction flows ("chattering controls") that could be used by the firm to overcome the problems associated with the non-convexity of cost functions.[2] In previous literature, the statements that infinitely rapid variations should be ruled were not precisely formalized in mathematical terms.[3] Rather it was implicitly assumed that standard dynamic optimization methods can be used, which would require preventing very rapid variations in order to be to be correct. Formally speaking, however, with non convex cost and no explicit constraints preventing chattering controls, all so-called "optimal paths" derived using "first-order condition methods" are dominated by (possibly smooth) paths which would use sufficiently rapid variations. Thus, for the analysis to be rigorous, the constraints that prevent chattering controls have to be made explicit. We know that some of these constraints will necessarily be binding. One possibility could be to move to discrete time, that is to impose that the extraction function is step-wise constant. But the solution would then depend on the length of the time period, about which it would be difficult to make convincing arguments. Other possibilities would involve putting bounds on the derivatives of extraction path, but again the solution would depend on the values given to such bounds.

[1] ELH explain in detail: "Assume that the industry consists of a sufficiently large number of identical firms so as to warrant price-taking behavior. For simplicity, we presume that all firms have the same initial reserves [...] and face the same costs of extraction [...]", see ELH p. 156/7.

[2] By varying infinitely rapidly the extraction quantities, the firm could get infinitely close to the case where it would face a convex cost function, equal to the convex envelope of the true cost function.

[3] See e.g. ELH p. 157 or Lozada (1996, p. 436)

In order to avoid making a difficult and arbitrary choice about the length of the time period or the bounds of the derivatives of the extraction path, we focus on a simple framework with non-convex costs where chattering controls can be ruled out in a straightforward manner. It is assumed in our model that there is a maintenance cost to be paid in order to keep the firm active independent of the level of extraction, which reflects costs for keeping a minimum level of security, replacing aging material, or non-compressible labor cost. In addition, we posit that, once the firm is closed, it is not feasible to reopen it. The rationale is that the maintenance of a resource extraction site cannot be disabled temporarily, while dismantling and cleaning-up are costly and hence only done once. These assumptions indeed appear to be more realistic than allowing for infinitely rapid fluctuations of extraction.[4]

Specifically, we posit that the extraction of resource quantity $q_t > 0$ at any date t implies a cost $f(q_t) + c_0$, where $c_0 > 0$ is the maintenance cost and f is continuously differentiable and (strictly) convex, with $f(0) = 0$. The maintenance cost is only supported when the firm is active ($q_t > 0$), but disappear when the firm is closed ($q_t = 0$).[5] The maintenance cost is not a fixed cost that has to be paid whatever happens; it can be avoided by closing the firm. However, its level is independent from the extraction level (when positive) and will be referred as a "fixed maintenance cost" to emphasize that property. Lewis, Matthews and Burnes stress that such costs "often constitute a substantial operation of operating resources" (Lewis, Matthews and Burnes, 1979, p. 227). The assumption that prevents chattering is as follows:

Assumption 1 *Reopening of the resource firm is not possible after a shutdown; formally*

$$q_t = 0 \Rightarrow q_\tau = 0 \text{ for all } \tau \geq t$$

The assumed cost function is represented in Figure 2. The non convexity exclusively comes from the fixed maintenance cost $c_0 > 0$, with Assumption 1 making it impossible to "bypass" it with chattering controls. To be more precise, Assumption 1 is needed to rule out the possibility of having an extraction that would infinitely rapidly alternate between points O and A. Such rapid variations would indeed make it possible to mimic the convex-envelope of the cost function, represented by the dotted line in Figure 2. We would be back to the convex extraction cost problem about which everything is well-known.

Our setting, with a fixed maintenance cost, is described as a "type 1 nonconvexity" in ELH. Their contribution extends to more general nonconvexities, but do not formalize the constraints that would rule out chattering controls. Rather than

[4]Literature admitting chattering control uses models with economies of scale but without fixed costs like we assume here, see Cairns (1991) and Asheim (1992).

[5]To lighten notation, the dependency in time is denoted by a subscript, i.e. we denote f_t for $f(t)$.

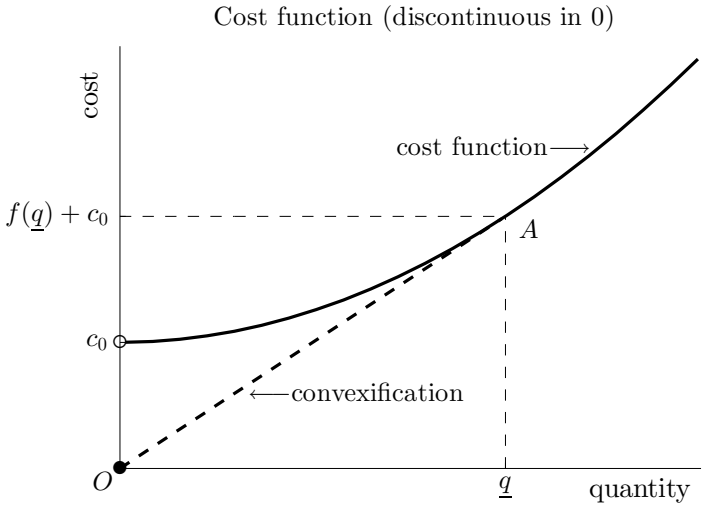

Figure 2: Extraction cost

following this hazardous path which consists in discussing an optimization problem without making all binding constraints explicit, we prefer to restrict the analysis to this so-called type 1 nonconvexity. Moreover, since our objective is to demonstrate the possibility of equilibrium existence, there is not much loss to focus on this setting which ELH view as the one where equilibrium non-existence (ELH, p. 155) is the most obvious. In fact, as will become clear below, existence and non existence of market equilibrium with that kind of nonconvexity is related to the choice of the closing time, just as in ELH.

We denote by $\underline{q}$ the extraction level that minimizes the average extraction cost. That is:

$$\underline{q} = \arg\min_{q} \frac{c_0 + f(q)}{q} \tag{1}$$

Looking at the derivative of $\frac{c_0+f(q)}{q}$, we see that $\underline{q}$ could also be defined as the solution to the equation:

$$f'(\underline{q}) = \frac{c_0 + f(\underline{q})}{\underline{q}}, \tag{2}$$

which means that $\underline{q}$ is at the same time the level of extraction where average cost equals marginal cost.

Resource stock (below the ground) at time t is denoted by R_t, while the level of resources stored above the ground at time t is I_t. By observing matter conservation

we have

$$\frac{\partial R_t}{\partial t} = -q_t,$$
$$\frac{\partial I_t}{\partial t} = q_t - s_t.$$

It will be assumed in all cases that for inventories we have $I_{t=0} = 0$ and $I_t \geq 0$ for all t. When storage is allowed it is assumed to be costless, like in the contribution of ELH.

For a given price path P_t, the problem of the firms involves finding a flow of extraction and sale that solves

$$\max_{q,s} \left\{ J[q] = \int_0^{+\infty} e^{-\delta t} \left(P_t s_t - f(q_t) - c_0 1_{q_t>0} \right) dt \right\} \tag{3}$$
$$\text{s.t.} \quad \frac{\partial R_t}{\partial t} = -q_t,$$
$$\frac{\partial I_t}{\partial t} = q_t - s_t,$$
$$R_{t=0} = R_0,$$
$$R_t \geq 0,\ I_t \geq 0,\ q_t \geq 0 \text{ and } s_t \geq 0 \text{ for all } t,$$

where $1_{q_t>0}$ denotes a dummy variable equal to 1 if $q_t > 0$ and to zero if $q_t = 0$. The no-storage and storage cases will only differ by the assumption on whether I_t is constrained or not to be equal to zero; in the no-storage case, the additional assumption $I_t = 0$ for all t is introduced. The demand side of the economy does not need to be modeled explicitly. We will simply assume that a demand for s_t will only be met for a price $g(s_t)$. The inverse demand function $g(.)$ is assumed to be decreasing; $g(0)$ is finite.

The paper is about the existence of market equilibria. Formally:

Definition 1 (Market equilibrium) *A market equilibrium is a price trajectory P_t^* and flows of extractions (q_t^*) and sales (s_t^*) such that:*

- *(Rationality) Extraction flows and sales (q_t^* and s_t^*) are those that maximize firm profit when the price path is P_t^*,*
- *(Market clearing) For all t we have $P_t^* = g(s_t^*)$.*

We investigate whether such equilibria may exist, considering in turn the cases where storage is ruled out or allowed. In both settings an obvious answer can be provided when $\frac{c_0+f(q)}{q} \geq g(0)$. In this case, there are no potential gains from trade, and there always exists a unique equilibrium. It corresponds to the case where the firm never starts operations and nothing is extracted. To exclude this degenerate case, we introduce the following assumption.

Assumption 2 *There are potential gains from trade; i.e. formally we have*

$$g(0) > f'(\underline{q}).$$

This assumption is a minimal requirement for the question we address to be of interest. The next sections provides formal statements and proofs for the results discussed above. We look in turn at the no-storage and the storage case.

3 Extraction without storage

3.1 The setup

Ruling out storage means that I_t has to equal zero at all periods of time, and thus $s_t = q_t$. The firm's problem involves finding a flow of extraction (or sale) that solves:

$$\begin{aligned} &\max_q \left\{ J[q] = \int_0^\infty e^{-\delta t} \left(P_t q_t - f(q_t) - c_0 1_{q_t>0} \right) dt \right\} \qquad (4) \\ \text{s.t.} \quad &\frac{\partial R_t}{\partial t} = -q_t, \\ &R_{t=0} = R_0, \\ &R_t \geq 0,\ q_t \geq 0 \text{ for all } t. \end{aligned}$$

It is possible to show that there cannot be a price path P_t^* and a solution q_t^* of the the problem (4) such that $P_t^* = g(q_t^*)$ for all t. That leads to following statement:

Proposition 1 (ELH, no existence without storage) *In the absence of storage a market equilibrium does not exist.*

Proof. The proof is split into two parts. In the first, given in Section 3.2, we use necessary conditions to restrict the set of potential equilibrium paths. In the second, provided in Section 3.3, we show that none of these potential equilibrium paths can be an equilibrium. ■

3.2 Potential equilibrium paths

Formally, we call a "potential equilibrium path" a path that fulfills the market clearing conditions and the first order conditions associated to the firm's optimization problem. If an equilibrium were to exist it would necessarily be one of these potential equilibrium paths. However, due to the non-convexity of the cost function, fulfilling the first order conditions is not sufficient to insure the optimality of the firms strategy. The question about whether these potential equilibrium paths are indeed equilibrium paths will be explored in Section 3.3.

We provide a detailed analysis of the case without storage in order to show our procedure as clear as possible and to motivate the later case with inventories. The first order conditions associated with the firm's problem are straightforward to establish. In the system (4), there is one (real) state variable, R, and one (real) control variable, q. Denoting by p the (unique) real co-state variable, the Hamiltonian of the system (4) reads as follows:

$$H(R,p,q,t) = -qp + e^{-\delta t}\left(P_t q - f(q) - c_0 1_{q>0}\right).$$

The maximum principle yields that the three following equations hold:

$$\dot{p} = -\frac{\partial H}{\partial R}(R^*,p,q^*,t) = 0, \tag{5}$$

$$\dot{R^*} = \frac{\partial H}{\partial p}(R^*,p,q^*,t) = -q^*, \tag{6}$$

$$H(R_t^*,p_t,q_t^*,t) = \max_a \left(-ap_t + e^{-\delta t}\left(P_t a - f(a) - c_0 1_{a>0}\right)\right), \tag{7}$$

where the $*$ superscript is used to denote the solution of the optimization problem.

Equation (5) implies that $p_t = p_0$ for all t. Equation (7) is strictly concave for $a > 0$, which implies that q^* solves:

$$\begin{aligned} e^{-\delta t}\left(P_t - f'(q_t^*)\right) &= p_0, \\ \text{or: } q_t^* &= 0. \end{aligned} \tag{8}$$

We check that the Hamiltonian is zero whenever $q_t^* = 0$. For any other date we obtain:

$$H(R,p,q^*,t) = e^{-\delta t}(q_t^* f'(q_t^*) - f(q_t^*) - c_0 1_{q_t^*>0}).$$

The case that extraction lasts forever can be ruled out as it would imply that there exist long periods with very low extraction levels, for which the average extraction cost would be very high. But this cannot be optimal for the firm, since the price is known to remain below or equal $g(0)$, which is finite. We thus denote by t_f the final extraction date. When $q_t^* = 0$ (that is for $t > t_f$) we have $H(R,p,q^*,t) = 0$. Also at the terminal date the Hamiltonian has to be equal to zero, $H(R,p,q^*,t_f) = 0$. Moreover, the map $q \mapsto qf'(q) - f(q) - c_0$ is strictly increasing over $\mathbb{R}_+$ and equal to zero when $q = \underline{q}$. Therefore $\underline{q} > 0$ is the only solution to $H(R,p,\underline{q},t) = 0$, implying $q_{t_f}^* = \underline{q}$. At all other dates, the Hamiltonian has to be positive (otherwise the resource firm prefers $q^* = 0$, which is an absorbing state), meaning that $q_t^* \geq \underline{q}$ as long as the extraction period is not over.

From (8) we can deduce that, for any t between 0 and t_f, we have for prices and

marginal extraction costs:

$$P_t - f'(q_t^*) = (P_{t_f} - f'(\underline{q}))e^{-\delta(t_f - t)}. \tag{9}$$

In equilibrium, the price of the resource must verify $P_t = g(q_t^*)$. We define the function for marginal extraction rent $\pi : \mathbb{R}^+ \to \mathbb{R}$ as follows:

$$\pi(q) = g(q) - f'(q). \tag{10}$$

The function π is strictly decreasing in q. In equilibrium, equation (9) implies that:

$$\pi(q_t^*) = \pi(\underline{q})e^{-\delta(t_f - t)},$$

$$q_t^* = \pi^{-1}\left(\pi(\underline{q})e^{-\delta(t_f - t)}\right).$$

In an equilibrium resource stocks have to be fully exhausted. Otherwise the firm would have interest in postponing its shutdown. The resource constraint imposes that t_f is thus defined as the unique date such that the following equality holds:

$$R_0 \ = \int_0^{t_f} q_t^* dt = \int_0^{t_f} \pi^{-1}\left(\pi(\underline{q})e^{-\delta(t_f - t)}\right) dt. \tag{11}$$

Finally, the optimal price path P^* is defined by:

$$\forall t \in [0, t_f], \ P_t^* = g(\pi^{-1}(\pi(\underline{q})e^{-\delta(t_f - t)})).$$

3.3 Profitable deviation from equilibrium

In order to show that an equilibrium does not exist, we will derive that, when the price path is $P^*(t) = g(\pi^{-1}(\pi(\underline{q})e^{-\delta(t_f - t)}))$ for $0 \leq t \leq t_f$ and $g(0)$ afterwards, it is indeed in the interest of the firm to deviate from the strategy depicted above. The deviation is simple and involves lowering extraction by dq during dt period of time before t_f, and extracting the $dt.dq$ amount of resources left at rate $\underline{q}$ during $\frac{dt.dq}{\underline{q}}$ periods of time after t_f.

The loss of income due to the sales that no longer occur before t_f is given by $e^{-\delta t_f} g(\underline{q}) dt.dq$ while the gain in income for sales after t_f is $e^{-\delta t_f} g(0) dt.dq$. As for the costs, they are diminished by $e^{-\delta t_f} f'(\underline{q}) dt.dq$ before t_f and increased by $e^{-\delta t_f}\left(f(\underline{q}) + c_0\right)\frac{dt.dq}{\underline{q}}$ after t_f. So the overall variation in profit is proportional to:

$$g(0) - g(\underline{q}) + f'(\underline{q}) - \frac{f(\underline{q}) + c_0}{\underline{q}} = g(0) - g(\underline{q}) > 0.$$

The deviation is thus profitable, which means that the path obtained in Section

3.2 cannot be an equilibrium. But that path was shown to be the only potential equilibrium. The conclusion is that no equilibrium exists in the no-storage case, which proves Proposition 1.

4 Extraction with storage

4.1 The setup

We now consider the case where storage becomes possible (i.e., the constraint $I_t = 0$ is relaxed) and investigate whether a market equilibrium exists. The strategy will be very similar to the no-storage case, with - however - very different results in the end. In the first section we look at all potential equilibrium paths, just like in Section 3.2. The problem is of course more complex, as there are now two control variables and two stocks instead of only one, but as can be seen in Proposition 3, we are able to provide a full characterization of the potential equilibrium paths. They have to be like the ones depicted in Figure 1. The difference with the no-storage case mainly comes from the second part of the analysis where we discuss whether these potential equilibrium paths could be real equilibrium paths. In contradistinction with the no-storage case, where no equilibrium exists, we will show that, under some broad technical conditions, these potential equilibrium paths are in fact true equilibrium paths.

As dealing with the storage case significantly increases the complexity of the problem, we replace Assumption 2 by a slightly stronger statement, which is

Assumption 3 *The inverse demand and cost functions are such that:*

$$g(\underline{q}) > f'(\underline{q}).$$

This assumption rules out the possibility of having an equilibrium path where some but not all resources are extracted.[6] In case of storage, there are still some sales (from inventories) after the firm closes. The market price is then typically lower than $g(0)$ just after the end of the extraction period. Assumption 2 is then no longer sufficient to guarantee that all resources must be extracted. However, it will be shown that sales from inventory only occur at a rate lower than $\underline{q}$, the market price being then above $g(\underline{q})$. Whenever Assumption 3 is verified, the firm would prefer to extract the remaining resources at rate $\underline{q}$ rather than closing and leaving them under the ground for ever.

[6] In the no-storage case, the assumption $g(0) > f'(\underline{q})$ was sufficient to show that all resources are extracted, the intuition being that rather than stopping the extraction the firm should continue it at rate $\underline{q}$ and sell it at the market price, $g(0)$.

In presence of storage, the problem of the firm is given in Equation (3) with no additional constraint. As our objective is to prove that general equilibrium may exist we restrict, without loss of generality, our attention to a price path that is continuous.[7] We consider the optimal extraction problem when storage is possible and start with a result on the different time periods.

Proposition 2 (Different periods) *If an equilibrium exists, optimal extraction with storage is characterized by three dates* $0 \leq t_1^* < t_2^* < t_f^*$ *such that*

- *between dates* 0 *to* t_1^**, there is extraction without storage;*
- *between dates* t_1^* *and* t_2^**, there is positive extraction, inventories build up, and the extraction stops at date* t_2^**;*
- *between dates* t_2^* *and* t_f^**, the inventory is sold and is exhausted at date* t_f^**.*

Proof. Looking at whether inventories I_t and the extraction flows q_t are positive or equal to zero we can distinguish four states.

State 1. $I_t = 0$ and $q_t = 0$: The firm is no longer extracting, has no positive inventory and, because of Assumption 1, cannot reopen. Such a state is thus an absorbing state.

State 2. $I_t > 0$ and $q_t = 0$: There is no extraction but the firm has positive inventories, which it is selling observing demand $g(.)$. Inventory is exhausted when sales become zero. As the firm cannot reopen, State 2 is necessarily followed in finite time by State 1.

State 3. $I_t > 0$ and $q_t > 0$: Given that average extraction costs tend to $+\infty$ when q tends to zero, such a state with positive extraction cannot last for ever in an equilibrium. Moreover, as will be shown in Section 4.2.3, in an equilibrium one must have $\frac{\partial I_t}{\partial t} > 0$ in a period where $I_t > 0$ and $q_t > 0$. Thus State 3 can only be followed by State 2. Moreover, since reopening is not possible, the only other period that may precede State 3, if any, is State 4.

State 4. $I_t = 0$ and $q_t > 0$: Again, such a state cannot last for ever in an equilibrium since that average extraction costs tend to $+\infty$ when q tends to zero. From the previous section, we know it cannot be followed by State 1 in an equilibrium (otherwise we would have a contradiction of Proposition 1). Nor can it be followed by State 2, since building inventories requires having $q_t > 0$. Thus, if such a state exists, it is necessarily followed by State 3. ■

From Proposition 2 we conclude that only two types of trajectories are possible. In the first, State 4 never exists, and we have the sequence State 3 - State 2 - State 1. Alternatively, we start with State 4 and we have the sequence State 4 - State 3

[7] It can be proved, using similar arguments that those used to establish Proposition 1, that no general equilibrium may be supported by discontinuous price paths.

- State 2 - State 1. We can thus define dates $0 \leq t_1^* < t_2^* < t_f^*$ such that we have a phase of extraction without storage between time 0 and t_1^*, extraction and storage between t_1^* and t_2^* and sale out of the storage between t_1^* and t_f^*. When $t_1^* = 0$ the first phase is inexistent.

4.2 Potential equilibrium paths

Proposition 3 (Characterization of optimal paths with storage) *If the equilibrium with possible storage exists, we have that*

- *resource sales are continuously decreasing and necessarily reach zero in finite time,*
- *the price is continuously increasing up to the maximal value $g(0)$,*
- *extraction quantities are decreasing in the initial period without storage (if this period exists) but unambiguously increasing throughout the period in which inventories are built up,*
- *the size of initial resource stock unambiguously determines whether the initial period without inventories exists.*

Proof. We present the detailed proof in the following Subsections 4.2.1-4.2.5. Specifically, we first characterize the optimal controls in Subsection 4.2.1, then explain all the different States, as defined in Proposition 2, in Subsections 4.2.2-4.2.4 and, finally, prove the statements on initial resource conditions in Subsection 4.2.5. ■

4.2.1 Optimal controls

We start our proof by formally characterizing the controls for the two possible trajectories $t_1^* = 0$ and $t_1^* > 0$.

Case 1. $t_1^* = 0$. There is no period of extraction without storage; the dynamics of optimal controls are given by:

$$\forall t \in [0, t_2^*],\ q_t^* = f'^{-1}(e^{\delta t} f'(q_0^*)), \tag{12}$$

$$\forall t \in [0, t_f^*],\ s_t^* = g^{-1}(e^{\delta t} g(s_0^*)), \tag{13}$$

where the dates t_2^* and t_f^* are determined as follows:

$$e^{\delta t_2^*} = \frac{f'(\underline{q})}{f'(q_0^*)}, \tag{14}$$

$$e^{\delta t_f^*} = \frac{g(0)}{g(s_0^*)}. \tag{15}$$

The resource constraint pins down the values s_0^* and q_0^*:

$$R_0 = \frac{1}{\delta}\int_{\underline{q}}^{q_0^*} u\frac{f''(u)}{f'(u)}du = -\frac{1}{\delta}\int_0^{s_0^*} u\frac{g'(u)}{g(u)}du \tag{16}$$

The optimal price path P^* is determined as follows:

$$P_t^* = \begin{cases} g(s_t^*) & \text{if } t \in [0, t_f^*], \\ g(0) & \text{if } t \geq t_f^*. \end{cases} \tag{17}$$

Case 2. $t_1^* > 0$. There is a period of extraction prior to building up storage. The dynamics of optimal controls are then given by:

$$\forall t \in [0, t_1^*],\ q_t^* = s_t^* = \pi'^{-1}(e^{\delta t}\pi'(q_0^*)), \tag{18}$$

$$\forall t \in [t_1^*, t_2^*],\ q_t^* = f'^{-1}(e^{\delta t}f'(q_1^*)), \tag{19}$$

$$\forall t \in [t_1^*, t_f^*],\ s_t^* = g^{-1}(e^{\delta t}g(q_1^*)), \tag{20}$$

where the dates t_1^*, t_2^* and t_f^* are determined as follows:

$$e^{\delta t_1^*} = \frac{\pi(q_1^*)}{\pi(q_0^*)}, \tag{21}$$

$$e^{\delta(t_2^* - t_1^*)} = \frac{f'(\underline{q})}{f'(q_1^*)}, \tag{22}$$

$$e^{\delta(t_f^* - t_1^*)} = \frac{g(0)}{g(q_1^*)}. \tag{23}$$

The resource constraint between dates t_1^* and t_2^* pins down the values q_1^*:

$$R_1^* = \frac{1}{\delta}\int_{q_1^*}^{\underline{q}} u\frac{f''(u)}{f'(u)}du = -\frac{1}{\delta}\int_0^{q_1^*} u\frac{g'(u)}{g(u)}du,$$

while the resource constraint between dates 0 and t_1^* pins down q_0^*:

$$R_0 - R_1^* = \frac{1}{\delta}\int_{q_1^*}^{q_0^*} u\frac{f''(u)}{f'(u)}du. \tag{24}$$

The optimal price path P^* is determined as follows:

$$P_t^* = \begin{cases} g(q_t^*) & \text{if } t \in [0, t_1^*], \\ g(s_t^*) & \text{if } t \in [t_1^*, t_f^*], \\ g(0) & \text{if } t \geq t_f^*. \end{cases} \tag{25}$$

We are now ready to analyze the different States in the following Subsections.

4.2.2 Extraction without storage (State 4)

Before date t_1, the inventory is zero, which implies that $s_t = q_t$. The firm's program is

$$\begin{aligned} &\sup_{q\in\Omega\times\Omega}\left\{J[q] = \int_0^{t_1} e^{-\delta t}\left(P_t q_t - f(q_t) - c_0 1_{q_t>0}\right) dt\right\} \qquad (26)\\ \text{s.t.}\quad &\frac{\partial R_t}{\partial t} = -q_t \\ &R_0 = R_0 > 0 \\ &R_{t_1} = R_1 > 0 \\ &R_t \geq 0. \end{aligned}$$

During this time interval the situation is very similar to extraction studied in Section 3.1. There is a single control q and it is straightforward to prove that the co-state variable associated to the state variable R is constant and equal to $e^{-\delta t}(P_t^* - f'(q_t^*))$, where the optimal extraction path is denoted q^* and the optimal price path P^*. Moreover, the transversality condition at date t_1 imposes that the co-state associated to the state variable R is continuous in t_1, when exiting the zone of a binding constraint on the state variable I. Since the co-state is constant both before and after t_1, its value must be the same for all t between 0 and t_2:

$$\forall t \in [0, t_2],\ e^{-\delta t}(P_t^* - f'(q_t^*)) = e^{-\delta t_2}(P_{t_2}^* - f'(q_{t_2}^*)).$$

More specifically, between 0 and t_1, the optimal extraction and the optimal price at the equilibrium are characterized as follows:

$$\forall t \in [0, t_1],\ e^{-\delta t}\pi(q_t^*) = e^{-\delta t_1}\pi(q_{t_1}^*), \qquad (27)$$

$$P_t^* = g(q_t^*). \qquad (28)$$

Equation (27) together with the continuity of the price (that we impose for the equilibrium to hold), implies that the optimal extraction is continuous over $[0, t_2]$, which in turns implies that at date t_1, the optimal sale flow is equal to the optimal extraction flow (otherwise the price would be discontinuous):

$$s_{t_1}^* = q_{t_1}^* = q_1,$$

where $q_1 \in \mathbb{R}^+$ denotes the common value.

4.2.3 Extraction and storage (State 3)

State 3 is the central phase of the problem exhibiting simultaneous extraction and storage. In order to use standard notations in optimal control, we define $B_I = \{(R, I) \in \mathbb{R}^2 | -I \leq 0\}$ and $B_R = \{(R, I) \in \mathbb{R}^2 | -R \leq 0\}$. As long as state variables do not belong to the frontier of one of this set for a positive measure of time, the firm can manage two controls: the extraction flow q and the sale flow s. We denote by $[t_1, t_2]$ the time interval for which the two controls are available. The frontier of B_R is absorbing and as explained below, it is not possible to reach the frontier of B_I once in $[t_1, t_2]$. It must therefore be the case that at date t_1, the inventory is null and that at date t_2, the resources are exhausted.

More precisely, total resource stock R_t varies between R_1 and 0, where R_1 is the remaining level of resources at date t_1, which obviously depends on the firm's behavior before t_1. The date t_2 is defined as the first date at which the resources are exhausted. Inventory I is zero at the initial date t_1 and needs to remain positive between both dates (the firm cannot sell resources that have not been extracted).

The resource firm's program between dates t_1 and t_2 is as follows:

$$\begin{aligned} & \sup_{\{q,s\}} \left\{ J[q,s] = \int_{t_1}^{t_2} e^{-\delta t} \left(P_t s_t - f(q_t) - c_0 1_{q_t>0} \right) dt \right\} \qquad (29) \\ \text{s.t.} \quad & \frac{\partial R_t}{\partial t} = -q(t), \\ & \frac{\partial I_t}{\partial t} = q_t - s_t, \\ & R_{t_1} = R_1 > 0, \\ & R_{t_2} = 0, \\ & I_{t_1} = 0, \\ & I_t \geq 0, \ R_t \geq 0. \end{aligned}$$

The optimal control program (29) features two state constraints: inventory and resources both have to remain non-negative over the period $[t_1, t_2]$. The period $[t_1, t_2]$ is precisely defined as the period of time during which the two controls are available.

If we denote by $p = [p_R, p_I] : \mathbb{R}^+ \to \mathbb{R}^2$ the vector of co-state variables corresponding to the two state variables R and I, the Hamiltonian of the system (29) can be expressed as follows:

$$H(R, I, p, q, s, t) = -q p_R + (q - s) p_I + e^{-\delta t} \left(P_t s - f(q) - c_0 1_{q>0} \right).$$

The Pontryagin Maximum Principle tells us that state and co-state variables verify

the following equations for all $t \in [t_1, t_2]$:

$$\begin{aligned}
\frac{\partial p_{R,t}}{\partial t} &= -\frac{\partial H}{\partial R}(R, I, p, q, s, t) = 0, \\
\frac{\partial p_{I,t}}{\partial t} &= -\frac{\partial H}{\partial I}(R, I, p, q, s, t) = 0, \\
\frac{\partial R_t}{\partial t} &= \frac{\partial H}{\partial p_x}(R, I, p, q, s, t) = -q, \\
\frac{\partial I_t}{\partial t} &= \frac{\partial H}{\partial p_y}(R, I, p, q, s, t) = q - s,
\end{aligned}$$

and that the Hamiltonian solves:

$$H(R_t, I_t, p_t, q_t, s_t, t) = \max_{a,b} \left(-a p_{R,t} + (a-b) p_{I,t} + e^{-\delta t} \left(P_t b - f(a) - c_0 1_{a>0} \right) \right).$$

We deduce that co-state variables are constant and that for all $t \in [t_1, t_2]$, the optimal controls (q^*, s^*) verify the following equations.

$$\begin{aligned}
p_{R,t} &= p_R \\
p_{I,t} &= p_I \\
e^{-\delta t} f'(q_t^*) &= p_I - p_R \\
s_t^* &= \begin{cases} \infty & \text{if } e^{-\delta t} P_t > p_I \\ 0 & \text{if } e^{-\delta t} P_t < p_I \\ \geq 0 & \text{if } e^{-\delta t} P_t = p_I \end{cases}
\end{aligned}$$

This implies that we must have $q_t^* > 0$ at the equilibrium. Moreover, at the equilibrium, we cannot have $s_t^* = \infty$ or $s_t^* = 0$: It must therefore be the case that the optimal price path P^* is such that for all $t \in [t_1, t_2]$, we have:

$$e^{-\delta t} P_t^* = p_I. \tag{30}$$

and

$$e^{-\delta t} f'(q_t^*) = p_I - p_R = e^{-\delta t_2} f'(q_{t_2}^*). \tag{31}$$

From this expression we derive that q_t^* is increasing over $[t_1, t_2]$, while s_t^* is decreasing. Increasing extraction over time emerges because only extension of the extraction quantity allows the firm to keep discounted marginal cost constant, as requested by Eqn. (31). This proves the middle part of Proposition 3. Moreover since $I_{t_1} = 0$ and stocks cannot be negative, we must have $q_{t_1^*}^* \geq s_{t_1^*}^*$. It follows that we can never obtain a null inventory in $[t_1, t_2]$: the set B_I described above cannot be reached. This confirms that the no-inventory phase can only occur at the beginning of the extraction.

The Hamiltonian expression therefore becomes:

$$\begin{aligned} H(R_t^*, I_t^*, p_t, q_t^*, s_t^*, t) &= q_t^*(p_I - p_R) - s_t^* p_I + e^{-\delta t}\left(P_t^* s_t^* - f(q_t^*) - c_0\right), \\ &= e^{-\delta t}\left(q_t^* f'(q_t^*) - f(q_t^*) - c_0\right). \end{aligned} \tag{32}$$

4.2.4 Depletion of the storage (State 2)

Starting from date t_2 there is no more extraction. Since re-opening is impossible, the firm is left with a single control s. The firm's program becomes:

$$\begin{aligned} &\sup_{s\in\Omega}\left\{J[s] = \int_{t_2}^{t_f} e^{-\delta t} P_t s_t dt\right\} \\ \text{s.t.} \quad &\frac{\partial I_t}{\partial t} = -s_t, \\ &I_{t_2} > 0, \\ &I_t \geq 0. \end{aligned} \tag{33}$$

Using the Maximum principle, we obtain that $e^{-\delta t}P_t^*$ must be constant at the equilibrium for any $t \in [t_2, t_f]$. Since from (30), we already know that $e^{-\delta t}P_t^*$ is also constant for $t \in [t_1, t_2]$. We deduce that for the price to be continuous, we must have:[8]

$$\forall t \in [t_1, t_f],\ e^{-\delta t}P_t^* = p_I.$$

Moreover, for the price to be continuous after date t_f, the sale value at date t_f must be zero: $s_{t_f}^* = 0$, which implies that:

$$\forall t \in [t_1, t_f],\ e^{-\delta t} g(s_t^*) = p_I = e^{-\delta t_f} g(0), \tag{34}$$

where the continuity of the price implies that the optimal sale flow s^* is also continuous over $[t_1, t_f]$.

We remark that the Hamiltonian is zero in $[t_1, t_f]$. The continuity of the Hamiltonian in t_1 implies from (32) that $q_{t_2}^* = \underline{q}$ (where $\underline{q}$ is defined in (1)) and (31) becomes

$$\forall t \in [t_1, t_2],\ e^{-\delta t} f'(q_t^*) = e^{-\delta t_2} f'(\underline{q}). \tag{35}$$

4.2.5 Initial resource endowment

We have two conditions for resource quantities: the first states that total extracted quantity is equal to R_0, while the second says that everything that is sold should

[8] Even without the assumption of a continuous price, the transversality condition at date t_2 imposes that the co-state p_I associated to the state variable I is continuous in t_2, when entering the zone of a binding constraint on the state variable R. Since the costate is constant both before and after t_2, its value must be the same for all t between t_1 and t_f.

actually be extracted. This last condition implies that:

$$R_0 = \int_{t_1}^{t_2} q_t^* dt = \int_{t_1}^{t_f} s_t^* dt. \tag{36}$$

Using equations (34) and (35), we obtain that the above condition becomes:

$$\int_{t_1}^{t_2} f'^{-1}(e^{\delta(t-t_2)} f'(\underline{q}))dt = \int_{t_1}^{t_f} g^{-1}(e^{\delta(t-t_f)} g(0))dt.$$

Using a change of variable ($x = f'^{-1}(e^{\delta(t-t_2)} f'(\underline{q}))$ in the first integral and $x = g^{-1}(e^{\delta(t-t_f)} g(0))$ in the second one), we obtain:

$$\int_{q_{t_1}}^{\underline{q}} x \frac{f''(x)}{f'(x)} dx = \int_0^{s_{t_1}} x \frac{-g'(x)}{g(x)} dx,$$

with $q_{t_1} \geq s_{t_1} \geq 0$ (the sold quantity should be greater than the extracted since there is no inventory at t_1).

We define a maximal quantity R_1^* that can be extracted between dates t_1^* and t_2^* when $q_{t_1^*} = s_{t_1^*} = q_1^*$, i.e. during the period where the firm holds a positive inventory. The flow q_1^* denotes the minimal rate of extraction that can be observed in an equilibrium with storage. Both variables R_1^* and $q_1^* \in [0, \underline{q}]$ are formally given by:

$$R_1^* = \frac{1}{\delta} \int_{q_1^*}^{\underline{q}} x \frac{f''(x)}{f'(x)} dx = \frac{1}{\delta} \int_0^{q_1^*} x \frac{-g'(x)}{g(x)} dx. \tag{37}$$

Since $q \mapsto \frac{1}{\delta} \int_q^{\underline{q}} x \frac{f''(x)}{f'(x)} dx - \frac{1}{\delta} \int_0^q x \frac{-g'(x)}{g(x)} dx$ is strictly decreasing, positive in 0 and negative in $\underline{q}$, it is clear from the second part of equality that q_1^* is well-defined, unique and in $[0, \underline{q}]$. Moreover, the quantity $R_1^* > 0$ is strictly positive.

We get the two cases of Subsection 4.2.1, that is either $t_1^* = 0$ or $t_1^* > 0$, depending on whether R_0 is greater than R_1^* or not. If $R_0 \leq R_1^*$, there is no period of extraction without storage. If $R_0 > R_1^*$, there is a period of extraction without storage, whose length corresponds to the time needed to extract the quantity $R_0 - R_1^*$. Given that R_1^* and q_1^* are defined in equation (37) we can formally distinguish our two cases. If $R_0 \leq R_1^*$, optimal controls s^* and q^* are characterized by their initial values $0 \leq s_0^* \leq q_0^* \leq \underline{q}$ defined in equation (16), while dates t_2^* and t_f^* are defined in equations (14) and (15). In the second case when $R_0 > R_1^*$, the optimal control q^* is characterized by its initial value $q_0^* \geq q_1^*$ defined in equation (24), while dates t_1^*, t_2^* and t_f^* are defined in equations (21), (22) and (23).

Since we have by now fully characterized extraction and sales paths in the different States and derived the impact of initial resource endowment the proof of Proposition 3 is completed.

4.3 Equilibrium existence

Having obtained a precise description of all potential equilibrium paths, we now investigate whether these paths may actually correspond to market equilibria. The question is whether the strategy of the firm depicted in Proposition 3 is actually the optimal strategy of the firm (prices being considered as given). We know that the optimal strategy must fulfill the first order conditions. However there may be several trajectories that fulfill the first order conditions but assume different terminal dates of extraction (which themselves determine the optimal date at which inventories start to be built).

4.3.1 General functions

We first state the following result:

Proposition 4 (Equilibrium existence) *There exists a resource level $\overline{R} > R_1^*$ (possibly equal to infinity), such that if $R_0 < \overline{R}$, an equilibrium to the optimal extraction problem always exists.*

Proof. The proof that can be found in Appendix A involves looking at whether there could be profitable deviations from the potential equilibrium path for the firm. Actually, three deviations have to be considered: (i) the firm may choose to extract but to leave resources below the ground, (ii) the firm may choose not to start the extraction and (iii) the firm may choose to end the extraction at another date than t_2^* while extracting everything.

First, the firm may choose to extract but not everything. In Appendix, we rule out the possibility of such marginal deviations and show that, once the firm has started the extraction, it will only stop when resources below the ground are exhausted.

Second, the firm may choose not to start the extraction, if the intertemporal profit associated to extracting R_0 with the price path P^* is negative. Indeed, due to the payment of a maintenance cost c_0, it is not always the case that the extraction generates positive intertemporal profits (as it is the case with standard convex costs). However, Assumption 3 guarantees that at least when the amount of initial resources is not too large, the intertemporal profit is positive, so that the firm always opts for extracting. The intuition is as follows. Assumption 3 (i.e., $\pi(\underline{q}) > 0$) and the fact that the marginal profit π is a decreasing function of the extraction flow imply that $\underline{q}$ should not be too large. Moreover, using the definition of $\underline{q}$ in equation (2), the maintenance cost can be expressed as an increasing function of $\underline{q}$. Therefore, Assumption 3 can also be interpreted as the maintenance cost c_0 being not too large. However, even though the maintenance cost is not too high, the intertemporal profit might be negative when the payment for the maintenance cost lasts for too long, i.e., when the quantity R_0 to extract is very large. In consequence, we need to

impose an upper bound $\overline{R}_a$ on this initial quantity R_0 to guarantee the positivity of intertemporal profits. We can show that the bound $\overline{R}_a$ is strictly greater than R_1^*. The case of $\overline{R}_a$ being infinite corresponds to the case when it is always optimal to fully exhaust the resource.

Third, the firm may extract the whole stock R_0 but opt for an extraction date which is different from the optimal date t_2^*. We need to distinguish two cases depending on whether R_0 is smaller than R_1^* or not. In the first case when $R_0 \leq R_1^*$, we prove that if the extraction stops before (resp. after) the optimal date t_2^*, the firm will extract less than (resp. more) than R_0. Therefore, there is no possible deviation and the only extraction date that is compatible with the resource constraint is t_2^*. The firm follows thus the optimal extraction path. Regarding the sales path, there is no deviation that provides a strictly greater profit when following the optimal sales path. In short, when $R_0 \leq R_1^*$, the firm follows optimal extraction paths both for extracting and selling. When $R_0 > R_1^*$, as for the deviation with respect to the extraction quantity, we cannot rule out all deviations with respect to the final extraction date. The firm may opt for an earlier or a later final extraction date. The intuition is as follows. On the one hand, since the optimal extraction path is non-monotonic when $R_0 > R_1^*$ (see Figure 1 or Proposition 3 for example), it is possible that the firm stops extracting earlier, while it leaves nothing in the ground and extracts R_0. Stopping the extraction earlier may yield a higher profit because of the fixed maintenance cost c_0 that will be paid for a shorter period of time. On the other hand, an extraction stopping later than the optimal date may also yield a higher profit, even if it is slightly less intuitive. Indeed, since the price path is an increasing function of time and the optimal extraction path non-monotonic, the firm may choose to postpone the extraction date to take advantage of the higher price at later dates that may compensate for paying the maintenance cost for a longer period of time. A later final extraction date may yield a higher profit. As a consequence, to circumvent these difficulties, we define a threshold $\overline{R}_b$ for the amount of initial resources, below which the firm always follows the optimal extraction path and never deviates. In the appendix we prove that such a threshold exists and is strictly greater than R_1^*. The case of $\overline{R}_b$ being infinite corresponds to the case when the firm always follows the optimal extraction path.

Finally, as long as the total stock of resources is below the level $\overline{R} = \min(\overline{R}_a, \overline{R}_b)$, there is no possible deviation and the equilibrium always exists. ■

The fact that $\overline{R} > R_1^*$ indicates that both equilibrium paths discussed in Proposition 3 and illustrated in Figure 1 can be observed.

4.3.2 Specifying cost and demand functions

As a complement to Proposition 4, we now demonstrate, that under some additional assumptions on the cost and inverse demand function, we have $\overline{R} = +\infty$, which implies the existence of an equilibrium whatever the initial size of resource stocks.

Proposition 5 (Equilibrium existence for convex marginal costs) *Provided that the marginal cost function is convex and null in zero ($f''' \geq 0$ and $f'(0) = 0$) and that the function $q \to qg'(q)$ is decreasing an equilibrium exists for any value of R_0.*

Proof. The proof strategy consists in proving that the deviations that are possible when the resource quantity R_0 is greater than R_1^* can be ruled out under some technical conditions. First, regarding the deviation with respect to the extraction quantity, when the marginal cost function is convex and null in zero, we show that the only terminal date which is compatible with the resource constraint is the date t_2^* of the optimal extraction path. Second, regarding the deviation with respect to the extraction date, when $q \to qg'(q)$ is decreasing, we prove that the intertemporal profit is always positive and the firm always prefers to extract all resources than none. Both proofs are rather technical. ■

Proposition 5 states than an equilibrium may always exist under certain technical assumptions, no matter what is the quantity of available resources. The first condition has a straightforward interpretation in terms of convexity of the marginal cost. Moreover, even if the second condition may look fairly technical, it also has a straightforward and intuitive economic implication. Indeed, this condition translates into a condition on the "instantaneous" profit. If $q \to qg'(q)$ is decreasing, then the "instantaneous" profit $g(q)q - f(q)$ is a concave function of the extracted quantity, which is a standard condition in the industrial organization literature.

5 Conclusions

The present paper shows that equilibrium on non-renewable resource markets exists even when extraction costs are non-convex, once we realistically include resource inventories in the analysis. The setup does not rely on narrow assumptions so that all the proofs can be given under very general conditions. The optimum extraction path is non monotonic, which emerges to be perfectly compatible with constant resource stocks. Extraction necessarily involves increasing quantities at the end of the extraction period, which is a novelty in literature, before demand is met by sales from inventories. The underlying reason is that resource extractors need to increase extraction quantities in order to obtain discounted resource rents which are constant.

Two different development paths are found to exist, depending on whether initial resource stock is larger or smaller than a threshold value depending on cost and

demand functions. If initial stock is low, the firm starts building up inventories from period zero on. The extraction path is increasing over time up to the point where firms stop extraction and start to sell out of inventories. If initial resource stock is high, exceeding the threshold, the previous pattern is preceded by a period in which extraction is positive but storage is not yet used. In that period, the extraction and sale flows are equal and decreasing over time.

The present paper addresses one of the main puzzles in resource economics: Why do we see many well functioning resource markets in reality despite non convex extraction costs? The paper complements alternative explanations for this empirical fact in earlier literature but avoids very specific assumptions like chattering controls or capacity constraints. It goes back to the original idea of including inventories and finds that this generally valid assumption is sufficient to prove the existence of equilibrium.

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Appendix

A Proof of Proposition 4

In this proof, we check that the necessary conditions of equations (12)–(25) define an equilibrium. To do so, we verify that no deviation is profitable. Deviations can be of three types: (i) the firm may choose to extract but to leave resources below the ground, (ii) the firm may choose not to start the extraction and (iii) the firm may choose to stop extracting at another date than the optimal date t_2^*. These possible deviations are discussed in Sections A.1 to A.3.

A.1 First deviation: Extracting but less than R_0

We prove that given the price path P^*, once the extraction has started, the firm will extract everything and leaves no resources below the ground.

We consider a deviation (q, s) corresponding to the extraction of the amount $R \leq R_0$, while the firm faces the optimal price path P^*. We denote t_2 the end of the extraction and t_f the end of the sale flow. Remark that the optimal price path P^* depends on R_0 but is independent of R. The plan (q, s) is characterized using a program similar to (3) and involves distinguishing two cases depending on $R_0 \geq R_1^*$ or not.

First case: $R_0 \leq R_1^*$. The intertemporal profit associated to this deviation is:

$$J_R[q, s] = \int_0^{t_f} e^{-\delta t} P_t^* s_t dt - \int_0^{t_2} e^{-\delta t}(f(q_t) + c_0)dt, \tag{38}$$

where we use the subscript R to highlight the dependence in R and where the controls (q, s) and the optimal price P^* can be (indirectly for controls – directly for the price)

derived from equations (12)–(17) and can be expressed as follows:

$$\begin{aligned} e^{-\delta t} g(s_t) &= g(s_0), \\ e^{-\delta t} f'(q_t) &= f'(q_0), \\ \int_0^{t_f} s_t dt &= \int_0^{t_2} q_t dt = R \leq R_0, \\ e^{-\delta t} P_t^* &= g(s_0^*). \end{aligned}$$

Using the above definitions, equation (38) simplifies into $J_R[q,s] = g(s_0^*)R - \int_0^{t_2} e^{-\delta t}(f(q_t) + c_0)dt$, whose derivation with respect to R yields:

$$\begin{aligned} \frac{\partial J_R[q,s]}{\partial R} &= g(s_0^*) - e^{-\delta t_2}(f(\underline{q}) + c_0)\frac{\partial t_2}{\partial R} - \int_0^{t_2} e^{-\delta t} f'(q_t)\frac{\partial q_t}{\partial R} dt \\ &= g(s_0^*) - e^{-\delta t_2}\underline{q} f'(\underline{q})\frac{\partial t_2}{\partial R} - e^{-\delta t_2} f'(\underline{q}) \int_0^{t_2} \frac{\partial q_t}{\partial R} dt \\ &= g(s_0^*) - e^{-\delta t_2} f'(\underline{q}) \end{aligned}$$

since the resource constraint $\int_0^{t_f} q_t dt = R$ implies after derivation $\underline{q}\frac{\partial t_f}{\partial R} + \int_0^{t_f} \frac{\partial q_t}{\partial R} dt$.

Since t_2 is an increasing function of R, $R \mapsto \frac{\partial J_R[q,s]}{\partial R}$ admits at most one zero on $[0, R_0]$ and $R \mapsto J_R[q,s]$ is either maximal in 0 or in R_0. Once the extraction has started, the firm will extract R_0. The firm will therefore extract either everything or nothing.

Second case: $R_0 \geq R_1^*$. The extraction q is assumed to stop at a date $t_2 \leq t_2^*$. Controls (q,s) and the optimal price P^* can be derived from equations (18)–(25). We distinguish again two cases, depending on $t_2 \geq t_1^*$ or not.

Case $t_2 \leq t_1^*$. The intertemporal profit $J_R[q,s]$ can be expressed as follows:

$$J_R[q,s] = \int_0^{t_f} e^{-\delta t}(P_t^* q_t - f(q_t) - c_0)dt, \tag{39}$$

where q follows:

$$\forall t \in [0, t_f],\ e^{-\delta t}(P_t^* - f'(q_t)) = e^{-\delta t_f}(P_{t_f}^* - f'(\underline{q})), \int_0^{t_f} q_t dt \qquad = R. \tag{40}$$

The derivation of (39) with respect to R yields:

$$\begin{aligned}\frac{\partial J_R[q,s]}{\partial R} &= e^{-\delta t_f}(P^*_{t_f}\underline{q} - f(\underline{q}) - c_0)\frac{\partial t_f}{\partial R} \\ &+ \int_0^{t_f} e^{-\delta t}(P^*_t - f'(q_t))\frac{\partial q_t}{\partial R}dt \\ &= e^{-\delta t_f}(P^*_{t_f} - f'(\underline{q}))\left(\underline{q}\frac{\partial t_f}{\partial R} + \int_0^{t_f}\frac{\partial q_t}{\partial R}dt\right) \\ &= e^{-\delta t_f}(P^*_{t_f} - f'(\underline{q})), \end{aligned} \tag{41}$$

since the derivation of (40) wrt R implies $\underline{q}\frac{\partial t_f}{\partial R} + \int_0^{t_f}\frac{\partial q_t}{\partial R}dt = 1$.

Case $t_2 \geq t_1^*$. We have:

$$J_R[q,s] = \int_0^{t_1^*} e^{-\delta t}(P^*_t q_t - f(q_t) - c_0)dt + \int_{t_1^*}^{t_f} e^{-\delta t}P^*_t s_t dt - \int_{t_1^*}^{t_2} e^{-\delta t}(f(q_t) + c_0)dt, \tag{42}$$

where controls (q,s) verify:

$$\begin{aligned} e^{-\delta t}(P^*_t - f'(q_t)) &= e^{-\delta t_1^*}(P^*_{t_1^*} - f'(q_1)) \\ e^{-\delta t}g(s_t) &= e^{-\delta t_1^*}g(q_1), \\ e^{-\delta t}f'(q_t) &= e^{-\delta t_1^*}f'(q_1), \\ \int_{t_1^*}^{t_f} s_t dt &= \int_{t_1^*}^{t_2} q_t dt = R - \int_0^{t_1^*} q_t dt. \end{aligned} \tag{43}$$

The derivation of (42) yields:

$$\begin{aligned}\frac{\partial J_R[q,s]}{\partial R} &= \int_0^{t_1^*} e^{-\delta t}(P^*_t - f'(q_t))\frac{\partial q_t}{\partial R}dt + e^{-\delta t_1^*}P^*_{t_1^*}(1 - \int_0^{t_1^*}\frac{\partial q_t}{\partial R}dt) \\ &- e^{-\delta t_2}(f(\underline{q}) + c_0)dt\frac{\partial t_2}{\partial R} - \int_{t_1^*}^{t_2} e^{-\delta t}f'(q_t)\frac{\partial q_t}{\partial R}dt \\ &= e^{-\delta t_1^*}(P^*_{t_1^*} - f'(q_{t_1^*}))\int_0^{t_1^*}\frac{\partial q_t}{\partial R}dt + e^{-\delta t_1^*}P^*_{t_1^*}(1 - \int_0^{t_1^*}\frac{\partial q_t}{\partial R}dt) \\ &- e^{-\delta t_1^*}f'(q_{t_1^*})\left(\underline{q}\frac{\partial t_2}{\partial R} + \int_{t_1^*}^{t_2}\frac{\partial q_t}{\partial R}dt\right)\end{aligned}$$

Moreover, we have by derivation of (43), $\underline{q}\frac{\partial t_2}{\partial R} + \int_{t_1^*}^{t_2}\frac{\partial q_t}{\partial R}dt = 1 - \int_0^{t_1^*}\frac{\partial q_t}{\partial R}dt$, which implies:

$$\frac{\partial J_R[q,s]}{\partial R} = e^{-\delta t_1^*}(P^*_{t_1^*} - f'(q_{t_1^*})) = e^{-\delta t_f}(P^*_{t_f} - f'(\underline{q})) \tag{44}$$

Conclusion of the second case $R_0 \geq R_1^*$. From (41) and (44), we deduce that for any $R \in [0, R_0]$ and thus for any $t_2 \in [0, t_2^*]$, we have $\frac{\partial J_R[q,s]}{\partial R} = e^{-\delta t_f}(g(q^*_{t_f}) - f'(\underline{q}))$.

Since $\pi(q_1^*) > 0$, we know from Proposition 3 that $t \mapsto P_t^*$ is increasing. Thus $R \mapsto \frac{\partial J_R[q,s]}{\partial R}$ admits at most one zero for $R \in [0, R_0]$ and $R \mapsto J_R[q,s]$ is maximal either for $R = 0$ ($t_2 = 0$, i.e., no extraction) or $R = R_0$ ($t_2 = t_2^*$, i.e., full extraction). Once the extraction has started, the firm will extract everything and leaves no resources below the ground.

A.2 Second deviation: Not starting the extraction

We now check that not starting the extraction cannot be a possible deviation. We denote $J_{R_0}[q^*, s^*]$ the intertemporal profit associated to the extraction of R_0, when the firm faces the price path P^* and follows the plan (q^*, s^*). Note that in this case (compared to $R \mapsto J_R$), both the optimal plan (q^*, s^*) and the price P^* depend on R_0. These quantities are defined in equations (12)–(17).

Since no extraction corresponds to a zero intertemporal profit: $J_{R_0=0}[q^*, s^*] = 0$, we need to prove that $J_{R_0}[q^*, s^*] > 0$ for any $0 < R_0 \leq R_1^*$.

The intertemporal profit can also be expressed as follows:

$$J_{R_0}[q^*, s^*] = g(s_0^*)R_0 - \int_0^{t_2^*} e^{-\delta t}(f(q_t^*) + c_0)dt$$

Since $e^{-\delta t} f'(q_t^*) = e^{-\delta t_2^*} f'(\underline{q}) = f'(q_0^*)$ for any $0 \leq t \leq t_2^*$, we obtain:

$$\begin{aligned} J_{R_0}[q^*, s^*] &= \frac{1}{\delta}(g(s_0^*)\delta R_0 + \varphi(q_0^*) - \varphi(\underline{q})), \\ \text{where: } \varphi(q) &= qf'(q) - f(q). \end{aligned}$$

Since $\delta R_0 = \int_{q_0^*}^{\underline{q}} u \frac{f''(u)}{f'(u)} du$, we have:

$$J_{R_0}[q^*, s^*] = \frac{1}{\delta}(g(s_0^*) \int_{q_0^*}^{\underline{q}} u \frac{f''(u)}{f'(u)} du + \varphi(q_0^*) - \varphi(\underline{q})). \tag{45}$$

Using that f is convex, we obtain:

$$\begin{aligned} \int_{q_0^*}^{\underline{q}} u \frac{f''(u)}{f'(u)} du &\geq \frac{1}{f'(\underline{q})} \int_{q_0^*}^{\underline{q}} u f''(u) du \\ &\geq \frac{\varphi(\underline{q}) - \varphi(q_0^*)}{f'(\underline{q})} \geq \frac{\varphi(\underline{q}) - \varphi(q_0^*)}{g(\underline{q})}, \end{aligned}$$

where the last equality comes from the fact that $\pi(\underline{q}) \geq 0$ and $\underline{q} \geq q_0^*$. We deduce:

$$J_{R_0}[q^*, s^*] \geq \frac{1}{\delta} \frac{g(s_0^*) - g(\underline{q})}{g(\underline{q})} (\varphi(\underline{q}) - \varphi(q_0^*)) \geq 0,$$

where the inequality sign is strict for any $R_0 > 0$.

To conclude, we define $\overline{R}_a = \inf\{R_0 \geq R_1^*,\ J_{R_0}[q^*, s^*] > 0\}$. Using our above result, we deduce that $\overline{R}_a > R_1^*$ by continuity, which proves the result.

A.3 Third deviation: Extracting R_0 but opting for other extraction and sale paths

We prove that there exists a resource level $\overline{R}_b > R_1^*$ (possibly equal to infinity), such that if $R_0 < \overline{R}_b$, a firm extracting R_0 always follows the optimal extraction plan (q^*, s^*) and no deviation is profitable. The proof involves two steps: (i) we start with proving that there is no deviation when $R_0 \leq R_1^*$; (ii) we find a threshold $\overline{R}_b > R_1^*$ such that no deviation holds when $R_0 < \overline{R}_b$.

A.3.1 No deviation when $R_0 \leq R_1^*$.

We denote (q, s) the optimal plan of the firm. We denote τ_2 the date at which the extraction q stops. The plan (q, s) is characterized by equation similar to (12)–(16). We distinguish two cases depending on whether τ_2 is smaller than t_f^* or not.

The extraction stops before t_f^*: $\tau_2 \leq t_f^*$. The extraction plan is characterized as follows:

$$\forall t \in [0, \tau_2],\ f'(q_t) = f'(\underline{q})e^{\delta(t-\tau_2)}.$$

We know that if $\tau_2 = t_2^*$ or equivalently $q_0 = q_0^*$, we have $q_t = q_t^*$. Let us consider $\tau_2 \mapsto \int_0^{\tau_2} q_t dt$. After a change of variable and using resource constraint in equation (16), we have:

$$\int_0^{\tau_2} q_t dt - R_0 = \frac{1}{\delta} \int_{q_0}^{q_0^*} u \frac{f''(u)}{f'(u)} du.$$

The resource constraint therefore imposes $q_0 = q_0^*$ and therefore $q_t = q_t^*$.

The extraction stops after t_f^*: $\tau_2 > t_f^*$. The extraction plan q is characterized as follows:

$$\begin{aligned}
&\forall t \in [0, t_f^*],\ f'(q_t) = f'(q_{t_f^*})e^{\delta(t-t_f^*)},\\
&\forall t \in [t_f^*, \tau_2],\ f'(q_t) = g(0) + (f'(\underline{q}) - g(0))e^{\delta(t-\tau_2)}.
\end{aligned}$$

Since $f'(\underline{q}) - g(0) \leq -\pi(\underline{q}) < 0$, q_t is decreasing over $[t_f^*, \tau_2]$. It implies that if $\tau_2 > t_f^*$, then $q_{t_f^*} > \underline{q}$ and for any $t \in [0, t_f^*]$, $f'(q_t) > f'(q_t^*)$: the resource constraint cannot hold when $\tau_2 > t_f^*$.

Conclusion for the extraction. The extraction must therefore stop at date $\tau_2 = t_2^*$, which implies that the firm must follow the optimal extraction: $\forall t \in [0, t_2^*],\ q_t = q_t^*$.

The inventory. We assume that the inventory is exhausted at date τ_f (remark: we do not prevent $q = s$ and $\tau_f = t_2^*$). Since $e^{-\delta t} P_t^*$ is decreasing for $t \geq t_f^*$, it is optimal for the firm to exhaust inventories before t_f^*: $\tau_f \leq t_f^*$.

We denote $J_{R_0}[q, s]$ the intertemporal profit associated to the plan (q, s).

$$J_{R_0}[q, s] = \int_0^{\tau_f} e^{-\delta t} P_t^* s_t dt - \int_0^{t_2^*} (f(q_t) + c_0) dt.$$

Using that $q = q^*$ and $e^{-\delta t}P_t^* = e^{-\delta t_f^*}g(0)$ for $0 \leq t \leq t_f^*$, we deduce that the difference with the intertemporal profit derived from the optimal extraction plan $J_{R_0}[q,s] - J_{R_0}[q^*,s^*]$ can be expressed as follows:

$$J_{R_0}[q,s] - J_{R_0}[q^*,s^*] = e^{-\delta t_f^*}g(0)(\int_0^{\tau_f} s_t dt - \int_{t_1^*}^{t_f^*} s_t^* dt) = 0,$$

because of resource constraints. Therefore, the deviation never strictly dominates the initial allocation when $R_0 \leq R_1^*$.

A.3.2 Ruling out deviations when $R_0 \geq R_1^*$

Dealing with the case where $R_0 \geq R_1^*$ is more complex and involves three steps: (i) we show that no storage occurs before date t_1^*; (ii) we show that for a deviation to hold, the extraction must stop after date t_1^*; (iii) for deviations whose extraction stops after date t_1^*, we show that they cannot occur provided that $R_0 \leq \overline{R}_b$, where $\overline{R}_b > R_1^*$ is a threshold that we define.

We consider that a firm facing the optimal price path P^* defined in (25) chooses the plan (q,s). We assume that the extraction stops at date τ_2.

No storage before date t_1^*. We start with the following lemma.

Lemma 1 (No storage before t_1^*) *Given a price path P^*, it is never optimal for the firm to store before date t_1^*.*

Proof. Using the expression (25) of P^* together with (18), we obtain for any $t \in [0, t_1^*]$:

$$\begin{aligned} e^{-\delta t}P_t^* &= e^{-\delta t}g(\pi^{-1}(e^{\delta(t-t_1^*)}\pi(q_1^*))) \\ &= e^{-\delta t}f'(\pi^{-1}(e^{\delta(t-t_1^*)}\pi(q_1^*))) + e^{-\delta t_1^*}\pi(q_1^*) \end{aligned}$$

Since $t \mapsto e^{-\delta t}f'(\pi^{-1}(e^{\delta(t-t_1^*)}\pi(q_1^*)))$ is a decreasing function (product of two positive decreasing functions), the function $t \mapsto e^{-\delta t}P_t^*$ is decreasing over $[0, t_1^*]$. It implies that the firm prefers to sell as much as possible, rather than to save. There is therefore no storage before t_1^*. ■

No deviation when the extraction stops before date t_1^*: $\tau_2 \leq t_1^*$. Since there is no storage before date t_1^* (Lemma 1), using a similar technique as for the problem (4) of optimal extraction without storage implies that the extraction path q is such that $e^{-\delta t}(f'(q_t) - P_t^*)$ is constant for any $t \in [0, \tau_2]$:

$$f'(q_t) - P_t^* = e^{\delta(t-\tau_2)}(f'(\underline{q}) - P_{\tau_2}^*)$$

Since $e^{-\delta t}(f'(q_t^*) - P_t^*)$ is also constant for any $t \in [0, \tau_2]$ (equation (18)), we have:

$$f'(q_t) = f'(q_t^*) + e^{\delta(t-\tau_2)}(f'(\underline{q}) - f'(q_{\tau_2}^*)). \tag{46}$$

Using the definition (46) of the deviation q_t for $t \in [0, \tau_2] \subset [0, t_1^*]$, we obtain:

$$\int_0^{\tau_2} q_t dt = \int_0^{\tau_2} f'^{-1}(f'(q_t^*) + e^{\delta(t-\tau_2)}(f'(\underline{q}) - f'(q_{\tau_2}^*))dt. \tag{47}$$

We know that for any $t \in [0, t_1^*]$, $e^{-\delta t}(P_t^* - f'(q_t^*))$ is constant. We have already proved (see proof of Lemma 1) that $t \mapsto e^{-\delta t} P_t^*$ is decreasing on $[0, t_1^*]$. Therefore $t \mapsto -e^{-\delta t} f'(q_t^*)$ is increasing. This implies that $\tau_2 \mapsto f'^{-1}(f'(q_t^*) + e^{\delta(t-\tau_2)}(f'(\underline{q}) - f'(q_{\tau_2}^*))$ is increasing and strictly positive on $[0, t_1^*]$. From (47), we deduce that $\tau_2 \mapsto \int_0^{\tau_2} q_t dt$ is strictly increasing on $[0, t_1^*]$. It implies that if we guarantee that $\int_0^{t_1^*} q_t dt < R_0$, no deviation is possible if $\tau_2 \leq t_1^*$.

Deviations whose extraction stops after t_1^*: $\tau_2 > t_1^*$. Using a similar technique as in the optimal control problem (3), we obtain that the extraction q is defined as follows:

$$\begin{cases} \forall t \in [0, t_1^*], & f'(q_t) = f'(q_t^*) + e^{\delta(t-t_1^*)}(f'(q_1) - f'(q_1^*)), \\ \forall t \in [t_1^*, \tau_2], & f'(q_t) = e^{\delta(t-t_1^*)} f'(q_1) = e^{\delta(t-\tau_2)} f'(\underline{q}). \end{cases} \tag{48}$$

We conclude the proof in two steps. First we define $\overline{R}_b$ and shows that it exists and is strictly larger than R_1^*. Second, we show that no deviation exists when the resource level is below $\overline{R}_b$.

Definition and existence of $\overline{R}_b$. We define $\overline{R}_b \in \mathbb{R}^+ \cup \{\infty\}$ as follows:

$$\overline{R}_b = \sup \left\{ R_0 \geq R_1^* | \exists t_2^{**} \geq t_2^*, \ \left\{ \tau_2 \geq t_1^* | \int_0^{\tau_2} q_t dt = R_0 \right\} = \{t_2^*, t_2^{**}\} \right\}. \tag{49}$$

Notice that we allow for $\overline{R}_b$ to be infinite. Note that the definition (49) hides several implicit dependencies. First, as defined in (48), the plan q depends on τ_2. Second, q also depends on R_0 through t_1^* and q_t^*. Finally both dates t_1^* and t_2^* depends on R_0.

The meaning of the level $\overline{R}_b$ is the following one. For any resource level R_0 smaller than $\overline{R}_b$, there are at most two dates (t_2^* and t_2^{**}) at which the resource constraint holds. It implies first that it is not possible to find a deviation which extracts resources faster than the optimal extraction q^*: the other possible deviation will stop extracting at t_2^{**} after the date t_2^*. Because of the time preference and of the fixed-cost c_0, the firm may prefer deviations that extract faster than q^* to q^*. It is therefore important to rule them out.

Lemma 2 (Existence of $\overline{R}_b$) *The maximal resources level $\overline{R}_b$ defined in (49) exists, is strictly larger than R_1^* and may be equal to infinity.*

Proof.

First, it is clear from the case $R_0 \leq R_1^*$ treated above that the set $\{R_0 \geq R_1^* | \exists t_2^{**} \geq t_2^*, \ \{\tau_2 \geq t_1^* | \int_0^{\tau_2} q_t dt = R_0\} = \{t_2^*, t_2^{**}\}\}$ is not empty, since R_1^* belongs to it (in that case, t_2^* is the unique date).

Second, we can show than whenever $R_0 - R_1^*$ is not too large but positive, $\tau_2 \mapsto \int_0^{\tau_2} q_t dt$ is increasing for $\tau_2 \leq t_2^*$. We have from (48):

$$\begin{aligned}\int_0^{\tau_2} q_t dt &= \int_0^{t_1^*} q_t dt + \int_{t_1^*}^{\tau_2} q_t dt \\ &= \int_0^{t_1^*} q_t dt + \int_{q_1}^{\underline{q}} u \frac{f''(u)}{f'(u)} du = Q(q_1)\end{aligned} \tag{50}$$

with $e^{\delta \tau_2} = \frac{f'(\underline{q})}{f'(q_1)}$: proving that $\tau_2 \mapsto \int_0^{\tau_2} q_t dt$ is increasing for $\tau_2 \in [t_1^*, t_2^*]$ is equivalent to prove that Q is decreasing for $q_1 \in [q_1^*, \underline{q}]$. We have

$$Q'(q_1) = \int_0^{t_1^*} \frac{\partial q_t}{\partial q_1} dt - q_1 \frac{f''(q_1)}{f'(q_1)} \leq t_1^* \sup_{t \in [0, t_1^*]} \frac{\partial q_t}{\partial q_1} - q_1 \frac{f''(q_1)}{f'(q_1)}. \tag{51}$$

From (48), $f''(q_t)\frac{\partial q_t}{\partial q_1} = e^{\delta(t-t_1^*)} f''(q_1)$, so $0 < \sup_{t \in [0,t_1^*]} \frac{\partial q_t}{\partial q_1} \leq \frac{f''(q_1)}{\inf_{u \in [q_1, q_0]} f''(u)}$.

Moreover, from equations (24) and (21), we have:

$$R_0 - R_1^* = \int_{q_1^*}^{q_0^*} u \frac{-\pi'(u)}{\pi(u)} du > (q_0^* - q_1^*) \inf_{u \in [q_1^*, q_0^*]} u \frac{-\pi'(u)}{\pi(u)}$$

and

$$t_1^* = \frac{1}{\delta} \ln \frac{\pi(q_1^*)}{\pi(q_0^*)} \leq \frac{1}{\delta} \ln \frac{\pi(q_1^*)}{\pi(q_1^* + \frac{R_0 - R_1^*}{\inf_{u \in [q_1^*, q_0^*]} u \frac{-\pi'(u)}{\pi(u)}})}.$$

We deduce that

$$Q'(q_1) \leq \frac{1}{\delta} \ln\left(\frac{\pi(q_1^*)}{\pi(q_1^* + \frac{R_0 - R_1^*}{\inf_{u \in [q_1^*, q_0^*]} u \frac{-\pi'(u)}{\pi(u)}})}\right) \frac{f''(q_1)}{\inf_{u \in [q_1, q_0]} f''(u)} - q_1 \frac{f''(q_1)}{f'(q_1)},$$

which can be made negative for any $q_1 \in [q_1^*, \underline{q}]$ provided that $R_0 - R_1^*$ is made sufficiently small. Remark that it is not fully obvious because q_0 and q_0^* depends (negatively) on $R_0 - R_1^*$. However, since both q_0 and q_0^* decrease with $R_0 - R_1^*$, it is possible to assume without loss of generality that for not too large $R_0 - R_1^*$, q_0 and q_0^* are bounded by some $\overline{q}_0$ and $\overline{q}_0^*$ independent of $R_0 - R_1^*$, such that $\inf_{u \in [q_1, q_0]} f''(u) \geq \inf_{u \in [q_1, \overline{q}_0]} f''(u)$. This latter lower bound is independent of $R_0 - R_1^*$ and strictly positive since f'' is continuous and $[q_1, \overline{q}_0]$ compact. We can then obtain an explicit bound on $R_0 - R_1^*$.

When $Q'(q_1)$ is negative for any $q_1 \in [q_1^*, \underline{q}]$, $\tau_2 = t_2^*$ is the first date at which $\int_0^{\tau_2} q_t dt = R_0$. We further need to prove that if $R_0 - R_1^*$ is not too large, there exists at most one other date $t_2^{**} \geq t_2^*$ (or another $q_1^{**} < q_1^*$), such that $Q(q_1^{**}) = R_0$. From (51), the sign of Q' is determined by the following expression $Q_1(q_1) = f'(q_1) \int_0^{t_1^*} \frac{e^{\delta(t-t_1^*)}}{f''(q_t)} dt - q_1$. Provided that f'' is continuously derivable, one can use the same technique as above and prove that when $R_0 - R_1^*$ is not too large, $Q_1(q_1)$ is decreasing for any $q_1 \leq q_1^*$, which guarantees that Q_1 admits at most one zero, smaller than q_1^*. Since

$Q(q_1^*) = R_0$, there therefore exists at most one other $q_1^{**} < q_1^*$, such that $Q(q_1^{**}) = R_0$.

We therefore deduce that $\overline{R}_b$ exists and is such that $\overline{R}_b > R_1^*$. ■

No deviation when $R_0 \leq \overline{R}_b$. From the definition (49) of $\overline{R}_b$, it is sufficient to prove that the (possible) deviation that stops at date t_2^{**} is dominated by the optimal extraction stopping at t_2^*. We formalize in the following lemma.

Lemma 3 (Optimality of q^*) *If there exists another extraction plan q^{**} stopping at date $t_2^{**} > t_2^*$ such that $\int_0^{t_2^{**}} q_t^{**} dt = R_0$, then the plan q^{**} is never preferred to the optimal plan q^*.*

Proof.

We still consider a plan (q, s), in which the extraction stops at date $\tau_2 \geq t_2^*$ and the sale stops at date $\tau_f \geq \tau_2$. The intertemporal profit $J[q, s]$ can be expressed as follows:

$$J[q,s] = \int_0^{t_1^*} e^{-\delta t}(P_t^* q_t - f(q_t) - c_0)dt + e^{-\delta t_1^*} P_{t_1^*}^* \int_{t_1^*}^{\tau_f} s_t dt - \int_{t_1^*}^{\tau_2} e^{-\delta t}(f(q_t) + c_0)dt$$

We know from (48) that $e^{-\delta t} P_t^* = e^{-\delta t} f'(q_t) + e^{-\delta t_1^*}(P_{t_1^*}^* - f'(q_1))$ and from the resource constraint that $\int_{t_1^*}^{\tau_f} s_t dt = \int_{t_1^*}^{\tau_2} q_t dt$. We deduce:

$$\begin{aligned}
J[q,s] &= e^{-\delta t_1^*}(P_{t_1^*}^* - f'(q_1)) \int_0^{t_1^*} q_t dt + \int_0^{t_1^*} e^{-\delta t}(f'(q_t)q_t - f(q_t) - c_0)dt \\
&\quad - \int_{t_1^*}^{\tau_2} e^{-\delta t}(f(q_t) + c_0)dt + e^{-\delta t_1^*} P_{t_1^*}^* \int_{t_1^*}^{\tau_2} q_t dt \\
&= e^{-\delta t_1^*}(P_{t_1^*}^* - f'(q_1)) \int_0^{\tau_2} q_t dt + \int_0^{t_1^*} e^{-\delta t}(f'(q_t)q_t - f(q_t) - c_0)dt \\
&\quad - \int_{t_1^*}^{\tau_2} e^{-\delta t}(f(q_t) + c_0)dt - e^{-\delta t_1^*} f'(q_1) \int_{t_1^*}^{\tau_2} q_t dt
\end{aligned}$$

Since (48) implies that $e^{-\delta t} f'(q_t) = e^{-\delta t_1^*} f'(q_1)$ for $t \geq t_1^*$, we have using the expression (50) of $Q(q_1)$:

$$J[q,s] = e^{-\delta t_1^*}(P_{t_1^*}^* - f'(q_1))Q(q_1) + \int_0^{\tau_2} e^{-\delta t}(f'(q_t)q_t - f(q_t) - c_0)dt,$$

where this expression is valid no matter the value of $Q(q_1)$. The derivation with respect to q_1 yields:

$$\begin{aligned}
\frac{\partial J[q,s]}{\partial q_1} &= e^{-\delta t_1^*}(P_{t_1^*}^* - f'(q_1))Q'(q_1) - e^{-\delta t_1^*} f''(q_1)Q(q_1) \\
&\quad + \int_0^{\tau_2} e^{-\delta t} f''(q_t) q_t \frac{\partial q_t}{\partial q_1} dt,
\end{aligned}$$

since $q_{\tau_2} = \underline{q}$ and $f'(\underline{q})\underline{q} - f(\underline{q}) = c_0$. Moreover, from (48), we have for any $t \in [0, \tau_2]$, $e^{-\delta t} f''(q_t) \frac{\partial q_t}{\partial q_1} = e^{-\delta t_1^*} f''(q_1)$, which implies:

$$\frac{\partial J[q,s]}{\partial q_1} = e^{-\delta t_1^*}(P^*_{t_1^*} - f'(q_1))Q'(q_1). \tag{52}$$

Assume that there exists another plan q^{**} whose terminal date is $t_2^{**} > t_2^*$ and such that $\int_0^{t_2^{**}} q_t^{**} dt = R_0$. Noting $q_1^{**} = q^{**}_{t_1^*}$, we have from (48), $q_1^{**} < q_1^*$, since $t_2^{**} > t_2^*$. We denote J^{**} (J^*) the profit associated to q^{**} (q^*). By integration by parts of (52):

$$\begin{aligned} J^{**} - J^* &= e^{-\delta t_1^*} \int_{q_1^{**}}^{q_1^*} f'(q_1)Q'(q_1)dq_1 \\ &= e^{-\delta t_1^*} R_0 (f'(q_1^*) - f'(q_1^{**})) - e^{-\delta t_1^*} \int_{q_1^{**}}^{q_1^*} f''(q_1)Q(q_1)dq_1, \end{aligned}$$

Moreover, $Q(q_1) \geq R_0$ for any $q_1 \in [q_1^{**}, q_1^*]$. Indeed $R_0 = Q(q_1^*) = Q(q_1^{**})$: otherwise there would exists another q_1 in $[q_1^{**}, q_1^*]$, such that $R_0 = Q(q_1)$, which is not possible. Therefore, we deduce (the inequality is strict otherwise Q would be constant and equal to R_0 over $[q_1^{**}, q_1^*]$):

$$J^{**} - J^* < e^{-\delta t_1^*} R_0 (f(q_1^*) - f(q_1^{**})) - R_0 \int_{q_1^{**}}^{q_1^*} f''(q_1)dq_1 < 0.$$

We conclude that the extraction q^* is strictly preferred to q^{**}.

■

Lemma 2 guarantees that we cannot find any extraction plan that terminates before t_2^*, while Lemma 3 ensures that the extraction plan q^* is preferred to any extraction plan that stops after t_2^*: we deduce that there is no possible deviation for the extraction.

B Proof of Proposition 5

We prove that adding two assumptions on the shape of f and g implies that an equilibrium always exists, in other words that $\overline{R} = \infty$. The proof is in two steps: (i) when $q \to qg'(q)$ is decreasing, the upper bound $\overline{R}_a$ is infinite and it is always optimal to extract R_0; (ii) when the extraction cost function f is such that the marginal cost of extraction is convex and null for zero extraction ($f'(0) = 0$), the upper bound $\overline{R}_b$ is infinite and it is always optimal to follow the plan (q^*, s^*).

B.1 Decreasing $q \to qg'(q)$ and $\overline{R}_a = \infty$

Lemma 4 *If $q \to qg'(q)$ is decreasing, then $\overline{R}_a = \infty$.*

Proof. We first prove that the intertemporal profit is always positive for large values of R_0. Second, we show that the intertemporal profit (as a function of R_0) admits at most one maximum between $R_0 = R_1^*$ and $R_0 = \infty$.

The intertemporal profit $J_{R_0}[q^*, s^*]$ is positive for large values of R_0. We prove that $J_{R_0}[q^*, s^*] > 0$ for any $R_0 \to \infty$. For any $R_0 > R_1^*$, we have:

$$J_{R_0}[q^*, s^*] = \int_0^{t_1^*} e^{-\delta t}(P_t^* q_t^* - f(q_t^*) - c_0)dt + \int_{t_1^*}^{t_f^*} e^{-\delta t} P_t^* s_t^* dt - \int_{t_1^*}^{t_f^*} e^{-\delta t}(f(q_t^*) + c_0)dt. \tag{53}$$

We are interested in the behavior of $J_{R_0}[q^*, s^*]$ for large R_0. Note that in that case the strategies (q^*, s^*) as well as the optimal price P^* depend on R_0. Optimal plan (q^*, s^*) is defined in equations (18)–(25).

It is straightforward to show that for $R_0 \to \infty$, we have $\int_{t_1^*}^{t_f^*} e^{-\delta t} P_t^* s_t^* dt - \int_{t_1^*}^{t_f^*} e^{-\delta t}(f(q_t^*) + c_0)dt \to 0$. Writing $P_t^* = \pi(q_t^*) + f'(q_t^*)$, we obtain

$$\lim_{R_0 \to \infty} J_{R_0}[q^*, s^*] = \lim_{R_0 \to \infty} \pi(q_0^*)(R_0 - R_1^*) + \int_0^{t_1^*} e^{-\delta t}(f'(q_t^*)q_t^* - f(q_t^*) - c_0)dt \tag{54}$$

Since $\pi(q_1^*) > 0$, for large R_0, q_0^* converges towards $q_\infty = \inf\{q \geq 0, \pi(q) = 0\}$ (q_∞ is unique if π strictly decreasing) and t_1^* converges towards infinity. Therefore, when $R_0 \to \infty$, we have $q_t^* \to q_\infty + 1_{t=t_1^*}(q_1^* - q_\infty)$ for all $t \in [0, t_1^*]$. We assume without loss of generality that $q_\infty < \infty$. Indeed, in that case, the result holds since $f'(q_t^*)q_t^* - f(q_t^*) - c_0$ is positive for all t as soon as R_0 is sufficiently large.

First, let us look at $\pi(q_0^*)(R_0 - R_1^*)$. After integration by parts of (24), we obtain

$$\delta\pi(q_0^*)(R_0 - R_1^*) = (q_1^*\pi(q_0^*)\ln(\pi(q_1^*)) - q_0^*\pi(q_0^*)\ln(\pi(q_0^*)) + \frac{1}{\delta}\pi(q_0^*)\int_{q_1^*}^{q_0^*} u\ln(\pi(u))du$$

For large R_0, $\pi(q_0^*) \to 0$ and $q_0^* \nearrow q_\infty$, so $q_1^*\pi(q_0^*)\ln(\pi(q_1^*)) - q_0^*\pi(q_0^*)\ln(\pi(q_0^*) \to 0$. For R_0 sufficiently large, we have $\pi(q_0^*)|\int_{q_1^*}^{q_0^*} u\ln(\pi(u))du| \leq \pi(q_0^*)\ln(\pi(q_0^*))\frac{(q_0^* - q_1^*)^2}{2} \to 0$. We deduce that $\pi(q_0^*)(R_0 - R_1^*) \to 0$ for $R_0 \to \infty$.

Second look at $\int_0^{t_1^*} e^{-\delta t}(f'(q_t^*)q_t^* - f(q_t^*) - c_0)dt$. We have $e^{-\delta t}|f'(q_t^*)q_t^* - f(q_t^*) - c_0| \leq e^{-\delta t}(f'(q_\infty)q_\infty - f(q_\infty) + c_0)$, which is integrable on $\mathbb{R}^+$. Using the dominated convergence theorem, we have:

$$\begin{aligned}\int_0^{t_1^*} e^{-\delta t}(f'(q_t^*)q_t^* - f(q_t^*) - c_0)dt &\longrightarrow (f'(q_\infty)q_\infty - f(q_\infty) - c_0)\int_0^\infty e^{-\delta t}dt \\ &= \frac{f'(q_\infty)q_\infty - f(q_\infty) - c_0}{\delta}.\end{aligned}$$

Therefore:

$$\lim_{R_0 \to \infty} J_{R_0}[q^*, s^*] = \frac{(f'(q_\infty)q_\infty - f(q_\infty)) - (f'(\underline{q})\underline{q} - f(\underline{q}))}{\delta},$$

which is positive iff $q_\infty \geq \underline{q}$ or equivalently, $\pi(\underline{q}) \geq 0$ (Assumption 3).

The intertemporal profit $R_0 \mapsto J_{R_0}[q^*, s^*]$ admits at most one maximum for $R_0 \geq R_1^*$. When $R_0 \geq R_1^*$, $J_{R_0}[q^*, s^*] = \int_0^{t_2^*} e^{-\delta t}(f'(q_t^*)q_t^* - f(q_t^*) - c_0)dt +$

$e^{-\delta t_1^*}\pi(q_1^*)R_0$ and we can prove that we have:

$$\delta J_{R_0}[q^*,s^*] = \delta R_1^* e^{-\delta t_1^*} g(q_1^*) + e^{-\delta t_1^*}\pi(q_1^*)\int_{q_1^*}^{q_0^*} u\frac{-g'(u)}{\pi(u)}du \tag{55}$$

$$+ e^{-\delta t_1^*}\frac{(q_0^* f'(q_0^*) - f(q_0^*)) - (\underline{q} f'(\underline{q}) - f(\underline{q}))}{\pi(q_0^*)}$$

$$= \delta R_1^* \frac{g(q_1^*)}{\pi(q_1^*)}\pi(q_0^*) + \pi(q_0^*)\int_{q_1^*}^{q_0^*} u\frac{-g'(u)}{\pi(u)}du - \int_{q_0^*}^{\underline{q}} u f''(u)du \tag{56}$$

and after derivation:

$$\delta\frac{\partial J_{R_0}[q^*,s^*]}{\partial R_0} = \frac{\partial R_0}{\partial q_0^*}\left(\delta R_1^*\frac{g(q_1^*)}{\pi(q_1^*)}\pi'(q_0^*) + \pi'(q_0^*)\int_{q_1^*}^{q_0^*} u\frac{-g'(u)}{\pi(u)}du - q_0^*\pi'(q_0^*)\right)$$

$$= -\pi'(q_0^*)\frac{\partial R_0}{\partial q_0^*}\left(q_0^* - \int_{q_1^*}^{q_0^*} u\frac{-g'(u)}{\pi(u)}du - \delta R_1^*\frac{g(q_1^*)}{\pi(q_1^*)}\right). \tag{57}$$

Since the derivation of (24) implies $1 = \frac{q_0^*}{\pi(q_0^*)}(-\pi'(q_0^*))\frac{\partial R_0}{\partial q_0^*}$, we have:

$$\delta\frac{\partial J_{R_0}[q^*,s^*]}{\partial R_0} = \frac{\pi(q_0^*)}{q_0^*}\left(\int_{q_1^*}^{q_0^*}\frac{ug'(u)+\pi(u)}{\pi(u)}du + q_1^* - \delta R_1^*\frac{g(q_1^*)}{\pi(q_1^*)}\right). \tag{58}$$

We now distinguish two cases, depending on the sign of $q_1^* g'(q_1^*) + \pi(q_1^*)$.

First case: $q_1^* g'(q_1^*) + \pi(q_1^*) < 0$**.** Since by assumption $q \mapsto qg(q) - f(q)$ is concave and since $q_1^* g'(q_1^*) + \pi(q_1^*) < 0$, we have for any $q_1^* \leq q_0^* \leq \underline{q}$, $q_0^* g(q_0^*) - f(q_0^*) \geq \underline{q} g(\underline{q}) - f(\underline{q})$.

Moreover, from (56) and (57), any extremum of $R_0 \mapsto J_{R_0}[q^*,s^*]$ denoted $J_{R_{ext}}[q^*,s^*]$ is such that (with $q_1^* \leq q_0^* \leq \underline{q}$):

$$\delta J_{R_{ext}}[q^*,s^*] = q_0^*\pi(q_0^*) - \int_{q_0^*}^{\underline{q}} u f''(u)du$$

$$= q_0^* g(q_0^*) - f(q_0^*) - \underline{q} f'(\underline{q}) + f(\underline{q})$$

$$\geq (q_0^* g(q_0^*) - f(q_0^*)) - (\underline{q} g(\underline{q}) - f(\underline{q})) \geq 0,$$

which implies that the intertemporal profit $J_{R_{ext}}[q^*,s^*]$ is positive for any extremum R_{ext}. Since it is already positive for R_1^* and for values of $R_0 \to \infty$, it implies that $J_{R_0}[q^*,s^*] \geq 0$ for any value of R_0.

Second case: $q_1^* g'(q_1^*) + \pi(q_1^*) \geq 0$**.** We first prove that we have $q_1^* \geq \delta R_1^* \frac{g(q_1^*)}{\pi(q_1^*)}$ in this case. Indeed:

$$\begin{aligned}\delta R_1^* \frac{g(q_1^*)}{\pi(q_1^*)} &= \int_0^{q_1^*} \frac{-ug'(u)}{g(u)} du \frac{g(q_1^*)}{\pi(q_1^*)} \\ &\leq \frac{-q_1^* g'(q_1^*)}{g(q_1^*)} q_1^* \frac{g(q_1^*)}{\pi(q_1^*)}, \end{aligned} \tag{59}$$

since $0 \leq -ug'(u) \leq -q_1^* g'(q_1^*)$ ($q \mapsto qg'(q)$ is decreasing by assumption) and $g(u) \geq g(q_1^*) > 0$.

Moreover, $\frac{-q_1^* g'(q_1^*)}{g(q_1^*)} q_1^* \frac{g(q_1^*)}{\pi(q_1^*)} \leq q_1^*$ is equivalent to $q_1^* g'(q_1^*) + \pi(q_1^*) \geq 0$, which holds in our case. Therefore, we have from (59) that:$\delta R_1^* \frac{g(q_1^*)}{\pi(q_1^*)} \leq q_1^*$.

Remark that if $\frac{\partial J_{R_0}[q^*,s^*]}{\partial R_0}$ does not admit any zero, $R_0 \mapsto J_{R_0}$ is increasing for any $R_0 \geq R_1^*$ and is thus positive, since $J_{R_1^*} \geq 0$.

Consider now the case that $\frac{\partial J_{R_0}[q^*,s^*]}{\partial R_0}$ admits at least one zero. We consider the smallest extremum and we denote the resource quantity R_e and the associated initial extraction flow q_e. Since $q_1^* - \delta R_1^* \frac{g(q_1^*)}{\pi(q_1^*)} > 0$, the first extremum cannot be a minimum (i.e., a maximum or a saddle point). Assume that there exists another extremum $\hat{q}_e$. From (58), we must have $\int_{q_e}^{\hat{q}_e} \frac{ug'(u)+\pi(u)}{\pi(u)} du = 0$. Since $u \mapsto ug'(u) + \pi(u)$ is decreasing, it implies that $q_e g'(q_e) + \pi(q_e) \geq 0$ and $\hat{q}_e g'(\hat{q}_e) + \pi(\hat{q}_e) \leq 0$: therefore, if there exists a second extremum, the first one is a saddle point and the second one cannot be a minimum. As consequence, any extremum of $R_0 \mapsto J_{R_0}[q^*, s^*]$ cannot be a minimum and $R_0 \mapsto J_{R_0}$ admits at most one maximum and admits a minimum value for $R_0 = R_1^*$ or $R_0 = \infty$. Since these both values are positive, $J_{R_0}[q^*, s^*] \geq 0$ for any $R_0 \geq R_1^*$. ■

B.2 Convex marginal cost and $\overline{R}_b = \infty$

The following lemma summarizes our result.

Lemma 5 *If the marginal cost function is convex and null in zero: $f'(0) = 0$, then $\overline{R}_b = \infty$.*

Proof.

The proof strategy consists in proving that no deviation fulfills the resource constraint.

We consider as a deviation (q, s), whose extraction stops at date τ_2. We aim at proving that $\tau_2 \mapsto \int_0^{\tau_2} q_t dt$ is strictly increasing over $\mathbb{R}^+$. Since $Q(t_2^*) = R_0$, there is no possible deviation and we must have $q = q^*$. We already know that $\tau_2 \mapsto \int_0^{\tau_2} q_t dt$ is strictly increasing over $[0, t_1^*]$ (see equation (47) and the discussion below). We now assume that $\tau_2 \geq t_1^*$.

From equations (48) and (50), we obtain that

$$Q(q_1) = \int_0^{t_1^*} f'^{-1}(f'(q_t^*) + e^{\delta(t-t_1^*)}(f'(q_1) - f'(q_1^*))dt + \frac{1}{\delta} \int_{q_1}^{\underline{q}} u \frac{f''(u)}{f'(u)} du.$$

We already know that proving that $\tau_2 \mapsto \int_0^{\tau_2} q_t dt$ is increasing for $\tau_2 \geq t_1^*$ is equivalent to prove that Q is decreasing for $q_1 \leq q_1^*$ (see (51)). After derivation, we have:

$$\begin{aligned} Q'(q_1) &= \int_0^{t_1^*} \frac{e^{\delta(t-t_1^*)} f''(q_1)}{f''(f'^{-1}(f'(q_t^*) + e^{\delta(t-t_1^*)}(f'(q_1) - f'(q_1^*)))} dt - \frac{1}{\delta} q_1 \frac{f''(q_1)}{f'(q_1)} \\ &= \frac{1}{\delta} q_1 \frac{f''(q_1)}{f'(q_1)} \left(\frac{f'(q_1)}{q_1} \int_0^{t_1^*} \frac{\delta e^{\delta(t-t_1^*)}}{f''(f'^{-1}(f'(q_t^*) + e^{\delta(t-t_1^*)}(f'(q_1) - f'(q_1^*)))} dt - 1 \right) \end{aligned} \tag{60}$$

We now consider $Q_1(q_1) = \frac{f'(q_1)}{q_1} \int_0^{t_1^*} \frac{\delta e^{\delta(t-t_1^*)}}{f''(f'^{-1}(f'(q_t^*) + e^{\delta(t-t_1^*)}(f'(q_1) - f'(q_1^*))))} dt$. Since $t \mapsto q_t^*$ is decreasing on $[0, t_1^*]$, and f', f'^{-1} and f'' are increasing, we have:

$$Q_1(q_1) \leq \frac{f'(q_1)}{q_1} \int_0^{t_1^*} \frac{\delta e^{\delta(t-t_1^*)}}{f''(f'^{-1}(f'(q_1^*) + e^{\delta(t-t_1^*)}(f'(q_1) - f'(q_1^*))))} dt.$$

Making a change of variable $u = f'^{-1}(f'(q_1^*) + e^{\delta(t-t_1^*)}(f'(q_1) - f'(q_1^*))$, we have $du = (f'(q_1) - f'(q_1^*)) \frac{\delta e^{\delta(t-t_1^*)}}{f''(f'^{-1}(f'(q_1^*) + e^{\delta(t-t_1^*)}(f'(q_1) - f'(q_1^*))))} dt$ and

$$\begin{aligned} Q_1(q_1) &\leq \frac{f'(q_1)}{q_1(f'(q_1) - f'(q_1^*))} \int_{f'^{-1}(f'(q_1^*) + e^{-\delta t_1^*}(f'(q_1) - f'(q_1^*)))}^{q_1} du \\ &\leq \frac{f'(q_1)}{(f'(q_1) - f'(q_1^*))} \frac{q_1 - f'^{-1}(f'(q_1^*) + e^{-\delta t_1^*}(f'(q_1) - f'(q_1^*)))}{q_1} \\ &\leq \frac{f'(q_1)}{(f'(q_1) - f'(q_1^*))} \frac{q_1 - q_1^*}{q_1} = Q_2(q_1^*) \end{aligned} \tag{61}$$

We aim at proving that $Q_2(q_1^*) \leq 1$ for any $q_1 \leq q_1^*$. However, $Q_2(q_1^*) \leq 1$ iff:

$$\frac{f'(q_1^*) - f'(q_1)}{q_1^* - q_1} \geq \frac{f'(q_1) - f'(0)}{q_1 - 0},$$

which always holds for any $q_1 \leq q_1^*$ (remind that $f'(0) = 0$), since f' is convex.

This implies from (60)–(61) that $Q'(q_1) \leq 0$ for any $q_1 \leq q_1^*$, and therefore that $\tau_2 \mapsto \int_0^{\tau_2} q_t dt$ is increasing on $[t_1^*, \infty[$. ■

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