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## Comparative Risk Aversion: A Formal Approach with Applications to Saving Behaviors

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Applications to Saving Behaviors

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# Comparative Risk Aversion: A Formal Approach with Applications to Saving Behaviors<sup>☆</sup>

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## Abstract

We consider a formal approach to comparative risk aversion and applies it to intertemporal choice models. This allows us to ask whether standard classes of utility functions, such as those inspired by Kihlstrom and Mirman [15], Selden [26], Epstein and Zin [9] and Quiggin [24] are well-ordered in terms of risk aversion. Moreover, opting for this model-free approach allows us to establish new general results on the impact of risk aversion on savings behaviors. In particular, we show that risk aversion enhances precautionary savings, clarifying the link that exists between the notions of prudence and risk aversion.

*Keywords:* Risk aversion, Savings behaviors, Precautionary savings.

*JEL:* D11, D81, D91

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## 1. Introduction

A common approach to study the role of risk aversion is to consider a particular class of preferences, presumably well-ordered in terms of risk aversion, and then analyze the decisions that result from preferences within this class. In the context of intertemporal choice, a number of different classes of utility functions have been considered. The most popular choice consists of preferences à la Epstein and Zin [9], while the frameworks in Kihlstrom and Mirman [15] and Quiggin's [24] provide alternative settings.

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Predictions about the impact of risk aversion radically depends on the model that is chosen. For example, regarding the relation between risk aversion and precautionary savings in a simple two-period model, the preferences in Kihlstrom and Mirman [15] and Quiggin [24] lead to the conclusion that precautionary savings rise with risk aversion (Drèze and Modigliani [8], Yaari [28], and Bleichrodt and Eeckhoudt [3]). On the contrary, this relation is ambiguous when Epstein and Zin's [9] preferences are used (Kimball and Weil [17]).<sup>1</sup>

The current paper makes three contributions. First, it discusses the extent to which the utility classes mentioned above are well-ordered in terms of risk aversion. In particular, we show that, when we consider aversion to marginal increases in risk, Epstein and Zin preferences are not well-ordered. Second, we suggest a model-free approach that makes it possible to discuss the role of risk aversion without focusing on any specific model of rationality. Third, we apply this setup to establish new general results on the role of risk aversion. In particular, we show that risk aversion enhances precautionary savings, clarifying the link that exists between risk aversion and prudence.

Our paper relies on an abstract procedure to define comparative risk aversion, which assumes no particular structure for the set of consequences. This definition is inspired by the seminal work of Yaari [27]. It states that if a given increase in risk is perceived as worthwhile for a decision maker (because it yields a higher level of ex ante welfare), it should also be so for any less risk-averse decision maker.

A considerable number of papers have used Yaari's approach to define comparative risk- (or uncertainty-) aversion. This is explicitly the case in Kihlstrom and Mirman [15], Ghirardato and Marinacci [11] and Grant and Quiggin [12], but also implicit in the papers that have focused on certainty equivalents, such as Chew and Epstein [7] and Epstein and Zin [9], as well as in Pratt's [23]. In most cases, although Grant and Quiggin [12] is a noteworthy exception, Yaari's procedure was (implicitly or explicitly) implemented based on a minimalist risk ordering, where random objects are only compared to deterministic constructs. Our paper departs from this minimalist approach to provide novel insights. Instead of focusing on certainty equivalents to assess the individual's degree of risk aversion, we also account for individual preferences over marginal variations in risk.

The notion of comparative aversion that we derive when considering marginal risk variations is stronger than that focusing on certainty equivalents. In consequence, although preferences may be well ordered in terms of risk aversion when considering certainty equivalents, this may no longer hold when considering our more stringent comparison. This turns out to be the case for Epstein and Zin preferences. No similar case can be made against

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<sup>1</sup>More details on the meaning of the preferences *à la* Kihlstrom and Mirman, Quiggin, and Epstein and Zin are provided in Section 2.

Kihlstrom and Mirman or Quiggin’s anticipated utility functions. These latter utility classes actually seem to be well-suited to provide insights into the impact of risk aversion.

Abandoning these standard but somewhat restrictive frameworks, we establish a general result allowing us to make predictions about the impact of risk aversion without assuming any particular form of rationality. It is possible to determine the impact of risk aversion under relatively weak assumptions on ordinal preferences, as long as states of the world can be ranked from bad to good independently of the agent’s action. The intuition behind this result is that risk aversion enhances the willingness to redistribute from good to bad states.

We provide direct applications to savings under uncertainty. In particular we prove, under weak conditions on ordinal preferences, that risk aversion enhances precautionary savings. Moreover, we show that risk aversion has a negative (resp. positive) impact on savings when the rate of return is uncertain, as soon as the intertemporal elasticity of substitution is larger (resp. smaller) than one. Risk aversion is also found to have a negative impact on savings when the lifetime is uncertain, therefore underlining that the relation between time preference, risk aversion and mortality risk discussed in Bommier [4] is general and is not restricted to the expected-utility framework.

The remainder of the paper is organized as follows. In Section 2, we present several classes of utility functions that have been used to analyze the role of risk aversion in intertemporal models. The main theoretical contents appear in Section 3, which is split up into several subsections. Subsection 3.1 introduces the relevant concepts, and Subsection 3.2 then focuses on the simplest random objects that we can think of: “heads or tails” gambles, which are lotteries with two equally-probable outcomes. This is sufficient to provide the main intuition and to show that Epstein and Zin preferences are not well-ordered in terms of risk aversion. To increase applicability, the analysis is extended in Section 3.3 to general lotteries. We define a formal notion of comparative risk aversion and show how it can be used to obtain model-free results on the impact of risk aversion. A number of applications providing insights into the impact of risk aversion on savings behavior are then developed in Section 4.

To help the reader to grasp the paper’s main message, we restrict the use of the term *Proposition* to the most significant results. The paper also includes other statements, which are useful for general understanding, or for the relation of our work to that of others, but which are admittedly less important or original. These are labeled as *Result*.

## **2. Popular classes of utility functions disentangling risk aversion from intertemporal substitution**

We present in this section the main risk preferences that have been suggested in the literature to discuss the role of risk aversion in intertemporal frameworks. As in Selden [26]

and many other papers on precautionary savings, we restrict our attention to preferences over “certain  $\times$  uncertain” consumption pairs that we denote  $(c_1, \tilde{c}_2)$  – the tilde emphasizing that the second element is random.

Kihlstrom and Mirman [15] convincingly explain that the comparison of agents’ risk aversions is possible if and only if agents have identical preferences over certain prospects. We therefore focus on utility classes involving different risk attitudes, while leaving preferences over certain consumption paths unchanged. This rules out the standard class of expected-utility models assuming additively-separable utility functions. Under additive separability, it is impossible to change risk preferences, without affecting ordinal preferences.<sup>2</sup>

We consider three extensions of the standard additively-separable expected-utility model, where we can analyze risk aversion without affecting ordinal preferences. This is not of course an exhaustive review of what can be found in the literature, but rather focuses on the most popular specifications. The first setup, which assumes expected utility, was suggested by Kihlstrom and Mirman [15]. The second one was introduced by Selden [26], building on the framework in Kreps and Porteus [18], and was then extended by Epstein and Zin [9] to deal with infinitely-long consumption streams, and appears to be a very convenient way of studying many intertemporal problems. This has now become by far the most popular approach to the analysis of risk aversion in intertemporal frameworks. The third class, based on Quiggin’s [24] anticipated-utility theory, is developed in Yaari [28] for example.

The initial contributions of Kihlstrom and Mirman [15], Selden [26] and Quiggin [24] were very general, and not limited to the analysis of intertemporal choices. However, applied works on savings often assume that preferences over certain consumption paths are additively separable. The (ordinal) utility  $U(c_1, c_2)$  associated with the certain consumption profile  $(c_1, c_2)$  is expressed as the sum of the utilities associated with the first-period and second-period consumptions:  $U(c_1, c_2) = u_1(c_1) + u_2(c_2)$ . We include this assumption of the additive separability of ordinal preferences in our definitions of what we call “Kihlstrom and Mirman”, “Selden” or “Quiggin” utility functions that rank certain consumptions pair as  $U$  does.<sup>3</sup>

**Definition 1 (Utility classes).** *A utility function  $U(c_1, \tilde{c}_2)$  is called:*

- *A Kihlstrom and Mirman utility function  $U_k^{KM}$  if there exist continuous increasing real*

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<sup>2</sup>One very popular representation is the additive expected utility specification:  $U(c_1, \tilde{c}_2) = \frac{c_1^{1-\rho}}{1-\rho} + \mathbb{E} \left[ \frac{\tilde{c}_2^{1-\rho}}{1-\rho} \right]$ , where  $\rho$  is interpreted as reflecting the agent’s risk aversion. However, changing  $\rho$  involves changing ordinal preferences (in particular, the intertemporal elasticity of substitution  $\frac{1}{\rho}$ ) and cannot be used to analyze the impact of risk aversion. Kihlstrom and Mirman [15] and Epstein and Zin [9] among others discuss this.

<sup>3</sup>It should be clear that our terminology is only indicative of the general frameworks in which these particular specifications may be related. We do not aim to provide a complete account of the contributions of the corresponding papers, which consider both much broader utility classes and more complex settings.

functions  $u_1, u_2$  and  $k$  such that  $U_k^{KM}(c_1, \tilde{c}_2) = k^{-1} (\mathbb{E} [k (u_1(c_1) + u_2(\tilde{c}_2))])$ .

- A Selden utility function  $U_v^S$  if there exist continuous increasing real functions  $u_1, u_2$  and  $v$  such that  $U_v^S(c_1, \tilde{c}_2) = u_1(c_1) + u_2(v^{-1}(\mathbb{E}[v(\tilde{c}_2)]))$ .
- A Quiggin utility function  $U_\phi^Q$  if there exist continuous real functions  $u_1, u_2$  and a continuous increasing function  $\phi : [0, 1] \rightarrow [0, 1]$ , with  $\phi(0) = 0$  and  $\phi(1) = 1$ , such that  $U_\phi^Q(c_1, \tilde{c}_2) = u_1(c_1) + \mathbb{E}_\phi[u_2(\tilde{c}_2)]$ , where  $\mathbb{E}_\phi[\cdot]$  denotes the Choquet expectation operator associated with  $\phi$ . For a real random variable  $\tilde{z}$  characterized by the cumulative distribution function  $F$ , this operator is defined as  $\mathbb{E}_\phi[\tilde{z}] = - \int_{-\infty}^{+\infty} z d(\phi(1 - F(z)))$ .

One popular specification results from choosing isoelastic functions in the Selden utility function. Setting  $u_1(c) = u_2(c) = \frac{c^{1-\rho}}{1-\rho}$  and  $v(x) = \frac{x^{1-\gamma}}{1-\gamma}$  yields a class of utility functions assuming a constant intertemporal elasticity of substitution and homothetic preferences. Such utility functions are often called “Epstein and Zin utility functions”, although they indicate only imperfectly what can be found in Epstein and Zin [9], who consider preferences over infinitely-long consumption paths, which is a much more complex issue. Even so, this terminology has become very popular in the Economic literature, and we think it is more productive and less confusing to adhere to it, rather than introducing a new one.

**Definition 2.**  $U_\gamma^{EZ}(c_1, \tilde{c}_2)$  is called an Epstein and Zin utility function if there exist positive scalars  $\rho \neq 1$  and  $\gamma \neq 1$  such that:<sup>4</sup>  $U_\gamma^{EZ}(c_1, \tilde{c}_2) = \frac{c_1^{1-\rho}}{1-\rho} + \frac{1}{1-\rho} (\mathbb{E} [\tilde{c}_2^{1-\gamma}])^{\frac{1-\rho}{1-\gamma}}$ .

Certainty-equivalent arguments have been used to suggest that these Kihlstrom and Mirman, Selden, Epstein and Zin, and Quiggin utility functions are well-suited for the analysis of risk aversion. It can indeed easily be shown that the greater is the concavity of  $k$  (for Kihlstrom and Mirman utility functions), the greater is the concavity of  $v$  (for Selden utility functions), the greater is the scalar  $\gamma$  (for Epstein and Zin utility functions) and the greater is the convexity of  $\phi$  (for Quiggin utility functions), the smaller (in terms of ordinal utility) is the certainty equivalent assigned to any random element  $(c_1, \tilde{c}_2)$ . It is then generally considered that these utility classes are well-ordered in terms of “risk aversion”. Though, we come back on this statement in Section 3, once we have introduced our formal approach to comparative risk aversion.

We provide two examples to illustrate that relying on different utility classes may yield different conclusions regarding the impact of risk aversion. A first example comes from

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<sup>4</sup>The extension to the cases where  $\rho = 1$  or  $\gamma = 1$  could easily be considered, but is ruled out here to avoid the systematic discussion of these particular cases.

the literature on precautionary savings. In a two-period consumption model, precautionary saving is the optimal amount of saving when second-period income is uncertain minus savings when income risk can be fully insured. For Kihlstrom and Mirman utility functions, it is straightforward to conclude from Drèze and Modigliani [8] (at least for small risks) that precautionary savings increase with the concavity of  $k$  as long as first-period consumption is a normal good. Risk aversion would then increase precautionary saving. A similar result is obtained by Bleichrodt and Eeckhoudt [3] for Quiggin utility functions. On the contrary, Kimball and Weil [17] prove in their Proposition 7 that the amount of precautionary savings is not monotonic in  $\gamma$ , for Epstein and Zin preferences, suggesting that there is no simple relationship between risk aversion and precautionary savings.

A second example concerns savings when the rate of return is random. Kihlstrom and Mirman [15] prove that risk aversion increases or decreases optimal savings when the return on saving is uncertain, according to whether the intertemporal elasticity of substitution is smaller or greater than one. This finding is contradicted by Langlais [19], who shows that no such result holds in the Selden framework.

Both examples illustrate that the above classes of utility functions may lead to divergent conclusions regarding the impact of risk aversion on saving behavior. However, we shall see that for both problems, the role of risk aversion becomes unambiguous and particularly intuitive once a formal and general sense is given to comparative risk aversion.

### 3. Theory

#### 3.1. Common features

Many papers, including Pratt [23], Yaari [27], Kihlstrom and Mirman [15], Chew and Epstein [7], Epstein and Zin [9] define the notion of “more risk-averse than” by considering certainty equivalents and state that the fact that an agent systematically has lower certainty equivalents than another one means greater risk aversion. This approach is equivalent to Yaari [27]’s one, who compares agents in terms of risk aversion by stating that any risk increase preferred by a given individual should also be preferred by a less risk averse individual. He uses a minimalist but indisputable definition of “riskier than” where a lottery  $\ell_1$  is said to be riskier than a lottery  $\ell_2$  if and only if  $\ell_2$  is a degenerate lottery providing a given outcome with certainty. Our paper departs from Yaari [27]’s minimalist approach by stating “being more risk-averse” should mean “greater aversion to increases in risk”, and not only a greater willingness to avoid all uncertainty. We argue that there are many cases where two non-degenerate lotteries can be unambiguously compared in terms of riskiness. We account in this paper for some of these cases, which allows us to define a notion of comparative risk



aversion which is stronger than that in Yaari [27], and which leads to interesting predictions regarding the impact of risk aversion in many concrete problems.

We therefore need a definition of “being riskier than” which is not a trivial issue. As explained in Chateauneuf, Cohen, and Meilijson [6], the literature on monetary lotteries has not reached a consensus on what an increase in risk is.<sup>5</sup> They review different notions, which are shown to yield different predictions regarding the role of risk aversion. We overcome the difficulty by considering consensual risk comparisons. First, we restrict our attention to basic “heads or tails” gambles, which only have a good and a bad payoff. Risk comparison is then particularly indisputable. Second, we extend our analysis to more general lotteries allowing us to derive results with a broader scope. For sake of clarity, we name these random objects respectively *gambles* and *lotteries*.

### 3.1.1. The setting

This section sets out the common setting for both gambles and lotteries.

*State and lottery sets.* We consider an abstract space set  $X$  endowed with an ordinal preference relation  $\succeq$ . Uncertainty is represented by a probability space  $(\Omega, \mathcal{F}, Pr)$ , where  $\Omega$  is the sample space including all states of the world (it is countable in the case of heads or tails, but not for more general lotteries),  $\mathcal{F}$  is the  $\sigma$ -algebra of events, which are subsets of  $\Omega$ , and  $Pr$  is the associated probability measure. Lotteries are random variables, more precisely measurable functions from the sample space  $\Omega$  to the state space  $X$ . We denote by  $L(X)$  the set of lotteries with outcomes in  $X$ . The function  $\ell : \Omega \rightarrow X$  of  $L(X)$  is a random variable, while  $\ell(\omega) \in X$  with  $\omega \in \Omega$  represents the realization of the lottery when state  $\omega$  occurs. We denote by  $\delta_x \in L(X)$  a degenerate lottery, which pays off  $x \in X$  with certainty. At this stage, it noteworthy that we do not define the set of lotteries as the set of measures defined on the state space  $X$ , as it is often popular in the risk literature since Anscombe and Auman [1] and Fishburn [10]. However, our paper deals with the impact of comparative risk aversion in two period saving problems. Defining lotteries as random variables allows us to simplify the discussion of intuitions driving our results and notably to speak of lottery outcomes in ‘good’ and ‘bad’ states.

*Risk preferences.* We consider two agents  $A$  and  $B$  with respective preferences  $\succeq^A$  and  $\succeq^B$  over a subset  $Y$  of  $L(X)$ . This set  $Y$  may be equal to  $L(X)$  but for greater generality we

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<sup>5</sup>Chateauneuf, Cohen, and Meilijson only consider mean-preserving increases in risk, but the notions they discuss could easily be generalized to compare distributions with different means.

only assume that  $Y$  includes the set of degenerate gambles.

$$\{\delta_x | x \in X\} \subseteq Y \subseteq L(X)$$

We assume that the risk preferences  $\succeq^A$  and  $\succeq^B$  are consistent with ordinal preferences:

**Assumption 1 (Consistency with ordinal preferences).** *Preferences over gambles are consistent with ordinal preferences if:*

$$x \succeq y \Leftrightarrow \delta_x \succeq^i \delta_y \text{ for all } x, y \in X \text{ and } i = A, B$$

Agents  $A$  and  $B$  rank degenerate lotteries as ordinal preferences rank outcomes. Without reproducing the discussion in Kihlstrom and Mirman [15] and Epstein and Zin [9], we take for granted that agents are comparable in terms of risk aversion if and only if they have the same ordinal preferences.

Another natural property when considering risk preferences is *ordinal dominance*, as formalized for example in Chew and Epstein [7]. The intuition behind this property is simple: any agent should prefer a lottery providing always a better outcome than another lottery (whatever the state of the world). Formally:

**Definition 3 (Ordinal dominance).** *Preferences over gambles  $\succeq^i$  ( $i = A, B$ ) fulfill ordinal dominance when we have for any lotteries  $\ell, \ell' \in Y$ :*

- (i) *if for all  $\omega \in \Omega$ ,  $\ell(\omega) \succeq \ell'(\omega)$  then  $\ell \succeq^i \ell'$ ,*
- (ii) *moreover, if there exists  $\omega \in \Omega$  such that  $\ell(\omega) \succ \ell'(\omega)$ , then  $\ell \succ^i \ell'$ .*

According to this definition, a first-order stochastically dominated lottery should not be preferred. Moreover, a first-order stochastically dominating lottery is strictly preferred if and only if it pays off a strictly better outcome in at least one state of the world.

It can be argued that this is a reasonable requirement for defining rational risk preferences. However, as some popular preferences (such as Selden and Epstein and Zin ones) do not satisfy this property (see for example the discussion in Chew and Epstein [7]), we do not systematically make this assumption, but mention it whenever necessary.

### 3.1.2. A formal definition of comparative risk aversion

We now clarify the procedure we use to give a sense to risk-aversion comparisons, when we consider a general set of outcomes. Intuitively, as in Yaari [27], an agent  $A$  will be said to be more risk-averse than an agent  $B$ , if any increase in risk that is considered to be desirable

by  $A$  is also considered so by  $B$ . This procedure is general in the sense that it is valid in both the gamble and the lottery setups. However, the definition of an “increase in risk” is different from Yaari’s one and across these setups.

Formally, we suppose that there exists a binary relation  $R$  defined over the lottery set  $Y$ . This relation is interpreted as “riskier than” and more precisely as “at least as risky as”. For example, for  $\ell, \ell' \in Y$ ,  $\ell R \ell'$  means that the lottery  $\ell$  is (weakly) riskier than  $\ell'$ . The relationship  $R$  is supposed to be reflexive and transitive, and thus defines a partial preorder. We now set out our definition of comparative risk aversion.

**Definition 4 (Comparative risk aversion).** *Let  $R$  be a partial preorder “riskier than” defined over the lottery set  $Y$ .  $A$  is more (weakly) risk-averse than  $B$  with respect to  $R$  if for all  $\ell, \ell' \in Y$ :*

$$\ell R \ell' \text{ and } \ell \succeq^A \ell' \implies \ell \succeq^B \ell'$$

This definition states that any riskier lottery, which is preferred by the more risk-averse agent is also preferred by the less risk-averse agent. This definition is reflexive by construction, and has of course to be completed with a reasonable notion of “riskier than”.

### 3.2. Theory, Part 1: Heads or tails gambles

We precise the previous setting, when we restrict our attention to heads or tails gambles.

#### 3.2.1. The setting

We suppose that the sample space is reduced to heads or tails:  $\Omega = \{h, l\}$  and that both states  $h$  and  $l$  occur with the same probability. We denote by  $H(X)$  the set of heads or tails gambles with outcomes in  $X$ . An element of  $H(X)$  denoted  $(x^l; x^h)$  is the lottery yielding  $x^l \in X$  with probability 0.5 and  $x^h \in X$  with probability 0.5. For sake of simplicity and without loss of generality, we always suppose that the first outcome is not better than the second:  $x^h \succeq x^l$ . Remember that we call these simple binary lotteries *gambles*, while the more general ones are called *lotteries*.

We consider two agents  $A$  and  $B$  with preferences  $\succeq^A$  and  $\succeq^B$  over gambles in  $Y \subseteq H(X)$  that are consistent with ordinal preferences (Assumption 1).

#### 3.2.2. Comparative riskiness

Definition 4 of comparative risk aversion supposes a relation  $R$  comparing the riskiness of gambles. We take advantage of the very basic structure of gambles to derive reasonable properties of the partial preorder  $R$  and see that it is not necessary to make fully explicit this relation to derive non-trivial results. We consider two gambles  $(x_l; x_h)$  and  $(y_l; y_h)$  of

the set  $Y$ . Since outcomes are ordered ( $x_h \succeq x_l$  and  $y_h \succeq y_l$ ), we are left with four possible combinations:

Case 1: ( $x_h \succ y_h$  and  $x_l \succeq y_l$ ) or ( $x_h \succeq y_h$  and  $x_l \succ y_l$ )

Case 2: ( $y_h \succ x_h$  and  $y_l \succeq x_l$ ) or ( $y_h \succeq x_h$  and  $y_l \succ x_l$ )

Case 3:  $x_h \succeq y_h \succeq y_l \succeq x_l$

Case 4:  $y_h \succeq x_h \succeq x_l \succeq y_l$

In Cases 1 and 2, one gamble strictly first-order dominates the other one. There may be diverse views about the relative riskiness of  $(x_l; x_h)$  and  $(y_l; y_h)$ , depending on the values of  $x_l, x_h, y_l$  and  $y_h$ , but this does not really matter as we expect preferences between these gambles to be guided by monotonicity properties and unrelated to risk and risk aversion.

In Cases 3 and 4, the lucky outcome of one gamble is better than the lucky outcome of the other, while the reverse is true for the unlucky outcome. We characterize such cases using the notion of *spread* that we define below:

**Definition 5 (Gamble spread).** *The gamble  $(x_l; x_h)$  is a spread of  $(y_l; y_h)$  (outcomes are ordered) , which is denoted by  $(x_l; x_h) \vdash (y_l; y_h)$ , if the following relationship holds:*

$$(x_l; x_h) \vdash (y_l; y_h) \iff x_h \succeq y_h \succeq y_l \succeq x_l$$

If  $(x_l; x_h)$  is a spread of  $(y_l; y_h)$ , choosing  $(x_l; x_h)$  instead of  $(y_l; y_h)$  involves taking the chance of being in a better position if the odds are good, but ending up in a worse situation if the odds are bad. It is then indisputable that  $(x_l; x_h)$  is riskier than  $(y_l; y_h)$ . If the ordinal preference relation  $\succeq$  is represented by a utility function,  $(x_l; x_h) \vdash (y_l; y_h)$  implies that the distribution of ex-post utilities associated with the gamble  $(x_l; x_h)$  is more dispersed than that with the gamble  $(y_l; y_h)$  in the strong sense of Bickel and Lehman [2], whatever the utility function representing  $\succeq$ . The fact that  $(x_l; x_h)$  is considered to be riskier than  $(y_l; y_h)$  whenever  $(x_l; x_h) \vdash (y_l; y_h)$  is therefore particularly robust: it is not restricted to any particular choice of ex-post utility dispersion, and is independent of the cardinality.

While there may be disagreement about the relative riskiness of  $(x_l; x_h)$  and  $(y_l; y_h)$  in Cases 1 and 2, a minimal requirement for any “riskier than” relation  $R$  is that it respects the ranking of the spread relationship  $\vdash$ . We additionally impose that if the gamble  $(x_l; x_h)$  is a spread of  $(y_l; y_h)$  in a strict sense (that is with either  $x_h \succ y_h$  or  $y_l \succ x_l$ ) then  $(y_l; y_h)$  cannot be considered to be riskier than  $(x_l; x_h)$ . We call these requirements *spread compatibility*:

**Definition 6 (Spread compatibility).** *A partial-order “riskier than”  $R$  is spread compatible if and only if:*

1.  $(x_l; x_h) \vdash (y_l; y_h) \Rightarrow (x_l; x_h)R(y_l; y_h)$
2. *If  $(x_l; x_h) \vdash (y_l; y_h)$  and  $(x_h \succ y_h$  or  $y_l \succ x_l)$ , it cannot be the case that  $(y_l; y_h)R(x_l; x_h)$ .*

The next result proves that the ordinal dominance together with the agreement over the statement that a spread involves an increase in risk are sufficient to define an unambiguous measure of comparative risk aversion for gambles. It is then not necessary to fully explicit the relation  $R$  in order to obtain a universal sense for being “more risk-averse than”.

**Result 1.** *We consider two agents  $A$  and  $B$  with preferences satisfying the ordinal dominance. If agent  $A$  is more risk-averse than agent  $B$  with respect to a spread-compatible partial order  $R$ , then it will also hold for any other spread compatible partial order  $R'$ .*

**Proof.** We consider two gambles  $(x_l; x_h)$  and  $(y_l; y_h)$ , with  $(x_l; x_h)R'(y_l; y_h)$  and  $(x_l; x_h) \succeq^A (y_l; y_h)$ . We prove that  $(x_l; x_h) \succeq^B (y_l; y_h)$ . There are at most four possibilities:

1.  $(x_l; x_h)$  strictly first-order dominates  $(y_l; y_h)$ . Due to ordinal-dominance,  $(x_l; x_h) \succeq^B (y_l; y_h)$ .
2.  $(y_l; y_h)$  strictly first-order dominates  $(x_l; x_h)$ . We rule out this possibility, because ordinal-dominance implies  $(y_l; y_h) \succ^A (x_l; x_h)$ , which contradicts  $(x_l; x_h) \succeq^A (y_l; y_h)$ .
3.  $(x_l; x_h) \vdash (y_l; y_h)$ , which implies  $(x_l; x_h)R(y_l; y_h)$ , since  $R$  is spread compatible.  $A$  being more risk-averse than  $B$  relative to  $R$ , the Definition 4 of comparative risk aversion implies  $(x_l; x_h) \succeq^B (y_l; y_h)$ ;
4.  $(y_l; y_h) \vdash (x_l; x_h)$ . The spread compatibility implies  $(y_l; y_h)R'(x_l; x_h)$ . Since we also have  $(x_l; x_h)R'(y_l; y_h)$ , it implies  $x_l \sim y_l$  and  $x_h \sim y_h$  (part (ii) of the definition of spread compatibility), and thus  $(x_l; x_h) \succeq^B (y_l; y_h)$  due to ordinal-dominance.

■

### 3.2.3. Application to standard classes of preferences over certain $\times$ uncertain consumption pairs

We now examine whether the utility classes mentioned in Section 2 are well-ordered with respect to this comparative risk aversion relation. We specify our setting to ensure compatibility with these utility classes. The set of outcomes  $X$  is the set of admissible

two-period consumption profiles. We restrict our attention to gambles, where first-period consumption is certain. The sole source of uncertainty concerns second-period consumption, which may be either low (state  $l$ ) or high (state  $h$ ) with equal probability. We call  $Y$  the set of such gambles, with certain first-period consumption. An element of  $Y$  is denoted  $(c_1, (c_2^l, c_2^h))$ , where  $c_1$  is the certain first-period consumption, while  $(c_2^l, c_2^h)$  is the gamble over second-period consumption.

The preferences associated with Kihlstrom and Mirman, and Quiggin utility functions satisfy ordinal dominance. From Result 1, every spread-compatible relation  $R$  yields identical conclusions about comparative risk aversion within these utility classes. The following result characterizes the comparative risk aversion ordering in both frameworks.

**Result 2 (Standard risk preferences and risk aversion).** *For Kihlstrom and Mirman, and Quiggin utility functions, the following characterization holds:*

1. *An agent with utility function  $U_{k_A}^{KM}$  is more risk-averse than an agent with utility function  $U_{k_B}^{KM}$  with respect to any spread-compatible relation  $R$  if and only if  $k_A$  is more concave than  $k_B$ .*
2. *An agent with utility function  $U_{\phi_A}^Q$  is more risk-averse than agent with utility function  $U_{\phi_B}^Q$  with respect to any spread-compatible relation  $R$  if and only if  $\phi^A(\frac{1}{2}) \leq \phi^B(\frac{1}{2})$ .*

**Proof.** See the Appendix. ■

The above result states that both Kihlstrom and Mirman and Quiggin preferences are well-ordered in terms of risk aversion. However, such a simple characterization does not hold for Epstein and Zin utility classes. On the contrary, we prove the following negative result:

**Proposition 1 (Epstein and Zin utility functions and risk aversion).** *Let  $A$  and  $B$  be two agents with respective utility functions  $U_{\gamma_A}^{EZ}$  and  $U_{\gamma_B}^{EZ}$ , where  $\gamma_A \neq \gamma_B$ . There exists no spread-compatible relation  $R$ , such that  $A$  is more risk-averse than  $B$  with respect to  $R$ .*

**Proof.** Assume that  $\gamma_A > \gamma_B$  and consider a spread-compatible relation  $R$ . We show that neither  $A$  is more risk-averse than  $B$ , nor is  $B$  more risk-averse than  $A$ .

**Proof that  $A$  is not more risk-averse than  $B$ .**

We construct two gambles  $G^a \vdash G^b$  (and thus  $G^a R G^b$  for every spread-compatible relation  $R$ ), such that agent  $A$  is indifferent between both gambles, and agent  $B$  strictly prefers  $G^b$  to  $G^a$ . This generates a contradiction with  $A$  being more risk-averse than  $B$ .

With  $0 < \varepsilon \ll 1$ ,  $c_a, c_b > 0$ ,  $G^a$  and  $G^b$  are defined as follows:

$$G^a = \left( c_a, \left( 3^{\frac{1}{1-\rho}}(1-\varepsilon), 3^{\frac{1}{1-\rho}}(1+\varepsilon) \right) \right)$$

$$G^b = (c_b, (1-2\varepsilon, 1+2\varepsilon))$$

where:<sup>6</sup>

$$c_a^{1-\rho} - c_b^{1-\rho} = \left[ \frac{(1-2\varepsilon)^{1-\gamma_A} + (1+2\varepsilon)^{1-\gamma_A}}{2} \right]^{\frac{1-\rho}{1-\gamma_A}} - 3 \left[ \frac{(1-\varepsilon)^{1-\gamma_A} + (1+\varepsilon)^{1-\gamma_A}}{2} \right]^{\frac{1-\rho}{1-\gamma_A}}$$

1. Agent  $A$  is indifferent between  $G^a$  and  $G^b$ .  $U_{\gamma_B}^{EZ}(G^a) = U_{\gamma_B}^{EZ}(G^b)$  directly stems from the construction of  $c_a$  and  $c_b$ .
2. The gamble  $G^a$  is a spread of  $G^b$ ,  $G^a \vdash G^b$ , if:

$$\begin{aligned} \frac{c_a^{1-\rho}}{1-\rho} + \frac{3}{1-\rho}(1-\varepsilon)^{1-\rho} &< \frac{c_b^{1-\rho}}{1-\rho} + \frac{1}{1-\rho}(1-2\varepsilon)^{1-\rho} \\ &< \frac{c_b^{1-\rho}}{1-\rho} + \frac{1}{1-\rho}(1+2\varepsilon)^{1-\rho} < \frac{c_a^{1-\rho}}{1-\rho} + \frac{3}{1-\rho}(1+\varepsilon)^{1-\rho} \end{aligned} \quad (1)$$

Using Taylor expansions to express  $c_a^{1-\rho} - c_b^{1-\rho}$ , we show in the following that the inequality (1) holds when  $0 < \varepsilon \ll 1$ . First:

$$(j = 1, 2) \quad \left[ \frac{(1-j\varepsilon)^{1-\gamma_A} + (1+j\varepsilon)^{1-\gamma_A}}{2} \right]^{\frac{1-\rho}{1-\gamma_A}} = 1 - \frac{\gamma_A(1-\rho)}{2} j^2 \varepsilon^2 + O(\varepsilon^3), \quad (2)$$

where  $O(\varepsilon^3)$  denotes a function such that  $\frac{O(\varepsilon^3)}{\varepsilon^3}$  is bounded as  $\varepsilon$  tends to zero.

Using the above first-order approximations ( $j = 1, 2$ ), the difference  $c_a^{1-\rho} - c_b^{1-\rho}$  simplifies into:

$$\frac{c_a^{1-\rho}}{1-\rho} - \frac{c_b^{1-\rho}}{1-\rho} = -\frac{2}{1-\rho} - \frac{\gamma_A}{2} \varepsilon^2 + O(\varepsilon^3) \quad (3)$$

In addition, both of the following approximations hold:

$$\frac{3}{1-\rho}(1 \pm \varepsilon)^{1-\rho} - \frac{1}{1-\rho}(1 \pm 2\varepsilon)^{1-\rho} = \frac{2}{1-\rho} \pm \varepsilon + O(\varepsilon^2) \quad (4)$$

Combining Eq. (2)–(4), we obtain that condition (1) holds for  $0 < \varepsilon \ll 1$ .  $G^a$  is then a spread of  $G^b$ , which implies that  $G^a R G^b$  since  $R$  is spread compatible.

---

<sup>6</sup>It is always possible to find a pair of first-period consumptions  $(c_a, c_b)$  that satisfy this equality, whatever the value of  $\rho$ , since the range of  $x^{1-\rho} - y^{1-\rho}$  is  $\mathbb{R}$ , when  $x$  and  $y$  cover  $\mathbb{R}^+$ .

3. Agent  $B$  strictly prefers  $G^b$  to  $G^a$ . We have:

$$U_{\gamma_B}^{EZ}(G^b) - U_{\gamma_B}^{EZ}(G^a) = \frac{c_b^{1-\rho}}{1-\rho} - \frac{c_a^{1-\rho}}{1-\rho} \\ + \frac{1}{1-\rho} \left[ \frac{(1-2\varepsilon)^{1-\gamma_B} + (1+2\varepsilon)^{1-\gamma_B}}{2} \right]^{\frac{1-\rho}{1-\gamma_B}} - \frac{3}{1-\rho} \left[ \frac{(1-\varepsilon)^{1-\gamma_B} + (1+\varepsilon)^{1-\gamma_B}}{2} \right]^{\frac{1-\rho}{1-\gamma_B}}$$

Using approximations (2) and (3) where  $\gamma_A$  is replaced by  $\gamma_B$ , we obtain:

$$U_{\gamma_B}^{EZ}(G^b) - U_{\gamma_B}^{EZ}(G^a) = \frac{1}{2}(\gamma_A - \gamma_B)\varepsilon^2 + O(\varepsilon^3) > 0 \quad \text{since } \gamma_A > \gamma_B$$

As a conclusion,  $A$  cannot be more risk-averse than  $B$  with respect to  $R$ .

**Proof that  $B$  is not more risk-averse than  $A$ .**

We consider two gambles  $H^a$  and  $H^b$ , with  $c > 0$  and  $0 < \varepsilon \ll 1$ :

$$H^a = \left( c, \left( (1-\varepsilon)^{\frac{1}{1-\gamma_B}}, (1+\varepsilon)^{\frac{1}{1-\gamma_B}} \right) \right) \quad \text{and} \quad H^b = (c, (1, 1))$$

$H^b$  is a degenerate gamble paying off the consumption profile  $(c, 1)$  with certainty.

1. Agent  $B$  is indifferent between both gambles, since  $U_{\gamma_B}^{EZ}(H^a) = U_{\gamma_B}^{EZ}(H^b) = \frac{c^{1-\rho}}{1-\rho} + \frac{1}{1-\rho}$ .
2. The gamble  $H^a$  is obviously a spread of  $H^b$ . Thus  $H^a R H^b$  since  $R$  is spread compatible.
3. Agent  $A$  strictly prefers  $H^b$  to  $H^a$ , since we have:

$$U_{\gamma_A}^{EZ}(H^a) - U_{\gamma_A}^{EZ}(H^b) = \frac{1}{2} \frac{\gamma_B - \gamma_A}{(1-\gamma_B)^2} \varepsilon^2 + O(\varepsilon^3) < 0 \quad \text{since } \gamma_A > \gamma_B$$

As a conclusion,  $B$  cannot be more risk-averse than  $A$  with respect to  $R$ . ■

This latter proposition emphasizes that, unless we deny that a spread in a simple heads or tails gamble is an increase in risk, Epstein and Zin utility functions cannot be considered as appropriate tools for exploring the role of risk aversion. Changing the parameter  $\gamma$  in Epstein and Zin utility functions does involve changing cardinal preferences while holding ordinal preferences constant, but there is no direct relation between risk aversion and the  $\gamma$  parameter. An agent with a higher value of  $\gamma$  will exhibit greater aversion to some particular increases in risk (the second example in the proof), but also reduced aversion for some other kinds of increases in risk (the first example in the proof). Interpreting the results obtained from changes in the value of  $\gamma$  as reflecting the impact of risk aversion is therefore misleading.



Since Epstein and Zin utility functions are a particular case of Selden utility functions, Proposition 1 *a fortiori* implies that considerations about the concavity of the function  $v$  in Selden utility functions has no direct interpretation in terms of risk aversion.

The reason for which Epstein and Zin utility functions are not well ordered in terms of risk aversion is fairly intuitive. Rewrite the Epstein and Zin utility function as:

$$U_{\gamma}^{EZ}(c_1, \tilde{c}_2) = \frac{c_1^{1-\rho}}{1-\rho} + \frac{\mathbb{E}[\tilde{c}_2]^{1-\rho}}{1-\rho} \left( \mathbb{E} \left[ \left( \frac{\tilde{c}_2}{\mathbb{E}[\tilde{c}_2]} \right)^{1-\gamma} \right] \right)^{\frac{1-\rho}{1-\gamma}}$$

It is clear that a greater value of  $\gamma$  means greater relative risk aversion with respect to second-period consumption. But, there is no monotonic relation between relative risk over second-period consumption and aggregate risk over lifetime utility – with the latter being what matters for comparative risk aversion. A gamble may imply greater relative risk over second-period consumption than another, but at the same time less absolute risk over lifetime utility. This is actually the case when we compare the gambles  $G^b$  and  $G^a$  defined above. Even if the “relative” risk expressed as a share of average second-period consumption is larger in  $G^b$  than in  $G^a$ , the (absolute) risk embedded in  $G^a$  is greater than that in  $G^b$ . Agent  $B$  with  $\gamma_B < \gamma_A$  prefers lottery  $G^b$  with the greatest second-period “relative risk” while were he to be less risk-averse he should prefer lottery  $G^a$  with less aggregate risk.

The divergence between our conclusions and those of Epstein and Zin stems from that, according to our approach, the risk that matters to individuals is life-time risk, and not the risk over second period consumption. The difference becomes of course of importance when looking at behaviors that impact individual’s well being at different periods of time, like saving behaviors.

### 3.3. Theory, Part 2: General Lotteries

In the remainder of the paper, we consider the general case of lotteries. The cost for this generalization is that there are now many possible definitions for an increase in risk, implying different meanings for being “more risk-averse than”. We argue that this difficulty should be acknowledged, rather than ignored. Different definitions of what is an increase in risk provide different notions of comparative risk aversion. However, we will show that the consideration of simple spreads (which are just a generalization of the spread relation introduced above for gambles) allows us to derive a general model-free result that makes it possible to derive unambiguous conclusions regarding the impact of risk aversion in a wide variety of frameworks.

### 3.3.1. The setting

The setting is very similar to that initially described in Section 3.1.1, the only difference being that we no longer restrict the sample space  $\Omega$ , which is a priori uncountable, nor the probability  $Pr$ . The set  $X$  is endowed with a preference relation  $\succeq$ , and  $L(X)$  is the set of lotteries, defined over  $\Omega$  and paying off in  $X$ . As we only consider risks with well specified probabilities, there is no loss of generality to restrict to a canonical probability space, such that  $\Omega = [0, 1]$ ,  $\mathcal{F}$  is the Borel  $\sigma$ -algebra of subsets in  $[0, 1]$ , and  $Pr$  is the Lebesgue measure.

For simplicity's sake, we suppose that the ordinal preference relationship  $\succeq$  over  $X$  can be represented by a function  $U : X \rightarrow \mathbb{R}$ . We shall however insist on the fact that the results we derive do not depend on a particular utility representation. Any utility representation of  $\succeq$ , based on a different utility function would yield the same conclusions. The cumulative distribution function for a lottery  $\ell \in L(X)$  is denoted  $F_\ell$  and is defined over  $\mathbb{R}$ . For any real number  $u$ ,  $F_\ell(u)$  is simply the probability ( $Pr$  defined over the probability space) that the utility of the lottery realization (whose value is in  $X$ ) is smaller than a given  $u$ :

$$\forall u \in \mathbb{R}, \quad F_\ell(u) = Pr\{U(\ell(\omega)) \leq u | \omega \in \Omega\}$$

A lottery  $\ell$  will be said to first-order dominate a lottery  $\ell'$  if and only if  $F_\ell(u) \leq F_{\ell'}(u)$  for all  $u$ , and to strictly first-order dominate  $\ell'$ , if there additionally exists  $v$  such that  $F_\ell(v) < F_{\ell'}(v)$ . It is clear that this notion of dominance is independent of the utility function that is chosen to represent the preference relation  $\succeq$ .

The preferences of agents  $A$  and  $B \succeq^i$  ( $i = A, B$ ) over a subset  $Y \subseteq L(X)$  of lotteries are compatible with ordinal preferences (Assumption 1) and ordinal dominance (Property 3).

### 3.3.2. Comparative riskiness

In order to apply the general procedure for comparing risk aversion (Definition 4), we need a notion of “riskier than” that is valid for lotteries. We generalize the notion of spread introduced in Definition 5 as follows:

**Definition 7 ( $p$ -Spread).** *Given a scalar  $p \in ]0, 1[$ , a lottery  $\ell$  is said to be a  $p$ -spread of the lottery  $\ell'$  that we denote by  $\ell \vdash_p \ell'$ , if there exists  $u_0 \in \mathbb{R}$  such that:*

1. for all  $u < u_0$ ,  $p \geq F_\ell(u) \geq F_{\ell'}(u)$ ,
2. for all  $u \geq u_0$ ,  $p \leq F_\ell(u) \leq F_{\ell'}(u)$ .

This definition generalizes the previous notion of gamble spread. In particular, if  $G_x$  a gamble spread of  $G_y$  according to Definition 5,  $G_x$  is also a  $\frac{1}{2}$ -spread of  $G_y$  according to

Definition 7. This definition also means that  $F_\ell$  single-crosses  $F_{\ell'}$ , with the crossing occurring at the  $y$ -value of  $p$ . In Figure 1, lottery  $\ell$  is a  $p$ -spread of lottery  $\ell'$ .

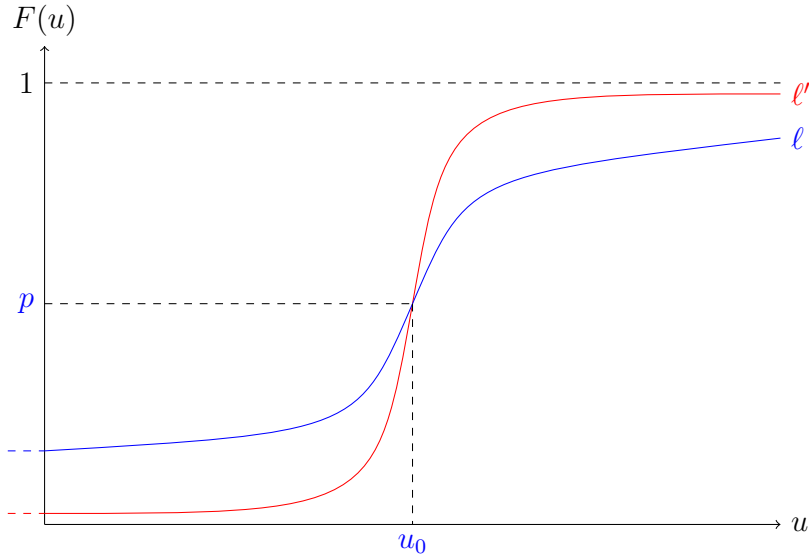


Figure 1:  $p$ -spread  $\ell \vdash_p \ell'$

It is worth noting that the above definition does not depend on the choice of the representation  $U$  of preferences, but only on ordinal preferences.<sup>7</sup> If a lottery  $\ell$  is a utility spread of another lottery  $\ell'$  for a given utility representation  $U$ , then it will also be so for any representation corresponding to the same ordinal preferences. The  $p$ -spread property is therefore an ordinal and not a cardinal concept.

We can then easily check that the  $p$ -spread relation is reflexive and transitive, and thus defines a partial preorder on  $Y$ . We also argue that if a lottery  $\ell$  is a  $p$ -spread of the lottery  $\ell'$ , then  $\ell$  is riskier than  $\ell'$ . Comparing  $\ell$  to  $\ell'$ , states of the world can be split up into “bad states” with measure  $p$ , and “good states” with measure  $1 - p$ , such that: (i) the outcome of  $\ell$  or  $\ell'$  obtained in any good state of the world is preferable to that which is obtained in bad states of the world; (ii) conditional on the state being good, the lottery  $\ell$  first-order dominates the lottery  $\ell'$ , while the reverse holds when states are bad. The lotteries  $\ell$  and  $\ell'$  can be seen as the result of binary gambles (determining whether the state of the world is bad, with probability  $p$ , or good, with probability  $1 - p$ ) with the good outcome of  $\ell$  dominating the good outcome of  $\ell'$ , and the bad outcome of  $\ell$  being dominated by the bad outcome of  $\ell'$ . In this sense, it seems clear that  $\ell$  is riskier than  $\ell'$ , since it pays off more in good states and less in bad states.

<sup>7</sup>This would not be the case for other notions of dispersion, such as that suggested by Bickel and Lehman [2], or for mean-preserving spreads, second-order stochastic dominance, etc.

It is possible to define a notion of spread as  $\ell \vdash \ell'$  if and only if  $\ell \vdash_p \ell'$  for some  $p \in ]0, 1[$ . A number of papers, such as Jewitt [13] and Johnson and Myatt [14], have used such spreads or single-crossing properties as a criterion of greater dispersion. This definition has many appealing features, but is not transitive and thus does not build a risk order. As it may seem unappealing to have a notion of “riskier than”, which is not transitive, we introduce the notion of a  $p$ -spread. Results relying on assumptions valid for all  $p$  can equivalently be expressed using the spread relation  $\vdash$ . It is noteworthy that taking the transitive closure of this single-crossing property is not a good alternative to the  $p$ -spread, since the succession of two single crossings may yield something infinitely close to an increase in risk. Formally, we can have  $\ell_1 \vdash \ell_2 \vdash \ell_3$  and  $\ell_3$  very close to a lottery  $\ell_0$  such that  $\ell_0 \vdash \ell_1$ .

When considering preferences over certain  $\times$  uncertain consumption pairs, both the Kihlstrom and Mirman and Quiggin utility functions can easily be ordered in terms of aversion for  $p$ -spread increases in risk. This result extends the characterization obtained in Result 2.

**Result 3 (Comparative risk aversion and standard utility classes).** *The following results hold for Kihlstrom and Mirman and Quiggin utility functions:*

- *An agent with a utility function  $U_{k^A}^{KM}$  is more risk-averse than an agent with a utility function  $U_{k^B}^{KM}$  with respect to the  $p$ -spread relation for every  $p \in ]0, 1[$  if and only if  $k^A$  is more concave than  $k^B$ .*
- *An agent with a utility function  $U_{\phi^A}^Q$  is more risk-averse than an agent with a utility function  $U_{\phi^B}^Q$  with respect to the  $p$ -spread relation for every  $p \in ]0, 1[$  if and only if  $\phi^A$  is more convex than  $\phi^B$ .*

**Proof.** See the Appendix. ■

Regarding the Epstein and Zin class, Proposition 1 states that Epstein and Zin utility functions are not properly ranked with respect to aversion for  $1/2$ -spread increases in risk, so that there is no chance of reaching a conclusion similar to those of Result 3.

### 3.3.3. A model-free result

Having provided a formal meaning of comparative risk aversion, we are now interested in deriving results for the impact of risk aversion on agents' behaviors. We suppose that agents may chose an action  $t \in I \subseteq \mathbb{R}$ , which modifies the payoff of a lottery. Such a lottery is noted  $\ell_t \in Y$  and its realization when state  $\omega \in \Omega$  occurs is  $\ell_t(\omega) \in X$ . With minimal assumptions, which are detailed below, we prove a very general result stating that the optimal action under uncertainty covaries monotonically with risk aversion.

Our first assumption is that the action  $t$  has a true effect on lotteries.

**Assumption 2 (Non-Constant).** *Consider two actions  $t_1 \in I$  and  $t_2 \in I$ . If  $\ell_{t_1}(\omega) \sim \ell_{t_2}(\omega)$  for all  $\omega \in \Omega$ , then  $t_1 = t_2$ .*

The above assumption is obviously a necessary condition for our model-free result. In the extreme case, when  $t$  does not have any influence on the lottery  $\ell_t$ , we would obviously be silent about the impact of the risk aversion on the choice of the action.

Second, we make an assumption of single-peakedness. For each  $\omega \in \Omega$ , the application  $t \mapsto \ell_t(\omega)$  is single-peaked, which implies that in a given state of the world  $\omega$ : (i) there exists a best action  $t_\omega$  and (ii) an action is all the more preferred the closer it is to  $t_\omega$ .

**Assumption 3 (Single-Peakedness).** *For all  $\omega \in \Omega$ :*

$$(i) \quad \exists t_\omega \in \Omega \text{ such that } \forall t \in I, \ell_{t_\omega}(\omega) \succeq \ell_t(\omega)$$

$$(ii) \quad t_1 \leq t_2 \leq t_\omega \leq t_3 \leq t_4 (\in I), \Rightarrow \begin{cases} \ell_{t_\omega}(\omega) \succeq \ell_{t_2}(\omega) \succeq \ell_{t_1}(\omega) \\ \ell_{t_\omega}(\omega) \succeq \ell_{t_3}(\omega) \succeq \ell_{t_4}(\omega) \end{cases}$$

Third, we assume that actions do not modify the initial order of the lottery outcomes. In other words, for any pair of actions  $t$  and  $t'$ , the lotteries  $\ell_t$  and  $\ell_{t'}$  are comonotonic.

**Assumption 4 (Comonotonicity).** *Consider two states  $\omega_1, \omega_2 \in \Omega$ . Lottery outcomes satisfy the following implication:*

$$(\ell_t(\omega_1) \succeq \ell_t(\omega_2) \text{ for some } t \in I) \Rightarrow (\ell_{t'}(\omega_1) \succeq \ell_{t'}(\omega_2) \text{ for all } t' \in I)$$

When Assumption 4 holds, the states of the world may be ranked from good to bad, independently of agents' actions. We then write  $\omega_1 \geq \omega_2$  if  $\ell_t(\omega_1) \succeq \ell_t(\omega_2)$  for all  $t \in I$ . This assumption holds whenever it is possible to tell what constitutes good news, without knowing agents' actions. It is for example the case when considering random income, random returns, provided that we suppose that agents' well-being increases with wealth. This assumption may not however hold in other circumstances, for example when action  $t$  involves betting on a particular horse, since in this case the action determines which outcome is preferred.

The last assumption we make for practical purposes is that the sequence of optimal actions  $(t_\omega)_{\omega \in \Omega}$  is ordered according to the states of the world  $\omega$ . The better the state of the world  $\omega$ , the greater is the optimal action  $t_\omega$ .

**Assumption 5 (Action order).** For any states  $\omega_1, \omega_2 \in \Omega$ ,  $\omega_1 \geq \omega_2 \implies t_{\omega_1} \geq t_{\omega_2}$ .

This last assumption is simply technical. We can always define a bijection  $\psi : I \rightarrow I$  such that  $\psi(t)$  is well-ordered and that Assumption 5 holds.

It is now possible to formalize a general result about the role of risk aversion:

**Proposition 2 (A general model-free result).** Consider two agents  $A$  and  $B$  who have to choose an action  $t$  providing them with a lottery satisfying Assumptions 2, 3, 4, and 5. We assume that the preferences of agents  $A$  and  $B$  satisfy ordinal dominance and define the respective single optimal actions  $t^A$  and  $t^B$ . The following implication then holds:

If agent  $A$  is more risk-averse than agent  $B$  with respect to the  $p$ -spread relation for every  $p \in ]0; 1[$  then  $t^A \leq t^B$ .

**Proof.** We assume that  $t^A > t^B$ . In order to obtain a contradiction, we prove that lottery  $\ell_{t^A}$  is a  $p$ -spread of  $\ell_{t^B}$  for some  $p \in ]0; 1[$ . In this case we have that  $\ell_{t^A} \succeq^A \ell_{t^B}$  and  $\ell_{t^A} \vdash_p \ell_{t^B}$ , which would imply that  $\ell_{t^A} \succeq^B \ell_{t^B}$  because  $A$  is more risk-averse than  $B$ , contradicting the optimality of  $t^B$  for agent  $B$ . Proving that lottery  $\ell_{t^A}$  is a  $p$ -spread of  $\ell_{t^B}$  involves showing that there exists  $u_0 \in \mathbb{R}$  and  $p \in ]0; 1[$ , such that  $p \geq F_{\ell_{t^A}}(u) \geq F_{\ell_{t^B}}(u)$  for  $u < u_0$  and  $p \leq F_{\ell_{t^A}}(u) \leq F_{\ell_{t^B}}(u)$  for  $u \geq u_0$ .

We define  $\xi^-$  as the subset of  $\mathbb{R}$ , where the cdf of  $\ell_{t^B}$  is larger than that of  $\ell_{t^A}$ . Conversely,  $\xi^+$  is the subset, where the cdf of  $\ell_{t^A}$  is larger than that of  $\ell_{t^B}$ .

$$\xi^- = \{u \in \mathbb{R}, F_{\ell_{t^B}}(u) \geq F_{\ell_{t^A}}(u)\} \quad \text{and} \quad \xi^+ = \{u \in \mathbb{R}, F_{\ell_{t^A}}(u) \geq F_{\ell_{t^B}}(u)\}$$

First, note that each  $u \in \mathbb{R}$  belongs either to  $\xi^+$  or  $\xi^-$ :  $\xi^+ \cup \xi^- = \mathbb{R}$ . We then distinguish four cases, depending on whether the sets  $\xi^+$  and  $\xi^-$  are included in each other or not.

1. Suppose that  $\xi^+ = \xi^- = \mathbb{R}$ . This means that for all  $u \in \mathbb{R}$ ,  $F_{\ell_{t^A}}(u) = F_{\ell_{t^B}}(u)$ , which implies that lotteries pay off the same outcomes in all states of the world. Assumption 2 implies that  $t^A = t^B$ , which contradicts the assumption that  $t^B < t^A$ .
2. Suppose that  $\xi^+ \subsetneq \xi^-$  (this means that  $\xi^+$  is either empty or contains only elements  $u$  such that  $F_{\ell_{t^A}}(u) = F_{\ell_{t^B}}(u)$ ). The cdf of lottery  $\ell_{t^B}$  is always larger than that of  $\ell_{t^A}$ , and is strictly larger at least once:  $\ell_{t^A}$  strictly first-order dominates the lottery  $\ell_{t^B}$ . Since preferences satisfy ordinal dominance (Property 3), agent  $B$  strictly prefers  $\ell_{t^A}$  to  $\ell_{t^B}$ , which contradicts the optimality of  $t^B$ .
3. Suppose that  $\xi^- \subsetneq \xi^+$ . It analogously contradicts the optimality of  $t^A$  for agent  $A$ .

4. We now necessarily have  $\xi^- \not\subseteq \xi^+$  and  $\xi^+ \not\subseteq \xi^-$ . There exists at least one element in each set, not belonging to the other one, which we denote  $u^+ \in \xi^+$  (and  $u^+ \notin \xi^-$ ) and  $u^- \in \xi^-$  (and  $u^- \notin \xi^+$ ).

We first focus on  $u^-$ . By definition,  $1 - F_{\ell_{t^B}}(u^-) < 1 - F_{\ell_{t^A}}(u^-)$ , or equivalently  $\{\omega \in \Omega | U(\ell_{t^B}(\omega)) \geq u^-\} \subsetneq \{\omega \in \Omega | U(\ell_{t^A}(\omega)) \geq u^-\}$ . There exists  $\omega_1 \in \Omega$  in the second set but not in the first:  $U(\ell_{t^B}(\omega_1)) \leq u^- \leq U(\ell_{t^A}(\omega_1))$ . Single-peakedness (Assumption 3) implies that there exists  $t_{\omega_1}$  such that:  $t_{\omega_1} \geq t^A \geq t^B$ .

We consider  $u \geq u^-$  and want to show that  $\{\omega \in \Omega | U(\ell_{t^B}(\omega)) \geq u\} \subseteq \{\omega \in \Omega | U(\ell_{t^A}(\omega)) \geq u\}$ . Let  $\omega^u \in \{\omega \in \Omega | U(\ell_{t^B}(\omega)) \geq u\}$ . Since  $U(\ell_{t^B}(\omega^u)) \geq u \geq u^- \geq U(\ell_{t^B}(\omega_1))$ , we deduce, from Assumption 4 of comonotonicity, that  $\omega^u \geq \omega_1$ . From Assumption 5, we deduce that  $t_{\omega^u} \geq t_{\omega_1} \geq t^A > t^B$ . Single-peakedness allows us to conclude that  $U(\ell_{t^A}(\omega^u)) \geq U(\ell_{t^B}(\omega^u)) \geq u$  and  $\omega^u \in \{\omega \in \Omega | U(\ell_{t^A}(\omega)) \geq u\}$ .

We have therefore proved that  $[u^-, +\infty[ \subseteq \xi^-$ . We can show analogously that  $] -\infty, u^+] \subseteq \xi^+$ .  $u^+$  (resp.  $u^-$ ) is a lower (resp. upper) bound for  $\xi^-$  (resp.  $\xi^+$ ) (otherwise  $u^+ \in \xi^-$ , which is contradictory). We thus define  $\bar{u} = \inf \xi^-$  and  $\underline{u} = \sup \xi^+$ , which satisfy  $\underline{u} \leq \bar{u}$  (otherwise  $\xi^+ \cup \xi^- \neq \mathbb{R}$ ). We define  $u_0$  as an element of the non-empty segment  $[\underline{u}; \bar{u}]$  and  $p$  as an element of  $[\lim_{u \rightarrow \bar{u}, u < \bar{u}} F_{\ell_{t^A}}(u); \lim_{u \rightarrow \underline{u}, u > \underline{u}} F_{\ell_{t^A}}(u)]$  (cdf are right-continuous and have left limits, both everywhere, and the last segment collapses to a singleton when  $F_{\ell_{t^A}}$  is continuous or when  $\underline{u} < \bar{u}$ ). The cdf  $F_{\ell_{t^A}}$  and  $F_{\ell_{t^B}}$  satisfy:

$$\forall u < u_0, F_{\ell_{t^B}}(u) \leq F_{\ell_{t^A}}(u) \leq p \quad \text{and} \quad \forall u \geq u_0, F_{\ell_{t^B}}(u) \geq F_{\ell_{t^A}}(u) \geq p$$

According to Definition 7, this therefore shows that lottery  $\ell_{t^A}$  is a  $p$ -spread of  $\ell_{t^B}$ , which terminates the proof.

■

This model-free result shows that, under some mild assumptions, the more risk-averse is the agent, the smaller is his optimal action. We can summarize the intuition as follows. Consider the optimal action  $t^B$ . We can group the states of the worlds into two subsets. The first consists of the states  $\omega$  for which the optimal actions  $t_\omega$  are smaller than  $t^B$ , while the second consists of the states for which the optimal actions are larger than  $t^B$ . Since without uncertainty optimal actions are assumed to be larger when the state of the world is better, we can qualify the former as “bad” states of the world and the latter as “good” states. Due to single-peakedness, choosing an action  $t$  smaller than  $t^B$  involves increasing the agent’s welfare in bad states and reducing it in good states. Opting for a smaller action is thus one way of redistributing welfare from good to bad states, and a way of reducing risk regarding agent welfare. Such a strategy is preferred by more risk-averse agents.

## 4. Applications

We use Proposition 2 to analyze in a very simple two-period framework the savings behavior of an agent facing uncertainty. We consider in turns three types of uncertainty: (i) second-period income is random; (ii) the savings interest rate is uncertain; and (iii) the agent faces a mortality risk, i.e. a risk of dying at the end of the first period. For simplicity, we simply write in this section “agent  $A$  is more risk averse than  $B$ ” to actually mean that “ $A$  is more risk-averse than agent  $B$  with respect to the  $p$ -spread relation for every  $p \in ]0, 1[$ ”.

### 4.1. Application to precautionary savings

We consider the case of agents who live for two periods, have random second-period incomes, and have to decide how much to save. This very simple problem has been the object of number of inspirational contributions, including Leland [20], Sandmo [25], Drèze and Modigliani [8], Caperaa and Eeckhoudt [5], Kimball [16], and Kimball and Weil [17]. These led to the development of the notion of prudence, whose link to risk aversion has not been clarified despite some impressive efforts (Kimball and Weil, [17]). We will see, however, that our general approach does lead to clear and simple conclusions.

To apply our general result, we specify the setting as in Section 3.2.3. The set  $X = (\mathbb{R}^+)^2$  is endowed with an ordinal preference relationship  $\succeq$  represented by a utility function  $u$ . The set of lotteries with outcomes in  $X$  is denoted  $L(X)$ , and  $Y \subseteq L(X)$  is the set of lotteries with deterministic first-period consumption. We consider two agents  $A$  and  $B$  with preferences  $\succeq^i$  ( $i = A, B$ ) defined over  $Y$ . We assume that these preferences satisfy the consistency assumption and ordinal dominance.

We now introduce two assumptions regarding ordinal preferences.

**Assumption 6 (Convexity of ordinal preferences).** *For all  $(c_1, c_2)$ ,  $(c'_1, c'_2)$ , and  $(c''_1, c''_2)$  in  $X$ , and all  $\lambda \in [0, 1]$ :*

$$(c'_1, c'_2) \succeq (c_1, c_2) \text{ and } (c''_1, c''_2) \succeq (c_1, c_2) \implies (\lambda c'_1 + (1 - \lambda)c''_1, \lambda c'_2 + (1 - \lambda)c''_2) \succeq (c_1, c_2)$$

**Assumption 7 (Normality of first-period consumption).** *Consider the agent's optimization problem  $\max_{c_1, c_2} u(c_1, c_2)$  subject to the budget constraint  $c_1 + \frac{1}{(1+R)}c_2 = I$ , where  $I \geq 0$  is the discounted total certain income and  $R > -1$  the gross certain interest rate. The ordinal preference relationship  $\succeq$  is such that this problem has a unique solution denoted  $(c_1(I, R), c_2(I, R))$ , where, additionally, first-period consumption  $c_1(I, R)$  increases with total income  $I$ .*



The assumption of preference convexity is fairly standard in the analysis of consumer behavior.<sup>8</sup> This implies Assumption 3 of single-peakedness which is required for our model-free result.<sup>9</sup> Assumption 7 of good normality is also very standard. In cases where the preference relation  $\succeq$  can be represented by a differentiable utility function  $u(c_1, c_2)$  over first-period  $c_1$  and second-period  $c_2$  consumptions, this assumption concerns the derivative of the marginal rate of substitution between consumption in both periods relative to second-period consumption (namely  $\frac{\partial}{\partial c_2} \left( \frac{\frac{\partial u}{\partial c_1}}{\frac{\partial u}{\partial c_2}} \right) > 0$ ). However, we believe that greater insight is gained by emphasizing that the requirement is good normality.

We can now express our finding with respect to precautionary savings:

**Proposition 3 (Precautionary savings).** *Consider two agents A and B, who choose first-period consumption  $c_1$  providing them with a certain  $\times$  uncertain income profile denoted  $(c_1, \tilde{y}_2 + (1 + R)(y_1 - c_1))$ , where  $y_1 > 0$  is certain first-period income,  $\tilde{y}_2$  random second-period income, and  $R > -1$  the certain interest rate. If:*

1. *The ordinal preference relationship  $\succeq$  satisfies Assumptions 6 and 7.*
2. *Risk preferences  $\succeq^A$  and  $\succeq^B$  satisfy ordinal dominance and define optimal first-period consumption levels of  $c_1^A$  and  $c_1^B$ .*

*Then, the following implication holds:*

$$\text{Agent A is more risk-averse than agent B} \implies c_1^A \leq c_1^B$$

**Proof.** This proposition comes via an application of the model-free result formulated in Proposition 2. We simply need to check that the required assumptions hold in this setting, when the action chosen by the agent is the first-period consumption  $c_1$ .

- Assumptions 2 and 4 hold by construction.
- The normality of first-period consumption (Assumption 7) ensures that Assumption 5 regarding optimal-action ordering holds. Indeed, the better the state of the world (i.e., the larger is second-period income), the greater is optimal first-period consumption.

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<sup>8</sup>See for example the discussion in Mas-Colell, Whinston and Green [21], page 44.

<sup>9</sup>This is formally shown below, in the proof of Proposition 3. In fact, single-peakedness and convexity are equivalent in the case of continuous preferences that we do not assume here.

- The convexity of the relation  $\succeq$  implies the single peakedness of preferences. Let  $s^*$  be the solution of  $\max_s u(y_1 - s, y_2 + (1 + R)s)$  and consider, for example,  $s' < s'' < s^*$ . By the definition of  $s^*$ , we have first that  $(y_1 - s^*, y_2 + (1 + R)s^*) \succeq (y_1 - s', y_2 + (1 + R)s')$  and also  $(y_1 - s', y_2 + (1 + R)s') \succeq (y_1 - s'', y_2 + (1 + R)s'')$ . Convexity then implies that for all  $\lambda \in [0, 1]$  we can deduce  $(y_1 - (\lambda s^* + (1 - \lambda) s'), y_2 + (1 + R)(\lambda s^* + (1 - \lambda) s')) \succeq (y_1 - s', y_2 + (1 + R)s')$ . As  $s' < s'' < s^*$ , we can choose  $\lambda \in [0, 1]$ , such that  $s'' = \lambda s^* + (1 - \lambda) s'$ , which proves single-peakedness.

■

This proposition makes it clear that the greater is risk aversion, the more the agent saves. The intuition behind this result is very simple. Take an agent who decides to save  $s(\tilde{y}_2)$  anticipating a random second-period income of  $\tilde{y}_2$ . For simplicity, we assume that this random income can take two values,  $\underline{y}_2$  and  $\bar{y}_2$ . The amount  $s(\tilde{y}_2)$  is an intermediate value between what he would have saved knowing that he would receive  $\bar{y}_2$  and what he would have saved knowing that he would earn  $\underline{y}_2$ :  $s(\bar{y}_2) < s(\tilde{y}_2) < s(\underline{y}_2)$ . By saving more than  $s(\tilde{y}_2)$  he increases his welfare in the bad state of the world, but reduces it in the good state. As this diminishes the degree of risk regarding his welfare, larger savings will therefore be preferred by more risk-averse agents.

This result clarifies the link between prudence and risk aversion. Agents may be prudent or imprudent in the sense that they may react positively or negatively to an increase in income uncertainty. Drèze and Modigliani [8] and Kimball [16] have established the conditions for prudence to occur in the expected utility framework. Our results complement their findings by showing that, for a given level of income uncertainty, increasing risk aversion leads to increased savings.

#### 4.2. Application to optimal savings with interest-rate uncertainty

We now raise the question of the relationship between optimal savings and risk aversion, not in the face of income uncertainty, but rather interest-rate uncertainty. This question was addressed by Kihlstrom and Mirman [15] in the expected-utility framework, and Langlais [19] for Selden utility functions, with diverging conclusions.

The formal setting of this question (the structure of  $X$ , etc.) is exactly the same as in the previous section. However, the ordinal properties that are required to obtain results regarding risk aversion are different.

In a deterministic setting, increasing the interest rate is equivalent to changing the price of second-period consumption, generating both income and substitution effects. A higher interest rate means a lower price for second-period consumption, with a positive income effect yielding higher first-period consumption and lower savings. The substitution effect

reduces first-period consumption, and therefore increases savings. The income and substitution effects thus have opposing effects on optimal savings, and the overall effect may be either positive or negative. For the sake of clarity, we define the optimal savings function  $s(y_1, y_2, R) = \arg \max_s (y_1 - s, y_2 + (1 + R)s)$ . This function may either rise or fall with respect to  $R$ , depending on ordinal preferences. This sign is key for the determination of the effect of risk aversion on savings when interest rates are non-deterministic.

**Proposition 4 (Savings with an uncertain interest rate).** *Consider two agents A and B, who choose first-period consumption  $c_1$  providing them with a certain  $\times$  uncertain income profile  $(c_1, y_2 + (1 + \tilde{R})(y_1 - c_1))$ , where  $y_1 > 0$  ( $y_2 > 0$ ) is certain first- (second-) period income, and  $\tilde{R}$  is the random interest rate. If:*

1. *The ordinal preference relationship  $\succeq$  satisfies Assumption 6.*
2. *Risk preferences  $\succeq^A$  and  $\succeq^B$  satisfy the ordinal dominance and define optimal first-period consumptions  $c_1^A$  and  $c_1^B$ .*

*Then, the following implication holds:*

$$\text{Agent A is more risk-averse than agent B} \implies \begin{cases} c_1^A \leq c_1^B \text{ if } R \mapsto s(y_1, y_2, R) \text{ is decreasing} \\ \text{or} \\ c_1^B \leq c_1^A \text{ if } R \mapsto s(y_1, y_2, R) \text{ is increasing} \end{cases}$$

**Proof.** The proof is straightforward and is very similar to that in Proposition 3. When the substitution effect dominates, in order to directly use the result of Proposition 2, we may consider that the action is not  $c_1$ , but rather  $s = y_1 - c_1$ . ■

Our findings extend those in Kihlstrom and Mirman [15]. In an expected-utility framework, with differentiable utility functions, they derive a similar result as in Proposition 4. They express that the derivative  $\frac{\partial s}{\partial R}$  is positive (negative) if the intertemporal elasticity of substitution is greater (less) than one. The preceding proposition could then be expressed by referring to the value of the intertemporal elasticity of substitution, although this would be slightly less general (as differentiability would then be required). Our results however contradict those in Langlais [19], who considers Selden utility functions. Our explanation is that these latter utility functions are not well-ordered in terms of risk aversion.

#### 4.3. Application to optimal savings with lifetime uncertainty

In our last application, we consider the effect of an uncertain lifetime on optimal savings. The traditional view in Economics is that risk aversion and time preference are orthogonal

aspects of preferences. However, Bommier [4] has underlined that as soon as we take lifetime uncertainty into account, there is a strong direct relationship between risk aversion and time discounting, with significant implications for savings behavior.<sup>10</sup> Bommier’s results were however derived in an expected-utility framework, omitting some aspects of preferences such as bequests. Here we show how the impact of risk aversion on savings with lifetime uncertainty can be addressed without assuming expected utility but allowing for bequests.

We consider an agent who has an initial endowment  $W$  and who may live for one or two time periods. This agent chooses his consumption  $c_1$  in the first period. In the second period, either he survives and consumes his wealth, or dies, and his wealth is transmitted to his heirs, for whom he may care. To account for the potential existence of annuities, we assume that the return to saving may depend on whether the agent survives or not. More precisely, if we denote by  $c_2^a$  the second-period consumption in the case where the agent is alive, and by  $c_2^d$  the amount transmitted to his heirs if he dies, we have:  $c_2^a = (1 + R_a)(W - c_1)$  and  $c_2^d = (1 + R_d)(W - c_1)$ , where  $R_a$  and  $R_d$  are the savings returns in the case of survival and death respectively. We assume that  $R_a > -1$  and  $R_d \geq -1$ , but make no assumptions about the relative values of  $R_a$  and  $R_d$ . When there are no annuities or taxes on bequests we have  $R_a = R_d$ , while with perfect annuities we have  $R_a > R_d = -1$ . There are also many intermediary situations (e.g. when there are taxes on bequests) or contracts (life insurance) with  $R_d$  greater than  $R_a$ .

We now apply our model-free result to show that risk aversion generates an unambiguous result. Formally, the set  $X$  has to be defined to reflect the specificity of the context. The second-period outcome can no longer be described by a scalar variable  $c_2$ , but by a pair  $(c_2, \sigma)$  of a scalar  $c_2$  and a binary variable  $\sigma \in \{a, d\}$  indicating whether the individual is dead or alive. This means that  $X = (\mathbb{R}^+)^2 \times \{a, d\}$ . The notation  $(c_1, c_2)_a$  and  $(c_1, c_2)_d$  will however be used instead of the cumbersome  $(c_1, c_2, a)$  and  $(c_1, c_2, d)$ . The index  $a$  or  $d$  thus indicates whether the individual is alive or not in the second period.

We make three assumptions regarding ordinal preferences. First, we suppose that the agent always prefers to live in the second period and to consume, rather than to die and bequeath his wealth. Second, we suppose that optimal saving conditional on living for two periods is greater than optimal saving conditional on living one period. In other words, in a deterministic setting, the propensity to consume falls with life duration. This seems a very natural assumption in this setting, where agents have no second-period income. Last, we introduce a convexity assumption similar to Assumption 6, but taking into account that  $X$

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<sup>10</sup>See also Bommier’s working paper “Rational Impatience?”, downloadable at <http://hal.archives-ouvertes.fr/hal-00441880/fr/> where the relation between risk aversion and time discounting is formalized in Proposition 4.

is convex, and not even connected.

**Assumption 8.** *Ordinal preferences satisfy the following properties:*

- *The agent is always better off when alive:  $\forall c_1, (c_1, (1 + R_a)(W - c_1))_a \succeq (c_1, (1 + R_d)(W - c_1))_d$*
- *For  $i = a, d$ , all  $W > 0$  and  $R_a, R_d \geq -1$ , the problem  $\max_s u(W - s, (1 + R_i)s)_i$  has a unique solution  $s_i$ .*
- *Optimal saving when surviving is always greater than that when dying:  $s_a > s_d$ .*
- *Ordinal preferences are convex over both  $(\mathbb{R}^+)^2 \times \{a\}$  and  $(\mathbb{R}^+)^2 \times \{d\}$ .*

Given Assumption 8, Proposition 2 allows us to determine how the savings of an agent facing an uncertain lifetime depend on risk aversion.

**Proposition 5 (Saving when lifetime is uncertain).** *We consider two agents  $A$  and  $B$ , who face an (identical) exogenous risk of dying after the first period. They have to choose a saving level of  $s$  providing them with a consumption profile of  $(W - s, (1 + R_a)s)_a$  if they survive and a consumption-bequest profile of  $(W - s, (1 + R_d)s)_d$  if they die. If:*

1. *The ordinal preference relationship  $\succeq$  satisfies Assumption 8.*
2. *Risk preferences  $\succeq^A$  and  $\succeq^B$  satisfy the ordinal dominance and define optimal savings  $s^A$  and  $s^B$ .*

*Then the following implication holds:*

$$\text{Agent } A \text{ is more risk-averse than agent } B \implies s^A \leq s^B$$

**Proof.** Assumptions 2 and 3 hold by construction of the consumption profile and the convexity of preferences. Assumptions 4 and 5 directly stem from Assumption 8. The result is then straightforward from Proposition 2. ■

The more risk-averse agent saves less. In other words, when mortality is taken into account, there is a positive relationship between risk aversion and impatience. We shall however emphasize that Proposition 5 assumes that agents  $A$  and  $B$  have the same probability of dying. The relation between risk aversion and impatience holds when comparing

agents with identical mortality, but can not be applied to any correlations obtained from individuals with different mortality risks.<sup>11</sup>

## 5. Conclusion

The most common approach to quantifying (and comparing) agents' risk aversion involves focusing on how individuals compare lotteries with certain outcomes. In this paper we argue that this is not sufficient, as risk aversion also reflects agents' willingness to marginally reduce risks. We therefore consider a formal procedure to compare agents' aversion to (marginal) increases in risk. This procedure can be applied to many settings, since it does not presuppose any kind of structure for the set of consequences. Moreover, it makes it possible to derive general predictions about the impact of risk aversion in a wide variety of problems, as illustrated with our "model free" result.

To demonstrate the interest of our general approach we apply it to three intertemporal problems, where the role of risk aversion is not well understood. We first identify what are the relevant classes of utility functions to study the role of risk aversion in intertemporal settings. Interestingly enough, we find that relying on Epstein and Zin utility functions is inadequate, since these functions are not well ordered in terms of risk aversion. Kihlstrom and Mirman preferences, or those arising from Quiggin's anticipated utility theory are better alternatives. Though, in a number of cases, predictions on the role of risk aversion are easier to obtain through our general approach than when relying on specific models of preferences. In particular we clarify the link between risk aversion and prudence, showing that precautionary savings increase with risk aversion. We also prove that when considering lifetime uncertainty, greater risk aversion should lead to lower savings.

Our approach could also be relevant for a number of problems that we could not consider in a single paper. An interesting possibility, which we leave for further contributions, is to consider  $X$  as being a set of lotteries and discuss preferences over two-stage lotteries as it is done in many applied papers on ambiguity aversion. We would then obtain model-free results on the impact of ambiguity aversion.

## Appendix A. Proofs

### *Appendix A.1. Proof of Result 2*

We consider each utility class in turn.

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<sup>11</sup>In particular, the fact that women might seem to be more patient and more risk-averse than men should not be interpreted as contradicting this proposition. This could in fact follow from gender differences in mortality.

1. Kihlstrom and Mirman utility functions.

We define  $k_A = k \circ k_B$  where  $k$  is increasing and continuous. The utility associated to a gamble  $(x_l; x_h)$  for agent  $A$  is:  $U_{k_A}^{KM}(x_l; x_h) = k_A^{-1} \left( \frac{k_A(x_l) + k_A(x_h)}{2} \right) = k_A^{-1} \left( \frac{k(k_B(x_l)) + k(k_B(x_h))}{2} \right)$ .

- We assume that  $k$  is concave. Let  $(x_l; x_h)$  and  $(y_l; y_h)$  be two gambles. By definition:

$$(x_l; x_h) \succeq_A (y_l; y_h) \text{ iff: } \frac{k(k_B(x_l)) + k(k_B(x_h))}{2} \geq \frac{k(k_B(y_l)) + k(k_B(y_h))}{2} \quad (\text{A.1})$$

Since  $x_l < y_l \leq y_h < x_h$  and  $k_B$  is increasing, the inequality in (A.1) becomes:

$$\frac{k(k_B(x_h)) - k(k_B(y_h))}{k_B(x_h) - k_B(y_h)} \geq \frac{k(k_B(y_l)) - k(k_B(x_l))}{k_B(y_l) - k_B(x_l)} \frac{k_B(y_l) - k_B(x_l)}{k_B(x_h) - k_B(y_h)}$$

(if  $k_B(x_h) = k_B(y_h)$  or  $k_B(x_l) = k_B(y_l)$ , the result is straightforward)

$k$  is concave, which implies that  $0 \leq \frac{k(k_B(x_h)) - k(k_B(y_h))}{k_B(x_h) - k_B(y_h)} \leq \frac{k(k_B(y_l)) - k(k_B(x_l))}{k_B(y_l) - k_B(x_l)}$ . We deduce from both previous inequalities that:

$$1 \geq \frac{k_B(y_l) - k_B(x_l)}{k_B(x_h) - k_B(y_h)} \quad \text{or} \quad \frac{k_B(x_l) + k_B(x_h)}{2} \geq \frac{k_B(y_l) + k_B(y_h)}{2} \quad (\text{A.2})$$

which implies that by similarity with (A.1) that  $B$  prefers  $(x_l; x_h)$  to  $(y_l; y_h)$ .

- We now assume that  $A$  is more risk-averse than  $B$ , and that the inequality in (A.1) and  $(x_l; x_h) \vdash (y_l; y_h)$  imply the inequality in (A.2). We choose a level of income  $y = y_l = y_h$  such that the inequality in (A.1) holds with equality for a given pair  $(x_l, x_h)$ . The inequality in (A.2) then implies that  $k\left(\frac{k_B(x_l) + k_B(x_h)}{2}\right) \geq k(k_B(y)) = \frac{k(k_B(x_l)) + k(k_B(x_h))}{2}$ , or that  $k$  is concave.<sup>12</sup>

2. Quiggin anticipated utility function.

We consider two gambles  $(x_l; x_h) \vdash (y_l; y_h)$  ( $x_l < y_l \leq y_h < x_h$ ; the result is straightforward if there is equality), such that agent  $A$  prefers  $(x_l; x_h)$  to  $(y_l; y_h)$ . It means  $U_{\phi^A}^Q(x_l; x_h) \geq U_{\phi^A}^Q(y_l; y_h)$  or  $\phi^A(\frac{1}{2}) \geq \frac{y_l - x_l}{x_h - y_h + y_l - x_l}$ , since the utility associated with the gamble  $(x_l; x_h)$  for  $A$  is  $U_{\phi^A}^Q(x_l; x_h) = x_l + (x_h - x_l)\phi^A(1/2)$ .

It is then straightforward that agent  $B$  prefers  $(x_l; x_h)$  to  $(y_l; y_h)$  iff  $\phi^B(\frac{1}{2}) \geq \phi^A(\frac{1}{2})$ .

*Appendix A.2. Proof of Result 3*

We prove the result for each utility class.

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<sup>12</sup>A continuous function  $f$  is concave iff for all  $x_1, x_2$ ,  $f\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f(x_1) + f(x_2)}{2}$ .

1. Kihlstrom and Mirman utility functions.

1. a. The first implication directly stems from our previous Result 2.

1. b. Suppose now that  $k^A$  is more concave than  $k^B$ , i.e. that  $k^A = k \circ k^B$ , with  $k$  continuous, increasing, and concave. We consider two lotteries  $\ell_1$  and  $\ell_2$ , such that:  $\ell_1 \vdash_p \ell_2$  for some  $p \in ]0, 1[$  and  $\ell_1 \succeq^A \ell_2$ . As agent  $A$  prefers  $\ell_1$  to  $\ell_2$ , we have:

$$\int_{-\infty}^{\infty} k(k^B(u)) dF_{\ell_1}(u) \geq \int_{-\infty}^{\infty} k(k^B(u)) dF_{\ell_2}(u) \quad (\text{A.3})$$

Since  $\ell_1 \vdash_p \ell_2$  there exists  $u_0 \in \mathbb{R}$ , such that  $p \geq F_{\ell_1}(u) \geq F_{\ell_2}(u)$  for  $u < u_0$  and  $p \leq F_{\ell_1}(u) \leq F_{\ell_2}(u)$  for  $u \geq u_0$ . Any concave function defined over an open set admits left and right derivatives everywhere. Both are equal to each other and the function is differentiable, except on a countable set.<sup>13</sup> In consequence, we deduce that there exists a (countable) partition  $\{s_j, j \in \mathbb{Z}\}$  of  $\mathbb{R}$ , such that  $k$  and  $k^B$  are differentiable on every interval  $]s_j, s_{j+1}[$  and that  $s_0 \equiv u_0$  (for sake of simplicity). We deduce:

$$\begin{aligned} \int_{-\infty}^{\infty} k(k^B(u)) dF_{\ell_1}(u) - \int_{-\infty}^{\infty} k(k^B(u)) dF_{\ell_2}(u) &= \sum_{j=-\infty}^{\infty} \int_{s_j}^{s_{j+1}} k(k^B(u)) (dF_{\ell_1}(u) - dF_{\ell_2}(u)) \\ &= \sum_{j=-\infty}^{\infty} (k(k^B(s_{j+1})) (F_{\ell_1}(s_{j+1}) - F_{\ell_2}(s_{j+1})) - k(k^B(s_j)) (F_{\ell_1}(s_j) - F_{\ell_2}(s_j))) \\ &\quad - \sum_{j=-\infty}^{\infty} \int_{s_j}^{s_{j+1}} k^{B'}(u) k'(k^B(u)) (F_{\ell_1}(u) - F_{\ell_2}(u)) du \end{aligned} \quad (\text{A.4})$$

Since we both have  $F_{\ell_1}(-\infty) - F_{\ell_2}(-\infty) = F_{\ell_1}(\infty) - F_{\ell_2}(\infty) = 0$ , the inequality (A.3) simplifies using (A.4) into:

$$\begin{aligned} \sum_{j=0}^{\infty} \int_{s_j}^{s_{j+1}} k^{B'}(u) k'(k^B(u)) (F_{\ell_2}(u) - F_{\ell_1}(u)) du &\geq \sum_{j=-\infty}^0 \int_{s_j}^{s_{j+1}} k^{B'}(u) k'(k^B(u)) (F_{\ell_1}(u) - F_{\ell_2}(u)) \\ \text{or: } \int_{u_0}^{\infty} k^{B'}(u) k'(k^B(u)) (F_{\ell_2}(u) - F_{\ell_1}(u)) du &\geq \int_{-\infty}^{u_0} k^{B'}(u) k'(k^B(u)) (F_{\ell_1}(u) - F_{\ell_2}(u)) \end{aligned} \quad (\text{A.5})$$

Since  $k$  is increasing, concave and admits left and right derivatives everywhere, we also have for  $u \geq u_0 = s_0$ ,  $0 \leq k'(k^B(u)) \leq k'^{,r}(k^B(u_0))$  and for  $u \leq u_0 = s_0$ ,  $0 \leq k'^{,l}(k^B(u_0)) \leq k'(k^B(u))$ , where  $k'^{,r}$  (resp.  $k'^{,l}$ ) is the right (resp. left) derivative of  $k$ . Because  $F_{\ell_1}(u) -$

<sup>13</sup>This result stems for example from Theorem 1.3.7 p. 23 in Nicolescu and Persson [22]



$F_{\ell_2}(u) \geq 0$  for  $u < u_0$  and  $F_{\ell_2}(u) - F_{\ell_1}(u) \geq 0$  for  $u \geq u_0$ , we obtain after simplification:

$$k'^r(k^B(u_0)) \int_{u_0}^{\infty} k^{B'}(u)(F_{\ell_2}(u) - F_{\ell_1}(u))du \geq k'^l(k^B(u_0)) \int_{-\infty}^{u_0} k^{B'}(u)(F_{\ell_1}(u) - F_{\ell_2}(u))$$

and:

$$\int_{u_0}^{\infty} k^{B'}(u)(F_{\ell_2}(u) - F_{\ell_1}(u))du \geq \int_{-\infty}^{u_0} k^{B'}(u)(F_{\ell_1}(u) - F_{\ell_2}(u)),$$

since  $k'^l(k^B(u_0)) \geq k'^r(k^B(u_0)) \geq 0$  and both integrals are positive. This inequality is similar to (A.5) and states that agent  $B$  prefers  $\ell_1$  to  $\ell_2$ , proving the result.

2. Quiggin anticipated utility functions. We use a proof strategy which is similar to Chateauneuf, Cohen and Meilijson [6].

2.a. First implication. Assume that agent  $A$  is more risk-averse than agent  $B$ . We consider lotteries with four possible outcomes.  $\ell_1$  pays  $x_1 < x_2 < x_3 < x_4$  with respective probabilities  $p_1, p_2, p_3$ , and  $p_4 = 1 - p_1 - p_2 - p_3$ .  $\ell_2$  pays  $x_1, x_2 - \varepsilon_2, x_3 + \varepsilon_3, x_4$  with the same probabilities, and with  $\varepsilon_2, \varepsilon_3 > 0$  small enough to respect the initial outcome ranking.  $\ell_2$  is a  $(p_1 + p_2)$ -spread of  $\ell_1$ . The utility of  $A$  associated with  $\ell_1$  expresses as:

$$\begin{aligned} U_{\phi^A}^Q(\ell_1) &= -x_1(\phi^A(1 - p_1) - \phi^A(1)) - x_2(\phi^A(1 - p_1 - p_2) - \phi^A(1 - p_1)) \\ &\quad - x_3(\phi^A(1 - p_1 - p_2 - p_3) - \phi^A(1 - p_1 - p_2)) - x_4(\phi^A(0) - \phi^A(1 - p_1 - p_2 - p_3)) \\ &= x_1 + (x_2 - x_1)\phi^A(q_2) + (x_3 - x_2)\phi^A(q_3) + (x_4 - x_3)\phi^A(q_4) \end{aligned}$$

where:  $p_j = q_j - q_{j+1}$  with  $1 = q_1 \geq q_2 \geq q_3 \geq q_4 \geq q_5 = 0$

We choose  $\varepsilon_3$  such that agent  $A$  is indifferent between  $\ell_2$  and  $\ell_1$ . Agent  $A$  being more risk-averse than  $B$ ,  $B$  prefers  $\ell_2$  to  $\ell_1$ . Noting  $\phi^A = \phi \circ \phi^B$  (which implies that  $\phi$  is increasing and continuous), we have the following two relationships:

$$\begin{aligned} \varepsilon_3 (\phi \circ \phi^B(q_3) - \phi \circ \phi^B(q_4)) &= \varepsilon_2 (\phi \circ \phi^B(q_2) - \phi \circ \phi^B(q_3)) \\ \varepsilon_3 (\phi^B(q_3) - \phi^B(q_4)) &\geq \varepsilon_2 (\phi^B(q_2) - \phi^B(q_3)) \end{aligned}$$

Substituting the first equality ( $\phi^B(q_2) \geq \phi^B(q_3) \geq \phi^B(q_4)$  since  $\phi^B$  is increasing) yields:

$$\begin{aligned} \frac{\phi \circ \phi^B(q_2)}{\phi^B(q_2) - \phi^B(q_3)} + \frac{\phi \circ \phi^B(q_4)}{\phi^B(q_3) - \phi^B(q_4)} &\geq \phi \circ \phi^B(q_3) \frac{\phi^B(q_2) - \phi^B(q_4)}{(\phi^B(q_3) - \phi^B(q_4))(\phi^B(q_2) - \phi^B(q_3))} \\ \text{and: } \frac{\phi^B(q_3) - \phi^B(q_4)}{\phi^B(q_2) - \phi^B(q_4)} \phi \circ \phi^B(q_2) + \frac{\phi^B(q_2) - \phi^B(q_3)}{\phi^B(q_2) - \phi^B(q_4)} \phi \circ \phi^B(q_4) &\geq \phi \circ \phi^B(q_3) \end{aligned}$$

Since  $\frac{\phi^B(q_3) - \phi^B(q_4)}{\phi^B(q_2) - \phi^B(q_4)} > 0$  and  $\frac{\phi^B(q_2) - \phi^B(q_3)}{\phi^B(q_2) - \phi^B(q_4)} > 0$ , the last inequality states that  $\phi$  is convex.

2.b. Second implication. We suppose that  $\phi^A$  is more convex than  $\phi^B$ , i.e. that  $\phi^A =$

$\phi \circ \phi^B$ , with  $\phi$  continuous, increasing, convex and  $\phi(0) = \phi(1) = 0$ .  $\ell_1$  and  $\ell_2$  are two lotteries, such that:  $\ell_1 \vdash_p \ell_2$  for some  $p \in ]0, 1[$  and  $\ell_1 \succeq^A \ell_2$ , which implies:

$$- \int_{-\infty}^{\infty} u d(\phi(\phi^B(1 - F_{\ell_1}(u))) - \phi(\phi^B(1 - F_{\ell_2}(u)))) \geq 0$$

or:  $\int_{-\infty}^{\infty} (\phi(\phi^B(1 - F_{\ell_1}(u))) - \phi(\phi^B(1 - F_{\ell_2}(u)))) \geq 0$  because  $F_1(\pm\infty) = F_2(\pm\infty)$

Since  $\ell_1 \vdash_p \ell_2$  there exists  $u_0 \in \mathbb{R}$ ,  $1 - F_{\ell_2}(u) \geq 1 - F_{\ell_1}(u) \geq 1 - p$  for  $u \leq u_0$  and  $1 - F_{\ell_2}(u) \leq 1 - F_{\ell_1}(u) \leq 1 - p$  for  $u \geq u_0$ . We obtain:

$$\int_{-\infty}^{u_0} [\phi(\phi^B(1 - F_{\ell_2}(u))) - \phi(\phi^B(1 - F_{\ell_1}(u)))] du \leq \int_{u_0}^{\infty} [\phi(\phi^B(1 - F_{\ell_1}(u))) - \phi(\phi^B(1 - F_{\ell_2}(u)))] du \quad (\text{A.6})$$

Focusing on the left-hand side, we deduce:

$$\int_{-\infty}^{u_0} [\phi(\phi^B(1 - F_{\ell_2}(u))) - \phi(\phi^B(1 - F_{\ell_1}(u)))] du = \int_{-\infty}^{u_0} (\phi^B(1 - F_{\ell_2}(u)) - \phi^B(1 - F_{\ell_1}(u))) \frac{\phi(\phi^B(1 - F_{\ell_2}(u))) - \phi(\phi^B(1 - F_{\ell_1}(u)))}{\phi^B(1 - F_{\ell_2}(u)) - \phi^B(1 - F_{\ell_1}(u))} du$$

We use a similar argument as in the Kihlstrom and Mirman case. We denote  $\phi^B(1 - F_{\ell_i}(u)) = 1 - t_i$  (for  $i = 1, 2$ ). Since  $\phi^B$  is increasing,  $0 \leq t_2 \leq t_1 \leq 1 - \phi^B(1 - p)$  for  $u \leq u_0$ . We then focus on  $\frac{\psi(t_1) - \psi(t_2)}{t_1 - t_2}$ , where  $\psi(t) = -\phi(1 - t)$  is increasing and concave. We can find a lower bound for this expression, which is  $\psi'^l(1 - \phi^B(1 - p)) = \phi'^r(\phi^B(1 - p))$ . Since  $\phi^B(1 - F_{\ell_2}(u)) \geq \phi^B(1 - F_{\ell_1}(u))$  for  $u \leq u_0$ , we deduce:

$$\int_{-\infty}^{u_0} [\phi(\phi^B(1 - F_{\ell_2}(u))) - \phi(\phi^B(1 - F_{\ell_1}(u)))] du \geq \phi'^r(\phi^B(1 - p)) \int_{-\infty}^{u_0} [\phi^B(1 - F_{\ell_2}(u)) - \phi^B(1 - F_{\ell_1}(u))] du$$

Focusing on the right-hand side of (A.6), we similarly obtain:

$$\int_{u_0}^{\infty} [\phi(\phi^B(1 - F_{\ell_1}(u))) - \phi(\phi^B(1 - F_{\ell_2}(u)))] du \leq \phi'^l(\phi^B(1 - p)) \int_{u_0}^{\infty} (\phi^B(1 - F_{\ell_1}(u)) - \phi^B(1 - F_{\ell_2}(u))) du$$

The inequality (A.6) becomes:

$$\begin{aligned} & \phi'^{r}(\phi(1-p)) \int_{-\infty}^{u_0} [\phi^B(1-F_{\ell_2}(u)) - \phi^B(1-F_{\ell_1}(u))] du \leq \\ & \phi'^{l}(\phi^B(1-p)) \int_{u_0}^{\infty} (\phi^B(1-F_{\ell_1}(u)) - \phi^B(1-F_{\ell_2}(u))) du \\ \text{or: } & \int_{-\infty}^{u_0} [\phi^B(1-F_{\ell_2}(u)) - \phi^B(1-F_{\ell_1}(u))] du \leq \int_{u_0}^{\infty} (\phi^B(1-F_{\ell_1}(u)) - \phi^B(1-F_{\ell_2}(u))) du \end{aligned}$$

since  $0 \leq \phi'^{l}(1 - \phi^B(p)) \leq \phi'^{r}(1 - \phi^B(p))$  and both integrals are positive. Carrying out the same manipulations in reverse order yields:

$$- \int_{-\infty}^{\infty} u d(\phi^B(1 - F_{\ell_1}(u))) \geq - \int_{-\infty}^{\infty} u d(\phi^B(1 - F_{\ell_2}(u)))$$

Agent  $B$  therefore prefers lottery  $\ell_1$  to  $\ell_2$ , which completes the proof.

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