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The Dark Side of the Force: Evolutionary Equilibrium in Contests with Stochastic Entry*

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Abstract

In this paper we study evolutionarily stable strategy (ESS) in contests where participation is stochastic. We find under ESS players exert more effort than under Nash for given participation probabilities. We show that with ESS there is ex-ante overdissipation for sufficiently large participation probability and sufficiently high discriminative power of the contest success function. With costly endogenous entry, players are also more likely to enter the contest, and hence they incur higher total cost and obtain lower absolute payoff under ESS than Nash. Ex-ante overdissipation also occurs for concave impact functions. From a contest designer's point of view, implementing ESS induces players to exert higher total effort and thus yields higher total expected revenue.

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1 Introduction

In many situations, a player in a contest does not know the number of opponents. For example, in a job interview, an applicant may not know the number of short-listed interviewees. Likewise, a lobbyist may not know how many others he/she is lobbying against. Animals competing for food or mating opportunities may not be able to perceive the actual number of competitors or to tune their effort level according to the number of contestants. In this paper, we consider a situation where players are drawn to play stochastically without knowing the actual number of opponents.

Differing from the standard Nash solution concept employed in the literature, in this paper, we investigate the evolutionarily stable strategy (ESS) in such contests. The reason is twofold. Firstly, compared to Nash equilibrium, ESS focuses on the behaviour itself rather than on the choice of behaviour by the players. Thus ESS demands neither players' rationality nor the consistency between behaviour and beliefs, an essential element to Nash equilibrium.¹ Secondly, insofar as the behaviour itself is concerned, the concept of ESS in finite population, as noted in Schaffer (1988), amounts to a (symmetric) Nash equilibrium with relative payoff maximization in the standard rational choice paradigm. Hence, a study of ESS in finite populations will also shed light on equilibrium outcomes with rational players whose preferences display a concern for relative payoff.²

We consider two scenarios. First, we study stochastic contests with exogenous participation probabilities. For this case, we show that findings in Hehenkamp et al. (2004) on the ESS in deterministic contests, where players participate with certainty, generalize to stochastic contests, where other players' participation is uncertain. In particular, we show that the ESS in

¹See e.g. Perea (2012) for an excellent textbook treatment on the subject of epistemic game theory.

²Seminal contributions on this type of preferences are e.g., Messick and Thorngate (1967) as well as Bolton and Ockenfels (2000) and Fehr and Schmidt (1999) among many others.

stochastic contests has each potential contestant exerting a higher level of effort than in Nash equilibrium. More precisely, individual effort in the ESS is exactly $n/(n-1)$ of its Nash counterpart in a contest with n potential contestants.

Naturally, ex-ante total expected spending increases in the number of players as well as in the participation probability. While Lim and Matros (2009) demonstrate that under Nash equilibrium overdissipation is only possible ex-post, we show that under the economic evolutionary approach, the ESS entails ex-ante as well as ex-post overdissipation for sufficiently large participation probability and sufficiently high discriminative power of the contest success function.

To further understand the role of uncertainty on effort expenditure, one can compare contests of the same (expected) number of players. It is known that under Nash equilibrium the expected total expenditure is lower when participation is stochastic rather than deterministic (Lim and Matros, 2009). That is, within the same equilibrium concept, uncertainty about the actual number of contestants reduces total effort. However, across equilibrium concepts and for the same (expected) number of players, the expected total expenditure under stochastic participation can be higher under the ESS than the total expenditure under Nash equilibrium in the corresponding deterministic contest. This suggests that when participation is stochastic, a contest designer could try to induce ESS behaviour to mitigate the loss in revenue caused by uncertainty.³

Second, we consider endogenous entry. We find that, for a given entry cost, ESS participation probabilities strictly exceed Nash participation probabilities whenever the latter fall strictly below one, i.e. when endogenous entry is truly stochastic in Nash equilibrium. Otherwise, endogenous entry is deterministic under both concepts otherwise. Ex-ante total expenditure as well as total entry costs are also higher under ESS. Accordingly, players behave more aggressively under ESS than under Nash along both dimensions, entry and effort.

³To this end, insights gained from experimental economics could be used. For instance, Huck et al. (1999) show that information on strategies and profits induces imitative and hence more competitive behaviour in a Cournot oligopoly. There, the ESS corresponds to (symmetric) Walrasian equilibrium. See e.g. Vega-Redondo (1996).

The papers most closely related to the present one are Lim and Matros (2009), Fu et al. (2015), and Hehenkamp et al. (2004). Lim and Matros (2009) (LM henceforth), characterize the Nash equilibrium of stochastic contests with exogenous participation probabilities. They show that individual spending is single-peaked in the participation probability while total spending is monotonically increasing in the participation probability and the number of players. We establish a precise relationship between Nash spending and ESS spending in stochastic contests and show that these properties of Nash equilibrium spending are also present in ESS.

Fu et al. (2015) (hereafter FJL) study contests with endogenous entry employing the Nash solution concept. The authors show that a Tullock contest can be optimal for the contest designer and identify the conditions under which the optimum can be achieved by solely setting the right discriminatory power in a Tullock contest with a single fixed prize.⁴ In contrast, we investigate the evolutionary stability of entry probabilities and effort levels in Tullock contests.

Hehenkamp et al. (2004) (hereafter HLP) use the ESS concept in a finite population to study deterministic Tullock contests and demonstrate the possibility of rent overdissipation. We establish that their results on ESS existence and overdissipation extend to stochastic contests both under exogenous and under endogenous entry.

Stochastic participation in contest has also been featured in Myerson and Wärneryd (2006), Münster (2006) and Fu et al. (2011). Myerson and Wärneryd (2006) study contests in which a player is uncertain about the actual size of the contest. They do not assume a particular distribution of the contest size and it can potentially be infinitely large. In Münster (2006), the size of a contest follows a binomial distribution as in the current paper. However, Münster (2006) compares risk-neutral players with CARA players. Fu et al. (2011) study how an effort-maximizing contest organizer should disclose the information on the actual number of contestants in a contest with stochastic participation. The present paper departs from these papers and from LM by exploring a different equilibrium concept, viz. evolutionary stable strategies.

⁴Fu and Lu (2010) study the optimal choice of prize size and entry fee/subsidy under a fixed budget when the contest designer maximizes effort and entry is endogenous.

ESS in finite population and relative payoff maximization are closely related, and the latter has been used to explain overbidding compared to theoretical Nash predictions one observes in laboratory settings. For instance, Herrmann and Orzen (2008) report evidence that can attribute subjects' investment decisions to spiteful preferences rather than fairness or reciprocity. Mago et al. (2016) present a behavioural model that incorporates a non-monetary utility of winning and relative payoff maximization that explains significant overexpenditure of effort in their controlled laboratory experiment.⁵

That preferences with a concern for relative payoff are relevant to the analysis of contests has also been argued from the perspective of the so-called indirect evolutionary approach (Güth and Yaari, 1992). Examining deterministic Tullock contests of the type investigated in HLP, Guse and Hehenkamp (2006) show that contestants with a relative payoff concern (so-called *status-seekers*) earn strictly higher payoff than do their counterparts who follow absolute payoff maximization. This gives them a material advantage so that status-seekers should prevail in the long-run. Leininger (2009) rounds off the indirect evolutionary approach by deriving the evolutionarily stable type of preferences for Tullock contests, which, as it turns out, displays a relative payoff concern. Both papers strongly warrant an analysis where contestants exhibit relative payoff concerns.

We proceed as follows. Section 2 introduces the model and presents the analysis for the case of exogenous entry. Section 3 is devoted to the case of endogenous entry. Finally, Section 4 concludes.

2 Stochastic contests with exogenous entry

We consider a stochastic contest of $n \geq 2$ potential players as in LM. Each potential player is drawn to play, i.e., becomes active, with an independent probability $p \in (0, 1]$. All active players compete for a single prize of value $V > 0$ by selecting an effort level $X_i \in [0, +\infty)$.

Conditional on being active, player i 's probability of winning the contest is

⁵For an excellent survey on experimental studies on contests see Dechenaux et al. (2015).

given by

$$P_i(X_i; \mathbf{M}) = \begin{cases} \frac{1}{|\mathbf{M}|+1} & \text{if } X_i = 0 \text{ and } X_j = 0 \text{ for all } j \in \mathbf{M}, \\ \frac{X_i^r}{X_i^r + \sum_{j \in \mathbf{M}} X_j^r} & \text{otherwise,} \end{cases}$$

where $0 < r \leq n/n-1$, \mathbf{M} is the set of active players *except* player i , and $|\mathbf{M}|$ denotes the cardinality of \mathbf{M} . The payoff of inactive players is 0.

Before studying the ESS outcome, we note the below unique Nash equilibrium in this stochastic contest.

Theorem 1. There exists a unique symmetric pure-strategy Nash equilibrium where each active player's equilibrium expenditure is

$$X^{\text{Nash}}(r, V, n, p) = rV \left[\sum_{i=1}^{n-1} C_i^{n-1} p^i (1-p)^{n-1-i} \frac{i}{(i+1)^2} \right], \quad (1)$$

where the binomial coefficient $C_i^{n-1} = \frac{(n-1)!}{i!(n-i-1)!}$.

Proof: Lim and Matros (2009) establish this result for $0 < r \leq 1$. For $1 < r \leq n/(n-1)$ see Gu and Hehenkamp (2016).

2.1 Evolutionary stable strategy with stochastic participation

We now proceed to characterise the evolutionarily stable strategy. For this purpose, we first adapt Schaffer's 1988 evolutionary stable equilibrium condition for games with stochastic participation.

Let us consider a finite population of $n \geq 2$ players each being drawn to play with an independent probability $p \in (0, 1]$. Instead of playing a fixed size contest, an active player in our context may face $k = 0, \dots, n-1$ active opponents. Thus the (expected) payoff of an *active* mutant playing \bar{s} when the rest of the population playing $s^{\text{ESS}}, \bar{\pi}$, is the sum of the payoffs when \bar{s} playing against $k = 0, \dots, n-1$ ESS strategists weighted by the probability of each case. On the other hand, for an *active* ESS strategist, with a probability p the mutant will be present and $(1-p)$ not. In the former case, the ESS strategist faces the mutant and possibly also $k = 0, \dots, n-2$ other ESS

strategists while in the latter only other ESS strategists. The (expected) payoff of an ESS strategist, π^{ESS} is thus the weighted average of the payoffs in those individual cases. For s^{ESS} to be evolutionary stable, following Smith and Price (1973) and Schaffer (1988), we need $p \cdot \pi^{ESS} \geq p \cdot \bar{\pi}$.⁶

We now apply this adapted definition to Tullock contests with stochastic participation. Let X denote the candidate strategy for an ESS and \bar{X} denote the mutant strategy. We consider invasions by a single mutant, say w. l. o. g. player 1. When being active, the expected payoff of player 1 is

$$\Pi_1(\bar{X}, X, \dots, X) = V \left[\sum_{M \in \mathcal{P}^{N_1}} p^{|M|} (1-p)^{|N_1 \setminus M|} \cdot \frac{\bar{X}^r}{\bar{X}^r + |M|X^r} \right] - \bar{X}, \quad (2)$$

where N_1 is the set of player 1's possible opponents and \mathcal{P}^{N_1} is the set of all subsets of N_1 .

The expected payoff of an ESS strategist conditional being active, say player 2, is

$$\begin{aligned} & \Pi_2(\bar{X}, X, \dots, X) \\ &= p V \left[\underbrace{\sum_{M \in \mathcal{P}^{N_2}} p^{|M|} (1-p)^{|N_2 \setminus M|} \cdot \frac{X^r}{\bar{X}^r + X^r + |M|X^r}}_{\text{The mutant being active}} \right] \\ &+ (1-p) V \left[\underbrace{\sum_{M \in \mathcal{P}^{N_2}} p^{|M|} (1-p)^{|N_2 \setminus M|} \cdot \frac{X^r}{X^r + |M|X^r}}_{\text{The mutant being inactive}} \right] - X, \quad (3) \end{aligned}$$

where N_2 is the set of player 2's possible opponents *except* the mutant player 1, and \mathcal{P}^{N_2} is the set of all subsets of N_2 .

As noted by Schaffer (1988), a strategy X is an ESS if and only if the relative payoff between a mutant and an ESS strategist, as a function of \bar{X} , reaches its maximum value of zero when $\bar{X} = X$. Let $\phi(\bar{X}, X) := \Pi_1(\bar{X}, X, \dots, X) -$

⁶Note that this definition differs from Schaffer's (1988) in that the contest size is not only variable but also stochastic.

$\Pi_2(\bar{X}, X, \dots, X)$. As $p > 0$, the ESS strategy X should solve

$$\max_{\bar{X}} \phi(\bar{X}, X). \quad (4)$$

To determine the solution to (4), consider the corresponding first order condition:

$$\begin{aligned} \frac{\partial \phi}{\partial \bar{X}} = & V \left[\sum_{\mathbf{M} \in \mathcal{P}^{N_1}} p^{|\mathbf{M}|} (1-p)^{|N_1 \setminus \mathbf{M}|} \cdot \frac{r \bar{X}^{r-1} |\mathbf{M}| X^r}{(\bar{X}^r + |\mathbf{M}| X^r)^2} \right] - 1 + \\ & pV \left[\sum_{\mathbf{M} \in \mathcal{P}^{N_2}} p^{|\mathbf{M}|} (1-p)^{|N_2 \setminus \mathbf{M}|} \cdot \frac{r \bar{X}^{r-1} X^r}{(\bar{X}^r + X^r + |\mathbf{M}| X^r)^2} \right] = 0. \end{aligned}$$

By symmetry, we have

$$\begin{aligned} X = & rV \left[\sum_{\mathbf{M} \in \mathcal{P}^{N_1}} p^{|\mathbf{M}|} (1-p)^{|N_1 \setminus \mathbf{M}|} \cdot \frac{|\mathbf{M}|}{(1 + |\mathbf{M}|)^2} \right] \\ & + prV \left[\sum_{\mathbf{M} \in \mathcal{P}^{N_2}} p^{|\mathbf{M}|} (1-p)^{|N_2 \setminus \mathbf{M}|} \cdot \frac{1}{(2 + |\mathbf{M}|)^2} \right]. \end{aligned} \quad (5)$$

Simplifying (5), verifying the second order condition, and comparing (5) with (1), we obtain the following result on the existence of ESS and its relationship with the Nash outcome.

Theorem 2 (Individual Spending). There exists a unique ESS in a Tullock contest with $r \leq n/n-1$ where each potential player becomes active with probability $p \in (0, 1]$. It is given by

$$X^{\text{ESS}}(r, V, n, p) = \frac{n \cdot rV}{n-1} \left[\sum_{i=1}^{n-1} C_i^{n-1} p^i (1-p)^{n-1-i} \frac{i}{(i+1)^2} \right]. \quad (6)$$

Moreover,

$$X^{\text{ESS}}(r, V, n, p) = \frac{n}{n-1} \cdot X^{\text{Nash}}(r, V, n, p). \quad (7)$$

Proof: See Appendix A.1.

Theorem 2 establishes the existence of the ESS strategy and characterises the individual equilibrium spending in this stochastic contest. Firstly, it generalizes the main insight in HLP. While in their paper $X^{\text{ESS}}(r, V, n, p) = \frac{n}{n-1} \cdot X^{\text{Nash}}(r, V, n, p)$ is shown for $p = 1$, we find this precise relationship holds much more generally, i.e., for all $p \in (0, 1]$. Thus, the relative aggressiveness of ESS behaviour is not affected by participation uncertainty.

Secondly, as expected, active players exert more effort when the prize size V is larger and the discriminative power r of the contest technology is higher. Due to the precise relationship (7), it follows from LM that $X^{\text{ESS}}(r, V, n, p)$ is single-peaked in p . In other words, individual spending in general reaches its maximum at a participation probability that lies strictly between 0 and 1.

Thirdly, when the population gets large, the difference between ESS and Nash equilibrium individual spending gets smaller. In accordance to Crawford (1990), as n goes to infinity, ESS converges to the Nash equilibrium level.

2.2 Total spending

In the contest literature, a variable that has received considerable interest is the equilibrium level of total effort because this variable may directly correspond to a contest organizer's objective. In this section, we therefore study ex-ante expected total spending.

Let T^{ESS} denote the ex-ante expected total expenditure. Note that the expected value of a random variable that follows the binomial distribution $B(n, p)$ is np . Consequently, ex-ante expected total ESS spending is given by

$$\begin{aligned} T^{\text{ESS}}(r, V, n, p) &:= np \cdot X^{\text{ESS}}(r, V, n, p) \\ &= rV \frac{n}{n-1} \left[\sum_{i=1}^n C_i^n p^i (1-p)^{n-i} \left(1 - \frac{1}{i} \right) \right]. \end{aligned}$$

It follows from Theorem 2 that the total ESS expenditure also corresponds to $n/(n-1)$ of its counterpart under Nash equilibrium denoted by $T^{\text{Nash}}(r, V, n, p)$.

Thus, from a contest organizer's point view, ESS yields strictly higher ex-ante expected total expenditure, which is due to more aggressive individual behaviour.

Although the relationship between individual spending and the participation probability p is non-monotonic, we find that ex-ante expected *total* spending increases in both n and p .

Theorem 3. Suppose r and V are given. Then,

- i) for any $n \geq 2$, the expected total spending increases in p ;
- ii) for any $p \in (0, 1]$, the expected total spending increases in n .

Proof: See Appendix A.4.

Part i) of Theorem 3 shows that total expected spending increases in players' participation probability. This observation follows directly from LM and intuitively the positive effect of p on the expected number of players, np , dominates any potentially negative effect on individual spending.

That total spending increases in n is not straightforward because T^{Nash} increases in n while the part $n/n-1$ decreases. Nevertheless, as Part ii) shows, with an increasing number of potential contestants ex-ante ESS total spending also increases. In other words, the increase in total Nash equilibrium expenditure dominates the decrease in players' aggressiveness. Hence, from the contest designer's point view, a reduction in either the participation probability or the number of potential players decreases expected total expenditure.

From the players' perspective, an important question is whether they gain ex ante from playing the contest. In other words, whether the expected revenue from winning the prize is more or less than the expected cost of effort. In a Nash equilibrium, ex ante no player will incur effort costs that exceed the expected revenue from winning the contest. The reason is that by exerting zero effort the player can always break even ex-ante. This is however not true when ESS is considered. As shown in deterministic contests by

HLP, ex ante overdissipation is an equilibrium outcome when ESS is considered. The next result generalizes their result to stochastic contests where the probability of participation is sufficiently high.

Theorem 4 (*Ex ante overdissipation*). For any given $n \geq 2$ and $r \in (1, n/(n-1))$, we have $T^{\text{ESS}} > V$ for p sufficiently large.

Proof: See Appendix A.5.

Theorem 4 shows that ex-ante overdissipation is present if both the discriminative power of the contest technology, r , and the probability of participation, p , are sufficiently high. Like in deterministic contests, overdissipation is a result of spiteful behaviour in the presence of increasing returns to expenditures ($r > 1$). However, as total expenditure decreases with the probability of participation, p has to be large enough to entail overdissipation. As it turns out, there is a substitution relationship between the discriminative power of the contest technology and the participation probability, i.e. the higher $r \in (1, n/(n-1)]$, the larger the overdissipation interval $(p, 1]$.

3 Stochastic contests with Endogenous Entry

In this section we let effort spending as well as participation probabilities be subject to evolutionary forces. The entry-bidding game is formulated as follows. There are $n \geq 2$ potential contestants. A strategy of player i is an ordered pair (p_i, X_i) where $p_i \in [0, 1]$ is the entry probability and $X_i \in [0, +\infty)$ the effort level.⁷ The payoffs are as before except that now to enter the contest a player has to pay a fixed cost of entry of $0 < c < V$.

3.1 (Bayesian) Nash Equilibrium

For convenience, here we restate the symmetric Nash equilibrium outcome which is studied in detail in FJL.

⁷For brevity, we do not consider mixed effort levels; according to FJL, the existence of pure strategy Nash equilibrium effort is ensured given the parameter values in the current paper.

Theorem 5 (Fu et al., 2015). There exists a unique symmetric equilibrium with pure-strategy bidding of the entry-bidding game in which (a) if $c \leq (n - r(n - 1))V/n^2$ the entry probability is $p^{\text{Nash}} = 1$; (b) otherwise it is implicitly determined by

$$V \frac{1 - (1 - p^{\text{Nash}})^n}{np^{\text{Nash}}} - X^{\text{Nash}}(r, V, n, p^{\text{Nash}}) = c \quad (8)$$

where individual spending $X^{\text{Nash}}(r, V, n, p)$ is given by (1).

Proof: See Appendix B.1.

Intuitively, when the expected payoff from entering exceeds the entry cost, it pays to enter, and hence the entry probability is 1. However, when the entry cost becomes high enough such that all players entering cannot be sustained, equilibrium competitiveness of the contest has to decrease - achieved by a reduced entry probability - to keep it worthwhile for players to play the contest. On the other hand, the equilibrium entry probability cannot be too low as this will lead to all players entering with probability 1. Thus, in equilibrium all players should be indifferent between entering and abstaining, and the required symmetric entry probability is implicitly determined by (8). This intuition indeed applies in the ESS case except that players would be concerned of relative payoff rather than absolute payoff.

3.2 ESS Outcome

Consider now a mutant, say player 1, which enters with probability q and spends \bar{X} . Conditional on entering, the mutant's expected payoff from the contest is given by (2) as before. The expected payoff of an ESS strategist conditional on entering, say player 2, now depends on the mutant's entry probability q :

$$\begin{aligned} & \Pi_2(\bar{X}, X, \dots, X; q) \\ &= q V \underbrace{\left[\sum_{M \in \mathcal{P}^{N_2}} p^{|M|} (1 - p)^{|N_2 \setminus M|} \cdot \frac{X^r}{\bar{X}^r + X^r + |M|X^r} \right]}_{\text{The mutant being active}} \end{aligned}$$

$$+(1-q)V \underbrace{\left[\sum_{M \in \mathcal{P}^{N_2}} p^{|M|} (1-p)^{|N_2 \setminus M|} \cdot \frac{X^r}{X^r + |M|X^r} \right]}_{\text{The mutant being inactive}} - X.$$

The ex-ante relative payoff denoted by Φ - after taking into account entry probabilities and entry cost - is

$$\Phi(q, \bar{X}; p, X; c) = q [\Pi_1(\bar{X}, X, \dots, X) - c] - p [\Pi_2(\bar{X}, X, \dots, X; q) - c]. \quad (9)$$

We note that exogenous entry represents a special case of the current setting in that $\Phi(p, \bar{X}; p, X; c) = p\phi(\bar{X}, X)$ for $p > 0$.

For a pair (p, X) to constitute an evolutionary stable strategy, no mutant can invade or, in other words, can obtain a higher payoff than ESS strategists. Thus, ESS requires $\Phi(q, \bar{X}; p, X; c) \leq 0$ for all $q \in [0, 1]$ and $\bar{X} \in [0, +\infty)$. To identify the ESS under endogenous entry, we consider the maximization problem

$$\max_{q, \bar{X}} \Phi(q, \bar{X}; p, X; c). \quad (10)$$

Applying symmetry to the first order condition with respect to \bar{X} , it reduces to (5), provided that $p > 0$. On the other hand, the first order derivative with regard to q is

$$\begin{aligned} \frac{\partial \Phi}{\partial q} = & V \left[\sum_{M \in \mathcal{P}^{N_1}} p^{|M|} (1-p)^{|N_1 \setminus M|} \cdot \frac{\bar{X}^r}{\bar{X}^r + |M|X^r} \right] - \bar{X} - c \\ & - pV \left[\sum_{M \in \mathcal{P}^{N_2}} p^{|M|} (1-p)^{|N_2 \setminus M|} \cdot \frac{X^r}{\bar{X}^r + X^r + |M|X^r} \right] \\ & + pV \left[\sum_{M \in \mathcal{P}^{N_2}} p^{|M|} (1-p)^{|N_2 \setminus M|} \cdot \frac{X^r}{X^r + |M|X^r} \right] \end{aligned} \quad (11)$$

Similarly, symmetry implies

$$\frac{\partial \Phi}{\partial q} = \frac{V}{n-1} \left[\frac{1 - (1-p)^n}{p} - (1-p)^{n-1} \right] - X - c. \quad (12)$$

These two necessary first order conditions identify the candidate ESS equi-

librium with endogenous entry. The next theorem shows they are also sufficient.

Theorem 6 (ESS with entry). An ESS strategist's entry probability p^{ESS} is (a) 1 if $c \leq (n - r(n - 1))V/(n(n - 1))$; (b) otherwise, it is implicitly determined by

$$\frac{V}{n - 1} \left[\frac{1 - (1 - p^{\text{ESS}})^n}{p^{\text{ESS}}} - (1 - p^{\text{ESS}})^{n-1} \right] - X^{\text{ESS}}(r, V, n, p^{\text{ESS}}) = c \quad (13)$$

where ESS individual spending $X^{\text{ESS}}(r, V, n, p^{\text{ESS}})$ is given by (6).

Proof: See Appendix B.2

As in the Nash equilibrium case, when the entry cost is sufficiently small, part (a) of the theorem says that all players enter with probability 1 in the ESS. In this degenerated case, individual spending $X^{\text{ESS}}(r, V, n, p^{\text{ESS}} = 1)$ takes the value of rV/n . Although individual spending is higher in ESS than in Nash, what matters in ESS is relative fitness and hence the marginal advantage of entering is higher. This can be seen as follows. Since $r < n/n-1$, $V/n-1 - rV/n > V/n - (n-1)rV/n^2$, for a given entry cost c , full participation is more likely under ESS than under Nash.

Consider part (b). In the proof, we establish that $\Phi(q, \bar{X}; p^{\text{ESS}}, X^{\text{ESS}}; c) \leq 0$ for all possible pairs of (q, \bar{X}) . The intuition follows from the exogenous case. Suppose, as a first step, that a mutant's entry probability is exogenously fixed at q . Then no other effort level than $X^{\text{ESS}}(r, V, n, p^{\text{ESS}})$ can give the mutant a higher relative fitness. In other words, for any possible exogenously given mutant entry probability, $X^{\text{ESS}}(r, V, n, p^{\text{ESS}})$ leads to highest relative payoff. On the other hand, given that all other players are ESS strategists, and the mutant is playing $X^{\text{ESS}}(r, V, n, p^{\text{ESS}})$, the mutant's relative payoff stays at zero for all entry probabilities which means it cannot obtain a higher absolute payoff than ESS strategists with any combinations of $q \in [0, 1]$ and $\bar{X} \geq 0$.⁸

For a given entry cost, by comparing part (a) in Theorem 5 and Theorem 6 respectively, we know full participation ($p = 1$) is more likely under ESS

⁸We formally show these results as Lemma 2 in Section B.2.

than Nash. The next theorem shows that for a given entry cost, the entry probability is in general higher in ESS than Nash.

Theorem 7. For a given $0 < c < V$, $p^{\text{ESS}} \geq p^{\text{Nash}}$. This inequality holds strictly when $\frac{V}{n} - \frac{n-1}{n^2}rV < c < V$, i.e., when $p^{\text{Nash}} < 1$.

Proof: See Appendix B.5.

The higher entry probability can also be explained by relative payoff maximization. Although when at the Nash probability level, entering results in a net loss in absolute payoff but it reduces opponents' even more. Therefore under ESS equilibrium entry is more aggressive.

It is also true that the total effort expenditure under endogenous entry is higher in ESS than in Nash.

Theorem 8. For a given $0 < c < V$, $T^{\text{ESS}}(r, V, n, p^{\text{ESS}}) > T^{\text{Nash}}(r, V, n, p^{\text{Nash}})$.

Proof: $T^{\text{ESS}}(r, V, n, p^{\text{ESS}}) \geq T^{\text{ESS}}(r, V, n, p^{\text{Nash}}) = \frac{n}{n-1}T^{\text{Nash}}(r, V, n, p^{\text{Nash}})$ where the inequality follows from Theorem 3 and that $p^{\text{ESS}} \geq p^{\text{Nash}}$. Q.E.D.

The intuition behind Theorem 8 follows from two observations. First, players enter more often under ESS than under Nash. Second, for a given entry probability players exert higher total effort in an ESS than in a Nash equilibrium. Thus although *a priori* it is not clear whether ESS total expected effort is higher or lower than its Nash counterpart due to endogenous entry, with the result of Theorem 7 and the monotonicity of total expected effort in participation probability ESS players do exert higher total effort.

Total costs incurred by the players consist of total effort cost and total entry costs. Theorems 7 and 8 together imply that players unequivocally spend more in ESS than in Nash: not only do they exert more effort upon entering, they on average also incur more entry cost. We also note that each player's absolute payoff in a Nash equilibrium with a non-degenerated entry probability has to be zero because of the indifference condition on entry choices. As a result, each player obtains a negative absolute payoff under ESS when entry is truly stochastic. The pressure of "survival of the fittest"

leads to more competitive behaviour and overall lower and absolute payoff than “rationality”.

Remark 1. When there is mixed entry under Nash ($0 < p^{\text{Nash}} < 1$), then the total cost of the players under ESS - including entry cost and effort cost - will be larger than the prize V . The reason is that, since under Nash players will just break even *ex ante*, under ESS their expected material payoff must be less than zero. This represents a different version of ex-ante “over-dissipation” result. than the one found in Hehenkamp et al. (2004). Interestingly, with endogenous entry, ex-ante overdissipation also occurs for concave impact functions X_i^r whenever $r < 1$ is sufficiently large. In two-player contests overdissipation results *for all* $r \in (0, n/(n-1))$.

4 Conclusion

In this paper we study evolutionarily stable behaviour in contests where participation is stochastic. We established the relationship between ESS and Nash equilibrium under exogenous stochastic entry as well as endogenous costly entry. We find players exert more effort and enter more often under ESS than Nash. In this sense, the evolutionary force of “survival of the fittest” selects more competitive and more aggressive behaviour in situations of conflicts. However, this also results in lower absolute payoff for players under ESS than under the “rational” Nash setting. This is instructive to our understanding of rational behaviour under Nash equilibrium when players are concerned of their relative payoff.

From a contest designer’s point of view, implementing ESS induces players to exert higher total effort and thus yields higher total expected revenue. In particular, implementing ESS can mitigate revenue losses from participation uncertainty.

A Proofs - Exogenous entry

A.1 Proof of Theorem 2

The plan of the proof is as follows. We first simplify (5) and establish its relationship with (1). Secondly, we verify that the solution to the first order condition indeed maximises the relative payoff (4).

We first derive the symmetric effort level implied by the first order condition (5):

$$\begin{aligned} X &= rV \left[\sum_{i=1}^{n-1} C_i^{n-1} p^i (1-p)^{n-1-i} \frac{i}{(i+1)^2} + \sum_{i=0}^{n-2} C_i^{n-2} p^{i+1} (1-p)^{n-2-i} \frac{1}{(i+2)^2} \right] \\ &= rV \left[\sum_{i=1}^{n-1} C_i^{n-1} p^i (1-p)^{n-1-i} \frac{i}{(i+1)^2} + \sum_{i=1}^{n-1} C_{i-1}^{n-2} p^i (1-p)^{n-1-i} \frac{1}{(i+1)^2} \right] \\ &= rV \left[\sum_{i=1}^{n-1} p^i (1-p)^{n-1-i} \frac{iC_i^{n-1} + C_{i-1}^{n-2}}{(i+1)^2} \right] \end{aligned} \quad (14)$$

$$\begin{aligned} &= \frac{n}{n-1} rV \left[\sum_{i=1}^{n-1} p^i (1-p)^{n-1-i} C_i^{n-1} \frac{i}{(i+1)^2} \right] \\ &= \frac{n}{n-1} \cdot X^{\text{Nash}} \end{aligned} \quad (15)$$

where from (14) to (15) we have used the identity $C_{i-1}^{n-2} = iC_i^{n-1}/n-1$. Individual spending in Nash equilibrium, X^{Nash} , is given by Theorem 1.

We now proceed to show that, for $0 < r \leq n/n-1$, the first order condition X^{ESS} as given by (15) indeed solves the maximisation problem (4). To this end, consider the second order derivative (17) we derived in Appendix A.2 below.

Obviously, for $r \leq 1$, (17) is negative and hence (4) is globally concave. Consider the remaining case $1 < r \leq n/n-1$. To show that X^{ESS} solves the maximization problem (4) also in this case, we proceed in three steps. First, note that the second order derivative is positive for \bar{X} close to 0. In fact this holds if $(r-1)(k+1)(X^{\text{ESS}})^r > (r+1)\bar{X}^r$ for all $k = 0, \dots, n-2$, i.e. for $\bar{X} < X^{\text{ESS}} \sqrt[r]{(r-1)(n-1)/r+1}$.

Second, the second order derivative is negative if $(r-1)(k+1)(X^{\text{ESS}})^r < (r+1)\bar{X}^r$ for all $k = 0, \dots, n-2$, i.e. for $\bar{X} > X^{\text{ESS}} \sqrt[r]{r-1/r+1}$. Moreover, as

shown in the next two paragraphs, the second order derivative is negative when evaluated at the candidate ESS strategy and it switches sign only once for $\bar{X} \in (0, X^{\text{ESS}})$. These two points imply (4) is first convex and then concave, and hence in this range the only two candidates for a global maximum of (4) are 0 and X^{ESS} .

To see these, we evaluate (17) when $\bar{X} = X^{\text{ESS}}$. Then (17) simplifies to

$$Vrn \left(X^{\text{ESS}} \right)^{-2} \left[\sum_{i=1}^{n-1} p^i (1-p)^{n-1-i} \frac{C_{i-1}^{n-2} [(r-1)(i+1) - 2r]}{(i+1)^3} \right]. \quad (16)$$

Note $(r-1)(i+1) < 2r$ for $i = 1 \dots n-1$ as long as $r \leq n/n-2$. But the latter holds true because $r \leq n/n-1$. Thus we have just established that (16) is negative.

Now we observe that because $r-2 \leq 2-n/n-1 \leq 0$, (17) clearly decreases monotonically in \bar{X} so long as (17) remains positive. On the other hand, by inspection we know once \bar{X} becomes large enough to turn (17) negative, it stays negative for all larger \bar{X} . Thus (4) is first convex and then concave in the range of $\bar{X} \in (0, X)$.

Third and finally, we show in Lemma 1 in Appendix A.3 that a mutant cannot increase its relative fitness by playing 0. This completes the proof.

A.2 Second derivative of relative payoff w.r.t. effort

$$\begin{aligned} & \frac{\partial^2 \phi}{\partial \bar{X}^2}(\bar{X}, X) \\ = & rVX^r \bar{X}^{r-2} \left[\sum_{M \in \mathcal{P}^{N_1}} p^{|M|} (1-p)^{|N_1 \setminus M|} \frac{|M| (r-1) (\bar{X}^r + |M| X^r) - 2r |M| \bar{X}^r}{(\bar{X}^r + |M| X^r)^3} \right] \\ & + prVX^r \bar{X}^{r-2} \left[\sum_{M \in \mathcal{P}^{N_2}} p^{|M|} (1-p)^{|N_2 \setminus M|} \frac{(r-1) (\bar{X}^r + |M+1| X^r) - 2r \bar{X}^r}{(\bar{X}^r + (|M|+1) X^r)^3} \right] \\ = & rVX^r \bar{X}^{r-2} \left[\sum_{k=1}^{n-1} C_k^{n-1} p^k (1-p)^{n-1-k} k \frac{(r-1) (\bar{X}^r + k X^r) - 2r \bar{X}^r}{(\bar{X}^r + k X^r)^3} \right] \end{aligned}$$

$$\begin{aligned}
& + prVX^r\bar{X}^{r-2} \left[\sum_{k=0}^{n-2} C_k^{n-2} p^k (1-p)^{n-2-k} \frac{(r-1)(\bar{X}^r + (k+1)X^r) - 2r\bar{X}^r}{(\bar{X}^r + (k+1)X^r)^3} \right] \\
& = rVX^r\bar{X}^{r-2} \left[\sum_{k=0}^{n-2} C_{k+1}^{n-1} p^{k+1} (1-p)^{n-2-k} (k+1) \frac{(r-1)(\bar{X}^r + (k+1)X^r) - 2r\bar{X}^r}{(\bar{X}^r + (k+1)X^r)^3} \right] \\
& \quad + rVX^r\bar{X}^{r-2} \left[\sum_{k=0}^{n-2} C_k^{n-2} p^{k+1} (1-p)^{n-2-k} \frac{(r-1)(\bar{X}^r + (k+1)X^r) - 2r\bar{X}^r}{(\bar{X}^r + (k+1)X^r)^3} \right] \\
& = rVX^r\bar{X}^{r-2} \left[\sum_{k=0}^{n-2} C_k^{n-2} p^{k+1} (1-p)^{n-2-k} (n-1) \frac{(r-1)(k+1)X^r - (r+1)\bar{X}^r}{(\bar{X}^r + (k+1)X^r)^3} \right] \\
& \quad + rVX^r\bar{X}^{r-2} \left[\sum_{k=0}^{n-2} C_k^{n-2} p^{k+1} (1-p)^{n-2-k} \frac{(r-1)(k+1)X^r - (r+1)\bar{X}^r}{(\bar{X}^r + (k+1)X^r)^3} \right] \\
& = rnVX^r\bar{X}^{r-2} \left[\sum_{k=0}^{n-2} C_k^{n-2} p^{k+1} (1-p)^{n-2-k} \frac{(r-1)(k+1)X^r - (r+1)\bar{X}^r}{(\bar{X}^r + (k+1)X^r)^3} \right] \tag{17}
\end{aligned}$$

where the first equality represents a transformation from set-wise to combinatorial notation, the second equality an index transformation, the third equality follows from inserting $(k+1)C_{k+1}^{n-1} = (n-1)C_k^{n-2}$, and the final equality results from taking the sum of the two previous expressions.

A.3 Relative payoff of zero-effort mutant

Lemma 1. Let $p \in (0, 1]$, $r \leq n/n-1$, and X be given by (6). Then $\phi(0, X) \leq 0$, with strict inequality for $n > 2$ and $p < 1$.

Proof: Let $p \in (0, 1]$ and $r \leq n/n-1$ be arbitrary. Then $\phi(0, X) \leq 0$ if $\Pi_2(0, X, \dots, X) \geq \Pi_1(0, X, \dots, X) = V(1-p)^{n-1}$. Using the identity $C_{\kappa/\kappa+1}^\nu = C_{\kappa+1}^{\nu+1}/\nu+1$ for $\kappa \leq \nu$ this inequality can be equivalently rewritten as

follows:

$$\begin{aligned}
V \left[\sum_{k=0}^{n-2} C_k^{n-2} p^k (1-p)^{n-k-2} \frac{1}{k+1} \right] &\geq X + V(1-p)^{n-1} \\
\frac{\sum_{k=0}^{n-2} C_{k+1}^{n-1} p^{k+1} (1-p)^{n-k-2}}{p(n-1)} &\geq \frac{rn \left[\sum_{k=1}^{n-1} C_k^{n-1} p^k (1-p)^{n-k-1} \frac{k}{(k+1)^2} \right]}{n-1} + (1-p)^{n-1} \\
\left[\sum_{k=0}^{n-2} C_{k+1}^{n-1} p^{k+1} (1-p)^{n-k-2} \right] &\geq r \left[\sum_{k=1}^{n-1} C_{k+1}^n p^{k+1} (1-p)^{n-k-1} \frac{k}{k+1} \right] + (n-1)p(1-p)^{n-1} \\
\underbrace{\left[\sum_{k=1}^{n-1} C_k^{n-1} p^k (1-p)^{n-k-1} \right]}_{=1-(1-p)^{n-1}} &\geq r \left[\sum_{k=2}^n C_k^n p^k (1-p)^{n-k} \frac{k-1}{k} \right] + (n-1)p(1-p)^{n-1}.
\end{aligned}$$

The last inequality holds for all $r \leq n/n-1$ if and only if it holds for $r = n/n-1$, i.e. if

$$\begin{aligned}
&(n-1)(1-(1-p)^{n-1}) + n \left[\sum_{k=2}^n C_k^n p^k (1-p)^{n-k} \frac{1}{k} \right] \\
&\geq n \left[1 - (1-p)^n - np(1-p)^{n-1} \right] + (n-1)^2 p(1-p)^{n-1}
\end{aligned}$$

which reduces to

$$\left[\sum_{k=2}^n C_k^n p^k (1-p)^{n-k} \frac{n}{k} \right] + np(1-p)^{n-1} + (1-p)^n \geq 1. \quad (18)$$

Because of $n/k \geq 1$, for $k = 2, \dots, n$, the left hand side (LHS) is bounded below by

$$\sum_{k=0}^n C_k^n p^k (1-p)^{n-k} = 1. \quad (19)$$

Thus, (18) represents a true statement. Moreover, the LHS of (18) strictly exceeds either side of (19) if $p < 1$ and $n > 2$. Q.E.D.

A.4 Proof of Theorem 3

Since $T^{\text{ESS}}(r, V, n, p) = n/n-1 \cdot T^{\text{Nash}}(r, V, n, p)$, part i) follows directly from Theorem 6 in LM.

To show Part ii), note that

$$\begin{aligned}
& \frac{1}{rV} \left[T^{\text{ESS}}(r, V, k+1, p) - T^{\text{ESS}}(r, V, k, p) \right] \\
&= \frac{1}{rV} \left[\frac{k \left[T^{\text{Nash}}(r, V, k+1, p) - T^{\text{Nash}}(r, V, k, p) \right]}{k-1} - \frac{T^{\text{Nash}}(r, V, k+1, p)}{k(k-1)} \right] \\
&= \frac{k \sum_{i=2}^{k+1} C_{i-1}^k p^i (1-p)^{k-i+1} \frac{1}{i(i-1)}}{k-1} - \frac{\sum_{i=2}^{k+1} C_i^{k+1} p^i (1-p)^{k+1-i} \frac{i-1}{i}}{k(k-1)} \\
&= \frac{1}{k(k-1)} \sum_{i=2}^{k+1} C_i^{k+1} p^i (1-p)^{k-i+1} \left(\frac{1}{i-1} \frac{k^2}{k+1} - \frac{i-1}{i} \right) \\
&= \frac{1}{k(k-1)} \sum_{i=2}^{k+1} C_i^{k+1} p^i (1-p)^{k-i+1} \left[\frac{((k+1)i-1)(k+1-i)}{i(i-1)} \right] > 0,
\end{aligned}$$

where in the step from the second to the third line we used the result for $[T^{\text{Nash}}(r, V, k+1, p) - T^{\text{Nash}}(r, V, k, p)]$ on page 596 in LM.

A.5 Proof of Theorem 4

We first show that ex-ante total spending is larger than the prize value V under sure participation ($p = 1$) and high r . Then the claim holds true by continuity. For this purpose let $p = 1$. The total spending is then

$$T^{\text{ESS}}(r, V, n, 1) = \frac{nrV}{n-1} \left(1 - \frac{1}{n} \right) = rV,$$

which confirms the result in HLP. It follows that there is ex-ante overdissipation when $1 < r \leq n/n-1$. By continuity, the same holds for p sufficiently close to 1.

B Proofs - Endogenous entry

B.1 Proof of Theorem 5

Firstly, we note that r_0 and \bar{r} in FJL satisfy $\bar{r} = n^{-1}/(n-2)$ and $r_0 > n/(n-1)$, after using the parameters in the current paper.⁹ It follows then $r \leq n/(n-1) \leq \min\{r_0, \bar{r}\}$. By Theorem 4 in FJL, there exists a unique symmetric equilibrium with pure-strategy bidding of the entry-bidding game characterised by their Lemma 2.

Secondly, it is verified that their *break-even condition* in **Lemma 2** reduces to (8) in the current paper.

B.2 Proof of Theorem 6

Part (a): Suppose $c \leq V/(n-1) - rV/n$. Note that $X^{\text{ESS}}(r, V, n, p = 1) = rV/n$. Firstly, consider a mutant using strategy $(q, \bar{X}) = (0, 0)$. The mutant obtains zero absolute payoff by staying out while ESS strategists - due to the absence of the mutant - obtains $V/(n-1) - rV/n \geq c$. Thus, in terms of relative payoff, the mutant cannot improve from the ESS strategy pair, $p = 1, X = \frac{rV}{n}$, which yields a zero relative payoff.

Now consider an arbitrary, (q, \bar{X}) . First note that we can decompose $\Phi(q, \bar{X}; p, X; c)$ as follows:

$$\Phi(q, \bar{X}; p, X; c) = q\Phi(1, \bar{X}; p, X; c) + (1 - q)\Phi(0, \bar{X}; p, X; c).$$

This can be verified by using (9) and the observation that

$$\Pi_2(\bar{X}, X, \dots, X; q) = q\Pi_2(\bar{X}, X, \dots, X; 1) + (1 - q)\Pi_2(\bar{X}, X, \dots, X; 0).$$

Note also that $\Phi(0, \bar{X}; p, X; c) = \Phi(0, 0; p, X; c)$. Then we have

$$\begin{aligned} \Phi(q, \bar{X}; 1, rV/n; c) &= q\Phi(1, \bar{X}; 1, rV/n; c) + (1 - q)\Phi(0, 0; 1, rV/n; c) \\ &\leq q\Phi(1, rV/n; 1, rV/n; c) + (1 - q)\Phi(0, 0; 1, rV/n; c) \\ &\leq q \cdot 0 + (1 - q) \cdot 0 = 0, \end{aligned}$$

⁹That is, $\alpha = 1$, $M = n$ and $\Delta = c$.

where the first inequality follows that r^V/n is ESS in the exogenous case and the above discussion that a mutant cannot improve relative payoff by playing $(0, 0)$.

Part (b): Let (q, \bar{X}) be an arbitrary mutant strategy and let the incumbents' strategy $(p^{\text{ESS}}, X^{\text{ESS}})$ satisfy the first order conditions (6) and (13). We show that the mutant's relative payoff $\Phi(q, \bar{X}; p^{\text{ESS}}, X^{\text{ESS}}; c)$ is indeed maximized at 0 by playing $(q, \bar{X}) = (p^{\text{ESS}}, X^{\text{ESS}})$ too. The proof consists of two steps:

$$\begin{aligned}\Phi(q, \bar{X}; p^{\text{ESS}}, X^{\text{ESS}}; c) &\leq \Phi(q, X^{\text{ESS}}; p^{\text{ESS}}, X^{\text{ESS}}; c) \\ &= \Phi(p^{\text{ESS}}, X^{\text{ESS}}; p^{\text{ESS}}, X^{\text{ESS}}; c) = 0\end{aligned}$$

The inequality is established in Lemma 2 below. The first equality holds because relative payoff $\Phi(q, X^{\text{ESS}}; p^{\text{ESS}}, X^{\text{ESS}}; c)$ is constant in q , which follows from equations (11), (12) and (13). The second equality is implied by symmetry.

Lemma 2. For arbitrary $q \in (0, 1]$, we have

$$\Phi(q, \bar{X}; p^{\text{ESS}}, X^{\text{ESS}}; c) \leq \Phi(q, X^{\text{ESS}}; p^{\text{ESS}}, X^{\text{ESS}}; c), \text{ for all } \bar{X} \geq 0.$$

Proof: Let $q \in (0, 1]$ be arbitrary. Using (21) in Appendix B.3 below, we can write the first order condition with respect to effort as

$$q \left[\frac{\partial \Pi_1}{\partial \bar{X}}(\bar{X}, X, \dots, X) - \frac{\partial \Pi_2}{\partial q \partial \bar{X}}(\bar{X}, X, \dots, X) \right] = 0. \quad (20)$$

As $q > 0$, it follows that (20) is equivalent to the first order condition of (4) with entry probability being exogenously given at p^{ESS} . Since the first and higher order derivatives of (10) wrt. \bar{X} with a fixed q represent multiples of the corresponding derivatives of maximizing (4), the remainder of the proof can be established along the lines of Theorem 2. The only difference to be taken into account is when we evaluate the relative fitness of the zero effort mutant. To compare the relative payoff of $\bar{X} = 0$ with that of $\bar{X} = X^{\text{ESS}}$, Lemma 3 in Appendix B.4 below shows that $\Phi(q, 0; p^{\text{ESS}}, X^{\text{ESS}}; c) \leq 0$, which completes the proof. Q.E.D.

B.3 Derivatives with respect to effort

As it turns out, maximizing (10) wrt. \bar{X} is equivalent to maximizing (4) wrt. \bar{X} . To show this, we start with deriving the corresponding first order conditions:

$$\frac{\partial \Phi}{\partial \bar{X}} = q \frac{\partial \Pi_1(\bar{X}, X, \dots, X)}{\partial \bar{X}} - p \frac{\partial \Pi_2(\bar{X}, X, \dots, X; q)}{\partial \bar{X}},$$

where

$$\begin{aligned} \frac{\partial \Pi_1(\bar{X}, X, \dots, X)}{\partial \bar{X}} &= V \left[\sum_{M \in \mathcal{P}^{N_1}} p^{|M|} (1-p)^{|N_1 \setminus M|} \frac{|M| r \bar{X}^{r-1} X^r}{(\bar{X}^r + |M| X^r)^2} \right] - 1 \\ \frac{\partial \Pi_2(\bar{X}, X, \dots, X; q)}{\partial \bar{X}} &= qV \left[\sum_{M \in \mathcal{P}^{N_2}} p^{|M|} (1-p)^{|N_2 \setminus M|} \frac{r \bar{X}^{r-1} X^r}{(\bar{X}^r + (|M| + 1) X^r)^2} \right] \end{aligned}$$

Note that $\frac{\partial \Pi_2(\bar{X}, X, \dots, X; q)}{\partial \bar{X}}$ is linear in q . In particular,

$$\begin{aligned} \frac{\partial \Pi_2(\bar{X}, X, \dots, X; q)}{\partial \bar{X}} &= qV \left[\sum_{M \in \mathcal{P}^{N_2}} p^{|M|} (1-p)^{|N_2 \setminus M|} \frac{r \bar{X}^{r-1} X^r}{(\bar{X}^r + (|M| + 1) X^r)^2} \right] \\ &= \frac{q}{p} \frac{\partial \Pi_2(\bar{X}, X, \dots, X)}{\partial \bar{X}}, \end{aligned}$$

where $\Pi_2(\bar{X}, X, \dots, X)$ is defined in (3) with entry probability being given exogenously at p .

We can hence rewrite

$$\begin{aligned} \frac{\partial \Phi}{\partial \bar{X}} &= q \frac{\partial \Pi_1(\bar{X}, X, \dots, X)}{\partial \bar{X}} - q \frac{\partial \Pi_2(\bar{X}, X, \dots, X)}{\partial \bar{X}} \\ &= q \left[\frac{\partial \Pi_1(\bar{X}, X, \dots, X)}{\partial \bar{X}} - \frac{\partial \Pi_2(\bar{X}, X, \dots, X)}{\partial \bar{X}} \right], \end{aligned} \quad (21)$$

such that the term in brackets is the first order derivative for (4). This implies that all results of the exogenous entry case that relate to the sign of the first and higher order derivatives of (4) wrt. effort level \bar{X} extend to the endogenous entry case, where (10) is to be maximized.

B.4 Relative payoff of zero effort mutant

Lemma 3. Let $r \leq n/n-1$ and suppose (p, X) are given by (13) and (6). Then $\Phi(q, 0; p, X; c) \leq 0$. More specifically, we have

$$(a) \quad \Pi_1(0, X \dots, X) \leq c \quad \text{and}$$

$$(b) \quad \Pi_2(0, X \dots, X) = c$$

where (a) holds strictly if $n > 2$ and $p < 1$.

Proof: Let $r \leq n/n-1$ and suppose (p, X) are given by (13) and (6). Then $\Phi(q, 0; p, X; c) \leq 0$ follows from (a) and (b).

We start with showing (a), which is equivalent to $V(1-p)^{n-1} \leq c$. Using (13), we can equivalently write this as

$$V(1-p)^{n-1} \leq \frac{V}{n-1} \left(\frac{1 - (1-p)^n}{p} - (1-p)^{n-1} \right) - X.$$

Multiplying with $p(n-1)/V$ and inserting (6), this reduces to

$$(n-1)p(1-p)^{n-1} \leq 1 - (1-p)^n - p(1-p)^{n-1} \\ - prn \left[\sum_{k=1}^{n-1} C_k^{n-1} p^k (1-p)^{n-1-k} \frac{k}{(k+1)^2} \right].$$

Inserting $nC_k^{n-1}/(k+1) = C_{k+1}^n$ and rearranging terms, we obtain

$$r \left[\sum_{k=1}^{n-1} C_{k+1}^n p^{k+1} (1-p)^{n-1-k} \frac{k}{k+1} \right] = r \left[\sum_{k=2}^n C_k^n p^k (1-p)^{n-k} \frac{k-1}{k} \right] \\ \leq 1 - (1-p)^n - np(1-p)^{n-1}$$

This inequality holds for arbitrary $r \leq n/n-1$ if it holds for $r = n/n-1$, i.e. if

$$n \left[\sum_{k=2}^n C_k^n p^k (1-p)^{n-k} \frac{k-1}{k} \right] \leq (n-1) \underbrace{\left[1 - (1-p)^n - np(1-p)^{n-1} \right]}_{\sum_{k=2}^n C_k^n p^k (1-p)^{n-k}}$$

$$\text{or equivalently } \left[\sum_{k=2}^n C_k^n p^k (1-p)^{n-k} \right] \leq \left[\sum_{k=2}^n C_k^n p^k (1-p)^{n-k} \frac{n}{k} \right].$$

Because $n/k \geq 1$, for $k = 2, \dots, n$, the above inequality is satisfied. Moreover, it holds strictly if $n > 2$ and $p < 1$.

To establish (b), we first rewrite (b) using (3):

$$\begin{aligned} X + c &= V \left[\sum_{k=0}^{n-2} C_k^{n-2} p^k (1-p)^{n-k-2} \frac{1}{k+1} \right] \\ &= \frac{V}{n-1} \frac{1 - (1-p)^{n-1}}{p}, \end{aligned} \quad (22)$$

where the second equality can be established along the lines of Lemma 1. On the other hand, the first order condition (13) implies that

$$X + c = \frac{V}{n-1} \frac{1 - (1-p)^n - p(1-p)^{n-1}}{p}. \quad (23)$$

Note that the right hand sides of (22) and (23) coincide (as do the left hand sides). Thus, (23) implies (22), which shows (b). *Q.E.D.*

B.5 Proof of Theorem 7

Let $c > V/n - n^{-1}/n^2 r V$. Then from Theorem 5 we know $p^{\text{Nash}} < 1$ and

$$\frac{1 - (1 - p^{\text{Nash}})^n}{p^{\text{Nash}}} = n \frac{c + X^{\text{Nash}}(r, V, n, p^{\text{Nash}})}{V}.$$

Evaluating (12) at $p = p^{\text{Nash}}$ leads to

$$\begin{aligned} &\frac{V}{n-1} \left[n \frac{c + X^{\text{Nash}}}{V} - (1 - p^{\text{Nash}})^{n-1} \right] - \frac{n}{n-1} X^{\text{Nash}} - c \\ &= \frac{c - V(1 - p^{\text{Nash}})^{n-1}}{n-1} > 0, \end{aligned}$$

where the last inequality holds because c is equal to the expected equilibrium surplus by bidding X^{Nash} and $V(1 - p^{\text{Nash}})^{n-1}$ is a deviation to bidding 0. The inequality indeed holds strictly by equation (22) on page 594 in LM. It thus follows that $p^{\text{ESS}} > p^{\text{Nash}}$.

If $c \leq V/n - n^{-1}/n^2 r V$, then $p^{\text{Nash}} = 1$. Note that $r \leq n/n-1$, it is readily

verified that

$$\frac{V}{n} - \frac{n-1}{n^2}rV \leq \frac{V}{n-1} - \frac{n}{n-1} \frac{n-1}{n^2}rV.$$

Thus, $c \leq V/n-1 - X^{\text{ESS}}(r, V, n, p = 1)$ and by Theorem 6, $p^{\text{ESS}} = 1$. This concludes the proof.

References

- Bolton, G. E. and Ockenfels, A. (2000). ERC: A theory of equity, reciprocity, and competition. *American Economic Review*, 90:166–193.
- Crawford, V. P. (1990). Nash equilibrium and evolutionary stability in large- and finite - population “playing the field” models. *Journal of Theoretical Biology*, 145(1):83 – 94.
- Dechenaux, E., Kovenock, D., and Sheremeta, R. M. (2015). A survey of experimental research on contests, all-pay auctions and tournaments. *Experimental Economics*, 18(4):609–669.
- Fehr, E. and Schmidt, K. M. (1999). A theory of fairness, competition and cooperation. *Quarterly Journal of Economics*, 114(3):817–868.
- Fu, Q., Jiao, Q., and Lu, J. (2011). On disclosure policy in contests with stochastic entry. *Public Choice*, 148(3-4):419–434.
- Fu, Q., Jiao, Q., and Lu, J. (2015). Contests with endogenous entry. *International Journal of Game Theory*, 44(2):387–424.
- Fu, Q. and Lu, J. (2010). Contest design and optimal endogenous entry. *Economic Inquiry*, 48(1):80–88.
- Gu, Y. and Hehenkamp, B. (2016). Rent dissipation in contests with stochastic number of players: the case of convex impact functions. Unpublished Manuscript.
- Guse, T. and Hehenkamp, B. (2006). The strategic advantage of interdependent preferences in rent-seeking contests. *Public Choice*, 129(3):323–352.
- Güth, W. and Yaari, M. (1992). An evolutionary approach to explain reciprocal behavior in a simple strategic game. In Witt, U., editor, *Explaining Pro-*

- cess and Change - Approaches to Evolutionary Economics*, pages 23–34. The University of Michigan Press.
- Hehenkamp, B., Leininger, W., and Possajennikov, A. (2004). Evolutionary equilibrium in tullock contests: spite and overdissipation. *European Journal of Political Economy*, 20(4):1045 – 1057.
- Herrmann, B. and Orzen, H. (2008). The appearance of homo rivalis: Social preferences and the nature of rent seeking. University of Nottingham.
- Huck, S., Normann, H.-T., and Oechssler, J. (1999). Learning in cournot oligopoly — an experiment. *Economic Journal*, 109:80–95.
- Leininger, W. (2009). Evolutionarily stable preferences in contests. *Public Choice*, 140(3):341–356.
- Lim, W. and Matros, A. (2009). Contests with a stochastic number of players. *Games and Economic Behavior*, 67(2):584 – 597.
- Mago, S. D., Samak, A. C., and Sheremeta, R. M. (2016). Facing your opponents: social identification and information feedback in contests. *Journal of Conflict Resolution*, 60(3):459 – 481.
- Messick, D. M. and Thorngate, W. B. (1967). Relative gain maximization in experimental games. *Journal of Experimental Social Psychology*, 3(1):85 – 101.
- Münster, J. (2006). Contests with an unknown number of contestants. *Public Choice*, 129(3-4):pp. 353–368.
- Myerson, R. B. and Wärneryd, K. (2006). Population uncertainty in contests. *Economic Theory*, 27(2):pp. 469–474.
- Perea, A. (2012). *Epistemic game theory: reasoning and choice*. Cambridge University Press.
- Schaffer, M. E. (1988). Evolutionarily stable strategies for a finite population and a variable contest size. *Journal of Theoretical Biology*, 132(4):469 – 478.
- Smith, J. M. and Price, G. (1973). The logic of animal conflict. *Nature*, 246:15.
- Vega-Redondo, F. (1996). *Evolution, Games, and Economic Behaviour*. Oxford University Press.