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Harvesting a Remote Renewable Resource

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Harvesting a Remote Renewable Resource

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Abstract

In standard models of spatial harvesting, the resource is distributed over the complete domain and the agent is able to control the harvesting activity everywhere all the time. In some cases, though, it is more realistic to assume that the resource is located at a single point in space and that the agent is required to travel there in order to be able to do harvesting. In this case, the agent faces a combined travelling-and-harvesting problem. We scrutinize this type of a two-stage optimal control problem, and illuminate the interdependencies between the solution of travelling and that of the harvesting sub-problem. In particular, bounds on either control, *i. e.* on acceleration respectively on the harvesting capacity, significantly affect the policies in both sub-problems. Since the model is parsimoniously parameterised, we are able to characterise the optimal policy of the complete travelling-and-harvesting problem analytically.

Keywords: Optimal travelling-and-harvesting decision; spatial renewable resource; remote location; continuous time; optimal control

JEL classification: Q20, Q22, C61

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1. Introduction

In the management of renewable natural resources the spatial dimension has attracted substantial attention in the last years. The focus of this literature is on the movement of the resource, such as fish or game, and on the optimal allocation of the harvesting effort over the domain (distributed control). In this paper, we reverse this view: we consider an agent required to move within the space in order to be able to harvest an immobile resource. While the standard approach is descriptive when the resource is able to cover large distances when compared with the agent, our approach is expedient if, the resource, when compared with the agent, is spatially rather immobile or even fixed.

The management of renewable natural resources is a central issue in economics since, at the latest, Gordon (1954) and Smith (1968) have revitalized this topic. In this regard, optimal control theory has proved as a suitable technique to design optimal harvesting strategies with respect to both the temporal and the spatial dimension. Notably, in their monographs Conrad and Clark (1987), Conrad (2010) and Clark (2010) nicely demonstrate how optimal control theory may fruitfully contribute to the management of fisheries. Subsequently, these textbook models have been extended and generalised in various respects. For example, Fan and Wang (1998) generalise the optimal harvesting policy of a autonomous harvesting problem with logistic growth (see, for example, Clark, 2010) to a non-autonomous case with periodic coefficients; Liski et al. (2001) accounting for costly changes of the harvesting rate explore the effects of increasing returns to scale for a standard fishery management model;¹ and Ainseba et al. (2003), Feichtinger et al. (2003), Hritonenko and Yatsenko (2006), Tahvonen (2008, 2009a,b), Li and Yakubu (2012), Skonhøft et al. (2012), Quaas et al. (2013), Tahvonen et al. (2013) and Belyakov and Veliov (2014) investigate harvesting of age-structured populations.²

While that work takes into account the temporal and the bioeconomic dimension, the spatial dimension—though already present in the literature of theoretical biology and applied mathematics—has entered the focus of economists relatively late; namely, it was Sanchirico and Wilen (1999) who brought the spatial dimension to the attention of resource economists. In their seminal paper Sanchirico and Wilen generalize the fundamental open-access models of Gordon (1954) and

¹In this way these authors demonstrate a link between stable limit cycle policies and increasing returns in harvesting; notably, they show that for moderate adjustment costs the harvest rate and thus the stock of fish may oscillate persistently.

²Notably, Ainseba et al. (2003), investigating the optimal harvesting problem for a non-linear age-dependent and spatially structured population dynamics model, prove the existence and uniqueness of a solution along with the existence of an optimal control, and provide necessary optimality conditions.

Smith (1968) with respect to the spatial dimension: they set up a bioeconomic model with a finite number of patches with migration of the biomass and reallocation of effort between patches. In this way the authors integrate within- and between-patch biological forces and economic forces, and demonstrate how these effects determine the process of bioeconomic convergence over space and time.³

Following Sanchirico and Wilen (1999), the early models in spatial resource economics feature discrete patches, where at each location of the resource the stock evolves according to an ordinary differential equation (ODE); migration of the biomass is then modelled as entry and exit of the biomass from one location to the other. The contemporary literature, though, models the migration and the spread of the biomass as diffusion described by partial differential equations.⁴ Notable contributions are Cañada et al. (1998), Montero (2000, 2001), Neubert (2003), Bai and Wang (2005), Brock and Xepapadeas (2008, 2010), Ding and Lenhart (2009), Joshi et al. (2009), Bressan et al. (2013), Uecker and Upmann (2016) and others.

In both strands of the literature it is the biomass which is mobile while the agent harvesting the resource is immobile. Metaphorically speaking, the agent is waiting for the resource approaching, catching it when passing by. In many instances this is a reasonable approach suitably describing the situation (*e.g.* shooting off game or coastal fishery), but in other cases it is not. For example, in fruit harvesting, forestry, extensive agriculture *etc.* it is the agent who is moving in order to access the resource. In this paper, we build on that observation and analyse the optimal behaviour of an agent who is required to travel in order to be able to get access to and to harvest a remote resource. Thus, when compared with the standard approach in spatial resource economics, we complement that literature by reversing the ability of movement. This reversal may enhance the realism in modelling natural resource extraction when the resource is rather immobile and is located at some distant or hardly accessible place.

In this paper, we assume that the resource is localised at some fixed and known patch. Since the agent's harvesting activity is limited to that single location, the agent is required to move to that location in order to get access to the resource, which is a prerequisite for any harvesting activity.⁵ Yet, harvesting not only requires the physical presence of the agent, but the process of harvesting also takes time. Thus, both processes, movement and harvesting, are time-consuming.

³In a subsequent paper Sanchirico and Wilen (2005) utilize the model of their 1999 paper to characterise the spatially differentiated landings and effort taxes suitable to implement a first-best allocation.

⁴A presentation of the basic population models with diffusion can be found, for example, in Anița (2000, sec. 1.2), Okubo and Levin (2001), Murray (2003) and the references therein.

⁵The realism of this approach is thus highest where the spatial domain is relatively large when compared with the region which can be harvested at a single instant of time.

In order to move from their initial location to the location of the resource and then to perform harvesting, the agent has first to control the navigation process and then, upon arrival at the resource, the harvesting process. Consequently, any admissible policy consists of a sequence of a travelling and a harvesting interval—and we are interested in this optimal travelling-and-harvesting policy, that is, in the interdependency of the travelling and the harvesting decision. In this way, this paper complements the contemporary literature on spatial resources economics by seriously taking into account the sequence of time periods required for travelling and harvesting, and by scrutinizing the interaction between the travelling and the harvesting decision.

Few papers only consider the travelling-and-harvesting problem of the agent in a spatial domain: notably, Behringer and Upmann (2014) and Belyakov et al. (2015) consider an immobile resource located on the periphery of a circle and an agent who gets out for a round trip, returning to their home after a turn. Both, Behringer and Upmann (2014) and Belyakov et al. (2015), assume that the agent is able to do *en passant* harvesting, so that the agent need not stop (at each location) in order to extract the resource; rather, the agent is able to extract the resource by passing by, implying that there is no need not reduce the travelling speed and to stop. As a consequence, the harvesting activity does not cost any time (over and above the time of travelling), but can be done during travelling. In this way, the travelling and the harvesting activity go in parallel and may even be identified with each other. This is quite opposite to our approach where travelling and harvesting are mutually exclusive, rival activities (with different cost functions): the more time is spent on travelling, the less time is left for harvesting, and *vice versa*.

Since in our approach travelling and harvesting are quite two different activities which take place at different locations at different times, we are confronted with two, yet interdependent, optimal control problems: the problem of travelling, where acceleration (or speed) has to be chosen to steer to the location of the resource; and the harvesting problem, where the harvesting rate has to be determined to maximise the yield. In order to solve this combined profit-maximising problem, we draw upon the literature of two-stage optimal control problems with finite time horizon: notably, Amit (1986), Tomiyama (1985), Tomiyama and Rossana (1989) provide optimality conditions for two-phase, finite time dynamic optimization problems similar to the one considered here.⁶

We solve this two-stage optimal control problem and derive the optimal travelling and harvesting policy, including the optimal point in time at which the agent

⁶An extension to infinite horizon can be found in Makris (2001); and applications of this theory to two-stage optimal control problems, in Grass et al. (2012), Bar-Ilan and Strange (1998), Tahvonen and Withagen (1996) and Boucekine et al. (2004), for example.

arrives at the location of the resource and commences harvesting. In particular, we demonstrate the interdependency between the travelling and the harvesting problem, a feature which has, to our knowledge, been left unnoticed and unexplored in the literature. To scrutinize the robustness of our finding, we consider two different specifications of the growth process of the resource: exponential growth and logistic growth. For both types of processes we derive the value function of the optimal harvesting policy. Finally, we explore the sensitivity of our results with respect to the rate at which future revenues and costs are discounted and with respect to the presence of bounds on the control of movement, and show that the fundamental insights are robust.

The rest of the paper is structured as follows. In Section 2 we set up the model. In Section 3 we decompose the travelling-and-harvesting problem into the two sub-problems, which are then analysed in Section 4 (harvesting) and in Section 5 (travelling). In Section 6 we explore the robustness of our results by inspecting the effects of changes in the discount rate and of bounds on the controls. Finally, we conclude in Section 7.

2. The Model

2.1. Sketch of the Model. We consider a renewable natural resource located at some fixed location. The agent can harvest the resource at their current location only, and is thus required to move to get access to and to be able to extract the resource. Consequently, the agent has to steer to the location of the resource, at may commence with harvesting upon arrival (at the earliest). The process of harvesting takes time, and so the stock diminishes gradually during harvesting takes place.⁷ Consequently the agent's problem is a combined travelling-and-harvesting problem where the speed of travelling, and thus the arrival time, and the harvesting rate have to be determined jointly in order to maximise the total profit composed of the revenue from harvesting net of harvesting and travelling cost.

2.2. Details of the Model. We consider a finite time horizon T with a planning period $\mathcal{T} \equiv [0, T]$. During this planning period, the economic agent has the exclusive right to harvest the renewable natural resource, which is located at some fixed and known position $x_1 > 0$. At time $t \in \mathcal{T}$ the position of the economic agent is $x(t) \in \mathcal{X} \equiv [0, \bar{x}]$, called the location of the agent; the initial location of the agent is $x(0) = 0$. We assume $\bar{x} \geq x_1$. Since the resource is located x_1 units of length remote from the agent, the agent is unable to commence harvesting until

⁷This contrasts with *en passant* harvesting of a continuously distributed resource where the agent is not required to stop but can collect the resource while he is moving.

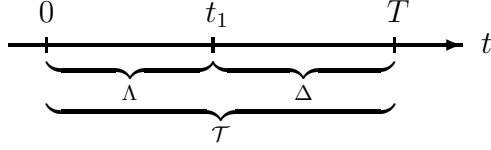


FIGURE 1. Travelling-and-harvesting period

they arrive at the location of the resource, *i. e.*, $x(t) = x_1$. Harvesting thus requires the agent to travel from 0 to x_1 , to stop there, and to commence with harvesting.

In order to move from one location to the next, the agent has to adjust the velocity of travelling $v(t)$, which we assume to be non-negative and finite so that $v(t) \in \mathcal{V} = [0, \bar{v}]$.⁸ Since speed cannot be chosen directly, but is controlled by means of acceleration $a(t) \in \mathcal{A}$ of the vehicle of movement or the harvesting machine we have⁹

$$\dot{x}(t) = v(t) \quad \text{and} \quad \dot{v}(t) = a(t) \quad \forall t \in \mathcal{T}. \quad (1)$$

There may be lower and upper bounds on acceleration; in Section 6, we shall assume that acceleration is bounded so that $a \in \mathcal{A} \equiv [\underline{a}, \bar{a}]$ with $\underline{a} < 0$ and $\bar{a} > 0$.¹⁰

Since harvesting, as well as travelling, takes time and the time horizon is finite, more time is left for harvesting the earlier the agent arrives at location x_1 . More precisely, let t_1 denote the arrival time of the agent at the location of the resource x_1 , that is $x(t) < x_1$ for all $t < t_1 \leq T$ and $x(t_1) = x_1$; if the agent does not arrive at x_1 by time T , such that $x(T) < x_1$, then we set $t_1 = +\infty$. Thus, we have $t_1 \in \mathcal{T}_{+\infty} \equiv \mathcal{T} \cup \{+\infty\}$. Consequently, $\Lambda \equiv [0, t_1 \wedge T]$ denotes the agent's travelling period;¹¹ and $\Delta \equiv (t_1 \wedge T, T]$, the resulting harvesting period. The total available time is then either spent for travelling or for harvesting, so that $\Lambda \cup \Delta = \mathcal{T}$ represents the *travelling-and-harvesting period*; this is visualized in Figure 1.

The size of the stock of the resource (*i. e.* the biomass) at time $t \in \mathcal{T}$ is denoted by $s(t) \geq 0$. We assume that the renewable resource is growing autonomously at rate g , and allow for the growth rate of the stock to depend on the size of the stock: $g(s)$ with $g(0) = 0$. Furthermore, the stock is reduced as a result of the harvesting

⁸The assumption of non-negative speed rules out that the agent moves backwards. Since moving backwards is economically unreasonable, this assumption can be made without reservations.

⁹Taking into account acceleration is made to avoid an unrealistic speed profile where the agent may instantaneously switch speed in a non-continuous way.

¹⁰The minimum acceleration \underline{a} is necessarily negative to allow for a slowdown of speed, as the agent would otherwise be unable to stop—and start harvesting.

¹¹We assume that the travelling period is convex. That is, once the agent has reached location x_1 , they will never start travelling again, and thus completes the planning period at x_1 .

activity. The harvest depends on the availability of the resource, *i.e.* on the stock s , and on the harvesting effort h exercised to acquire the resource. Suppose that effort is less productive the lower the stock, and that a given stock yields less harvest the lower the effort. According to this, we assume that harvest at time t amounts to $H(t) = h(t)s(t)$ provided that the agent's location is x_1 , and $H(t) = 0$ otherwise. Thus, the resulting growth of the stock is governed by the differential equation

$$\dot{s}(t) = g(s(t)) - h(t)s(t)\mathbf{1}_{\{x(t)=x_1\}}(t), \quad \forall t \in \mathcal{T}, \quad (2)$$

where the indicator function $\mathbf{1}_{\{x(t)=x_1\}}$ accounts for the fact that harvesting can only be effective if the agent's location at time t equals x_1 , *i.e.*, if $x(t) = x_1$. In other words, upon arrival at location x_1 , the agent starts the path of the harvesting activity $\{h(t)\}_{t \in \Delta}$.¹²

The process of harvesting gradually diminishes the stock, and the agent may decide to continue with harvesting until the stock is depleted: with $s(t) = 0$ it immediately follows that $H(t) = 0$ for any harvesting activity $h(t) \geq 0$. Also, once the stock is depleted, we have $\dot{s}(t) = 0$ due to our assumption $g(0) = 0$. Hence, $s = 0$ represents an absorbing barrier or an equilibrium of the stock dynamics. (Subsequently, we will consider the cases of exponential and logistic growth of the stock both of which satisfy these assumptions.) Owing to the immediate, negative effect of harvesting on growth, intensive harvesting leaves the stock with less beneficial conditions for future growth, and thus impairs the possibilities for future harvesting. There is thus a trade-off between present and future yield from harvesting.

Travelling and harvesting both are costly. We assume that harvesting cost C increase in the amount of the resource harvested. Accordingly, harvesting cost $C(H)$ is increasing and (weakly) convex, *i.e.* $C' > 0$ and $C'' \geq 0$ for all $H \in \mathbb{R}_+$, with $C(0) = 0$. Also, travelling is associated with some cost, which (generically) depends on both speed and acceleration: $K : \mathcal{V} \times \mathcal{A} \rightarrow \mathbb{R} : (v, a) \mapsto K(v, a)$. Naturally, pausing is costless, $K(0, 0) = 0$, while travelling cost increase with both speed and acceleration, and acceleration is more costly the higher the speed. Thus, the signs of the partial derivatives of the cost function are given by $K_v \geq 0$, $K_a \geq 0$ and $K_{va} \geq 0$.

Since the maximum speed \bar{v} is finite, travelling from 0 to x_1 takes at least x_1/\bar{v} units of time. For this reason, travelling is not only costly in pecuniary terms, but also in terms of foregone time, time that might otherwise been used for harvesting. Thus, while a short travelling time may be desirable in order to

¹²In principle, we allow the agent to choose $h(t) > 0$ for times $t < t_1$, but since this harvesting activity is sure to yield no return at any time $t \in \Delta$, the choice of $h(t) > 0$ is a futile action in this case. Mathematically, this is captured by the indicator function.

commence harvesting soon, a short travelling time is costly in terms of pecuniary expenses, as this requires travelling at a high speed. Also, a short travelling time implies that there is less time for the resource to grow before a new period of harvesting starts.

Let $\rho \geq 0$ denote the discount rate of the agent, and normalize the price of one unit of the harvested resource to unity. The problem of the agent is then to maximize the discounted profit flow consisting of instantaneous revenue net of harvesting cost and net of travelling cost for the planning period \mathcal{T} . Presupposing that the agent reasonably chooses $h(t) = 0, \forall t \in \Lambda$,¹³ we obtain the travelling cost

$$J_1(a) \equiv \int_0^{t_1} e^{-\rho t} K(v(t), a(t)) dt \quad (3)$$

and the profit from harvesting

$$J_2(h) \equiv \int_{t_1}^T e^{-\rho t} (h(t)s(t) - C(h(t)s(t))) dt \quad (4)$$

where the arrival time t_1 depends on the acceleration path $\{a\}_{t \in \Lambda}$. Putting pieces together, the agent's optimisation problem then reads as

$$\max_{\{a, h\}} J(a, h) \equiv -J_1(a) + J_2(h) \quad (5)$$

subject to the dynamics of movement (1), the stock dynamics of the resource (2), and their associated constraints $v(t) \in \mathcal{V}(t)$, $a(t) \in \mathcal{A}(t)$, $h(t) \in \mathcal{H}(t) \forall t \in \mathcal{T}$, as well as to the initial conditions $s(0) = s_0$, $x(0) = 0$ and $v(0) = 0$, the “arrival conditions” $t_1 \in \mathcal{T}_{+\infty}$ free, $x(t_1) = x_1$ and $v(t_1) = 0$ if $t_1 \in \mathcal{T}$, and the terminal condition $s(T) \geq 0$ free, $x(T) \in \mathcal{X}$ free, $v(T) = 0$. Note that the constraints $x(t) = x_1, \forall t \in \Delta$ and $v(t) = 0, \forall t \in \Delta$ are already implied by (2) and thus need not be added.¹⁴

3. Decomposition of the problem

In order to solve problem (5), we decompose the intertemporal optimal travelling–and–harvesting problem in the travelling and the harvesting sub-problem. In order to render the travelling–and–harvesting problem reasonable, we subsequently assume that the costs of travelling are not too high, so that the problem is meaningful and an arrival is before time T is expedient. In addition, because maximum speed is finite, the arrival time must be strictly positive. Therefore, the corner solutions

¹³We easily may include the case $h(t) = 0$ for some $t \in \Lambda$ by setting the lower limit of integration in J_2 equal to 0.

¹⁴In addition, we may also require the agent to terminate his trip at the point of departure, that is we may impose the requirement $x(T) = 0$, so that the agent can only terminate his trip “at home”. Also, we may add a ‘scrap’ value (‘salvage’ value) of the stock of the resource at time T : $\phi(s(T), T)$, or we may add the constraint of a fixed endpoint $s(T) = s_T$.

$t_1 = 0$ and $t_1 = T$ as well as $t_1 = +\infty$ can be ruled out, so that both the travelling period and the harvesting period are non-empty, $\Lambda, \Delta \neq \emptyset$. For those reasons, we subsequently presume $t_1 \in (0, T)$.

In the travelling problem we choose an acceleration path $\{a(t)\}_{t \in \Lambda}$ and thus the arrival time t_1 so as to move from location 0 to location x_1 at minimal cost:

$$\begin{aligned} \min_{\{a, t_1\}} J_1(a, t_1) &\equiv \int_0^{t_1} e^{-\rho t} K(v(t), a(t)) dt & (6) \\ \text{s. t.} \quad &\dot{x}(t) = v(t), & \forall t \in [0, t_1] \\ &\dot{v}(t) = a(t), & \forall t \in [0, t_1] \\ &a(t) \in \mathcal{A}(t), & \forall t \in [0, t_1] \\ &v(t) \in \mathcal{V}(t), & \forall t \in [0, t_1] \\ &\dot{s}(t) = g(s(t)), & \forall t \in [0, t_1] \\ &x(0) = x_0, & x(t_1) = x_1, \\ &v(0) = 0, & v(t_1) = 0, \end{aligned}$$

Since the travelling time t_1 can be chosen, we face a free-terminal-time problem. Then, t_1 represents the starting time of the harvesting period, and in the resulting harvesting problem we choose a path of the harvesting activity (effort) $\{h(t)\}_{t \in \Delta}$ to maximise profit from this activity (4):

$$\begin{aligned} \max_{\{h\}} J_2(h, t_1) &\equiv \int_{t_1}^T e^{-\rho t} [h(t)s(t) - C(h(t)s(t))] dt & (7) \\ \text{s. t.} \quad &\dot{s}(t) = g(s(t)) - h(t)s(t) & \forall t \in [t_1, T] \\ &h(t) \in \mathcal{H}(t), & \forall t \in [t_1, T] \\ &s(t_1) = s_1, \quad s(T) \geq 0, \text{ free.} \end{aligned}$$

During the travelling time the resource remains unimpaired and thus grows (at least) until the agent arrives at the location of the resource, x_1 . Consequently, upon arrival, the stock of the resource equals $s(t_1)$ representing the solution of the supposed growth process $\dot{s}(t) = g(s(t))$ with $s(0) = s_0$. In this way, the travelling decision determines $s(t_1)$ and thus the initial value of the stock of the harvesting problem. The fact that the travelling time of the agent also represents the growth time of the resource is the crucial link between the travelling problem (6) and the harvesting problem (7). As a consequence, when we decide about the speed of travelling, we have to take into account that a longer travelling time will result in a postponement of the profits from harvesting; while, in contrast, a lower speed of travelling makes travelling less expensive and gives the resource more time to grow, and thus provides the opportunity for a more abundant harvest

at later times. Thus, taking into account these interdependencies between sub-problems (6) and (7), then together they constitute the *travelling-and-harvesting problem*:

$$\begin{aligned}
V_1 \equiv \max_{\{h,a,t_1\}} & - \int_0^{t_1} e^{-\rho t} K(v(t), a(t)) dt + \int_{t_1}^T e^{-\rho t} [h(t)s(t) - C(h(t)s(t))] dt \quad (8) \\
\text{s. t.} \quad & \dot{x}(t) = v(t), \quad \forall t \in [0, t_1] \\
& \dot{v}(t) = a(t), \quad \forall t \in [0, t_1] \\
& a(t) \in \mathcal{A}(t), \quad \forall t \in [0, t_1] \\
& v(t) \in \mathcal{V}(t), \quad \forall t \in [0, t_1] \\
& \dot{s}(t) = g(s(t)) \quad \forall t \in [0, t_1] \\
& \dot{s}(t) = g(s(t)) - h(t)s(t) \quad \forall t \in [t_1, T] \\
& h(t) \in \mathcal{H}(t), \quad \forall t \in [t_1, T] \\
& x(0) = x_0, \quad v(0) = 0, \\
& x(t_1) = x_1, \quad v(t_1) = 0, \\
& s(0) = s_0, \\
& s(T) \geq 0 \text{ free} \\
& t_1 \in \mathcal{T} \text{ free}
\end{aligned}$$

As the one-season travelling-and-harvesting problem is composed of two phases, the travelling and the harvesting phase, an analysis of this problem requires the derivation of the optimality conditions of a two-phase optimization model. In particular we want to answer the following questions: at what point of time should the agent switch from one phase to the other, *i. e.* when should he arrive at the location of the resource and begin with its exploitation, and how long and at what rate should the resource be exploited?

To solve problem (8), we derive necessary conditions for an optimal control pair (a^*, h^*, t_1^*) , by applying the decomposition of the original problem into two standard problems. We first consider the harvesting problem of the second stage, and then the travelling problem of the first stage, acknowledging the dependence of the solution of the second stage on the decision of the first stage. The optimal control h^* of the harvesting problem depends on the choice of the starting value $s_1 = s(t_1)$ and the starting time t_1 . More precisely, assuming the existence of the optimal switching time t_1 in the interior of the time interval \mathcal{T} , we solve the second stage problem and calculate the maximised objective function J_2^* as a function of the initial state s_1 and the switching time t_1 . Then, we derive the optimal control a^* and the optimal switching time t_1 by solving the travelling problem of the first stage.

Second stage. Given the control time interval $[t_1, T]$ and the initial condition $s(t_1) = s_1$, we solve problem (7) for an admissible optimal control h^* . This problem is of a standard form and can be solved using the well-known Pontryagin maximum principle (see, for example, Kamien and Schwartz, 1991.) Then, using the solution of the second-stage problem, $h^*(t)$, $\lambda_2^*(t)$ and $s^*(t)$, which depends on the starting values s_1 and t_1 , we calculate $J_2^*(s(t_1), t_1)$. Then, with the help of J_2^* , the original problem can be reformulated as follows:

First stage. Given a the constraints in (6), we look for an admissible optimal control a^* defined on $[0, t_1^*]$ and an optimal arrival time $t_1^* \in (0, T)$ such that

$$\max_a V_1(a, t_1) \equiv -J_1(v, a) + J_2^*(s(t_1), t_1), \quad (9)$$

where J_1 is defined as in (6). Since by assumption $t_1^* \in (0, T)$, the constraint $t_1 \in (0, T)$ is irrelevant, and this problem reduces to a standard problem with ‘scrap’ (or ‘salvage’) value J_2^* , free terminal time t_1 and end point $s(t_1)$. (See, for example, Léonard and Long, 1992, sec. 7.2 and 7.6.)

The optimality conditions for this type of a two-phase dynamic optimization problem are available from the literature. Details can be found in Tomiyama (1985) and Amit (1986) who provide necessary conditions for a two-stage, finite-horizon switching problem with endogenous switching time; while Makris (2001) provides corresponding results for a two-stage switching problem with an infinite time horizon.¹⁵ We here deploy the results of Tomiyama (1985) and Amit (1986).

4. Second Stage: Harvesting

We now solve the travelling-and-harvesting problem in the suggested way, *i. e.* we solve the harvesting problem in this section, and then solve the travelling problem in Section 5. We consider two standard specifications of the growth process of the resource: exponential growth, in sub-section 4.1, and logistic growth, in sub-section 4.3. For both processes we derive the value function of the optimal harvesting policy. (Similar models can be found in Conrad and Clark, 1987; Hocking, 1991; Clark, 2010 and others.)

For concreteness we subsequently speak of fish and catch rather than of renewable resource and harvest. Yet, the analysis is fully applicable to any natural renewable resource. Also, we may equivalently think of controlling the grazing of herbivores.

¹⁵Tomiyama and Rossana (1989) and Grass et al. (2008, sec. 8.1.1) generalise the results of Tomiyama (1985) and Amit (1986) for a finite and an infinite time horizon, respectively, when the switch point appears as an argument of the integrands in each integral of the objective function.

4.1. Exponential growth. Suppose that the stock of a given species of fish, when left unimpaired, increases at a constant rate: $g(s(t)) = s(t)$ for all $t \in \Delta \equiv [t_1, T]$. The stock is reduced by the catch $H(t) \equiv s(t)h(t)$, where the harvesting effort is bounded: $h(t) \in \mathcal{H} = [0, \bar{h}]$. Consequently the stock evolves according to the differential equation

$$\dot{s}(t) = s(t) - h(t)s(t), \quad s(t_1) = s_1, \quad \forall t \in \Delta, \forall h(t) \in \mathcal{H}. \quad (10)$$

We follow the familiar Schaefer model (see Schaefer, 1954), and specify the revenue from fishing as a bi-linear function of effort and the stock $H(t) = qs(t)e(t)$, where q is the catchability coefficient, defined as the fraction of the population fished by an effort unit. For convenience set $q = 1$. Also, concordantly with the literature, we presuppose a constant price of the resource so that revenue amounts to $pH(t)$. To complete our definition of the profit function, we follow the specification of the effort cost function chosen by, for example, Puchkova et al. (2014) and Moberg et al. (2015) and assume that harvesting costs are linear in total catch, $C(H(t)) = cH(t) = ch(t)s(t)$, with $0 \leq c < p$.¹⁶ Then, instantaneous profit amounts to $(p - c)h(t)s(t)$. Finally, we normalize the per unit profit $p - c$ to unity, so that the objective function becomes

$$\max_{h \in \mathcal{H}} J_2 = \int_{t_1}^T h(t)s(t) dt \quad \text{s. t. (10)}. \quad (11)$$

According to a substantial part of the literature we abstract from discounting for the moment, and set $\rho = 0$. (For example, the majority of the references provided in fn. 16 abstracts from discounting.) This allows us to simplify the analysis, and we show later, in Section 6, how our results are affected by the presence of a positive discount rate.

The Hamiltonian of this problem is given by

$$\mathcal{H} = h(t)s(t) + \pi(t)s(t)(1 - h(t)), \quad (12)$$

¹⁶The Schaefer model is almost universally used in the literature, and most authors add either linear or quadratic effort cost. For example, Clark (2010, Sec. 1.4), Puchkova et al. (2014) and Moberg et al. (2015) presume linear cost yielding an instantaneous profit equal to $pqs(t)e(t) - ce(t)$; while He et al. (1994), Leung (1995), Cañada et al. (2001), Montero (2001), Fister and Lenhart (2004, 2006) and Chang and Wei (2012) presume a quadratic effort cost function, and Ding and Lenhart (2009) presume a linear-quadratic effort cost function. One exception to the prevalence of linear and quadratic cost functions is Liski et al. (2001) who suppose a concave-convex harvest cost. An alternative specification of the objective function is to disregard effort cost altogether and to maximise the sustainable yield; this approach is followed by, for example, Fan and Wang (1998), Neubert (2003), Bai and Wang (2005) and Kelly et al. (2016).—All of these authors assume a fixed price of the (harvested) resource.

and the maximum principle yields

$$0 = (1 - \pi(t))s(t), \quad (13)$$

$$\pi'(t) = h(t)\pi(t) - h(t) - \pi(t), \quad (14)$$

along with eq. (10). Apparently, the optimal strategy depends on whether π is less or greater than one. The maximum of \mathcal{H} is thus achieved by

$$h(t) = \begin{cases} 0 & \text{if } \pi(t) > 1 \\ \bar{h} & \text{if } \pi(t) < 1. \end{cases} \quad (15)$$

Since $s(T)$ is free, the transversality condition requires $\pi(T) = 0$. From that condition together with $h(t) = \bar{h}$ for $\pi(t) < 1$, we infer that we cannot end the period Δ with $h = 0$, *i.e.* we must have $h(T) = \bar{h}$. Moreover, the solution of eq. (14) must satisfy

$$\pi(t) = \begin{cases} A_0 e^{-t} & \text{if } h(t) = 0 \\ \frac{\bar{h}}{\bar{h} - 1} + A_1 e^{t(\bar{h}-1)} & \text{if } h(t) = \bar{h}. \end{cases} \quad (16)$$

Neither solution achieves the critical value $\pi = 1$ more than once. Consequently, there is a unique switching point τ ,¹⁷ implying that we either have (i) $h(t) = \bar{h}$ for all $t \in \Delta$, or (ii) $h(t) = 0$ for all $t_1 \leq t < \tau$ and $h(t) = \bar{h}$ for all $\tau \leq t \leq T$. Then, along any path with $h = \bar{h}$, the costate variable must satisfy

$$\pi(t) = \frac{\bar{h}}{\bar{h} - 1} \left(1 - e^{(1-\bar{h})(T-t)} \right) \quad (17)$$

where we determined $A_1 = \bar{h}e^{T(1-\bar{h})}/(1 - \bar{h})$ so as to satisfy the transversality condition. Now, the switching time τ has to be chosen according to the condition $\pi(\tau) = 1$. Hence, we obtain from eq. (17)

$$\tau = T - \delta, \quad \text{with} \quad \delta \equiv \frac{\log(\bar{h})}{\bar{h} - 1}. \quad (18)$$

Since δ is a positive, decreasing and convex function for all values of $\bar{h} \neq 1$, we define $\delta = 1$ for $\bar{h} = 1$ so as to make δ a continuous function of \bar{h} .¹⁸ Consequently, the larger \bar{h} , the longer the agent can wait and let the resource grow unimpaired, allowing for more intensive harvesting later.

Depending on the sign of τ , either of two cases may occur.

¹⁷Alternatively, this observation follows from eq. (14), which implies that evaluated at a switching point τ we have $\pi'(\tau) = -1$ since $\pi(\tau) = 1$ by definition.

¹⁸To see that $\delta = 1$ for $\bar{h} = 1$, apply l'Hôpital's rule.

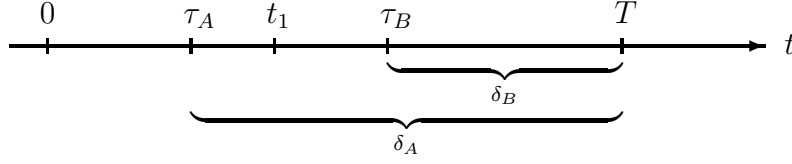


FIGURE 2. Cases A and B

4.1.1. Case A: $T < \delta + t_1$. In this case the maximal harvesting intensity \bar{h} is relatively low requiring a rather long period of extraction: $T - t_1 < \delta \Leftrightarrow \tau < t_1$. This implies that there is no switch in policy and for all $t \in \Delta$ we choose:

$$h(t) = \bar{h}, \quad (19)$$

$$s(t) = s_1 e^{(1-\bar{h})(t-t_1)}, \quad (20)$$

$$\pi(t) = \frac{\bar{h}}{\bar{h} - 1} \left(1 - e^{(1-\bar{h})(T-t)} \right), \quad (21)$$

and the maximised profit is given by

$$J_{2A}^* = s_1 \frac{\bar{h}}{\bar{h} - 1} \left(1 - e^{(1-\bar{h})(T-t_1)} \right). \quad (22)$$

4.1.2. Case B: $T > \delta + t_1$. In this case the maximal harvesting intensity \bar{h} is relatively high so that the agent may afford not to begin with harvesting immediately at time t_1 but at some point in time: $T - t_1 > \delta \Leftrightarrow \tau > t_1$. Here the agent begins with $h = 0$ and then, at time τ , switches to $h = \bar{h}$.

During the period $[t_1, \tau)$ the stock is left unimpaired and is thus given by

$$s(t) = s_1 e^{t-t_1},$$

so that at time τ the stock amounts to

$$s(\tau) = s_1 e^{\tau-t_1},$$

which is the starting value for the harvesting period $[\tau, T]$. During this period the stock equals

$$s(t) = A_2 e^{(1-\bar{h})t} = s(\tau) e^{(1-\bar{h})(t-\tau)} = s_1 e^{\bar{h}(\tau-t)+t-t_1}.$$

4.1.3. Optimal policy. Putting parts together, for all $\bar{h} \neq 1$ the optimal policy is thus given by

$$h(t) = \begin{cases} 0 & \text{for } t_1 \leq t < \tau \\ \bar{h} & \text{for } \tau \leq t \leq T \end{cases} \quad (23)$$

$$s(t) = \begin{cases} s_1 e^{t-t_1} & \text{for } t_1 \leq t < \tau \\ s_1 e^{\bar{h}(\tau-t)+t-t_1} & \text{for } \tau \leq t \leq T \end{cases} \quad (24)$$

$$\pi(t) = \begin{cases} e^{\tau-t} & \text{for } t_1 \leq t < \tau \\ \frac{\bar{h}}{\bar{h}-1} \left(1 - e^{(1-\bar{h})(T-t)}\right) & \text{for } \tau \leq t \leq T, \end{cases} \quad (25)$$

and the maximised profit amounts to

$$J_{2B}^* = \bar{h} \int_{\tau}^T s(t) dt = \frac{\bar{h}}{\bar{h}-1} s_1 e^{\tau-t_1} \left(1 - e^{(1-\bar{h})(T-\tau)}\right) = s_1 \bar{h}^{1/(1-\bar{h})} e^{T-t_1}, \quad (26)$$

where the last equality follows from the substitution of τ , given by eq. (18).

4.1.4. Optimal policy for $\bar{h} = 1$: In this case we may take the limits of Case A and Case B obtaining

$$(s, \pi, h) = \begin{cases} (s_1, T-t, 1) & \text{if } T \leq \delta + t_1, \\ (s_1 e^{\tau-t_1}, T-t, 1) & \text{if } T > \delta + t_1, \end{cases}$$

for all $t \in [\tau, T]$ and

$$J_{2B}^*|_{\bar{h}=1} = \begin{cases} s_1(T-t_1) & \text{if } T \leq \delta + t_1 \\ s_1 e^{T-t_1-1} & \text{if } T > \delta + t_1. \end{cases} \quad (27)$$

4.2. Discussion. In the optimal solution, the time during which fishing takes place is equal to $\delta = T - \tau = \log(\bar{h})/(\bar{h} - 1)$. If there is plenty of time in the sense that $T > \delta + t_1$, there will be no fishing during the initial period of length $T - t_1 - \delta$, while fishing will take place at the maximum rate \bar{h} during the final period. In case that there is not enough time available, that is if $T \leq \delta + t_1$, agent does fishing all the time at the maximum rate.

Whether the stock increases or decreases during fishing, depends on whether the harvesting capacity \bar{h} exceeds or falls short of the growth rate of the stock, which is assumed to be equal to one here. The situation when \bar{h} is larger than 1 is depicted in Figure 3; and the situation with $\bar{h} < 1$ in Figure 4 (both for $t_1 = 0$).

It is important to note the optimal length of the fishing period, δ , depends on \bar{h} but is independent of T . However, the maximised profit in Case A and B, given by eq. (22) and (26) respectively, depends on T . While J_{2B}^* is increasing and convex in T , J_{2A}^* is convex only if $\bar{h} < 1$, and is concave if $\bar{h} > 1$. Moreover, for

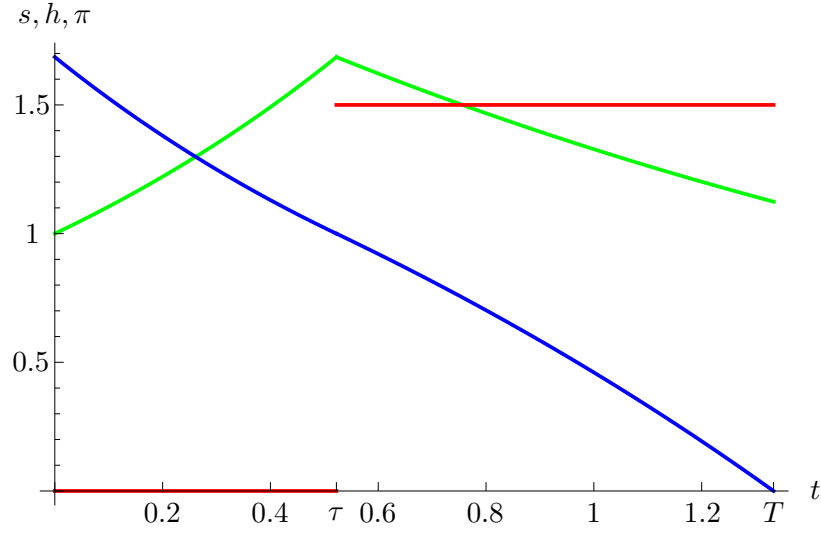


FIGURE 3. Optimal fishing in Case B, $T > \delta + t_1$, with $t_1 = 0$ and $\bar{h} > 1$.

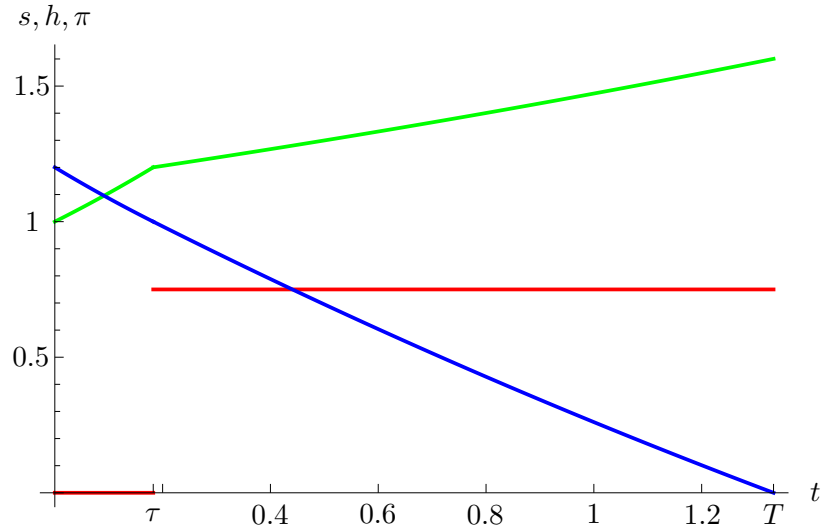


FIGURE 4. Optimal fishing in Case B, $T > \delta + t_1$, with $t_1 = 0$ and $\bar{h} < 1$.

any given values of t_1 and s_1 we have $J_{2B}^* \geq J_{2A}^*$. This is depicted in Figures 5 and Figure 6 for the case $t_1 = 0$. Therein, the vertical line represents the critical time $T = \delta + t_1$ for a given value of \bar{h} , and the red curve depicts the profit function for varying values of T . If time is scarce in the sense that $T - t_1 < \delta$, Case A applies and the blue curve represents the resulting maximised profit (covered by the red curve for values $T < \delta$, though). If there is plenty of time, in the sense that $T > \delta + t_1$, Case B applies and the green curve represents the resulting maximised profit (similarly covered by the red curve for values $T > \delta + t_1$). Note, however,

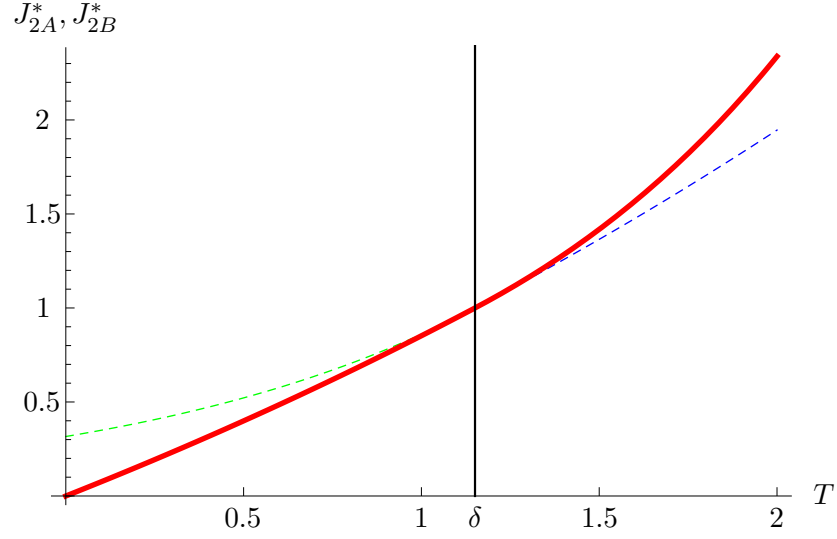


FIGURE 5. Maximised profit function for $\bar{h} = 3/4 < 1$, *i. e.* $\delta = 4 \log\left(\frac{4}{3}\right) = 1.15073$.

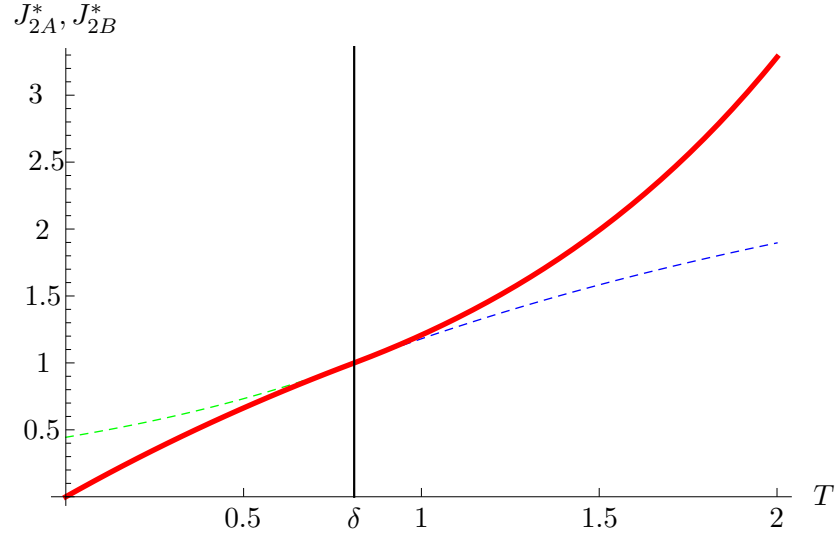


FIGURE 6. Maximised profit function for $\bar{h} = 3/2 > 1$, *i. e.* $\delta = 2 \log\left(\frac{3}{2}\right) = 0.81093$.

that in Case A, the Case B profit function is not attainable, so that the dashed green curve is merely hypothetical and cannot be reached for values of T lower than δ .

4.3. Logistic growth. In this section we modify the growth process of the resource and now assume that the stock obeys a logistic, rather than an exponential

growth process:

$$f(s(t)) = 2s(t) \left(1 - \frac{s(t)}{2}\right). \quad (28)$$

With this specification, the net-growth of the stock, *i. e.* after deduction of the harvest, is governed by the differential equation

$$\dot{s}(t) = f(s(t)) - h(t)s(t) = s(t) (2 - s(t) - h(t)) \quad (29)$$

If left unimpaired, the fish stock equilibrates at the level $s^* = 2$. Let us assume that the initial stock equals that level, *i. e.* $s(t_1) = 2$.—Everything else of the model is adopted from Section 4.1.

The Hamiltonian of the problem is given by

$$\mathcal{H} = h(t)s(t) + \pi(t)s(t) (2 - s(t) - h(t)),$$

and the maximum principle yields

$$0 = (1 - \pi(t))s(t), \quad (30)$$

$$\pi'(t) = -h(t) - \pi(t) (2 - 2s(t) - h(t)), \quad (31)$$

along with eq. (29); and since $s(T)$ is free, the transversality condition requires $\pi(T) = 0$.

Lemma 1. $\pi(t_1) < 1$.

Proof. Assume, on the contrary, that $\pi(t_1) > 1$. Then, since $s(t_1) = 2$, it follows from eq. (31) that $\pi'(t_1) = -h(t_1) + \pi(t_1) (2 + h(t_1)) > 0$. Since $h(t) = 0$ as long as $\pi(t) > 1$, the stock remains at its starting value $s(t_1) = 2$. Given this, there is no turning point in the evolution of π and thus π continues to grow, *i. e.* we have $\pi'(t) > 0$ for all t . Yet, this contradicts transversality condition $\pi(T) = 0$, and thus proves our claim $\pi(t_1) < 1$, and thus $h(t_1) = \bar{h}$. \square

It thus follows from Lemma 1 that optimal policy rule coincides with the rule obtained for exponential growth of the resource (15):

Lemma 2. *The maximum of the Hamiltonian \mathcal{H} is achieved by*

$$h(t) = \begin{cases} 0 & \text{if } \pi(t) > 1 \\ \bar{h} & \text{if } \pi(t) < 1. \end{cases} \quad (32)$$

Since $\pi(t_1) < 1$ by Lemma 1, it follows from eq. (32) that the optimal path begins with $h(t_1) = \bar{h}$. Intuitively, since the initial stock equals its maximum level, $s(t_1) = 2$, there is no case in beginning with $h = 0$, and thus we commence with $h(t_1) = \bar{h}$. Then, if time is relatively scarce, in relation to the harvesting capacity, we continue with $h(t) = \bar{h}$ for all $t \in \mathcal{T}$, or if there is plenty of time we have to reduce harvesting in the meantime because else we would have completed

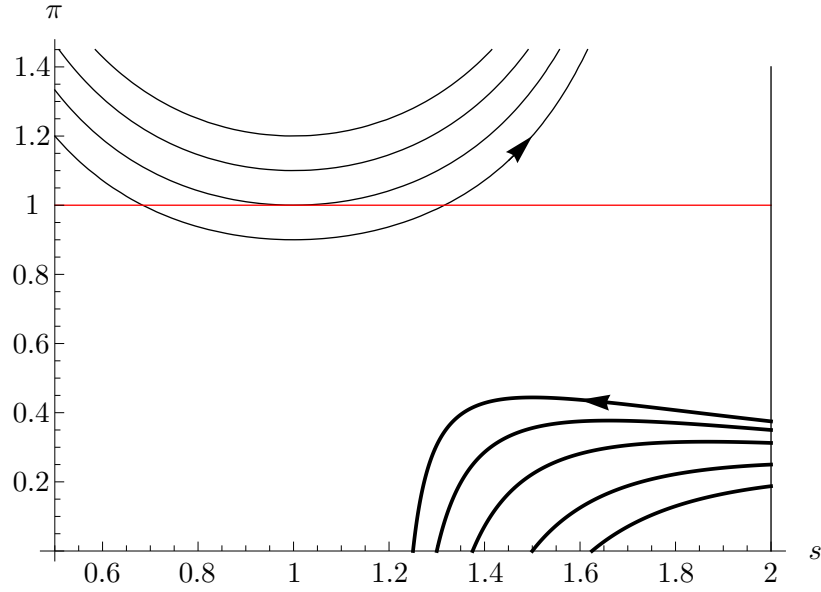


FIGURE 7. Trajectories for the case $h = \bar{h} = 0.8$ (solid curves) and the case $h = 0$ (thin curves).

harvesting too early, and the terminal condition $\pi(T) = 0$ could not be met. More precisely, the optimal harvesting policy is as follows.

Proposition 1. *The optimal harvesting policy is given by*

$$h(t) = \bar{h} \quad \text{if } \bar{h} \leq \bar{h}_c \quad (33)$$

$$h(t) = \begin{cases} \bar{h} & t_1 \leq t < t_2, \\ 1 & t_2 \leq t < t_3, \\ \bar{h} & t_3 \leq t < T. \end{cases} \quad \text{if } \bar{h} > \bar{h}_c, \quad (34)$$

with $\bar{h}_c \equiv \psi^{-1}(T)$, where $\psi : (1, 2) \rightarrow \mathbb{R}_+$ defines the critical length of the harvesting period

$$T_c = \psi(\bar{h}) \equiv t_1 + \frac{1}{2 - \bar{h}} \log \left(\frac{\bar{h}}{2(\bar{h} - 1)^2} \right). \quad (35)$$

Proof. Since the Hamiltonian is autonomous, it is constant along the optimal trajectory.¹⁹ We can therefore characterise the trajectories in the (s, π) plane for $h = 0$ and for $h = \bar{h}$. Let K denote the level of the Hamiltonian, then the optimal trajectories are characterised by the equations

$$\pi(t) = \frac{K}{2s(t) - s^2(t)} \quad \text{and} \quad \pi(t) = \frac{K - s(t)\bar{h}}{2s(t) - s^2(t) - s(t)\bar{h}} \quad (36)$$

¹⁹See, for example, Intriligator (1971, p. 355).

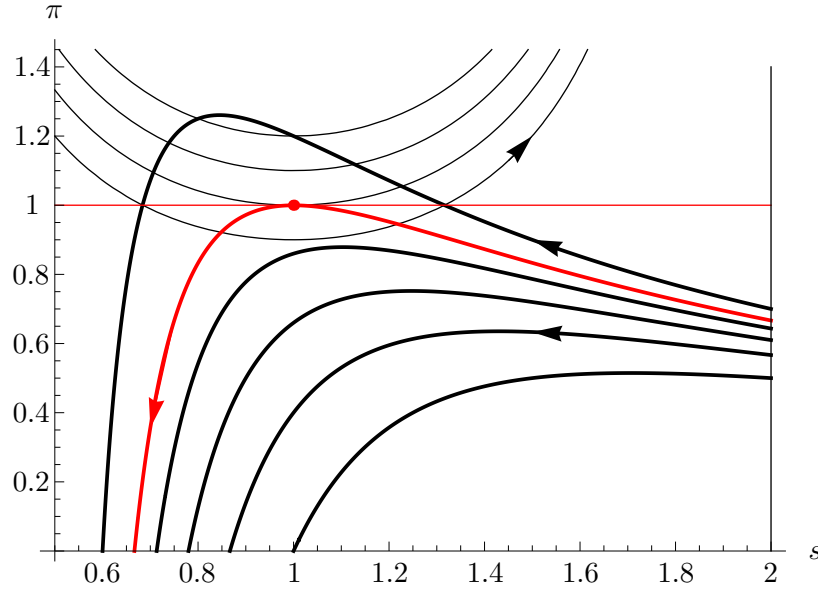


FIGURE 8. Trajectories for the case $h = \bar{h} = 1.5$ (solid curves) and the case $h = 0$ (thin curves), with critical trajectory (red).

for the cases $h = 0$ and $h = \bar{h}$, respectively. The $h = 0$ -curves have their minima at $s = 1$, and the curves with $h = \bar{h}$ have their maxima along the curve

$$\pi(t) = \frac{-\bar{h}}{2 - 2s(t) - \bar{h}} \quad \text{for } s > 1 - \frac{1}{2}\bar{h}. \quad (37)$$

Both of these curves are depicted in Figures 7 and 8. The curves starting from $s(t_1) = 2$ reach the horizontal axis at time T , *i.e.* $\pi(T) = 0$. Those curves with $\bar{h} < 1$ cross the horizontal axis at a point to the right of $2 - \bar{h}$, that is $s(T) > 2 - \bar{h}$. If \bar{h} is sufficiently small, the trajectory does not reach the $\pi = 1$ line (see Figure 7). Since the locus of maxima crosses the point $(1, 1)$, see eq. (36), the critical trajectory is that one which achieves its maximum at this point (see Figure 8). Because the trajectories do not cross the horizontal axis to the left of $2 - \bar{h}$, the critical trajectory must feature $\bar{h} > 1$, and is associated with a critical time horizon T_c . It thus follows that the critical harvesting capacity exceeds unity, $\bar{h}_c > 1$. Finally, that eq. (35) characterises the critical time horizon follows from Lemma 4. \square

4.3.1. Case A: either $\bar{h} < 1$ or $1 < \bar{h} < 2$ and $T \leq T_c$. In this case, the maximal fishing effort is relatively low, $\bar{h} < \bar{h}_c = \psi^{-1}(T)$, so that $h(t) = \bar{h}$ can be maintained throughout. Then, the optimal fishing strategy is given by

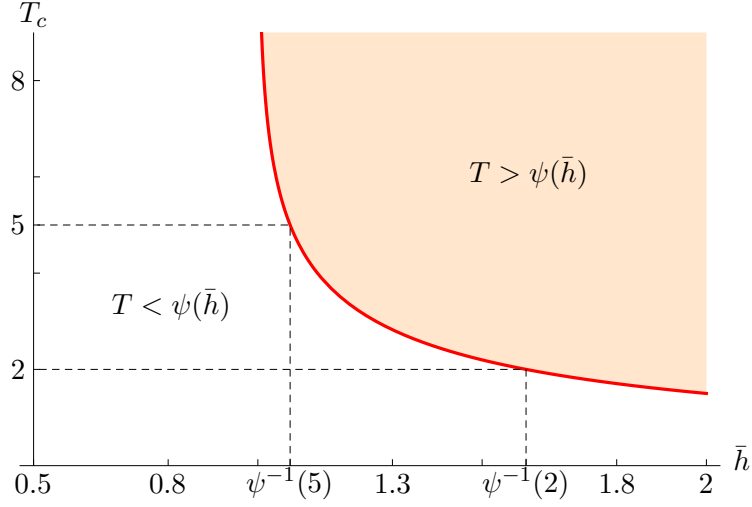


FIGURE 9. The critical value of T_c as function of \bar{h} .

Lemma 3. *Let either $\bar{h} < 1$ or $1 < \bar{h} < 2$ and $T \leq T_c$. Then the optimal fishing policy is given by*

$$h(t) = \bar{h}, \quad (38)$$

$$s(t) = \frac{2(\bar{h} - 2)}{\bar{h}e^{(\bar{h}-2)(t-t_1)} - 2}, \quad (39)$$

$$\pi(t) = \frac{\bar{h}(s(T) - s(t))}{2s(t) - s(t)^2 - \bar{h}s(t)}, \quad (40)$$

for all $t \in \Delta$. The resulting maximised profit amounts to

$$J_{2A}^* = \bar{h} \int_{t_1}^T x(t) dt = \bar{h} \log \left(\frac{2e^{(2-\bar{h})(T-t_1)} - \bar{h}}{2 - \bar{h}} \right). \quad (41)$$

Proof. We know from the proof of Proposition 1 that for all sub-critical cases $T < T_c$ (or $\bar{h} < \psi^{-1}(T)$) defined in eq. (35), we have $h(t) = \bar{h}$ for all $t \in \Delta$. Substituting this, together with initial condition $s(t_1) = 2$ and the terminal condition $\pi(T) = 0$, into eqs. (29)–(31), we obtain eqs. (38)–(41). \square

Lemma 4. *The critical time horizon is given by*

$$T_c = \psi(\bar{h}) = t_1 + \frac{1}{2 - \bar{h}} \log \left(\frac{\bar{h}}{2(\bar{h} - 1)^2} \right). \quad (42)$$

Proof. From eqs (39) and (40) we can calculate the critical time horizon T_c for which at some point in time t_c the trajectory goes through the point $(s(t_c), \pi(t_c)) = (1, 1)$. Using that information and evaluating eq. (40) at T_c yields $t_c = t_1 + \frac{1}{2-\bar{h}} \log \left(\frac{\bar{h}}{2(\bar{h}-1)} \right)$ and thus eq. (42). \square

The critical time horizon T_c is illustrated in Figure 9 (for $t_1 = 0$). In this critical case, the optimal profit amounts to

$$J_{2A}^c = 2\bar{h} \log \left(\frac{\bar{h}}{\bar{h} - 1} \right). \quad (43)$$

Remark 1. The profit for $\bar{h} = 1$ is given by

$$J_{2A}^*|_{\bar{h}=1} = \log(2e^{T-t_1} - 1)$$

4.3.2. Case B: $1 < \bar{h} < 2$ and $T > T_c$. In this case, the time available for harvesting $T - t_1$ is too long such that, given the maximal harvesting capacity \bar{h} , it is not optimal to do maximal harvesting all the time, as this would imply that $\pi = 0$ is reached before time T . Thus, harvesting cannot be maintained at rate \bar{h} throughout, but must be reduced during some interval—and the optimal fishing strategy is as follows.

Lemma 5. *Let $1 < \bar{h} < 2$ and $T > T_c$. Then the optimal fishing policy is given by*

$$h(t) = \begin{cases} \bar{h} & t_1 \leq t < t_2, \\ 1 & t_2 \leq t < t_3, \\ \bar{h} & t_3 \leq t < T, \end{cases} \quad (44)$$

with switching times

$$t_2 = t_1 + \frac{\log \left(\frac{\bar{h}}{2(\bar{h}-1)} \right)}{2 - \bar{h}} \quad \text{and} \quad t_3 = T - \frac{\log \left(\frac{1}{\bar{h}-1} \right)}{2 - \bar{h}}.$$

The resulting profit is given by

$$J_{2B}^* = T - t_1 + 2\bar{h} \log \left(\frac{\bar{h}}{\bar{h} - 1} \right) - \frac{1}{2 - \bar{h}} \log \left(\frac{\bar{h}}{2(\bar{h} - 1)^2} \right), \quad (45)$$

Proof. In the limiting case of $T = T_c$, we have $t_2 = t_3$ and the middle interval vanishes. That equation (44) is indeed the optimal fishing policy can be seen as follows. First note that $\pi = 1$ is a singular level. Since we already know that $\pi(t_1) < 1$, it follows that $\pi' \geq 0$ at the time the singular level $\pi = 1$ is reached. If we have $\pi = 1$ for some time interval with positive length, then $\pi' = 0$ and hence we must have $s = 1 \Rightarrow s' = 0 \Rightarrow h = 1$ from eq. (29).

Now, after completing the singular path we cannot have a path with $h = 0$. This can be seen as follows: $h = 0$ implies $s' > 0$, which in turn implies that, since $s = 1$ on the singular arc, $s > 1$ right after the singular arc. Together with $h = 0$ this in turn implies that $\pi' > 0$. Hence, we enter a path where both s and π are growing so that the transversality condition $\pi(T) = 0$ cannot be satisfied. We thus conclude that the optimal policy must proceed with $h = \bar{h}$ after completing singular path. in eq. (44) is indeed optimal.

Next we investigate the resulting profit. The fishing activity during the periods $[t_1, t_2)$ and $[t_3, T]$, the total length of which amounts to $T_c - t_1$, given by eq. (42), brings about the same profit as in Case A, that is J_{2A}^c as given in eq. (43); while during the time interval $[t_2, t_3)$ the profit collected is equal to the length of the fishing period: $t_3 - t_2 = T - T_c$. So we obtain

$$J_{2B}^* = J_{2A}^c + \int_{t_2}^{t_3} 1 \, dt = J_{2A}^c + T - T_c.$$

Then, substituting the definition of T_c , eq. (42), yields eq. (45). \square

Remark 2. The limit of the profit for $\bar{h} \rightarrow 1$ equals $T - t_1 + \log(2)$; and for $\bar{h} \rightarrow 2$, we obtain $T - t_1 - \frac{3}{2} + \log(16)$. In the limit when the fishing capacity becomes unbounded, we obtain $\lim_{\bar{h} \rightarrow \infty} J_{2B}^* = T - t_1 + 2$.

The intuition for the optimal strategy characterised in Lemma 5 is as follows. There is too much time for harvesting $T > T_c$, implying that if the agent followed the critical path (the red path in Figure 8), they would have reached the $\pi = 0$ -line too early. Thus, one might consider following a trajectory lying above the critical one, reaching the $\pi = 1$ -line at some value $s > 1$. But then one has to switch to $h = 0$ following an upward-sloping trajectory (a thin path in Figure 8), implying that both the stock and the costate variable increase—and there is no possibility of satisfying the terminal condition $\pi(T) = 0$. For that reason the optimal policy is as follows: pursue the critical path up to $(s, \pi) = 1$, which is reached at time t_2 ; then, upon arrival at $(s, \pi) = 1$ reduce harvesting to $h = 1$, which, in view of eqs (29) and (31), renders both s and π to be constant, for 1 is the natural growth rate of the resource; finally, to complete the optimal path, resume maximal harvesting so as to arrive at $\pi = 0$ at time T .

The maximised profit function is depicted in Figures 10–12 for $T = 2, 5$ and 20 , respectively. Therein, the vertical line represents the critical capacity $\psi^{-1}(T)$. For values of $\bar{h} < \psi^{-1}(T)$ Case A applies; for values of $\bar{h} > \psi^{-1}(T)$, Case B. The critical values $\bar{h}_c = \psi^{-1}(T)$ can be gathered from eq. (42) *viz.* from Figure 9.

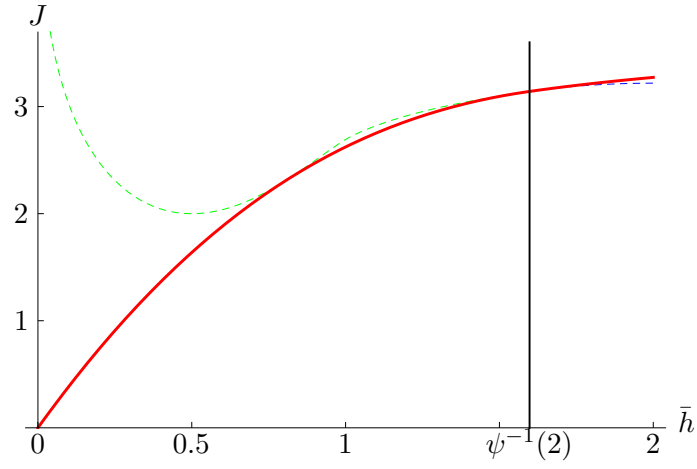


FIGURE 10. Profit in Case A and B for varying values of \bar{h} and $T = 2$.

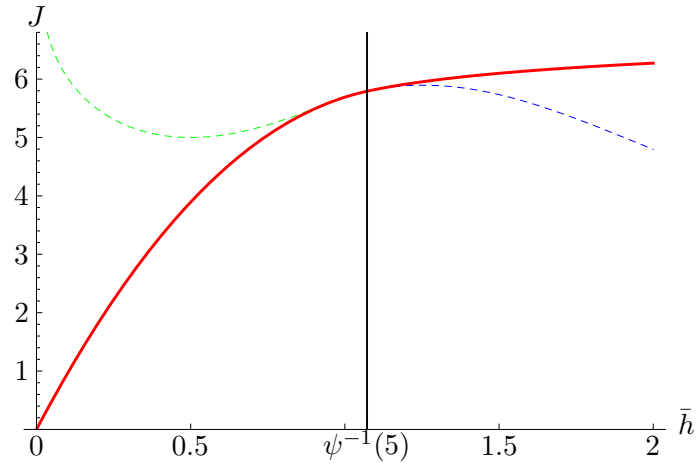


FIGURE 11. Profit in Case A and B for varying values of \bar{h} and $T = 5$.

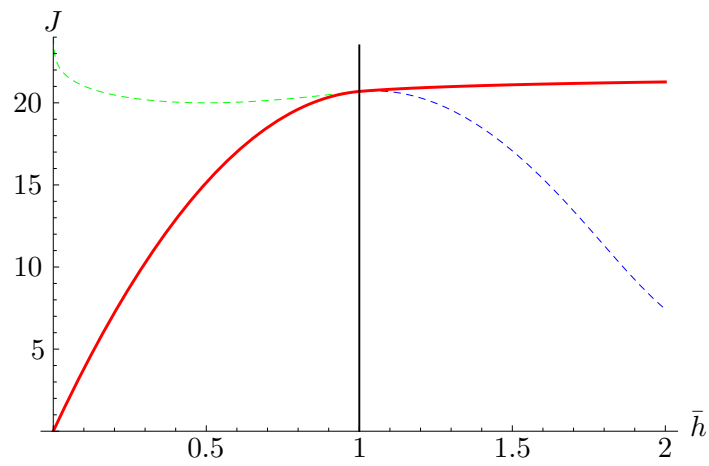


FIGURE 12. Profit in Case A and B for varying values of \bar{h} and $T = 20$.

5. First Stage: Travelling

Having solved the harvesting problem, we now go back in time and solve the travelling problem. We begin our analysis with the simple, in our framework hypothetical, case of a fixed travelling period (sub-section 5.1), and then continue with acknowledging the subsequent harvesting period and endogenising the arrival time t_1 in sub-section 5.2. In this way we are able to show which additional effects and which corresponding optimality conditions have to be added to the solution of former problem to obtain the solution of the latter. We proceed in this successive manner for this allows us to make very clear the differences between the solution of the isolated travelling problem (3) / (6) and the solution of the travelling-and-harvesting problem as formalised in (9).

5.1. Fixed travelling period. Assume that the cost of travelling depends linearly on speed v and quadratically on acceleration a :

$$K(v, a) = cv + a^2$$

Assuming $\rho = 0$ and $c = 1/10$, the resulting aggregated travelling cost amounts to

$$\int_0^{t_1} e^{-t\rho} (cv(t) + a(t)^2) dt = \int_0^{t_1} \left(\frac{v(t)}{10} + a(t)^2 \right) dt \quad (46)$$

Together with the constraints

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = a(t), \quad \dot{s}(t) = g(s(t)),$$

we obtain the Hamiltonian

$$\mathcal{H} = -\frac{v(t)}{10} - a(t)^2 + \pi_2(t)a(t) + \pi_1(t)v(t).$$

For ease of tractability, we assume that there are no bounds on the control a . Yet, we may drop this assumption later. Then, the familiar maximum principle then yields

$$\begin{aligned} x(t) &= \frac{t^2}{120} (30K_1 - 10K_2t + t), & v(t) &= \frac{t}{40} (20K_1 - 10K_2t + t), \\ \pi_1(t) &= K_2, & \pi_2(t) &= K_1 + \frac{t}{10} (1 - 10K_2), \end{aligned}$$

with K_1 and K_2 constants. Together with the boundary conditions $x(0) = v(0) = v(t_1) = 0$ and $x(t_1) = 1$, we obtain

$$\begin{aligned} x(t) &= \frac{t^2(3t_1 - 2t)}{t_1^3}, & v(t) &= \frac{6t(t_1 - t)}{t_1^3}, & a(t) &= \frac{6(t_1 - 2t)}{t_1^3}, \\ \pi_1(t) &= \frac{24}{t_1^3} + \frac{1}{10}, & \pi_2(t) &= \frac{12(t_1 - 2t)}{t_1^3}, \end{aligned}$$

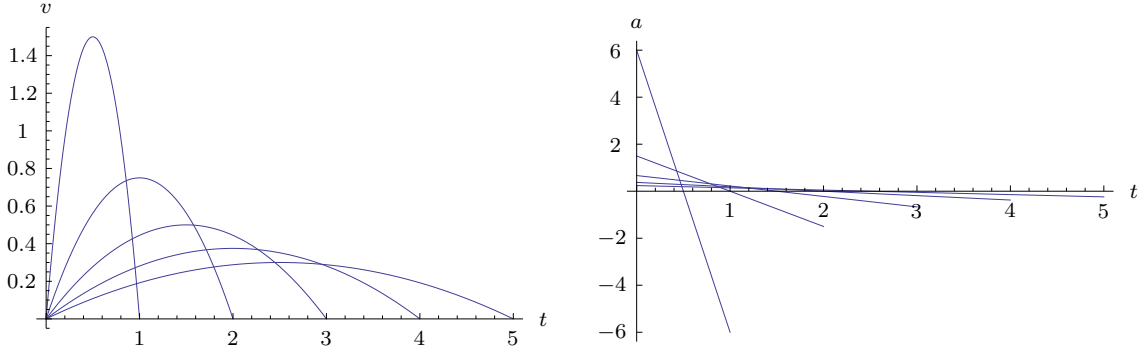


FIGURE 13. Speed and acceleration of the vehicle for $t_1 = 1, \dots, 5$.

and the maximised objective function equals

$$J_1^*(t_1) = \int_0^{t_1} \left(a(t)^2 + \frac{v(t)}{10} \right) dt = \frac{12}{t_1^3} + \frac{1}{10}. \quad (47)$$

Correspondingly, the value of the maximised Hamiltonian equals $\mathcal{H}^* = -\frac{36}{t_1^4}$. The acceleration of the vehicle and its resulting speed are depicted in Figure 13.

5.2. Optimal travelling-and-harvesting policy. In sub-section 5.1 we assumed that t_1 is fixed. But the agent may choose the length of the travelling period, and thus the beginning of the harvesting period to maximise the profit. In order to determine the optimal policy for the travelling-and-harvesting problem, three different effects must be acknowledged, and the associated necessary optimality conditions have to be added to those of the pure travelling decision.

First, the growth process of the resource during the travelling period must be acknowledged, and the associated necessary optimality condition must be added to the canonical system:

$$\dot{s}(t) = g(s(t)) = \begin{cases} 2s(t) - s^2(t) & \text{logistic growth} \\ s(t) & \text{exponential growth,} \end{cases} \quad (48)$$

$$\dot{\pi}(t) = -\frac{\partial \mathcal{H}_1}{\partial s(t)} = -\pi(t)g'(s(t)) = \begin{cases} -2\pi(t)(1 - s(t)) & \text{logistic growth} \\ -\pi(t) & \text{exponential growth.} \end{cases} \quad (49)$$

Next, the terminal time t_1 and the endpoint s_1 of the travelling problem are free and may be chosen in an optimal way. While the arrival time t_1 determines the length of the harvesting period Δ , the endpoint s_1 determines the initial value of the growth process in the harvesting problem. Together, both effects determine the maximal value $J_2^*(s_1, t_1)$ of the harvesting period, which in turn represents the scrap value of the compound problem (9). However, the endpoint $s_1 = s(t_1)$ is fully determined by the arrival time, as the stock of the resource cannot be controlled before time t_1 . For this reason, we do not have two, but only one transversality

condition taking into account both effects: the direct effect of the arrival time on the length of the harvesting period Δ , and the effect of t_1 on the stock at the beginning of that period $s(t_1)$.

Hence, to derive a necessary condition for the optimal choice of the arrival time t_1 , we first have to substitute the transversality condition (50), *i. e.* $s_1 = s(t_1) = s_0 e^{t_1}$, into J_{2A}^* . Then, this value function, which may be viewed as a scrap value function of the travelling problem, can be written, with slight abuse of notation, as $J_{2A}^*(t_1) \equiv J_{2A}^*(s(t_1), t_1)$. Using this, the associated necessary condition for the free terminal time of the travelling problem t_1 reads as²⁰

$$\mathcal{H}_1(s(t_1^*), c(t_1^*), \pi(t_1^*), t_1^*) + \frac{dJ_{2A}^*(t_1^*)}{dt_1} = 0. \quad (50)$$

With the help of condition (50) we are now able to calculate the optimal travelling- and harvesting policy. We do this for both growth functions specified above.

5.3. Optimal Travelling- and harvesting policy for exponential growth.

Acknowledging those transversality conditions, the following conditions have to be added

$$s(t) = s_0 e^t, \quad \pi(t) = \left(\frac{1}{\bar{h}}\right)^{\frac{1}{\bar{h}-1}} e^{T-t}. \quad (51)$$

It is obvious that t_1 must be equal to the switching time τ , so that harvesting commences immediately upon arrival. This is because arriving too early is costly without yielding any additional profit, as we have initially $h(t) = 0$ in Case B. So Case A applies (indeed, both Case A and Case B yield $h = \bar{h}$ at $t = t_1$), and the maximised value function of the harvesting problem is given by eq. (22):

$$J_{2A}^*(s_1, t_1) = s_1 \frac{\bar{h}}{\bar{h} - 1} \left(1 - e^{(1-\bar{h})(T-t_1)}\right). \quad (52)$$

After substitution of $s(t_1) = s_0 e^{t_1}$, we obtain the derivative of the value function $J_{2A}^*(t_1) \equiv J_{2A}^*(s(t_1), t_1)$:

$$\frac{dJ_{2A}^*(t_1)}{dt_1} = - \frac{s_0 e^{t_1} \bar{h} \left(\bar{h} e^{(\bar{h}-1)(t_1-T)} - 1\right)}{\bar{h} - 1}.$$

It is easy to show that the sign of this derivative depends on whether the switching point τ is before or after the arrival time t_1 :

$$\frac{dJ_{2A}^*(t_1)}{dt_1} \geq 0 \quad \Leftrightarrow \quad T - t_1 \geq \delta \equiv \frac{\log(\bar{h})}{\bar{h} - 1}.$$

Since δ is a decreasing function of \bar{h} , the derivative of dJ_{2A}^* is positive for large, and negative for small values of \bar{h} . If the harvesting capacity, when compared

²⁰Condition (50) represents a modification of the usual necessary condition for the free terminal time, as provided, for example, by Léonard and Long (1992, Theorem 7.6.1).

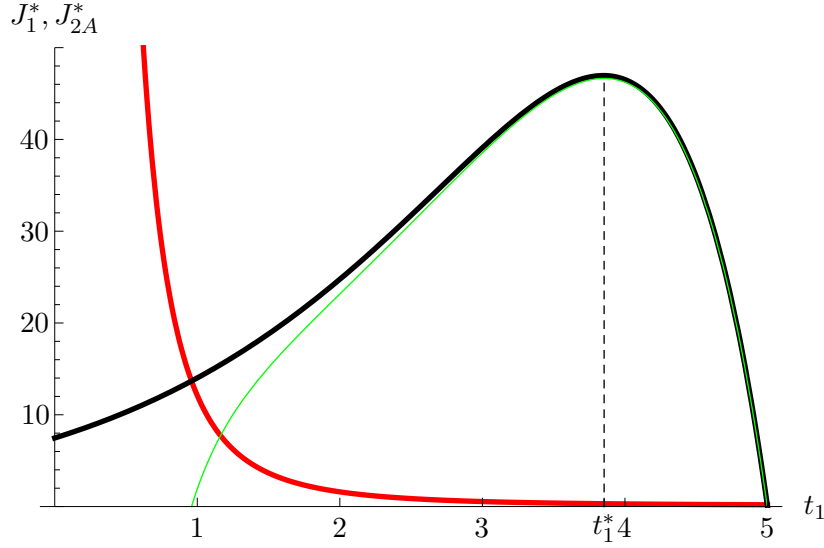


FIGURE 14. Value function J_{2A}^* (black), cost function J_1^* (red), and profit function $-J_1^* + J_{2A}^*$ (green), for $s_0 = 1$, $T = 5$ and $\bar{h} = 3/4$, yielding the optimal arrival time $t_1^* = 3.8539$.

with the length of the harvesting period $\Delta \equiv T - t_1$, is large, a later arrival time increases the yield from the harvesting period because it give the resource more time to grow while at the same time the harvesting capacity is large enough so as to harvest high volumes in a shorter time interval; in this case, the agent may wish to postpone the arrival. But when the harvesting capacity is relatively low, a postponing the start of the harvesting activity is unattractive, as the agent will be unable to benefit from the higher stock due to the constraint on the harvesting capacity. Hence, with negligible travelling cost the optimal arrival time will be equal to $t_1^* = T - \delta$. This arrival time balances the benefits from an earlier and a later arrival.

Now, using the maximised Hamiltonian of sub-section 5.1, $\mathcal{H}^* = -36/t_1^4$, the transversality condition (50) reads as

$$-\frac{36}{t_1^4} + s_0 e^{t_1} \frac{\bar{h}}{\bar{h} - 1} \left(\bar{h} e^{(\bar{h}-1)(t_1-T)} - 1 \right) = 0. \quad (53)$$

This condition determines t_1^* as a function of the exogenous parameters \bar{h} and T . Since \mathcal{H}^* is negative, the derivative $dJ_{2A}^*(t_1)/dt_1$ must be negative as well in order for (53) to have a solution t_1^* . Consequently, in the optimal travelling-and-harvesting policy, the length of the harvesting period $\Delta \equiv T - t_1$ is lower than the harvesting period the agent would have chosen in the absence of the need for travelling (assuming $T > \delta$); that is, in the optimal solution Case A prevails, *i. e.* $T - t_1 < \delta$. In other words, in the optimal solution arrival is relatively late given the harvesting capacity \bar{h} . We have already seen that in Case A the agent

begins with harvesting at the maximum rates immediately at time t_1 . This result also holds in the optimal solution, for an early arrival is associated with higher travelling cost, so that a premature arrival should be avoided.

The functions $J_{2A}(t_1)$ and $J_1^*(t_1)$, given by eq. (52) and (47) respectively, are depicted in Figure 14 for parameter values $s_0 = 1$, $T = 5$ and $\bar{h} = 3/4$. With these parameters the optimal solution, indicated by the dashed line in the figure, equals $t_1^* = 3.8539$, yielding a net profit equal to $J_{2A}(t_1^*) - J_1(t_1^*) = 46.6488$.

5.4. Optimal travelling-and-harvesting policy for logistic growth. By assumption $s(t_1)$ is fixed at $s_1 = 2$, so that we can choose only t_1 without affecting s_1 though.

With logistic growth we have to consider both Case A and Case B.

5.4.1. Case A: either $\bar{h} < 1$ or $1 < \bar{h} < 2$ and $T \leq T_c$. For $\bar{h} < 1$, we have $h(t) = \bar{h}$ for all $t \in \Delta$ and the resulting profit is given in eq. (41):

$$J_{2A}(t_1) = \bar{h} \int_{t_1}^T x(t) dt = \bar{h} \log \left(\frac{2e^{(2-\bar{h})(T-t_1)} - \bar{h}}{2 - \bar{h}} \right). \quad (54)$$

The time derivative of J_A equals

$$\frac{dJ_{2A}(t_1)}{dt_1} = -\frac{2(\bar{h} - 2)\bar{h}}{\bar{h}e^{(\bar{h}-2)(T-t_1)} - 2}.$$

This derivative is negative as the numerator and the denominator are both negative, as $\bar{h} < 2$.

The functions $J_{2A}(t_1)$ and $J_1^*(t_1)$, given by eq. (54) and (47) respectively, are depicted in Figure 15 for $T = 5$ and $\bar{h} = 3/4$. With these parameters the optimal solution, indicated by the dashed line in the figure, equals $t_1^* = 2.47928$ yielding a net profit equal to $J_{2A}(t_1^*) - J_1(t_1^*) = 1.81612$. Observe that Case A actually results for $T = 5$ and $\bar{h} = 3/4$, see Figure 11.

5.4.2. Case B: $1 < \bar{h} < 2$ and $T > T_c$. In case of a higher harvesting capacity, say $\bar{h} = 3/2$, Case B results (see again Figure 11). This scenario is depicted in Figure 16. With these parameters the optimal solution is given by $t_1^* = 2.44949$ yielding a net profit of $J_{2B}(t_1^*) - J_1(t_1^*) = 2.73263$.

Had we chosen some later starting time $t_1 > t_c \equiv T - T_c = 5 - 2 \log(3) = 2.8028$, then Case A would become relevant as the fisher had less than the required minimal time for fishing in Case B, $T_c = T - t_c = 2 \log(3) = 2.1972$.

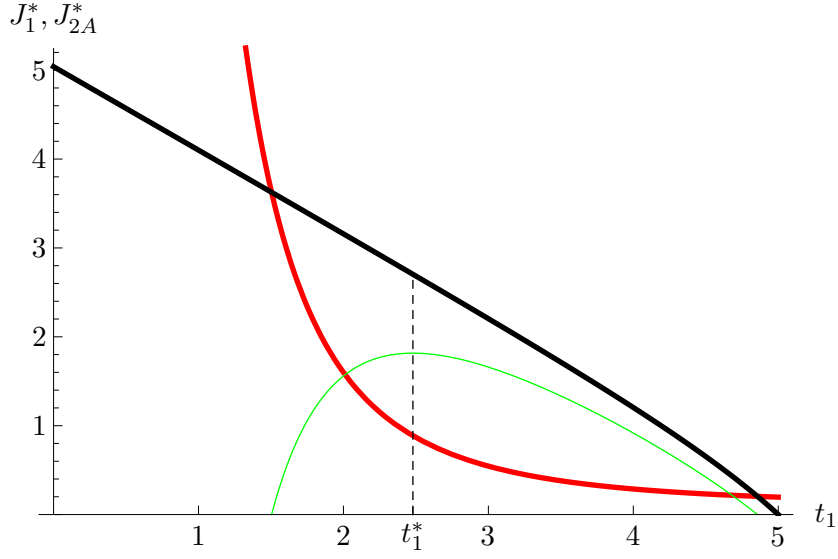


FIGURE 15. Value function J_{2A}^* (black), cost function J_1^* (red), and profit function $-J_1^* + J_{2A}^*$ (green), for $T = 5$ and $\bar{h} = 3/4$, yielding the optimal arrival time $t_1^* = 2.47928$.

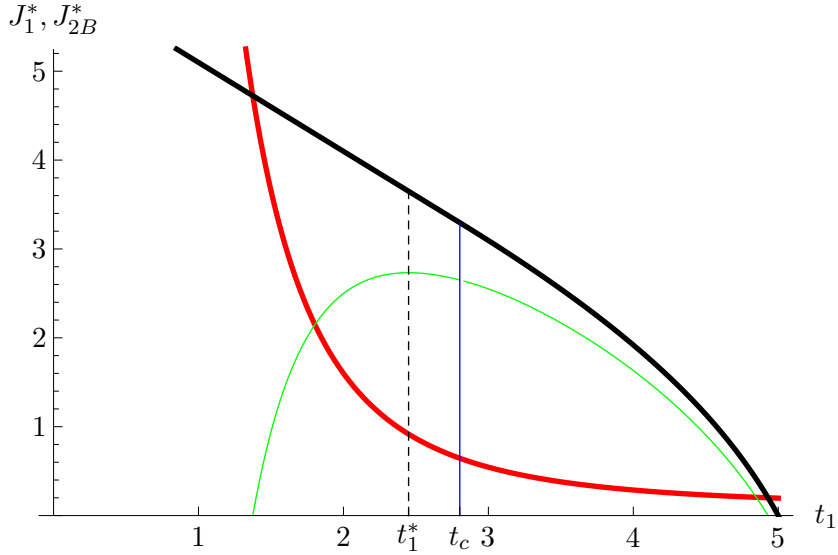


FIGURE 16. Value function J_{2B}^* (black) with the critical arrival time $t_c = 5 - 2\log(3) = 2.8028$ (blue), cost function J_1^* (red), and profit function $-J_1^* + J_{2B}^*$ (green), for $T = 5$ and $\bar{h} = 3/2$, yielding the optimal arrival time $t_1^* = 2.44949$.

6. Robustness of our results: a positive discount rate and bounds on the acceleration

We begin our investigation of the robustness of the model with the travelling problem. Using our previous specification Assume that the cost of travelling depends linearly on speed v and quadratically on acceleration a :

$$K(v, a) = cv + a^2$$

with $c = 1/10$, we now set $\rho = 1/20$ and, for the moment, $T = 40$. Also, we assume that acceleration is bounded to $\mathcal{A} = [\underline{a}, \bar{a}] = [-1, +1]$. With this specification the objective function is given by

$$J_1(a(t)) = \int_0^{40} e^{-t/20} \left(\frac{v(t)}{10} + a(t)^2 \right) dt.$$

Acknowledging the system of differential equations (1) governing the movement of the agent

$$\dot{x}(t) = v(t) \quad \text{and} \quad \dot{v}(t) = a(t) \quad \forall t \in \mathcal{T},$$

the Hamiltonian is given by

$$\mathcal{H} = -K(v(t), a(t)) + \psi_1(t)v(t) + \psi_2(t)a(t) = -\frac{v(t)}{10} - a(t)^2 + \psi_1(t)v(t) + \psi_2(t)a(t),$$

and the Lagrangean reflecting the restriction $\underline{u} \leq u \leq \bar{u}$ by

$$L = -\frac{v(t)}{10} - a(t)^2 + \pi_1(t)v(t) + \pi_2(t)a(t) + \lambda_1(t)(a(t) + 1) + \lambda_2(t)(1 - a(t)).$$

Note the Hamiltonian and the costate variable are now defined in current values.²¹ Accordingly, the necessary conditions are modified to account for the fact that the costate variables are defined in current values:

$$-2a(t) + \pi_2(t) + \lambda_1(t) - \lambda_2(t) = 0 \quad \Leftrightarrow \quad a(t) = \frac{1}{2} (\pi_2(t) + \lambda_1(t) - \lambda_2(t)). \quad (55)$$

In addition, we have the necessary conditions

$$\dot{\pi}_1(t) = -\frac{\partial L}{\partial x(t)} + \rho\pi_1 = \frac{\pi_1(t)}{20}, \quad (56)$$

$$\dot{\pi}_2(t) = -\frac{\partial L}{\partial v(t)} + \rho\pi_2 = -\pi_1(t) + \frac{\pi_2(t)}{20} + \frac{1}{10}. \quad (57)$$

²¹So, H should be read as \tilde{H} , and π as ψ in the notation of Léonard and Long (1992).

Analysis of the unbounded solution. In this case we have $\lambda_1(t) = 0 = \lambda_2(t)$, and equations (1) and (55) simplify to

$$x'(t) = v(t), \quad v'(t) = \frac{\pi_2(t)}{2} \quad (58)$$

together with equations (56) and (57). Using the, yet unspecified, initial values $\pi_1(0) = m_1$ and $\pi_2(0) = c_1$, we obtain

$$\pi_1(t) = m_1 e^{t/20}, \quad \pi_2(t) = e^{t/20}(c_1 - m_1 t + 2) - 2. \quad (59)$$

Substituting eq. (59) into (55) yields the system

$$x'(t) = v(t), \quad v'(t) = \frac{1}{2} (e^{t/20}(c_1 - m_1 t + 2) - 2), \quad x(0) = 0, \quad v(0) = 0.$$

Using the initial values and solving the resulting initial value problem, we obtain

$$\begin{aligned} x(t) &= \frac{1}{2} (20c_1 (-t + 20e^{t/20} - 20) - 400m_1 t \\ &\quad - 400e^{t/20}(m_1(t - 40) - 2) - 16000m_1 - t^2 - 40t - 800), \\ v(t) &= 10c_1 (e^{t/20} - 1) - 10e^{t/20}(m_1(t - 20) - 2) - 200m_1 - t - 20. \\ \pi_1(t) &= m_1 e^{t/20}, \\ \pi_2(t) &= e^{t/20}(c_1 - m_1 t + 2) - 2. \end{aligned}$$

Finally, using the terminal conditions $x(T) = 300$ and $v(T) = 0$ to determine the constants, we obtain

$$c_1 = \frac{-25 + 35e^2 - 4e^4}{2(1 - 6e^2 + e^4)}, \quad m_1 = \frac{13 + 3e^2}{40(1 - 6e^2 + e^4)}.$$

Thus, the solution is given by

$$x(t) = -\frac{1}{2(1 - 6e^2 + e^4)} (e^4 t^2 - 6e^2 t^2 + t^2 + 140e^2 t - 80t + 10e^{t/20} \quad (60)$$

$$(13t + e^2(3t - 340) - 100) + 3400e^2 + 1000), \quad (61)$$

$$v(t) = -\frac{e^{t/20} (13t + e^2(3t - 280) + 160) + 4(t + e^2((e^2 - 6)t + 70) - 40)}{4(1 - 6e^2 + e^4)}, \quad (62)$$

$$\pi_1(t) = \frac{(13 + 3e^2)e^{t/20}}{40(1 - 6e^2 + e^4)}, \quad (63)$$

$$\pi_2(t) = -\frac{e^{t/20} (13t + e^2(3t - 220) + 420)}{40(1 - 6e^2 + e^4)} - 2, \quad (64)$$

$$a(t) = \frac{e^{\frac{t}{20}+2}(220 - 3t) - e^{t/20}(13t + 420) - 80e^4 + 480e^2 - 80}{80(1 - 6e^2 + e^4)}. \quad (65)$$

Substituting this into the objective function yields

$$J_1^* = \frac{80 - 1725e^2 + 925e^4 - 80e^6}{4(e^2 - 6e^4 + e^6)} \approx 16.7095.$$

Analysis of the bounded solution. Now, assume that there are bounds on the control: $a(t) \in [\underline{a}, \bar{a}] \equiv [-1, 1]$. As we can see from Figure 17, the unbounded solution (blue case) hits the lower bound, at time $t = 34.2818$. Since the upper boundary $\bar{u} = 1$ is not binding, it suffices to consider the hitting of the lower bound only. We must have $a(t) = \underline{a} = -1$ for all t in the final interval $(\xi, T]$. Apparently, we must choose some $\xi < 34.2818$, for if $\xi = 34.2818$ the remaining time would only suffice to guarantee the terminal condition $v(T) = 0$, if we were able to set $u < \underline{u}$. Thus, during the final time interval $(\xi, T]$, the solution must satisfy

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = -1, \quad x(40) = 300, \quad v(40) = 0,$$

the solution of which is

$$x(t) = \frac{1}{2}(-t^2 + 80t - 1000), \quad v(t) = 40 - t, \quad a(t) = -1. \quad (66)$$

This is the optimal travelling policy and the end of the travelling period.

We must now calculate the optimal switching point ξ , which must be determined so that the following boundary conditions (for the first interval) are met:

$$x(0) = 0, \quad v(0) = 0, \quad x(\xi) = \frac{1}{2}(-\xi^2 + 80\xi - 1000), \quad v(\xi) = 40 - \xi.$$

Together with the optimality conditions for the unbounded problem, eqs (58), (56) and (57), this yields the system

$$\dot{x}(t) = v(t), \quad \dot{v}(t) = \frac{\pi_2(t)}{2}, \quad (67)$$

$$\dot{\pi}_1(t) = \frac{\pi_1(t)}{20}, \quad \dot{\pi}_2(t) = -\pi_1(t) + \frac{\pi_2(t)}{20} + \frac{1}{10}, \quad (68)$$

$$x(0) = 0, \quad v(0) = 0, \quad (69)$$

$$x(\xi) = \frac{1}{2}(-\xi^2 + 80\xi - 1000), \quad v(\xi) = 40 - \xi, \quad (70)$$

the solution of which gives the optimal travelling policy in the interval $[0, \xi]$:

$$\begin{aligned} a(t) = & \frac{1}{-e^{\xi/20}(\xi^2 + 800) + 400e^{\xi/10} + 400} \times \\ & \left(e^{\frac{t+\xi}{20}}(t(65 - 2\xi) + \xi(2\xi - 105) + 2100) - 5e^{t/20}(13t + 420) \right. \\ & \left. + e^{\xi/20}(\xi^2 + 800) - 400e^{\xi/10} - 400 \right), \end{aligned}$$

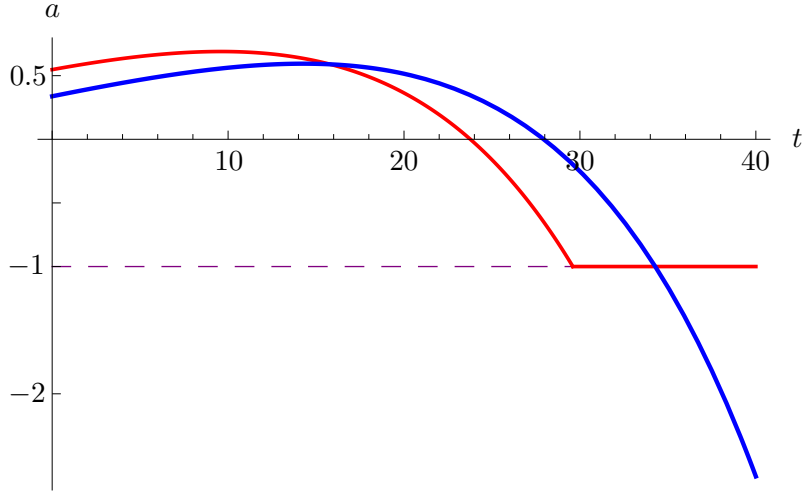


FIGURE 17. Optimal acceleration with (red) and without (blue) bounds.

which reaches the lower bound $\underline{a} = -1$ at time $\xi = 29.5984$. Using this value, the solution of system (67)–(70) is given by

$$x(t) = e^{t/20}(1455.52 - 20.9131t) - 0.5t^2 - 51.8628t - 1455.52, \quad (71)$$

$$v(t) = e^{t/20}(51.8628 - 1.04565t) - t - 51.8628, \quad (72)$$

$$\pi_1(t) = 0.104565e^{t/20}, \quad (73)$$

$$\pi_2(t) = e^{t/20}(3.09497 - 0.104565t) - 2, \quad (74)$$

and hence the optimal policy is given by

$$a(t) = \frac{1}{2} (e^{t/20}(3.09497 - 0.104565t) - 2) \quad (75)$$

Finally, composing both parts we obtain the optimal solution, which is illustrated in Figures 17, 18 and 19.

The optimal objective value equals $\overline{J}_1^*(40, 300) = 18.4648$ for the case of a bounded control, compared to $J_1^*(40, 300) = 16.7095$ for the case of an unbounded control. So, as expected, the presence of the bound on acceleration results in an increase in travelling cost. We may also compare our result with the case with a zero discount rate, explored in Section 5. Applying the specification $T = 40$, $x(T) = 300$ and $\rho = 0$, we obtain $J_1^*(40, 300)|_{\rho=0} = 375/8 = 46.875$. Clearly, discounting lets part of the cost disappear, so that here the cost without discounting are significantly exceed those with discounting, as the time horizon is larger. Also, as can be seen from Figures 17–19, discounting makes the agent initially move more slowly and speed up later so that part of the travelling cost is shifted to the future. In case of bounds on the control, such cost shifting becomes limited so that some part of the travelling cost must be incurred earlier.

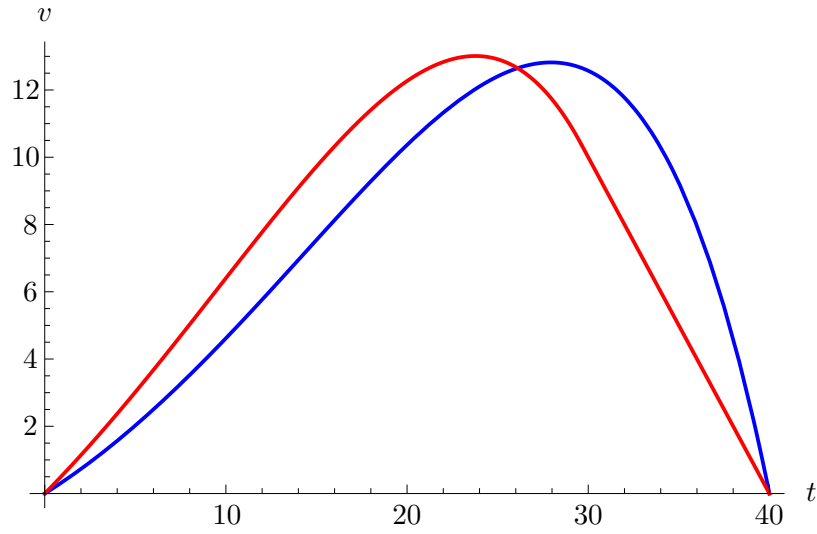


FIGURE 18. Optimal speed with (red) and without (blue) bounds on acceleration.

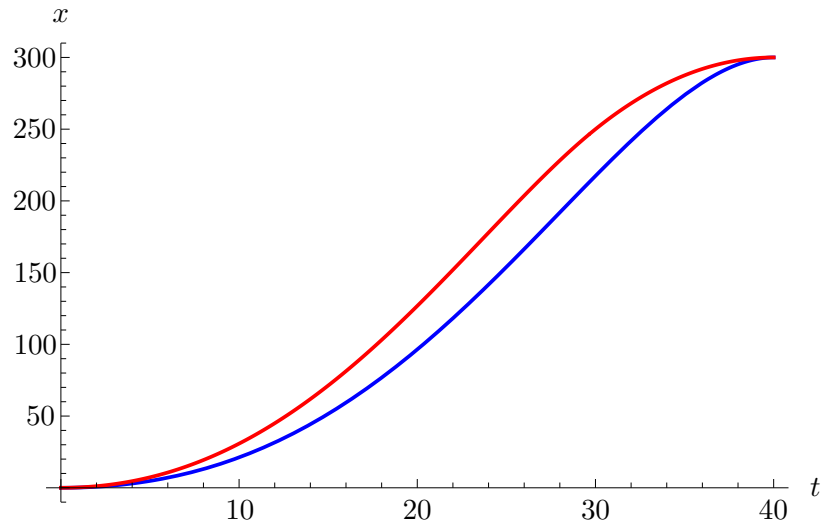


FIGURE 19. Optimal position with (red) and without (blue) bounds on acceleration.

We have thus shown that our results for the optimal travelling-and-harvesting policy for a resource that is located as some distance is robust to the introduction of both, a positive discount rate (as is commonly used in economics to take into account the opportunity costs) and bounds on the acceleration. Nevertheless, the introduction of a positive discount rate will affect the optimal acceleration path. Whereas the original path is decreasing linearly leading to a symmetric and concave velocity curve, the optimal acceleration under discounting is now concave too, first increasing then decreasing. This reflects the fact that present acceleration

and velocity costs have become more costly compared to future ones. As a consequence, such costs are shifted towards the future, leading to a concave acceleration profile. In case that acceleration is bounded from below, the slowing down is constrained, leading to an acceleration profile with a higher (and therefore costlier) acceleration at the start (leading to a slightly larger maximum speed) which may be afforded as later discounted costs of slowing down are exogenously limited. The velocity curves and the optimal position curves reflect these optimal acceleration patterns accordingly. However, irrespective of the particular acceleration profile, the qualitative way in which the travelling and the harvesting decision interact so as to determine the optimal arrival time, which coincides with the start of the harvesting activity, is unaffected by both the discount rate and possible bounds on acceleration.

7. Conclusion

In this paper we contribute to the theory of spatial resource economics. We explicitly take into account the fact that in many settings an agent has to travel to the location of the resource before being able to harvest that resource. Although some papers in the literature acknowledge the requirement of an agent to travel (*e.g.* Behringer and Upmann, 2014; Belyakov et al., 2015) the approach in this paper is novel in that the resource cannot be harvested in an *en passant* manner, *i.e.* the agent has to stop at the location of the resource in order to harvest. The travelling problem then is preceding the harvesting problem, and because of the growth process of the resource both sub-problems are linked by both the spatial and the temporal dimension making the arrival time at the resource an optimal control decision. We are able to fully characterize the control programme for the composed travelling-and-harvesting problem employing recent tools for two-phase dynamic optimization problems.

We investigate the consequences that different growth processes (exponential and logistic) have on the optimal harvesting control paths—which, in principle, allow the agent not to commence with harvesting immediately upon arrival at the resource, but to leave the resource unimpaired for some time—and characterize the implied optimal yields. Translating optimal yields into economic profits, we allow for a positive discount factor and investigate its consequences on the optimal travelling decision of the agent. We find that such a discount factor changes the optimal movement of the agent in an intuitive manner. Costs from acceleration and speed are shifted towards the future. We also allow for the fact that acceleration of the agent may be bounded by technical constraints which lead to yet another intuitive adjustment of the agent’s optimal control paths while obeying the travel cost minimizing objective.

Overall, we have demonstrated that acknowledging a spatial dimension in the classical renewable resource harvesting context can lead to tractable and economically relevant changes that even allow for the introduction of realistic periods of travelling and their associated economic costs. This extension, besides contributing to the call of introducing a spatial dimension and so making resource economics more realistic (*e. g.* Sanchirico and Wilen, 2005), allows for an even more precise extension of the theory into a realm where space implies that the agent also faces a transportation problem that is temporarily and spatially linked to the resource extraction problem.

An obvious extension of the above analysis consists of investigating sequences of travelling-and-harvesting problems that result from having multiple remote renewable resources. This is left for future research.

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