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Public Debt Sustainability and Defaults

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Public Debt Sustainability and Defaults.

Abstract

We offer a new methodology for the assessment of public debt sustainability in a stochastic economy when sovereign default taken into account. The default threshold differs from the no-Ponzi condition and depends on the post-default debt recovery rule. We distinguish sustainability and unsustainability conditions, related to alternative scenarios on the future sequence of shocks. We highlight the role of the debt recovery ratio on the whole dynamics of public debt. When a sovereign default occurs, the sustainability of the post-default debt is ensured when the haircut is sufficiently large. Lastly we provide an explanation of serial defaults.

JEL-Codes: E600, F400, H630.

Keywords: public debt, sovereign default, recovery rate, debt sustainability.

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1 Introduction.

Greece obtained three successive economic adjustment programs in 2010, 2011 and 2015, which were clear evidence that it could not get access to financial markets on its own since 2010. The first one provided for a write-off of 50% of the Greek bonds detained by commercial banks.[1] Thanks to these programs which de facto amounted to a partial default, the Greek public debt decreased from 356.3 billions euros in 2011 to 305.1 billions in 2012, ending with a figure of 311.4 billions in 2015.[2] Yet the ratio of Greek public debt to GDP for 2015 reached 176.9 %, a much larger amount than the corresponding level in 2010 (146.2 %), leading the IMF to call for an extra debt relief by European countries. Specifically, the IMF worries about "a public debt burden that remains unsustainable despite large debt relief already received".[3]

The ongoing Greek debt crisis[4] drammatically shows that defaulting on sovereign bonds is not sufficient to restore public finances and avoid the ballooning of public debt. Defaults impact on the dynamics of public debt and thus its sustainability for 2 reasons:

1. Ex post, a default implies a rescheduling of public debt and thus a new starting point for the path of public debt.

2. Ex ante, it has an expectational effect. When the prospect of a future default increases, this increases the risk premium charged on public bonds and thus contributes to the burden of public debt.

The standard view on public debt sustainability, based on the Intertemporal Government Budget Constraint (IGBC), does not address this issue, as it rules out sovereign default. According to this view, public debt at a given period t is said to be sustainable if the present value of expected future primary surpluses is at least equal to the initial

[1] Euro summit statement, 26 October 2011.

[2] Source on data: Eurostat.

[3] IMF, "Greece: Staff Concluding Statement of the 2016 Article IV Mission", September 23, 2016. http://www.imf.org/en/News/Articles/2016/09/23/MS092316-Greece-Staff-Concluding-Statement-of-the-2016-Article-IV-Mission. See also *Financial Times*, "IMF calls on eurozone for bolder Greece debt relief", September 23, 2016.

[4] Zettelmeyer, Trebesch and Gulati (2013) provide a thorough analysis of the Greek debt relief programme of 2012.

debt. This criterion is satisfied under two conditions: *i)* the government commits to repay its debt and strategic defaults are ruled out, *ii)* it has at its disposal a set of sufficiently flexible fiscal instruments. In this perspective the sustainability of public debt is the mirror of fiscal sustainability.

Two criticisms can be addressed to this view. First, governments are likely to be constrained in their fiscal policies: taxes and expenditures cannot always be manipulated at will. Fiscal policy may be so constrained that defaults cannot be avoided. Second, it does not relate the sustainability of public debt to its future dynamics. It is to be noticed that the assessment of public debt sustainability by rating agencies and international institutions focuses on the dynamics of public debt, depending on the financial and economic environment, including the policy stance. Rating agencies upgrade or downgrade public debts according to the prospect of defaults. Public institutions such as the IMF or the European commission develop toolkits or ready-to-use frameworks for assessing the macroeconomic conditions of a country and in particular the perils of unsustainability and default.[5] Implicitly the non-explosive behavior of public debt under alternative scenarios is used by these institutions as a criterion of its sustainability.

In this paper we provide a rigorous characterization of public debt sustainability addressing the interplay between (possible) defaults and the dynamics of public debt in a stochastic environment when fiscal policy is constrained. For this purpose we build a simple macroeconomic model of a closed economy without capital and money in which fiscal policy may be constrained. We rule out strategic or preemptive defaults and focus on defaults as market events. We introduce in the model a debt recovery rule which specifies the amount of the haircut applied to the defaulted debt. The model is simple enough to be tractable and allows full analytical resolution. Specifically, the pricing of public bonds is obtained, generating the (growth-adjusted) servicing of debt. Building on this step, we characterize the stochastic dynamics of public debt and the (possible) occurrence of defaults.

We prove that the default threshold is weakly lower than the solvency ratio defined

[5]See IMF (2013a) and European Commission (2016).

as the debt-to-GDP ratio strictly meeting the no-Ponzi condition. The former one depends on the debt recovery rule and is equal to the solvency ratio in the very special case, and clearly irrealistic, where the post-default debt is reset at the default threshold itself.

Public debt is to be analyzed in conjunction with macroeconomic shock realizations: Under which stochastic circumstances is public debt sustainable, that is, able to avoid meeting the default threshold? The IMF guidelines acknowledge the necessity to reason on "scenarios" as evidenced in the "Staff guidance note on public debt sustainability":

> *In general terms, public debt can be regarded as sustainable when the primary balance needed to at least stabilize debt under both the baseline and realistic shock scenarios is economically and politically feasible, such that the level of debt is consistent with an acceptably low rollover risk and with preserving potential growth at a satisfactory level.*[6]

In our framework, the reasoning on such scenarios is obtained by means of the use of truncated sets of admissible sequences of shocks over time. The extent of truncation reflects the strength of the (un)sustainability criterion imposed on public debt.

Specifically, a public debt is said to be "$\gamma-$sustainable" at date t when its trajectory does not reach the default threshold at any future date, assuming that there is no realization of the gross rate of growth lower than $\gamma \leq 1$. A public debt is said to be "$\gamma-$unsustainable" at date t when its trajectory reaches the default threshold at some finite date, assuming that there is no realization of the gross rate of growth higher than $\gamma \geq 1$. When a public debt is neither "$\underline{\gamma}-$sustainable" nor "$\overline{\gamma}-$unsustainable", with $\underline{\gamma} < 1 < \overline{\gamma}$, it is in a zone of "financial fragility".

We show in the paper that these definitions derive from the notion of a "$\gamma-$risky steady state" which is a generalization of a "risky steady state", recently developed by Coeurdacier, Rey and Winant (2011). This notion allows us to define sustainability and unsustainability thresholds. Both thresholds are always below the default threshold.

Turning to the post-default dynamics of public debt, the possibility of renewed default cannot be ruled out. For a given subset of admissible realizations of shocks, if

[6] IMF (2013b), p. 4.

the post-default debt set according to a rescheduling scheme is above the unsustainability threshold, the public debt eventually reaches default. This corresponds to a configuration characterized by "serial defaults".[7] We show that this happens when the reduction of debt generated by the debt recovery rule is too small.[8] On the other hand, we prove that, for a given sustainability criterion, the post-default debt is sustainable if the haircut implied by the debt recovery rule is sufficiently large, given the assumed truncation of realizations.

The paper is organized as follows. We discuss the related literature in the following section. Section 3 presents the macroeconomic framework. We first study the dynamics of expected public debt abstracting from the occurrence of default in section 4, defining the solvency ratio and emphasizing the importance of fiscal limits. In section 5, we study the market valuation of public debt in the presence of default risk. The dynamics of public debt when default is not ruled out is addressed in Section 6. Section 7 concludes.[9]

2 Related literature.

This paper combines together the economics of public debt sustainability and public default. We here provide a brief survey of the literature on these topics relevant for our research.

Aguiar and Amador (2014) and D'Erasmo, Mendoza and Zhang (2015) are recent surveys on sovereign debt. Regarding public debt sustainability, Bohn (1995) has convincingly argued that the traditional tests of the intertemporal government budget constraint (IGBC) are insufficiently rigorous. More precisely Bohn criticizes the neglect of the probability distribution of the various variables included in this constraint and thus the treatment of interest rates as risk-free. In an other influential paper, Bohn (2007) has demonstrated that the IGBC holds if either debt or revenue and spending inclusive of debt service are integrated of finite but arbitrarily high order. As noted by

[7] A seminal reference on the subject is Reinhard, Rogoff and Savastano (2003).

[8] This matches the observation that often the defaulting process leads to a "too small" haircut. See IMF (2013b).

[9] Proofs of propositions are contained in the Appendix section.

D'Erasmo et al. (2015), this condition is so easily fulfilled that sustainability tests are futile.

Reinhart and Rogoff (2008) constitutes the main empirical study of sovereign defaults, both external and domestic, based on a two-century analysis of historical episodes of defaults. Tomz and Wright (2013) and Reinhard and Trebesch (2014) are recent studies on the empirics of public debt and default. Das, Papaionnou and Trebesch (2012) focus on the empirics of debt restructuring. Sturzenegger and Zettelmeyer (2008) document the variability of haircuts in recent defaults, ranging from 13 percent (Uruguay external exchange) to 73 percent (2005 Argentina exchange). Cruces and Trebesch (2013) find that haircuts have a significant effect on public debt sustainability through the interest spread channel.

Aguiar and Amador (2013) and D'Erasmo, Mendoza and Zhang (2015) extensively address the issue of default, in particular from a theoretical point of view.[10] It is convenient to distinguish between strategic defaults and defaults as pure market events. A default is strategic when it is the outcome of a decision by the government. Most of the theoretical studies on default focus on solving the puzzle of the existence of sovereign debt contracts between fully rational agents when there is no or limited enforcement capacity, following Eaton and Gersovitz (1981). The issue is the designing of efficient contracts taking into account the incentive to default. Important references on the subject are Calvo (1988), Cole and Kehoe (2000), Aguiar and Gopinath (2006), Arellano (2008). Mendoza and Yue (2012) set up a DSGE model with strategic default which provides an explanation for the negative relationship between sovereign spreads and GDP growth but take as given the threshold levels linked to default.

Defaults as market events are less studied. Such defaults occur when the government, being unable to decrease its spending or raise taxes (for example, because of excessive fiscal distorsions), is facing lenders unwilling to lend it the needed sum at any rate. As shown by Bi (2013), Bi and Leeper (2012), Daniel and Shiamptanis (2013), fiscal limits arising from distorsions drastically modify the conditions on the sustainability of debt and contribute to defaults. Arellano, Atkeson and Wright (2015) show

[10] A previous survey is Eaton and Fernandez (1995).

how the spreads on public debt and thus public indebtedness depend on the flexibility of fiscal instruments. Lorenzoni and Werning's article (2014) is the closest to our setting as they investigate the gradual worsening of public debt position which is due to the presence of long-term debt. Yet they concentrate the risk in one period only. Thus default can only occur once. Collard, Habib and Rochet (2015), reasoning in a stochastic environment, relate the probability of default to the debt-to-GDP ratio. Yet they do not address the whole dynamics of public debt. Gosh et al. (2013), relating fiscal fatigue to public default, compute a "fiscal space", that is, the distance from the default threshold. Their focus being mainly empirically, using data on 23 advanced economies over 1970–2007, they estimate this fiscal space for each country.

An intermediate case of default is preemptive restructuring, which occurs when the government negotiates and obtains from lenders a restructuring of its existing debt - prior to a payment default. Asunoma and Trebesch (2016) is a first step in this direction.

3 The model.

We consider a closed economy with flexible prices and no capital. The government issues non-contingent bonds but savers have also access to a complete set of Arrow-Debreu contingent assets. The existence of complete financial markets allows effective hedging by savers and the pricing of public bonds reflecting the risk of default.

3.1 Private sector.

There is a representative agent whose preferences are described by the following utility function:

$$U_0 = E_0 \sum_{t=0}^{+\infty} \beta^t \left[u\left(C_t\right) - \ell\left(L_t\right) \right], \tag{1}$$

with: $u\left(C_t\right) = \ln C_t$ and $\ell\left(L_t\right) = \eta^{-1} L_t^{1+1/\sigma} / \left(1 + 1/\sigma\right)$, where C_t is consumption, L_t represents hours worked, and σ the Frisch elasticity of labor.

In each period the agent receives profits Π_t and labor income $W_t L_t$, where W_t denotes the real wage rate. Income, including profits but excluding other financial returns for sake of simplicity, is taxed at a proportional rate τ_t. The consumer can save by means of a portfolio of Arrow-Debreu state-contingent assets and one-period maturity Treasury bonds. The amount of new issued government bonds she chooses to buy in t is noted B_t and their unit price is q_t. The amount of redeemed debt is denoted by $h_t B_{t-1}$ where h_t denotes the fraction of debt actually reimbursed. It is less than 1 in the case of default. Denoting by $Q_{t,t+1}$ the price of a contingent asset which generates a real return of 1 in a given state of nature (and 0 in the others) divided by the probability (or density function) of such state,[11] and by D_{t+1} the quantity of this contingent asset,[12] the individual budget constraint at t writes:

$$C_t + q_t B_t + E_t \left(Q_{t,t+1} D_{t+1} \right) \leq (1 - \tau_t) \left(W_t L_t + \Pi_t \right) + h_t B_{t-1} + D_t. \tag{2}$$

The agent must also meet her intertemporal constraint on wealth:

$$h_{t+1} B_t + D_{t+1} \geq -E_{t+1} \sum_{s=t+1}^{\infty} Q_{t+1,s} \left(1 - \tau_s \right) \left(W_s L_s + \Pi_s \right) \qquad \forall t, \tag{3}$$

with $Q_{t+1,s} \equiv Q_{t+1,t+2} Q_{t+2,t+3} \cdots Q_{s-1,s}$ and $Q_{t+1,t+1} = 1$. This condition must hold for each possible state that may occur at date $t+1$.

Maximizing (1) under constraints (2) and (3), the following optimality conditions obtain for any period t:

$$Q_{t,t+1} = \beta \frac{u' \left(C_{t+1} \right)}{u' \left(C_t \right)} = \beta \frac{C_t}{C_{t+1}}, \tag{4}$$

$$q_t = E_t Q_{t,t+1} h_{t+1}, \tag{5}$$

$$(1 - \tau_t) W_t = \frac{v' \left(L_t \right)}{u' \left(C_t \right)} = \frac{L_t^{1/\sigma}}{\eta} C_t \tag{6}$$

[11] Which will be equal to the stochastic discount factor.

[12] For the sake of simplicity, we do not use notation for the states of Nature that may occur at each date. Remember that there are as many different values for D_{t+1} and $Q_{t,t+1}$ as possible states of nature in $t+1$. The contingent asset is indexed by $t+1$ since its return will depend on the state of nature realized in $t+1$. To the contrary the public bond emitted in t is indexed by t as it is not state-contingent in $t+1$.

and the transversality condition is given by:

$$\lim_{T\to\infty} E_t Q_{t,T} \left[h_T B_{T-1} + D_T\right] = 0. \tag{7}$$

(4) is the state contingent Euler equation for consumption. This condition must hold for each possible state that may occur at date $t+1$, given the state that has occurred at date t. (5) equates the price of the risky government bond to the expected discounted return of the reimbursed debt next period. The RHS of this equation can be interpreted as the value of a specific portfolio composed of contingent assets, each one bought in quantity h_{t+1}. Hence (5) is the no-arbitrage condition between the risky government bond and this particular portfolio. Finally, (6) is the intratemporal optimal condition between labor and consumption.

The good market is perfectly competitive and returns to scale are constant. The production technology is given by:

$$Y_t \leq A_t N_t \tag{8}$$

where Y_t denotes production, N_t is the quantity of labor hired by the firm, and A_t is the average (and marginal) productivity of labor. It is stochastic and the sole source of shock present in this economy. Profit maximization leads to standard results on returns: $W_t = A_t$, $\Pi_t = 0$ and (8) binds.

In order to simplify the analysis of this economy and fully characterize its dynamics thanks to the study of the debt-to-GDP ratio, we make the following assumption about the productivity shock:

Assumption 1.

$$A_t = a_t A_{t-1},$$

where a_t *is an* i.i.d. *random variable. The cumulative distribution function of* a_t *is denoted by* $G(a)$*, its density function by* $g(a)$ *and we assume that:*

1. *the support of* $g(a)$ *is bounded on the interval* $[a_{\text{inf}}, a^{\text{sup}}]$*. In addition,* $0 < a_{\text{inf}} <$

$1 < a^{\sup}$ *and*

$$E(a) = 1 \qquad \text{and} \qquad \beta E\left(\frac{1}{a}\right) < 1,$$

2. $g(a) > 0$; $\lim_{a \to a^{\sup}} g(a) = \lim_{a \to a_{\inf}} g(a) = \varepsilon$ *with* ε *arbitrarily small,*

3. *the elasticity of the density function* $g(a)$ *satisfies* $\frac{ag'(a)}{g(a)} > -1$.

Assumption **1.1** makes clear that the productivity follows a random walk and the growth rate of productivity is bounded. Assumption $\beta E\left(\frac{1}{a_t}\right) < 1$ will guarantee the existence of a positive risk-free interest rate for this economy when there is no risk of default.

Assumptions **1.2** and **1.3** are regularity assumptions which allow us to exclude the possibility of multiple equilibria as will be made explicit in Section 4.

3.2 Fiscal policy.

Government spends an amount $G_t = gY_t$, and collects taxes on income $\tau_t Y_t$. It balances its budget by issuing one-period maturity Treasury bonds at a price q_t. In case of default at t, it reimburses a fraction $h_t < 1$ of its debt contracted at $t-1$, B_{t-1}. The instantaneous government budget constraint writes:

$$q_t B_t = h_t B_{t-1} + (g - \tau_t) Y_t, \tag{9}$$

with $h_t \in (0, 1)$.

Fiscal rule and fiscal constraint.

Following Bi (2012), Daniel and Shiamptanis (2013) and Davig, Leeper and Walker (2011), we assume that the tax rate increases with the fraction of debt to GDP, up to a limit denoted by $\hat{\tau}$.[13] When the tax rate has reached its maximum value, we refer to

[13] An obvious candidate for this limit corresponds to the rate generating the highest point of the Laffer curve. More precisely we shall see below that there exist dynamic Laffer curves in the sense that the shape of the Laffer curve depends on the state of the economy, as in Bi (2012). Since we consider a non-stochastic fiscal policy, the maxima of these curves are obtained for a unique tax rate, in contrast with Bi (2012). This limit can also be the consequence of political economy or constitutional considerations.

the situation as *fiscally constrained* and we will say that the economy is in a *constrained fiscal regime.*

We assume that the tax rate depends on $\omega_t \equiv h_t B_{t-1}/Y_t$, the *actually redeemed* debt-to-output ratio,[14] as long as the upper limit $\hat{\tau}$ is not yet reached,

$$\tau_t = \min\left(\bar{\tau} + \theta \cdot (\omega_t - \bar{\omega})\,; \hat{\tau}\right), \tag{10}$$

and we make the following assumption:

Assumption 2. *$\theta > 1 - \beta$, $\bar{\omega} \geq 0$, and $\hat{\tau} > \bar{\tau} \equiv g + (1-\beta)\,\bar{\omega}$.*

Under Assumption 2, the term $\bar{\omega}$ can then be interpreted as a target value for the actually redeemed debt-to-output ratio. From (10), we define another debt-to-output ratio $\hat{\omega}$ at which the tax rate reaches its maximum $\hat{\tau}$:

$$\tau_t = \hat{\tau} \Longleftrightarrow \omega_t \geq \bar{\omega} + \frac{\hat{\tau} - \bar{\tau}}{\theta} \equiv \hat{\omega}. \tag{11}$$

Sovereign default and debt recovery rule.

Let us denote by $\Omega_t^{\max}$ the maximum debt level which can be redeemed by the Treasury without default in t: Default occurs when $B_{t-1} > \Omega_t^{\max}$. We refer to $\Omega_t^{\max}$ as the "default threshold" for period t. Note that it is *a priori* a random variable.

As we do not focus upon the strategic relationships between lenders and the public borrower, we assume a given debt recovery rule. In case of default, a simple rule, contingent on the level of contractual debt B_{t-1} and on the default threshold $\Omega_t^{\max}$, is applied. We use the following specification:

$$h_t = \mathcal{H}\left(B_{t-1}, \Omega_t^{\max}\right) \equiv \begin{cases} \mathbf{h} \cdot \Omega_t^{\max}/B_{t-1} < 1 & \text{if } \Omega_t^{\max} < B_{t-1}, \\ 1 & \text{if not,} \end{cases} \tag{12}$$

with $0 \leq \mathbf{h} \leq 1$.

According to this rule, any realization of the (stochastic) default threshold $\Omega_t^{\max}$ below the contractual level of debt triggers default and rescheduling. This rescheduling

[14] The redeemed debt is possibly affected by default when $h_t < 1$.

is such that the after-default (redeemed) debt-to-GDP ratio is a fraction of $\Omega_t^{\max}$, *i.e.*: $h_t B_{t-1} = \mathbf{h}\Omega_t^{\max}$. If we consider the limit case where the overrun is negligible ($B_{t-1} \to \Omega_t^{\max+}$), $\mathbf{h}$ can be interpreted as the maximum redemption ratio. By extension, $1 - \mathbf{h}$ is the minimal rate of default, or equivalently and loosely speaking, the lowest possible "haircut". This rule displays two important features.

1. This recovery rule has the property of ensuring that the government is immediately able to re-enter the bond market as its post-default initial debt is below $\Omega_t^{\max}$ and thus the economy functions again according to the set of equations characterizing its dynamics.

2. The possibility of future defaults is not ruled out. Nevertheless the rule allows the defaulting government to withstand adverse shocks in the future. The lower is $\mathbf{h}$, the more room there is to accommodate adverse shocks.

Finally it is also consistent with the evidence that the ratio of recovered to emitted debt h_t is not unique and varies according to countries and circumstances.[15]

3.3 Equilibrium conditions.

At this stage, we establish the equilibrium conditions of this economy taking as given the possible stochastic sequences of default threshold in each period: $\{\Omega_t^{\max}\}$.[16] A competitive equilibrium contingent to a sequence of default thresholds is defined as follows: It is a sequence of prices $\{W_t, q_t, \{Q_{t,t+1}\}\}_{t=0}^{\infty}$, policy instruments $\{\tau_t, h_t\}$, and quantities $\{N_t, Y_t, C_t, B_t, \{D_{t+1}\}\}_{t=0}^{\infty}$ such that, for all possible sequences of exogenous realizations $\{A_t\}_{t=0}^{\infty}$ and default thresholds $\{\Omega_t^{\max}\}_{t=0}^{+\infty}$, households and firms solve their respective optimization problems, the accumulation equation of public debt holds, the taxation and default rules hold, and all markets clear. The market clearing conditions for respectively the good market, the labor market and the contingent asset market are,

[15] Haircuts in sovereign debt restructuring for emerging market economies over 1998 and 2005 varied from 5% in Dominican republic to 72% in Argentina (see Sturzenegger and Zettelmeyer, 2008). See also Tomz and Wright (2012).

[16] In section 5 this sequence is endogenized.

for all t:

$$C_t = (1-g)\, Y_t, \tag{13}$$

$$L_t = N_t, \tag{14}$$

$$\{D_{t+1}\} = \{0\}. \tag{15}$$

Combining (6) where $W_t = A_t$, with (8) as an equality, (13) and (14) gives:

$$Y_t = \left(\eta \frac{1-\tau_t}{1-g}\right)^{\frac{\sigma}{1+\sigma}} A_t. \tag{16}$$

Combining (4), (5), and (13) generates the no-arbitrage condition:

$$q_t = \beta E_t \left(\frac{Y_t}{Y_{t+1}} h_{t+1}\right) \tag{17}$$

For given stochastic processes for the exogenous sequence $\{A_t\}_{t=0}^{\infty}$ and the sequence $\{\Omega_t^{\max}\}_{t=0}^{+\infty}$, the equilibrium conditions are reduced to equations (16) where τ_t is given by (10), (17) where h_{t+1} is given by the debt recovery rule (12), the GBC condition (9) as well as and the transversality condition:

$$0 = \lim_{T\to\infty} \beta^T E_t \omega_T. \tag{18}$$

Notice that when $\tau_t = \tau_{t-1} = \hat{\tau}$, the gross rate of output growth is equal to $Y_t/Y_{t-1} = A_t/A_{t-1}\,(\equiv a_t)$ and follows an exogenous stochastic process.

One can easily check that this economy displays a Laffer curve: the total amount of taxes collected by the government, $T_t = \tau_t Y_t$, is a non-monotone function of τ_t. In each period it is affected by the state of the economy, that is, the realization of the random variable A_t; however the tax rate for which it reaches its maximum is given by $\tau^{\max} = (1+\sigma)/(1+2\sigma)$ which is state-independent. $\tau^{\max}$ represents an upper value for the fiscal limit parameter $\hat{\tau}$.

4 The solvency ratio.

Substituting in the government budget constraint (9) the value of q_t given by (17), and dividing the result by Y_t, we get:[17]

$$\beta E_t \omega_{t+1} = \omega_t + g - \tau_t. \tag{19}$$

(19) corresponds the dynamic equation for expected actually redeemed debt-to-output ratio. Notice that the possibility of default does not appear explicitly in this equation. This is due to the combination of two elements: first, according to (17) the possibility of default is included into the pricing of public bond; second, in (19) we reason on the actually redeemed debt-to-output ratio which encompasses the possibility of default.

For a given value of the tax rate, $E_t\omega_{t+1}$ is a linear function of ω_t. This is due to the logarithmic specification of the utility function in consumption and the assumption that g is constant, which makes consumption proportional to output. Using the taxation rule (10) in equation (19) and using the definition of $\hat{\omega}$ given by (11), we get the following dynamic equation for expected actually redeemed debt-to-output ratio:

$$E_t\omega_{t+1} = \begin{cases} (1-\theta)\,\beta^{-1}\omega_t + (1-(1-\theta)\,\beta^{-1})\,\bar{\omega} & \text{for } \omega_t \leq \hat{\omega}, \\ \beta^{-1}\omega_t - \beta^{-1}\,(\hat{\tau} - g) & \text{for } \omega_t > \hat{\omega}. \end{cases} \tag{20}$$

(20) makes clear the consequence of a maximum tax rate. It creates a kink in the dynamics of expected debt. If the actually redeemed debt-to-output ratio ω_t is sufficiently low, negative shocks on output and the resulting reduction in tax receipts can partially be offset by an increase in the tax rate. When ω_t has reached the debt-to-output ratio $\hat{\omega}$ –at which the tax rate reaches its maximum $\hat{\tau}$–, then this possibility is foregone and a negative shock on output and the ensuing deficit can only be accommodated by an increase in public debt.

When the actually redeemed public debt ratio ω_t is less than $\hat{\omega}$, the expected actually redeemed debt ratio is obtained from a linear equation the slope of which $((1-\theta)\,\beta^{-1})$

[17] Remark that, using (17), the LHS of (9) divided by Y_t gives : $q_t \frac{B_t}{Y_t} = \beta E_t \left(\frac{Y_t}{Y_{t+1}} h_{t+1}\right) \frac{B_t}{Y_t} = \beta E_t \frac{h_{t+1} B_t}{Y_{t+1}}$, which is identical to $\beta E_t \omega_{t+1}$.

is less than one (from Assumption 2). When it is above it, the expected actually redeemed debt ratio is obtained from a linear equation the slope of which (β^{-1}) is more than one. Hence the kink at $\hat{\omega}$ creates two deterministic steady states, one of which is $\bar{\omega}$. The second deterministic steady state is defined by the following

$$\omega^{\text{sup}} \equiv \frac{\hat{\tau} - g}{1 - \beta}. \tag{21}$$

ω^{sup} is equal to the sum of expected discounted primary surpluses (relative to GDP), when they are set at their maximum value; hence it defines the conventional solvency limit of public debt(-to-output ratio). In the sequel, we will refer to ω^{sup} as the *solvency ratio* of sovereign debt. Using (11), (21) and $\bar{\tau} \equiv g + (1 - \beta)\,\bar{\omega}$, $\hat{\omega}$ can be expressed as:

$$\hat{\omega} = \left(1 - \frac{1 - \beta}{\theta}\right)\bar{\omega} + \frac{1 - \beta}{\theta}\omega^{\text{sup}} \tag{22}$$

From Assumption 2, as $\bar{\omega} < \hat{\omega} < \omega^{\text{sup}}$ the expected dynamics of the actually redeemed debt-to-output ratio is represented by Figure 1. The first deterministic steady state is stable, whereas the second one, ω^{sup}, is an unstable steady state in the following sense: If current debt ratio is less than ω^{sup}, it is expected to converge toward $\bar{\omega}$, absent of any future shock; if it were more than ω^{sup}, it is expected to grow indefinitely and violates the transversality condition.[18] Indeed, when $\omega_t > \hat{\omega}$, given (20) and using (21), the transversality condition (18) is written as:

$$\lim_{T \to \infty} \beta^T E_t \omega_T = \omega_t - \omega^{\text{sup}} = 0,$$

and therefore is violated when $\omega_t > \omega^{\text{sup}}$.

Note that $\omega_t = h_t B_{t-1}/Y_t$ is a stochastic variable which may "jump" in each period according to the growth rate innovation and the possibility of a sovereign default. Moreover the previous analysis based on the study of its expected dynamics is insufficient. As $E_t\omega_{t+1}$ is equal to $\beta^{-1} q_t B_t/Y_t$, it reflects the intertwined impacts of the ratio

[18] It is standard in many macroeconomic analyses to confuse the notions of solvency and sustainability of public debt. We shall see that this confusion is misleading.

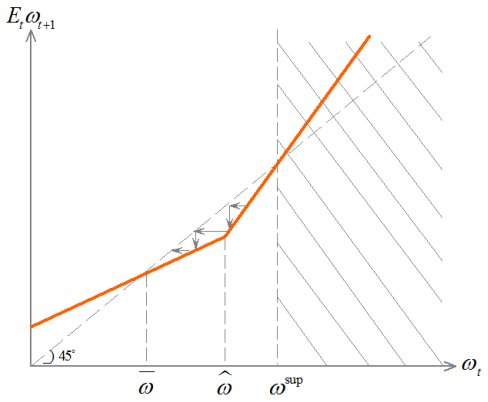

Figure 1: Deterministic Steady States

of emitted debt to GDP B_t/Y_t and the price at which it is sold on the market. We need to disentangle these two effects so as to obtain a proper understanding of the actual dynamics of B_t/Y_t.

5 Sovereign default and the market value of public debt.

In this section, we first study the determination of the market value of public debt, depending on the default threshold, and then we endogenize this default threshold. In the sequel, we restrict the analysis to configurations which fulfill the following assumption:

Assumption 3. *The economy in period t is such that:*

1. $min\,(\omega_{t-1}, \omega_t) > \hat{\omega}$,
2. $\exists \omega_t > \hat{\omega}$ *such that:* $prob\,\{default\ in\ t+1\,|\,\omega_t\} = prob\,\{\omega_{t+1} < \hat{\omega}\ |\,\omega_t\} = 0$.

Assumption 3 allows us to restrict the analysis of sovereign default to the fiscal constrained regime. 3.1 means that the economy in period t is already in the constrained fiscal regime since at least one period, implying that $\tau_t = \tau_{t-1} = \hat{\tau}$. 3.2 considers the case where, despite being in the constrained fiscal regime, that is $\omega_t > \hat{\omega}$, there exist some debt-to-output ratios such that the probability of sovereign default in $t+1$ is zero and the probability to stay in the constrained fiscal regime in $t+1$ is one.[19]

[19] We shall give conditions on the parameters set under which Assumption 3.**2** holds in section 5.2.

Let us denote by $b_t = B_t/Y_t$, the level of contractual government debt emitted today relative to GDP at t and by $\omega_t^{\max}$ the default threshold for period t as a percentage of GDP, that is: $\omega_t^{\max} \equiv \Omega_t^{\max}/Y_t$. Given the definition of ω_t, we get $\omega_t = h_t b_{t-1} Y_{t-1}/Y_t$. Under Assumption 3, we obtain from (16): $Y_t/Y_{t-1} = a_t$. The equilibrium conditions (16), (17) and (9) can be rewritten:

$$q_t = \beta E_t \left(\frac{h_{t+1}}{a_{t+1}} \right), \tag{23}$$

$$q_t b_t = h_t \frac{b_{t-1}}{a_t} + g - \hat{\tau}, \tag{24}$$

$$h_t = \begin{cases} \mathbf{h} \cdot a_t \omega_t^{\max}/b_{t-1} < 1 & \text{if } b_{t-1}/\omega_t^{\max} > a_t, \\ 1 & \text{if not.} \end{cases} \tag{25}$$

Taking the sequence $\{\omega_t^{\max}\}$ as given, these three equations are sufficient to analyze the valuation of public debt and the dynamics of emitted debt-to-output ratio b_t in the constrained fiscal regime.

5.1 Public debt valuation.

In order to solve the model consisting of equations (23) to (25), we conjecture that $\omega_{t+1}^{\max}$ is known in t.[20] Using (25) for $t+1$, and the probability distribution of a_{t+1}, the bond price given by (23) can be written : $q_t = \tilde{q}\left(b_t; \omega_{t+1}^{\max}, \mathbf{h}\right)$, where this last (pricing) equation is defined as

$$\tilde{q}\left(b_t; \omega_{t+1}^{\max}, \mathbf{h}\right) \equiv \beta \begin{cases} E\left(\frac{1}{a_{t+1}}\right) & \forall \frac{b_t}{\omega_{t+1}^{\max}} \leq a_{\inf}, \\ \mathbf{h} G\left(\frac{b_t}{\omega_{t+1}^{\max}}\right)\left(\frac{b_t}{\omega_{t+1}^{\max}}\right)^{-1} + \int_{b_t/\omega_{t+1}^{\max}} \left(\frac{1}{a_{t+1}}\right) dG\left(a_{t+1}\right) & \forall \frac{b_t}{\omega_{t+1}^{\max}} \in \left(a_{\inf}, a^{\sup}\right), \\ \mathbf{h}\left(\frac{b_t}{\omega_{t+1}^{\max}}\right)^{-1} & \forall \frac{b_t}{\omega_{t+1}^{\max}} \geq a^{\sup}. \end{cases} \tag{26}$$

Let us denote by v_t the market value of public debt relative to output: $v_t = q_t b_t$. It

[20] In the sequel we will restrict the analysis to a class of equilibria for which this conjecture holds.

can then be expressed as:

$$v_t = \tilde{q}\left(b_t; \omega_{t+1}^{\max}, \mathbf{h}\right) \cdot b_t \equiv V\left(b_t; \omega_{t+1}^{\max}, \mathbf{h}\right) \tag{27}$$

We refer to this last function as the "public debt valuation function". It allows to define three different regions:

1. When b_t is very low (less than $a_{\inf}\omega_{t+1}^{\max}$), there is no risk of default in $t+1$ and the value of emitted public debt is the quantity of bonds discounted at the risk-free gross interest rate $\beta E\left(1/a\right)$.

2. When b_t is in an intermediate range which happens to be $\left(a_{\inf}\omega_{t+1}^{\max}, a^{\sup}\omega_{t+1}^{\max}\right)$, the bond price q_t is a decreasing function of the emitted quantity of these bonds. Therefore the public debt value, v_t, is potentially non-monotone in b_t.

3. When b_t is very high (above $a^{\sup}\omega_{t+1}^{\max}$), default is certain in $t+1$. Therefore the value of sovereign bonds (in terms of GDP) is the discounted value of debt after rescheduling $\beta E_t\left(Y_t/Y_{t+1}\right) \cdot \mathbf{h}\Omega_{t+1}^{\max}/Y_t = \beta \mathbf{h}\omega_{t+1}^{\max}$.

The following proposition states the existence of a unique maximum value of public debt $v_t^{\max}$ for a given value of the future default threshold ratio $\omega_{t+1}^{\max}$:

Proposition 1. *Given $\omega_{t+1}^{\max}$, under Assumption 1, the valuation function reaches a unique maximum, denoted by $v_t^{\max}$, for a ratio $b_t = b_t^{\max}$. Both $v_t^{\max}$ and $b_t^{\max}$ are linearly increasing in $\omega_{t+1}^{\max}$: $v_t^{\max} = x_{\mathbf{h}}\omega_{t+1}^{\max}$ and $b_t^{\max} = \delta_{\mathbf{h}}\omega_{t+1}^{\max}$. The coefficients $x_{\mathbf{h}}$ and $\delta_{\mathbf{h}}$ are increasing functions of $\mathbf{h}$, satisfying $0 < x_{\mathbf{h}} \leq \beta$ for $0 \leq \mathbf{h} \leq 1$, and $1 < \delta_{\mathbf{h}} \leq a^{\sup}$ for $0 \leq \mathbf{h} \leq 1$.*

According to this proposition, the maximum value of public debt and the corresponding amount of emitted debt are simple functions of the expected default threshold.

Except in the region of no default, the price of the sovereign bond q_t is a decreasing function of b_t. Above $b_t^{\max}$, this negative effect overcomes the direct effect of increasing debt and makes the public debt value $v_t = q_t b_t$ starting to decrease. The higher the

debt recovery ratio $\mathbf{h}$, the higher the maximal market value: Lenders are ready to lend more as they receive more in case of default. Even in the extreme case of no debt recovery ($\mathbf{h} = 0$), lenders are potentially willing to lend to the government, despite possible default as they are compensated by a positive risk premium. In the extreme case of the highest recovery rate ($\mathbf{h} = 1$), the maximum public debt value is just equal to the discounted default threshold.

The valuation function $V\left(b_t; \omega_{t+1}^{\max}, \mathbf{h}\right)$ is represented when $\mathbf{h} < 1$ in Figure 2 by the non-linear curve displaying three different shapes over the three intervals defined above (see 27 and 26). The government's financing needs, given by the RHS of (24) with $h_t = 1$, $b_{t-1}/a_t - (\hat{\tau} - g)$, are also represented in Figure 2 by an horizontal straight line.

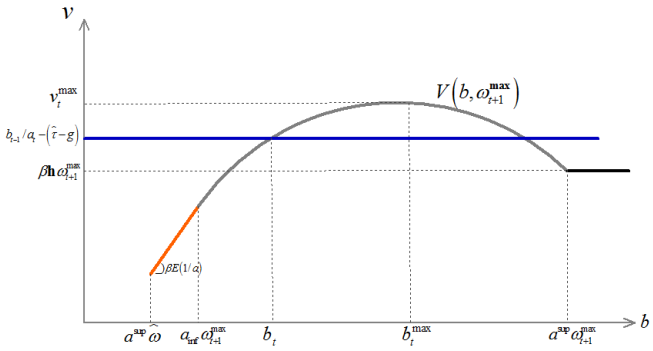

Figure 2: Public debt valuation in the no-default case

The first linear section of the curve corresponds to amounts of debt consistent with Assumption 3.**2.** Remember that under this assumption the economy remains in the constrained fiscal regime in $t+1$ with a probability 1:

$$\omega_{t+1} = b_t/a_{t+1} > \hat{\omega}, \forall a_{t+1} \Leftrightarrow b_t > a^{\sup}\hat{\omega} \tag{28}$$

and there exist some debt-to-GDP ratio in the constrained fiscal regime for which the economy does not face a risk of default in $t+1$, *i.e.* $b_t \leq a_{\inf}\omega_{t+1}^{\max}$.[21]

An equilibrium debt ratio b_t without default in t is such that (24) holds with $h_t = 1$ with $q_t b_t = V\left(b_t; \omega_{t+1}^{\max}\right)$. The equilibrium displayed in Figure 2 corresponds to the no-

[21] We shall give in section 6.1 a formal condition such that $a_{\inf}\omega_{t+1}^{\max} > a^{\sup}\hat{\omega}$.

default case. For financing needs between $\beta \mathbf{h} \omega_{t+1}^{\max}$ and $v_t^{\max}$, there are two values of b_t which meet this request (as shown in Figure 2). We can observe that the equilibrium situated on the decreasing side of the valuation function is "unstable" in the Walrasian sense. In the neighborhood of the high debt equilibrium, in the case of an excess demand a higher bond price increases the gap between demand and supply; the reverse is true in the case of an excess supply.[22] This leads us to select the low debt equilibrium, satisfying $b_t \leq b_t^{\max}$. Excluding the case of default (i.e. assuming $b_{t-1}/a_t \leq \omega_{t+1}^{\max}$), the equilibrium debt-to-output ratio is then given by:

$$b_t = \min \left(b \left| V \left(b_t; \omega_{t+1}^{\max}, \mathbf{h} \right) = -\left(\hat{\tau} - g \right) + b_{t-1}/a_t \right. \right). \tag{29}$$

5.2 The equilibrium default threshold.

Figure 2 helps us to graphically understand default as a market event. There is default in t when a sufficiently negative shock heightens the horizontal line above the $V \left(b_t; \omega_{t+1}^{\max}, \mathbf{h} \right)$ curve, that is, above $v_t^{\max}$.

Formally the condition corresponding to default can be written as:

$$\frac{b_{t-1}}{a_t} - \left(\hat{\tau} - g \right) \geq v_t^{\max}.$$

Up to now, the default condition used in (25) was written as $b_{t-1}/a_t > \omega_t^{\max}$. Thus the default threshold $\omega_t^{\max}$ is necessarily equal to:

$$\omega_t^{\max} = v_t^{\max} + \left(\hat{\tau} - g \right). \tag{30}$$

It is defined as the sum of the maximum value that the government can obtain from the market and the primary surplus of the period.

Since from Proposition 1 $v_t^{\max} = x_{\mathbf{h}}.\omega_{t+1}^{\max}$ and using the definition of (30), we get a

[22] Lorenzoni and Werning (2014) develop the same argument and give other reasons justifying the discarding of the "unstable" equilibrium.

dynamic expression for $\omega_t^{\max}$:

$$\omega_t^{\max} = x_{\mathbf{h}}.\omega_{t+1}^{\max} + (\hat{\tau} - g)\,. \tag{31}$$

It is a forward-looking equation: How much can at most be redeemed today depends on how much can at most be redeemed tomorrow, because this last term directly determines the opportunities for public funding.

Denoting by $\omega_{\mathbf{h}}^{\max}$ the stationary value of the default threshold in equation (31), we get the following proposition:

Proposition 2. *The equilibrium default threshold as a percentage of GDP, $\omega_t^{\max}$, is locally unique and equal to:*

$$\omega_t^{\max} = \frac{1-\beta}{1-x_{\mathbf{h}}}\omega^{\sup} \equiv \omega_{\mathbf{h}}^{\max}, \forall t. \tag{32}$$

$\omega_{\mathbf{h}}^{\max}$ is an increasing function of $\mathbf{h}$. If $\mathbf{h} = 1$, $x_{\mathbf{h}} = \beta$ and $\omega_{\mathbf{h}}^{\max} = \omega^{\sup}$.

Strikingly, even though we reason in a stochastic environment, the default threshold ratio is a constant, independent from the dynamics of public debt and thus from the history of shocks. This constancy will ease the study of this dynamics. However it depends on the debt recovery rule, that is, on $\mathbf{h}$. The lower the recovery rate, $\mathbf{h}$, the lower the default threshold $\omega_{\mathbf{h}}^{\max}$. This comes from the fact that $x_{\mathbf{h}}$ is an increasing function of $\mathbf{h}$, from Proposition 1.

Unless $x_{\mathbf{h}}$ is equal to β – its upper limit corresponding to the case $\mathbf{h} = 1$, the default threshold is lower than $\omega^{\sup}$, the solvency ratio.

Given this constant threshold we deduce from Proposition 1 that $v_t^{\max} = x_{\mathbf{h}}\omega_{\mathbf{h}}^{\max} \equiv v_{\mathbf{h}}^{\max}, \forall t$, that is, using (32)

$$v_{\mathbf{h}}^{\max} = \frac{(1-\beta)\,x_{\mathbf{h}}}{1-x_{\mathbf{h}}}\omega^{\sup}, \forall t, \tag{33}$$

and $b_t^{\max} = b_{\mathbf{h}}^{\max}, \forall t$, where $b_{\mathbf{h}}^{\max}$ denotes the amount of the debt-to-output ratio for which $V\left(b; \omega_{\mathbf{h}}^{\max}, \mathbf{h}\right)$ reaches its maximum.

6 Public debt dynamics and sustainability.

Now that the default threshold is known, the issues addressed in this section are threefold: 1) we study the dynamics of the emitted public debt ratio at time t, 2) offering new concepts of sustainability, we characterize the (un-) sustainability of public debt, 3) we emphasize the impact of the debt recovery rule on the post-default debt dynamics, highlighting the possibility of serial defaults.

6.1 Dynamics in t.

We first consider a period t where the random variable realization a_t and the debt amount to be redeemed b_{t-1} are such that no default occurs in t ($h_t = 1$). Defining $\upsilon(b_t; \mathbf{h}) \equiv V(b; \omega_{\mathbf{h}}^{\max}, \mathbf{h})$, the dynamics of public debt implicitly defined by (29) when $b_{t-1}/a_t \leq \omega_{\mathbf{h}}^{\max}$ can be written as:

$$\upsilon(b_t; \mathbf{h}) = \frac{b_{t-1}}{a_t} - (\hat{\tau} - g) \tag{34}$$

The function $\upsilon(b_t; \mathbf{h})$ is defined over the interval $[a^{\sup}\hat{\omega}, b_{\mathbf{h}}^{\max}]$ as monotonously increasing, continuous and thus invertible. Remember that the lower bound of the interval is the lowest debt ratio so that the economy remains in the constrained fiscal regime in $t+1$ with a probability 1 (see (28)). The upper bound corresponds to the debt-to-output ratio maximizing $\upsilon(b_t; \mathbf{h})$.

Inverting (34) for b_t belonging to the interval $[a^{\sup}\hat{\omega}, b_{\mathbf{h}}^{\max}]$, we get:

$$b_t = \upsilon^{-1}\left(\frac{b_{t-1}}{a_t} - (\hat{\tau} - g); \mathbf{h}\right). \tag{35}$$

This equation is well defined for b_{t-1}/a_t belonging to $[\omega_1, \omega_{\mathbf{h}}^{\max}]$ where $\omega_1 = \upsilon(a^{\sup}\hat{\omega}; \mathbf{h}) + (\hat{\tau} - g)$. The condition $b_{t-1}/a_t > \omega_1$ is equivalent to $b_t > a^{\sup}\hat{\omega}$, that is $b_t/a_{t+1} > \hat{\omega}, \forall a_{t+1}$, which corresponds to the certainty to stay in the constrained fiscal regime in $t+1$.

Let us define $\omega_2 = \upsilon(a_{\inf}\omega_{\mathbf{h}}^{\max}; \mathbf{h}) + (\hat{\tau} - g)$ as the highest debt ratio b_{t-1}/a_t for which the risk premium is nil in t, *i.e.* the probability of default in $t+1$ equals to 0.

The corresponding condition for b_t is then $b_t/a_{t+1} < \omega_{\mathbf{h}}^{\max}, \forall a_{t+1}$, *i.e.* $b_t < a_{\inf}\omega_{\mathbf{h}}^{\max}$. From (26) and (27) we can thus write ω_1 and ω_2 as:

$$\omega_1 = \beta E\left(1/a\right) a^{\sup}\hat{\omega} + \left(\hat{\tau} - g\right), \tag{36}$$

$$\omega_2 = \beta E\left(1/a\right) a_{\inf}\omega_{\mathbf{h}}^{\max} + \left(\hat{\tau} - g\right), \tag{37}$$

and Assumption 3.2 is equivalent to the condition $\omega_1 < \omega_2$, that is:[23]

$$a^{\sup}\hat{\omega} < a_{\inf}\omega_{\mathbf{h}}^{\max}. \tag{38}$$

The stochastic dynamics for a given realization of the random variable a_t is represented on Figure 3.

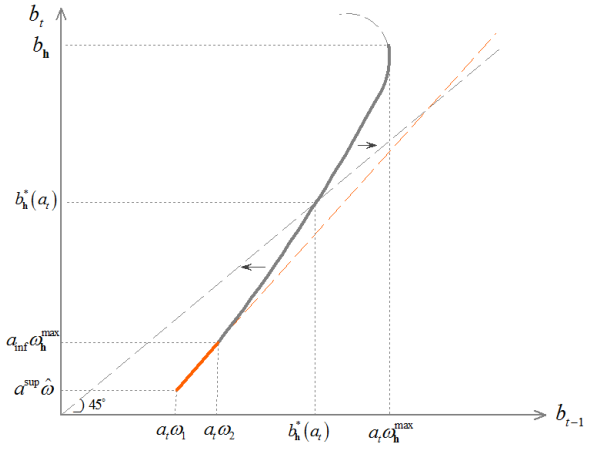

Figure 3: Dynamics

The first section of the curve is a linear segment over the interval $[a_t\omega_1, a_t\omega_2]$. The second section over the interval $[a_t\omega_2, a_t\omega_{\mathbf{h}}^{\max}]$ relates to positive probabilities of default. Its convexity is consistent with the concavity of the function $\upsilon\left(b_t; \mathbf{h}\right)$ on the relevant interval and reflects the increase of the risk premium with the emitted amount of debt. The turning point $(a_t\omega_{\mathbf{h}}^{\max}, b_{\mathbf{h}}^{\max})$ corresponds to the default threshold where $a_t\omega_{\mathbf{h}}^{\max}$ is the maximum amount of debt that can be redeemed when the shock is equal to a_t (thus

[23] As, from Proposition 2, $\omega_{\mathbf{h}}^{\max} \leq \omega^{\sup}$ for $\mathbf{h} \leq 1$, the inequality $a^{\sup} < \left(\omega^{\sup}/\hat{\omega}\right) a_{\inf}$ is a necessary condition for $\omega_1 < \omega_2$ when $\mathbf{h} < 1$ and is a necessary and sufficient condition when $\mathbf{h} = 1$. Using the definition of $\hat{\omega}$ given by (22), we check that the ratio $\omega^{\sup}/\hat{\omega}$ is superior to 1.

satisfying $b_{t-1}/a_t = \omega_{\mathbf{h}}^{\max}$).

As is clear from Figure 3, the following inequalities

$$\frac{a^{\sup}\hat{\omega}}{\omega_1} \equiv a_1 \leq a_t \leq \frac{b_{\mathbf{h}}^{\max}}{\omega_{\mathbf{h}}^{\max}} \equiv a_2(\mathbf{h}) \tag{39}$$

guarantee that the curve crosses the 45' line within $[a_t\omega_1, a_t\omega_{\mathbf{h}}^{\max}]$. Such an intersection is characterized by $b_t = b_{t-1} = b_{\mathbf{h}}^*(a_t)$ in (35). We define $b_{\mathbf{h}}^*(a_t)$ as a "conditional stationary equilibrium". It is contingent on a_t. At $b_{\mathbf{h}}^*(a_t)$ the exogenous primary surplus is used to pay the (growth adjusted) servicing of the emitted debt and thus the debt-to-GDP ratio remains constant from $t-1$ to t. From Figure 3, we observe that b_t is higher than b_{t-1} when the latter is superior to $b_{\mathbf{h}}^*(a_t)$ and thus the debt ratio heads toward the default threshold. A higher debt leads to a higher default risk premium and thus a lower bond price, leading the government to emit a higher debt in the subsequent period.

When $a_t = a_1$, the "conditional stationary equilibrium" $b_{\mathbf{h}}^*(a_1)$ is equal to the lowest limit of the constrained fiscal limit at which the probability of default is null, $a^{\sup}\hat{\omega}$. When $a_t = a_2(\mathbf{h})$, $b_{\mathbf{h}}^*(a_2(\mathbf{h}))$ is equal to the default threshold $b_{\mathbf{h}}^{\max}$; as such, it depends on the debt recovery ratio $\mathbf{h}$. For $a_t < a_1$, that is, for a sufficiently large and negative realization of the growth rate, the curve is above the 45' line and thus the public debt ratio increases for any level of b_{t-1} above $a_t\omega_1$, i.e. such that the prospect to remain in the constrained regime is certain. For $a_t > a_2(\mathbf{h})$, that is, for a sufficiently large and positive realization of the growth rate, the curve is below the 45' line and thus any level of redeemable debt ratio b_{t-1} lower than $a_t\omega_{\mathbf{h}}^{\max}$, i.e. such that there is no default in t, guarantees a decreasing debt ratio. Notice that the whole curve shifts rightward when a_t increases and $b_{\mathbf{h}}^*(a_t)$ is increasing in a_t.

As $b_{\mathbf{h}}^*(a_t) < a_t\omega_{\mathbf{h}}^{\max}$, the public debt emitted in $t-1$ can be such that $b_{\mathbf{h}}^*(a_t) < b_{t-1} < a_t\omega_{\mathbf{h}}^{\max}$ without provoking default: Lenders are ready to buy a larger amount of public debt as default is not certain and the increase in the risk premium compensates for the higher risk of default. The resulting increase of the debt ratio ($b_t > b_{t-1}$) formalizes the well-known "snowball" effect.

Notice that the public debt ratio may increase (when $b_{t-1} > b^*_{\mathbf{h}}(a_t)$) even though it is still true that $E_t\omega_{t+1}$ is inferior to ω_t. The former result indicates that the situation deteriorates as the *emitted* public debt ratio gets closer to the default threshold whereas the latter one shows that there is a decreasing behavior of the *expected* public debt ratio $E_t\left(h_{t+1}B_t/Y_{t+1}\right)$ (see Figure 1).

6.2 The public debt sustainability issue.

Given this unstable dynamics, the sustainability of public debt is at stake. Here, we offer a new methodology for assessing this issue in a stochastic environment where defaults as market events are possible.

As shown in the previous subsection, an ever increasing debt ratio eventually reaches the default threshold. To build upon this insight, let us consider the following simple scenario: the realization of the shock at any date is equal to its mean value: $a_t = E(a) = 1, \forall t$. The "conditional stationary equilibrium" $b^*_{\mathbf{h}}(1)$ is then consistent with the concept of "risky steady state", introduced by Juillard (2011) and Coeurdacier, Rey and Winant (2011). A risky steady state is a stationary equilibrium of the dynamic system when agents form their anticipations of future shocks knowing their probability distribution but the realizations of shocks are assumed to be at their mean values.

The dynamics corresponding to this scenario is illustrated by Figure 3 when $a_t = 1, \forall t$. If b_{t-1} is larger than $b^*_{\mathbf{h}}(1)$, as the debt ratio grows over time and heads toward the default threshold, this matches the loose concept of unsustainability given above: b_{t-1} is "unsustainable", if it is larger than $b^*_{\mathbf{h}}(1)$. Conversely, b_{t-1} would be qualified as "sustainable", if it is lower than $b^*_{\mathbf{h}}(1)$.

However these definitions are weak for the following reason. Assuming b_{t-1} is larger than but close to $b^*_{\mathbf{h}}(1)$, thus labelled as "unsustainable" in t, a small good realization of the shock would make b_t lower than $b^*_{\mathbf{h}}(1)$ and thus "sustainable" in $t+1$.[24] Thus these simple notions of "(un)sustainability" obtained in the case of the simple scenario are not operationally relevant for a debt sustainability analysis.

[24] Of course, the reverse reasoning applies for b_{t-1} lower than but close to $b^*_{\mathbf{h}}(1)$.

To overcome this weakness, we need to distinguish between the notions of sustainability and unsustainability. To do so, we offer the two following definitions:

Definition 1. *A public debt is said to be "$\gamma-$sustainable" at date t when its trajectory does not reach the default threshold at any future date, assuming that there is no realization of the (gross) rate of output growth a_{t+s} lower than γ.*

Definition 2. *A public debt is said to be "$\gamma-$unsustainable" at date t when its trajectory reaches the default threshold at some finite date, assuming that there is no realization of the (gross) rate of output growth a_{t+s} higher than γ.*

The first definition refers to the following "not-too-pessimistic" scenario: no (present and) future realizations of the shock can be lower than γ. Of course the interesting case for sustainability is when $\gamma < 1$, that is, when this value is below the mean value of the shock ($E(a) = 1$). The period t public debt is *"$\gamma-$sustainable"* if, under this scenario, a market-triggered default does not occur in the future. The second one refers to the following "not-too-optimistic" scenario: no (present and) future realizations of the shock can be higher than γ even though this value is above the mean value of the shock. The period t public debt is *"$\gamma-$unsustainable"* if, under this scenario, a market-triggered default will occur in the future. Here the interesting case for unsustainability is when $\gamma > 1$.

To be able to use these definitions, we generalize the concept of a risky steady state, introducing the notion of a "$\gamma-$*risky steady state*". This notion will allow us to distinguish between a "sustainability threshold" and an "unsustainability threshold".

Definition 3. *A $\gamma-$risky steady state is a stationary equilibrium of the dynamic system when agents form their anticipations of future shocks knowing the probability distribution whereas the realization of the shock is equal to a given (admissible) value γ at any period.*

Applying this definition to our problem, the $\gamma-$*risky steady state* level of debt denoted by $b^*_{\mathbf{h}}(\gamma)$ is the stationary level of the debt-to-GDP ratio b in equation (34) with $h_t = 1$, $a_t = \gamma$:

$$\upsilon(b; \mathbf{h}) = \gamma^{-1} b - (\hat{\tau} - g). \tag{40}$$

In the special case $\gamma = 1$ (the growth rate realization a_t equal to $\gamma = 1$) we are back to the study of a standard "risky steady state" and $b^*_{\mathbf{h}}(1)$ corresponds to the risky steady state debt-to-GDP ratio.

We are now able to offer the following

Proposition 3. *In the constrained fiscal regime (i.e. under Assumption 3),*

1. *for a given debt recovery ratio* $\mathbf{h}$*, there exists a pair* $\gamma_{\inf}$ *and* $\gamma^{\sup}_{\mathbf{h}}$ *satisfying* $a_{\inf} \leq \gamma_{\inf} < \gamma^{\sup}_{\mathbf{h}} \leq a^{\sup}$ *with* $\gamma_{\inf} < 1$ *if* $a^{\sup} < \frac{1-\beta}{1-\beta E\left(\frac{1}{a}\right)} \frac{\omega^{\sup}}{\hat{\omega}}$*, and* $1 < \gamma^{\sup}_{\mathbf{h}}$*, such that, for any* $\gamma \in (\gamma_{\inf}, \gamma^{\sup}_{\mathbf{h}})$

   (a) *there exists a unique* $\gamma-$*risky steady state* $b^*_{\mathbf{h}}(\gamma)$ *satisfying (40),*

   (b) $b^*_{\mathbf{h}}(\gamma)$ *is increasing in* γ*,*

   (c) *for* $\mathbf{h} \leq 1$*, we get* $a^{\sup}\hat{\omega} < b^*_{\mathbf{h}}(\gamma) < \gamma\omega^{\max}_{\mathbf{h}} < b^{\max}_{\mathbf{h}}$.

2. *for a given value of* γ*,* $b^*_{\mathbf{h}}(\gamma)$ *is increasing in* $\mathbf{h}$.

Proposition 3 states that a $\gamma-$*risky steady state* (in the sequel, a $\gamma-$*rss*) exists and is unique in the constrained fiscal regime for admissible values of γ.

When $\gamma < \gamma^{\sup}_{\mathbf{h}}$, according to Proposition 3, we get $b^*_{\mathbf{h}}(\gamma) < \gamma\omega^{\max}_{\mathbf{h}}$, the latter term being the maximum amount of debt that can be redeemed when the shock is equal to γ (thus satisfying $b/\gamma = \omega^{\max}_{\mathbf{h}}$). Thus the emitted public debt can be above the $\gamma-$rss without provoking default: b_t may be such that $b^*_{\mathbf{h}}(\gamma) < b_t < \gamma\omega^{\max}_{\mathbf{h}}$.

A higher constant realization of the growth rate γ increases the $\gamma-$*rss* $b^*_{\mathbf{h}}(\gamma)$. This comes from the fact that a higher gross rate of output growth γ alleviates the burden of the debt to be redeemed in each period relative to the current output. Thus the steady state debt can be higher when the constant growth rate γ is higher.

The $\gamma-$*rss* $b^*_{\mathbf{h}}(\gamma)$ is also increasing in the debt recovery ratio $\mathbf{h}$. As long as the default probability is positive, a higher $\mathbf{h}$ increases the market value for any b_t. Thus the curve displayed in Figure 3 is moved to the right and the $\gamma-$*rss* increases.

Applying the analysis of the dynamics of public debt to the case $a_t = \gamma, \forall t$, we deduce that the $\gamma-$*risky steady state* is dynamically unstable.[25] A higher growth path

[25] Referring to the transitory dynamics studied above, the dynamics of debt is decreasing (increasing) as long as it is lower (higher) than $b^*_{\mathbf{h}}(\gamma)$.

increases the upper value of the debt ratio such that this dynamics is decreasing. Given this instability result and using the definitions 1 and 2, we offer the following

Proposition 4. *For any* $\underline{\gamma}$ *and* $\overline{\gamma}$ *such that* $\gamma_{\inf} \leq \underline{\gamma} \leq 1 \leq \overline{\gamma} \leq \gamma_{\mathbf{h}}^{\sup}$,

1. *The emitted amount of public debt at* t, B_t, *is* "$\underline{\gamma}-$ sustainable" if $b_t \equiv B_t/Y_t < b_{\mathbf{h}}^{*}\left(\underline{\gamma}\right)$.

2. *The emitted amount of public debt at* t, B_t, *is* "$\overline{\gamma}-$ unsustainable" if $b_t \equiv B_t/Y_t > b_{\mathbf{h}}^{*}\left(\overline{\gamma}\right)$.

For a given pair $\left\{\underline{\gamma}, \overline{\gamma}\right\}$ when $\underline{\gamma} < 1 < \overline{\gamma}$, we refer to $b_{\mathbf{h}}^{*}\left(\underline{\gamma}\right)$ as the "$\underline{\gamma}-$sustainability threshold", and $b_{\mathbf{h}}^{*}\left(\overline{\gamma}\right)$ as the "$\overline{\gamma}-$unsustainability threshold". Figure 4 represents the curves corresponding the the dynamics of public debt for two values of γ, $\underline{\gamma}$ and $\overline{\gamma}$ and displays the corresponding thresholds.[26]

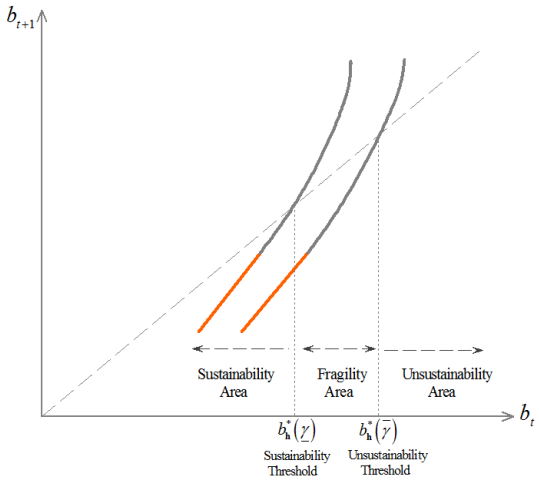

Figure 4: The Debt-to-GDP Area

Let us comment on the first part of the proposition. An initial amount of public debt b_t is $\underline{\gamma}-$*sustainable* when it is below the $\underline{\gamma}-$sustainability threshold level. In this case, conditional on a sequence of random events excluding the most unfavorable realizations (that is, excluding $a_{t+s} < \underline{\gamma}, \forall s \geq 1$), three features are worth mentioning:

[26] When $\underline{\gamma} = 1 = \overline{\gamma}$, the two thresholds are confounded, as we have just seen. But this configuration is exposed to the weakness we analyzed above. It is thus not enticing and has weak operational relevance.

1. the debt-to-output ratio decreases over time.

2. The probability of default and therefore the risk premium decrease over time.

3. The economy exits in finite time from the constrained fiscal regime.

The decrease of debt-to-output ratio can be explained using Figure 4. Consider the (left) curve corresponding to $\underline{\gamma}$. Similar curves could be drawn at its right for any realization of the shock satisfying $a_{t+s} > \underline{\gamma}$ (including $a_{t+s} = \overline{\gamma}$). When $b_t < b^*_{\mathbf{h}}\left(\underline{\gamma}\right)$, we get $b_{t+1} < b_t$ for any sequence such that $a_{t+s} \geq \underline{\gamma}$.

We label the region for b_t satisfying $b_t < b^*_{\mathbf{h}}\left(\underline{\gamma}\right)$ as the $\underline{\gamma}-$sustainability area. In this region, as long as $a_{t+s} \geq \underline{\gamma}$, the public debt ratio keeps decreasing and the economy diverges away from $b^*_{\mathbf{h}}\left(\underline{\gamma}\right)$. This means that at some date the redeemed public debt will pass below the critical value $\hat{\omega}$. Thus the economy will leave the constrained fiscal regime. It implies that a $\underline{\gamma}-$ sustainable public debt ratio is consistent with a gradual restoration of normal times. Eventually the economy regains fiscal margins to accommodate shocks.

Turning to the second part of the proposition, an initial amount of public debt b_t is $\overline{\gamma}-$*unsustainable* when it is above the $\overline{\gamma}-$unsustainability threshold level. In this case, conditional on a sequence of random events excluding the most favorable realizations (that is, excluding $a_{t+s} > \overline{\gamma}, \forall s \geq 1$), we get three symmetric features:

1. the debt-to-output ratio increases over time.

2. The probability of default and therefore the risk premium increase over time.

3. The economy hits in finite time the default threshold.

Consider a sequence of shocks ($a_{t+s} \leq \overline{\gamma}, \forall s \geq 1$). Assume that the initial level of debt b_t is too high: $b_t > b^*_{\mathbf{h}}\left(\underline{\gamma}\right)$. The same factors underlying the dynamics described for t in subsection 6.1 occur here repeatedly and nothing counters the evolution toward default. We label the region for b_t satisfying $b_t > b^*_{\mathbf{h}}\left(\overline{\gamma}\right)$ as the $\overline{\gamma}-$unsustainability area.

According to Proposition 3, when $\underline{\gamma} < 1 < \overline{\gamma}$, we get : $b^*_{\mathbf{h}}\left(\underline{\gamma}\right) < b^*_{\mathbf{h}}\left(\overline{\gamma}\right)$ and thus the two areas previously defined are not contiguous (see Figure 4). Therefore there exists

an intermediate region between these thresholds for which public debt can increase or decrease even under a not-too-pessimistic or a not-too-optimistic scenario. We label the region for b_t satisfying $b^*_{\mathbf{h}}\left(\underline{\gamma}\right) \leq b_t \leq b^*_{\mathbf{h}}\left(\overline{\gamma}\right)$ the "$\left\{\underline{\gamma}, \overline{\gamma}\right\}$- fragility area".

This partitioning may be made more complex for practical purpose. An international organization such as the IMF or a rating agency may define more than 2 thresholds, using the same methodology, and thus define a more elaborate set of thresholds for a set of values for γ. This suggests that our analytical reasoning could be developed so as to meet the needs of rating agencies which empirically develop complex rating formulas.

6.3 The debt recovery rule and public defaults.

The previous analysis shows that market triggered default cannot be ruled out in this economy. The issue is about understanding the impact of the debt recovery rule on the sustainability of the post-default public debt.

Let us assume that default has just occurred and public debt is rescheduled according to the debt recovery ratio, that is, $\omega_t = h_t b_{t-1}/a_t = \mathbf{h}\omega^{\max}_{\mathbf{h}}$. Reasoning on the extreme case of $\mathbf{h} = 1$, where lenders are minimally affected by the occurrence of default, the post-default debt ratio is equal to $\omega^{\max}_1 = \omega^{\sup}$ (from Proposition 2). Even when a very weak criterion of $\gamma-$sustainability is used with $\gamma = 1$,[27] this ratio is above $b^*_1(1)$ and is not sustainable: default is looming again. In other words, the default which led to the rescheduling of public debt has not been able to solve the public finance problem. Hence the economy is threatened to be engulfed in a sequence of defaults, what is referred to in the literature as "serial defaults". Reinhard and Rogoff (2008) provide evidence on the frequency of serial defaulting episodes. This example illustrates the point that a too low haircut does not solve the sustainability issue for good.

Facing this prospect, we look for the conditions on the debt reduction rule such that the post-default economy is $\gamma-$sustainable and thus both serial defaults and the fragility zone are avoided. Within our model, this issue can be more precisely formulated as follows: given a value of γ, for which values of $\mathbf{h}$ is the post-default debt ratio $\mathbf{h}\omega^{\max}_{\mathbf{h}}$ in the $\gamma-$sustainability area, that is, below $b^*_{\mathbf{h}}(\gamma)$? Notice that both ratios decrease when

[27] That is, no post-default realization of the growth shock is below the mean value.

h decreases (from Propositions 2 and 3). The effect on $\mathbf{h}\omega_{\mathbf{h}}^{\max}$ is the debt reduction effect, the effect on $b_{\mathbf{h}}^{*}(\gamma)$ is the "risk premium effect" as the prospect of possible losses due to default increases this premium and thus the $\gamma-$*r*ss debt level. So, once default occurs, the influence of the debt recovery ratio on the post-default situation is a priori ambiguous. The following proposition answers this question:

Proposition 5.

For a given γ *such that* $\gamma_{\inf} \leq \gamma \leq 1/\beta E\left(\frac{1}{a}\right)$*, there exists a critical value* $\mathbf{H}_{\gamma}$ *satisfying* $0 < \mathbf{H}_{\gamma} < \gamma$*, implicitly defined by:*

$$\mathbf{H}_{\gamma}\omega_{\mathbf{H}_{\gamma}}^{\max} = b_{\mathbf{H}_{\gamma}}^{*}(\gamma)$$

which is an increasing function of γ*.*
In case of default,

1. *for* γ *satisfying* $\gamma_{\inf} \leq \gamma < 1$*, the post-default debt-to-GDP ratio* $\mathbf{h}\omega_{\mathbf{h}}^{\max}$ *is* $\gamma-$*sustainable (i.e. satisfies:* $\mathbf{h}\omega_{\mathbf{h}}^{\max} < b_{\mathbf{h}}^{*}(\gamma)$*) if and only if* $\mathbf{h} < \mathbf{H}_{\gamma}$ *;*

2. *for* γ *satisfying* $1 \leq \gamma < 1/\beta E\left(\frac{1}{a}\right)$*, the post-default debt-to-GDP ratio* $\mathbf{h}\omega_{\mathbf{h}}^{\max}$ *is* $\gamma-$*unsustainable (i.e. satisfies:* $\mathbf{h}\omega_{\mathbf{h}}^{\max} > b_{\mathbf{h}}^{*}(\gamma)$*) if and only if* $\mathbf{h} > \mathbf{H}_{\gamma}$*.*

According to the first point of this proposition, once default occurs, the debt recovery ratio **h** must be sufficiently low (i.e. the haircut high enough) so as to set the post-default debt below the $\gamma-$sustainability ratio and thus ensures that the post-default debt is $\gamma-$sustainable. The proof shows that the difference $\mathbf{h}\omega_{\mathbf{h}}^{\max} - b_{\mathbf{h}}^{*}(\gamma)$ is monotonically increasing in **h**, despite the ambiguity noted above. Thus for a sufficiently small value of **h** (corresponding to a large haircut), the post-default debt is $\gamma-$sustainable. This proposition also states that a more demanding criterion of $\gamma-$sustainability (a lower value of γ) implies a lower debt recovery ratio. The requirement that the minimum haircut be larger (a lower $\mathbf{H}_{\gamma}$) implies a sufficiently low post-default debt in order to protect the dynamics of public debt from more adverse realizations of the growth rate and thus maintain its $\gamma-$sustainability.

On the other hand, if the debt recovery ratio is too high ($\mathbf{h} > \mathbf{H}_\gamma$), the possibility of a future default cannot be ruled out even when a "not-too-pessimistic scenario" is considered, that is a value of γ larger than 1. This proposition highlights a condition potentially leading to serial defaults.

A simple case will help us to understand what is at stake, when the sustainability criterion is simply $\gamma = 1$.[28] From proposition 5, the critical value for $\mathbf{H}_1$ is less than one. Suppose $\mathbf{h} = \mathbf{1}$ $(> \mathbf{H_1})$: the default rule is such that the post-default debt is at the highest level consistent with the reentering of the government on financial markets by emitting new public bonds. Consider then the scenario of the Risky Steady State, that is $a_t = 1$, $\forall t = 0, 1, ...$ It implies that there will be default in every period as the post-default debt is precisely at the default threshold and the dynamics of debt is diverging. This is the extreme case of serial defaults.

Proposition 5 is the analytical counterpart of the historical experience of serial defaults and the fact that post-default debt reductions often happen be "too little".[29]

7 Conclusion.

In this paper we offer a methodology for the assessment of public debt sustainability which is consistent with default episodes and the likelihood of future defaults. We tackle it within a macro dynamic stochastic (general equilibrium) model which allows for infrequent defaults and encompasses a debt recovery rule which defines the post-default reset public debt. Defaults are conceived as market events: there is default when no equilibrium price can be found for the redeemable public debt. The model is such that we are able to analytically solve it. Default occurs when the redeemable public debt trespasses a default threshold which we fully characterize. Based on this understanding of defaults we show the need to distinguish public debt sustainability and unsustainability conditions consistent with the stochastic nature of the economy.

Our analysis is embedded in a very simple macro-model which allows us to reach an

[28] Remember that this case is such that there is no distinction between the sustainability threshold and the unsustainability threshold.

[29] see IMF (2013b).

analytical solution. However the various concepts and tools provided here can be used in more complex macro-settings possibly solved by means of numerical methods.

The default threshold is different from the "solvency ratio". The solvency (debt-to-GDP) ratio is commonly defined as the extreme ratio satisfying the no-Ponzi condition. The standard view on public debt sustainability, ruling out a priori the possibility of default, has used this ratio as the criterion of public debt sustainability. When uncertainty is introduced and the assumption that a government will always be able to service its debt is relaxed, the pricing of public debt incorporates a default risk premium. As this premium feeds into the growth of public debt and may lead to a snowball effect, the default threshold is lower than the solvency ratio. Thus the debt recovery rule which defines the post-default public debt level affects the default threshold since it impacts on the risk premium and the public debt pricing. Consequently a rigorous and empirically relevant analysis of public debt sustainability cannot be based solely on the solvency ratio.

We provide an analysis of the sustainability of public debt consistent with a stochastic environment able to generate a risk premium linked to default by offering the notion of $\gamma-$risky steady state. It generalizes the notion of risky steady state. A $\gamma-$risky steady state of an economy is a steady state obtained when the realization of the shock (assuming for simplicity the existence of a unique shock) is always γ, even when agents base their behavior on the full distribution of the shock.

This allows us to make the distinction between public debt sustainability and unsustainability. We define two thresholds, the $\gamma-$sustainability threshold and the $\gamma-$unsustainability threshold. At a given time, if public debt is below the $\gamma-$sustainability threshold, the public issuer will not suffer default in the future provided future shocks on the growth rate are never lower than γ. Inversely, if public debt is above the $\gamma-$unsustainability threshold, the public issuer will encounter default at some future date provided future shocks on the growth rate are never larger than γ. When these thresholds are defined for two values of γ, the interval between these two values corresponds to a "financial fragility" zone for which it is impossible to ascertain the future course of public debt toward or away from default. These thresholds depend on the

debt recovery rule as this rule impacts on the service of public debt. We prove that the post-default public debt is $\gamma-$sustainable when the rescheduling scheme entails a sufficient reduction of public debt for a given γ. On the contrary a too low haircut ratio leads to serial defaults, as the post-default debt is $\gamma-$unsustainable.

Despite the theoretical nature of this paper, the various concepts we have offered have a realistic flavor as the paper explicitly puts to the fore the relationship between the pricing of public bonds, the amount of emitted debt and the likelihood of default. We think that operational counterparts of this relationship could be developed within empirically relevant models of countries and give some foundations to the process of rating sovereign debt, as done by private or public institutions such as the IMF or the European Commission.

This paper can be extended in different directions. Two issues seem particularly relevant. First, the maturity structure of public debt would be worth a thorough investigation. Second, the debt recovery rule is subject to strategic reasoning. In particular it raises an interesting time inconsistency issue. Prior to default it is tempting to announce a low haircut so as to reduce the risk premium and thus the prospect of unsustainability, But once default has occurred, a large haircut has the advantae to increase the capacity of the restructured public debt to be sustainable. This is left to further research.

References

[1] Aguiar, Mark, and Manuel Amador, 2014. "Sovereign debt." *Handbook of International Economics*, (North-Holland, vol. 4, 2014, 647–87).

[2] Aguiar, Mark, and Gita Gopinath, 2006. "Defaultable debt, interest rates and the current account." *Journal of International Economics*, 69 (2006), 64–83.

[3] Arellano, Cristina, 2008. "Default risk and income fluctuations in emerging economies." *The American Economic Review*, 98 (2008), 690–712.

[4] Arellano, Cristina, Andrew Atkeson, and Mark Wright, 2015. "External and public debt crises", National Bureau of Economic Research w21456 (2015).

[5] Asunoma, Tamon, and Christoph Trebesch, (2016). "Sovereign debt restructurings: Preemptive or post-Default," *Journal of the European Economic Association*, 14 (2016), 175-214.

[6] Bi, Huixan, 2012. "Sovereign default risk premia, fiscal limits, and fiscal policy", *European Economic Review,* 56 (2012), 389–410.

[7] Bi, Huixan, and Eric Leeper, 2012. "Analyzing fiscal sustainability". University of Indiana, unpublished manuscript.

[8] Bohn, Henning, 1995. "The Sustainability of Budget Deficits in a Stochastic Economy", *Journal of Money, Credit and Banking*, 27 (1995), 257-271.

[9] Bohn, Henning, 2007. "Are stationarity and cointegration restrictions really necessary for the intertemporal budget constraint?", *Journal of Monetary Economics*, 54 (2007), 1837–1847.

[10] Calvo, Guillermo, 1988. "Servicing the Public Debt: The Role of Expectations", *The American Economic Review*, 78 (1988), 647-661.

[11] Coeurdacier, Nicolas, Hélène Rey, and Pablo Winant, 2012. "The Risky Steady-State", *American Economic Review*, 101 (2011), 398-401.

[12] Cole, Harold, and Timothy Kehoe, 2000. "Self-Fulfilling Debt Crises", *Review of Economic Studies,* 67 (2000), 91-116.

[13] Collard, Fabrice, Michel A. Habib and Jean-Charles Rochet, 2015. "Sovereign debt sustainability in advanced economies", *Journal of the European Economic Association*, 13 (2015), 381–420.

[14] Cruces, Juan J., and Christoph Trebesch, 2013. "Sovereign Defaults: The Price of Haircuts", *American Economic Journal: Macroeconomics*, 5 (2013), 85–117.

[15] Cuadra, Gabriel, Juan M. Sanchez, Horacio Sapriza, 2010. "Fiscal policy and default risk in emerging markets", *Review of Economic Dynamics*, 13 (2010), 452–469.

[16] Daniel, Betty C., and Christos Shiamptanis, 2013. "Fiscal policy in the European monetary union", *Journal of Economic Dynamics & Control*, 37 (2013), 2307-2321.

[17] Das, Udaibir, Michael G. Papaioannou, and Christoph Trebesch, C. (2012). *Sovereign debt restructurings 1950-2010: literature survey, data, and stylized facts.* International Monetary Fund.

[18] Davig, Troy, Eric Leeper, and Todd B. Walker, 2011. "Inflation and the fiscal limit", *European Economic Review,* 55 (2011), 31-47.

[19] D'Erasmo, Pablo, Enrique G. Mendoza and Jhing Zhang, 2015. "What is a sustainable public debt?", Harald Uhlig and John Taylor (eds), *Handbook of Macroeconomics*, (North-Holland, vol. 2, 2015).

[20] Eaton, Jonathan, and M. Gersovitz, 1981. "Debt with potential repudiation: theoretical and empirical analysis", *Review of Economic Studies,* 48 (1981), 289–309.

[21] Eaton, Jonathan, and Rachel Fernandez, 1995. "Sovereign Debt" in *Handbook of International Economics*, G. M. Grossman and K. S. Rogoff, eds. (North-Holland, vol. 3, 1995, 2031-77).

[22] European Commission, 2016. *"Fiscal sustainability report 2015", Institutional paper 018,* January 2016.

[23] Ghosh, Atish. R. , Jun I. Kim, Enrique G. Mendoza, Jonathan D. Ostry and Mahvash S. Qureshi, 2013. "Fiscal Fatigue, Fiscal Space And Debt Sustainability In Advanced Economies", *The Economic Journal*, 123 (2013), 4–30.

[24] IMF, 2013a. "Staff guidance note for public debt sustainability analysis in market-access countries", May 9 2013.

[25] IMF, 2013b. "Sovereign debt restructuring - Recent developments and implications for the fund's legal and policy framework", April 26 2013.

[26] Juillard, Michel, 2011. "Local approximation of DSGE models around the risky steady state", mimeo, October 20, 2011.

[27] Lorenzoni, Guido, and Ivan Werning, 2014. "Slow Moving Debt Crises", *NBER Working Paper* No. 19228 (2014).

[28] Mendoza, Enrique G., and Vivian Z. Yue, "A general equilibrium model of sovereign default and business cycles", *The Quarterly Journal of Economics*, 127 (2012), 889–946.

[29] Reinhart, Carmen M., and Kenneth Rogoff, 2008. *This time is different*, (Princeton University Press, 2008).

[30] Reinhart, Carmen M., Kenneth Rogoff S., and Miguel A. Savastano, 2003. "Debt intolerance", *Brooking Papers on Economic Activity,* (2003), Issue 1, 1-74.

[31] Reinhart, Carmen M., and Christoph Trebesch, 2014. "A distant mirror of debt, default and relief", NBER Working Paper w20577 (2014).

[32] Sturzenegger, Federico, and Jeromin Zettelmeyer, 2008. "Haircuts: Estimating investor losses in sovereign restructurings, 1998–2005", *Journal of International Money and Finance*, 27 (2008), 780–805.

[33] Tomz, Michael, and Mark L. J. Wright, 2013. "Empirical research on sovereign debt and default", *National Bureau of Economic Research* Working Paper w18855 (2013).

[34] Uribe, Martin, 2006. "A fiscal theory of sovereign risk", *Journal of Monetary Economics,* 53 (2006), 1857-1875.

[35] Yue, Vivian Z., 2010. "Sovereign default and debt renegotiation", *Journal of International Economics*, 80 (2010), 176-187.

[36] Zettelmeyer, Jeromin, Christoph Trebesch, C., and Mitu Gulati, 2013. "The Greek debt restructuring: an autopsy", *Economic Policy*, 28 (2013), 513-563.

A Appendix

A.1 Proof of Proposition 1

By denoting $\delta_t = b_t/\omega_{t+1}^{\max}$, from (26) and (27) we can rewrite v_t as:

$$v_t = x_t \omega_{t+1}^{\max}, \tag{A.1}$$

with

$$x_t = \beta \begin{cases} E\left(1/a\right)\delta_t & \forall \delta_t \leq a_{\inf}, \\ \chi\left(\delta_t, \mathbf{h}\right) & \forall \delta_t \in \left(a_{\inf}, a^{\sup}\right), \\ \mathbf{h} & \forall \delta_t \geq a^{\sup}, \end{cases} \tag{A.2}$$

where $\chi\left(\delta, \mathbf{h}\right)$ is a non-monotonic function defined by:

$$\chi\left(\delta_t, \mathbf{h}\right) \equiv E\left(1/a\right)\delta_t - \int^{\delta_t} \left(\delta_t/a - \mathbf{h}\right) \cdot dG\left(a\right). \tag{A.3}$$

Let us define $\Phi\left(\delta, \mathbf{h}\right) = \partial\chi\left(\delta, \mathbf{h}\right)/\partial\delta$, the derivative of $\chi\left(\delta, \mathbf{h}\right)$ with respect to δ, that is:

$$\Phi\left(\delta, \mathbf{h}\right) \equiv E\left(1/a\right) - \int^{\delta} \frac{1}{a} dG\left(a\right) - \left(1 - \mathbf{h}\right) g\left(\delta\right). \tag{A.4}$$

Assume that there exists a value $\delta_{\mathbf{h}} \in \left(a_{\inf}, a^{\sup}\right)$ such that:

$$\Phi\left(\delta_{\mathbf{h}}, \mathbf{h}\right) = 0, \tag{A.5}$$

then, using (A.4) and (A.5) in (A.3), $\chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right)$ can be written:

$$\chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right) = \mathbf{h}G\left(\delta_{\mathbf{h}}\right) + \left(1 - \mathbf{h}\right)\delta_{\mathbf{h}} g\left(\delta_{\mathbf{h}}\right), \tag{A.6}$$

and gives:

$$x_t = \beta\chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right) \equiv x_{\mathbf{h}}. \tag{A.7}$$

We obtain the following results: $v_t^{\max} = x_{\mathbf{h}}\omega_{t+1}^{\max}$, and $b_t^{\max} = \delta_{\mathbf{h}}\omega_{t+1}^{\max}$.

By denoting $\Phi_z(\delta, \mathbf{h}) \equiv \partial\Phi(\delta, \mathbf{h})/\partial z$, the partial derivatives of $\Phi(\delta, \mathbf{h})$ for $z = \delta, \mathbf{h}$, we get, $\forall\ \delta \in (a_{\inf}, a^{\sup})$:

$$\Phi_{\mathbf{h}}(\delta, \mathbf{h}) = g(\delta) > 0, \tag{A.8}$$

$$\Phi_{\delta}(\delta, \mathbf{h}) = -\frac{1}{\delta}\left[g(\delta) + (1 - \mathbf{h})\,\delta g'(\delta)\right] < 0, \tag{A.9}$$

where the last inequality is satisfied if and only if:

$$\frac{\delta g'(\delta)}{g(\delta)} > -\frac{1}{1 - \mathbf{h}},$$

this condition being implied by Assumption *1*.3 for any $\mathbf{h} \in [0, 1)$.[30] From the definition of $\delta_{\mathbf{h}}$, implicitly given by (A.5) and satisfying $\delta_{\mathbf{h}} \in (a_{\inf}, a^{\sup})$, we then have:

$$\frac{\partial \delta_{\mathbf{h}}}{\partial \mathbf{h}} = -\frac{\Phi_{\mathbf{h}}(\delta_{\mathbf{h}}, \mathbf{h})}{\Phi_{\delta}(\delta_{\mathbf{h}}, \mathbf{h})} > 0. \tag{A.10}$$

Looking for the values $\underline{\mathbf{h}}$ and $\overline{\mathbf{h}}$ such that $\delta_{\underline{\mathbf{h}}} = a_{\inf}$ and $\delta_{\overline{\mathbf{h}}} = a^{\sup}$, we find from (A.4) and (A.5):

$$\underline{\mathbf{h}} \;=\; 1 - \frac{E(1/a)}{g(a_{\inf})}, \quad \overline{\mathbf{h}} = 1.$$

As it is assumed that $E(a) = 1$ (Assumption 1), by the Jensen Inequality $E(1/a) > 1$ and $\underline{\mathbf{h}} < 0$. As $\mathbf{h} \geq 0$, this value is irrelevant.

When $\mathbf{h}$ =1, we get from (A.6) and (A.7):

$$x_{\mathbf{1}} = \beta G(\delta_{\mathbf{1}})$$

with $\delta_{\mathbf{1}}$ given by (A.4) and (A.5) with $\mathbf{h} = 1$, or equivalently:

$$E\left(\frac{1}{a}\right) = \int^{\delta_{\mathbf{1}}} \frac{1}{a} dG(a)$$

implying $\delta_{\mathbf{1}} = a^{\sup}$ and therefore $x_{\mathbf{1}} = \beta G(a^{\sup}) = \beta$.

[30] The elasticity of the density function is higher than -1.

When $\mathbf{h} = 0$, we get from (A.6) and (A.7):

$$x_{\mathbf{0}} = \beta \delta_{\mathbf{0}} g\left(\delta_{\mathbf{0}}\right) \tag{A.11}$$

with $\delta_{\mathbf{0}}$ given by (A.4) and (A.5) with $\mathbf{h} = 0$, and is such that:

$$g\left(\delta_{\mathbf{0}}\right) = \int_{\delta_{\mathbf{0}}}^{a^{\sup}} \frac{1}{a} dG\left(a\right). \tag{A.12}$$

This value is positive using Assumption 1.2 and unique using Assumption 1.3. From (A.11), it implies that $x_{\mathbf{0}}$ is strictly positive.

Let us assume there exists $\check{\mathbf{h}}$ is such that $\delta_{\check{\mathbf{h}}} = 1$. From (A.4), it satisfies

$$\int_{1} \frac{1}{a} dG\left(a\right) - \left(1 - \check{\mathbf{h}}\right) g\left(1\right) = 0,$$

hence:

$$\check{\mathbf{h}} = 1 - \frac{\int_{1} \frac{1}{a} dG\left(a\right)}{g\left(1\right)} \tag{A.13}$$

From Assumption 1.3, we know that: $\frac{1}{a} g\left(a\right) > -g'\left(a\right), \forall a$, which implies:

$$\int_{1} \frac{1}{a} g\left(a\right) da > -\int_{1} g'\left(a\right) da = g\left(1\right) - g\left(a^{\sup}\right)$$

From Assumption 1.2, $\left(\lim_{a \to a^{\sup}} g\left(a\right) = \varepsilon\right)$, therefore

$$\int_{1} \frac{1}{a} g\left(a\right) da > g\left(1\right) - \varepsilon$$

It follows from (A.13) that $\check{\mathbf{h}} < \varepsilon / g\left(1\right)$. As we know from (A.10) that $\delta_{\mathbf{h}}$ is an increasing function of $\mathbf{h}$, $\delta_{\mathbf{h}} > 1, \forall \mathbf{h} \geq \varepsilon / g\left(1\right)$. Since ε is arbitrarily close to 0, this is true for $\mathbf{h}$ arbitrarily close to 0.

Finally, we prove that $\chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right)$ is increasing in $\mathbf{h}$. From (A.3) and (A.5), for $\delta_{\mathbf{h}} \in \left(a_{\inf}, a^{\sup}\right)$, we get:

$$\frac{d\chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right)}{d\mathbf{h}} = \frac{\partial \chi\left(\delta_{\mathbf{h}}, \mathbf{h}\right)}{\partial \mathbf{h}} = \beta G\left(\delta_{\mathbf{h}}\right) > 0. \tag{A.14}$$

A.2 Proof of Proposition 2.

Using the definition of $\omega^{\sup}$ given by (21), equation (31) can be rewritten as:

$$\omega_t^{\max} = x_{\mathbf{h}}.\omega_{t+1}^{\max} + (1-\beta)\,\omega^{\sup} \tag{A.15}$$

whose stationary value is given by:

$$\omega_t^{\max} = \frac{1-\beta}{1-x_{\mathbf{h}}}\omega^{\sup} \equiv \omega_{\mathbf{h}}^{\max}, \forall t. \tag{A.16}$$

From Proposition 1, $x_{\mathbf{h}} \leq \beta$, $\forall\ \mathbf{h} \leq 1$ (with $x_{\mathbf{1}} = \beta$) implying that the forward-looking equation (A.15) has an unstable dynamics around the unique stationary equilibrium, $\omega_{\mathbf{h}}^{\max}$, which is determinate and locally unique. From Proposition 1, $x_{\mathbf{h}}$ is an increasing function of $\mathbf{h}$, thus $\omega_{\mathbf{h}}^{\max}$ is an increasing function of $\mathbf{h}$ too, satisfying $\omega_{\mathbf{h}}^{\max} < \omega^{\sup}$, $\forall$ $\mathbf{h} < 1$, and $\omega_{\mathbf{1}}^{\max} = \omega^{\sup}$.

A.3 Proof of Proposition 3.

In order to establish this proposition, we first represent the $\gamma-rss$ on Figure 5 for different values of γ. When it exists, the $\gamma-rss$ corresponds to the intersection of the two curves associated with the LHS and RHS of (40) restated here:

$$\upsilon\,(b;\mathbf{h}) = \left(\gamma^{-1}b - (\hat{\tau}-g)\right). \tag{A.17}$$

We know from Section 5.1 that the bounding values of the public debt ratio corresponding to the constrained fiscal regime are $a^{\sup}\hat{\omega}$ and $b_{\mathbf{h}}^{\max}$. The dashed line $\gamma_l^{-1}b - (\hat{\tau}-g)$ corresponds to the lowest possible value of γ, γ_l, for which the $\gamma-rss$ is consistent with the constrained fiscal regime. The dashed line $\left(\gamma_{\mathbf{h}}^{\sup}\right)^{-1}b - (\hat{\tau}-g)$ correspond to the largest possible value of γ, $\gamma_{\mathbf{h}}^{\sup}$.

Let us characterize these two extreme values.

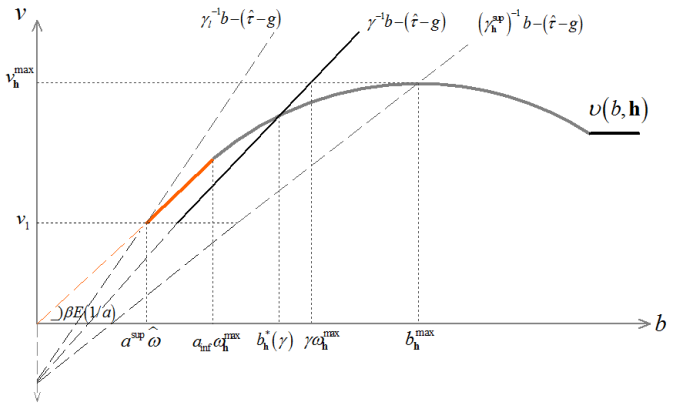

Figure 5: $\gamma-$Risky steady state

- γ_l is such that $b_{\mathbf{h}}^{*}(\gamma_l) = a^{\sup}\hat{\omega}$ and is implicitly given by:

$$\upsilon\left(a^{\sup}\hat{\omega};\mathbf{h}\right) = \gamma_l^{-1}a^{\sup}\hat{\omega} - (\hat{\tau}-g).$$

Hence, as $\upsilon\left(a^{\sup}\hat{\omega};\mathbf{h}\right) = \beta E\left(\frac{1}{a}\right)a^{\sup}\hat{\omega}$ from (27), we get:

$$\gamma_l = \frac{\hat{\omega}}{\beta E\left(\frac{1}{a}\right)a^{\sup}\hat{\omega} + (\hat{\tau}-g)}a^{\sup} = \frac{\hat{\omega}}{\omega_1}a^{\sup}. \tag{A.18}$$

Using the first equality and the definition of $\omega^{\sup}$ given by (21) $\gamma_l < 1$ is equivalent to:

$$a^{\sup} < \frac{1-\beta}{1-\beta E\left(\frac{1}{a}\right)}\frac{\omega^{\sup}}{\hat{\omega}}. \tag{A.19}$$

If γ_l is lower than $a_{\inf}$ which is the lowest value of the distribution support, then $\gamma_{\inf} = a_{\inf}$. Formally:

$$\gamma_{\inf} = \max\left(a_{\inf},\gamma_l\right) = \max\left(a_{\inf},\frac{\hat{\omega}}{\omega_1}a^{\sup}\right) < 1, \tag{A.20}$$

where the last inequality is satisfied under condition (A.19).

- $\gamma_{\mathbf{h}}^{\sup}$ is such that $b_{\mathbf{h}}^{*}(\gamma_{\mathbf{h}}^{\sup}) = b_{\mathbf{h}}^{\max}$ and is implicitly given by $\upsilon\left(b_{\mathbf{h}}^{\max};\mathbf{h}\right) = v_{\mathbf{h}}^{\max} = \left(\gamma_{\mathbf{h}}^{\sup}\right)^{-1}b_{\mathbf{h}}^{\max} - (1-\beta)\omega^{\sup}$. Using the definition of $v_{\mathbf{h}}^{\max}$given by (33), we get:

$$\gamma_{\mathbf{h}}^{\sup} = \frac{b_{\mathbf{h}}^{\max}}{\omega_{\mathbf{h}}^{\max}} = \delta_{\mathbf{h}}.$$

Given Proposition 1, we get that $1 < \gamma_{\mathbf{h}}^{\sup} = \delta_{\mathbf{h}} \leq a^{\sup}$ for $0 \leq \mathbf{h} \leq 1$.

We are now in capacity to prove Proposition 3:

1.

(a) Let us consider these two extreme values of γ, $\gamma_{\inf}$ and $\gamma_{\mathbf{h}}^{\sup}$. Based on these values, we obtain a cone defined by two straight lines depending on b, corresponding to equations $b/\gamma_{\inf} - (\hat{\tau} - g)$ and $b/\gamma_{\mathbf{h}}^{\sup} - (\hat{\tau} - g)$ respectively. For any γ such that $\gamma_{\inf} \leq \gamma \leq \gamma_{\mathbf{h}}^{\sup}$, the line of equation $b/\gamma - (\hat{\tau} - g)$ belongs to this cone and there exists a unique level $b_{\mathbf{h}}^{*}(\gamma)$ satisfying (A.17), given the continuity and concavity properties of $\upsilon(b; \mathbf{h})$. Figure 5 illustrates that there exists a unique $\gamma - rss$ for any $\gamma \in (\gamma_{\inf}, \gamma_{\mathbf{h}}^{\sup})$.[31]

(b) As the RHS of (A.17) is decreasing in γ and $\upsilon(b; \mathbf{h})$ is continuously increasing on $[a^{\sup}\hat{\omega}, b_{\mathbf{h}}^{\max}]$, $b_{\mathbf{h}}^{*}(\gamma)$ is an increasing function of γ (see Figure 5).

(c) For any $\gamma \in [\gamma_{\inf}, \gamma_{\mathbf{h}}^{\sup}]$, the slope of $\upsilon(b; \mathbf{h})$ is always lower than γ^{-1} in the neighborhood of $b_{\mathbf{h}}^{*}(\gamma)$. Using (30) and (32), we get:

$$\omega_{\mathbf{h}}^{\max} = v_{\mathbf{h}}^{\max} + (\hat{\tau} - g).$$

This allows us to get the value of b corresponding to $\gamma^{-1}b - (\hat{\tau} - g) = v_{\mathbf{h}}^{\max}$, that is $b = \gamma\omega_{\mathbf{h}}^{\max}$. As can be seen on Figure 5, for $\gamma = \gamma_{\mathbf{h}}^{\sup}$, we have $b_{\mathbf{h}}^{*}(\gamma) = \gamma\omega_{\mathbf{h}}^{\max} = b_{\mathbf{h}}^{\max}$ and for $\gamma < \gamma_{\mathbf{h}}^{\sup}$, we get: $a^{\sup}\hat{\omega} < b_{\mathbf{h}}^{*}(\gamma) < \gamma\omega_{\mathbf{h}}^{\max} < b_{\mathbf{h}}^{\max}$.

2. From (A.1), (A.2), (A.16), and Proposition 2, v_t is increasing in $\mathbf{h}$, $\forall b_t > a_{\inf}\omega_{t+1}^{\max}$. Thus, for a given value of γ the value of b for which $\upsilon(b; \mathbf{h})$ intersects with $\gamma^{-1}b - (\hat{\tau} - g)$, that is $b_{\mathbf{h}}^{*}(\gamma)$, is shifted rightward when $\mathbf{h}$ increases.

A.4 Proof of Proposition 4.

1. Assuming that the future realizations of the output growth rate satisfy $a_{t+s} \geq \underline{\gamma}$, $\forall s$, and $b_t < b_{\mathbf{h}}^{*}(\underline{\gamma})$, the public debt-to-GDP dynamics is decreasing (see Figure

[31] This figure is obtained in the case where $a_{\inf} < \gamma_l = \gamma_{\inf}$.

3). Therefore it cannot reach the default threshold at any future date. Thus it is $\underline{\gamma}-$*sustainable* according to Definition 1.

2. Similarly, assuming that the future realizations of the output growth rate satisfy $a_{t+s} \leq \overline{\gamma}$, $\forall s$, and $b_t > b^*_{\mathbf{h}}(\overline{\gamma})$, the public debt-to-GDP dynamics is increasing (see Figure 3). Therefore it reaches the default threshold at some finite future date. Thus it is $\overline{\gamma}-$*unsustainable* according to Definition 2.

A.5 Proof of Proposition 5

We assume $\gamma_{\inf} \leq \gamma \leq 1/\beta E\left(\frac{1}{a}\right)$. In order to prove the first part of this proposition, we have to show that $\Psi(\mathbf{h},\gamma) \equiv \mathbf{h}\omega^{\max}_{\mathbf{h}} - b^*_{\mathbf{h}}(\gamma)$ is monotonously increasing in $\mathbf{h}$, with $\Psi(\mathbf{H}_\gamma,\gamma) = 0$ for a value $\mathbf{H}_\gamma$ such that $0 < \mathbf{H}_\gamma < \gamma$. We define the function $\psi(\mathbf{h},\gamma)$ such that:

$$\psi(\mathbf{h},\gamma) \equiv \frac{\Psi(\mathbf{h},\gamma)}{\omega^{\max}_{\mathbf{h}}} = \mathbf{h} - \delta^*_{\mathbf{h}}(\gamma) \tag{A.21}$$

where $\delta^*_{\mathbf{h}}(\gamma) \equiv b^*_{\mathbf{h}}(\gamma)/\omega^{\max}_{\mathbf{h}}$. Since $\omega^{\max}_{\mathbf{h}} > 0$, a sufficient condition to get a value $\mathbf{H}_\gamma$ such that $\psi(\mathbf{H}_\gamma,\gamma) = 0$ is that the function $\psi(\mathbf{h},\gamma)$ be a function continuously increasing in $\mathbf{h}$, $\forall \mathbf{h} \in [0,\gamma]$ and such that $\psi(0,\gamma) < 0 < \psi(\gamma,\gamma)$.

By differentiating $\psi(\mathbf{h},\gamma)$ with respect to $\mathbf{h}$, we find:

$$\frac{\partial\psi(\mathbf{h},\gamma)}{\partial\mathbf{h}} = 1 - \frac{\partial\delta^*_{\mathbf{h}}(\gamma)}{\partial\mathbf{h}}. \tag{A.22}$$

By dividing the RLS and the RHS of (A.17), written for $b = b^*_{\mathbf{h}}$, by $\omega^{\max}_{\mathbf{h}}$ which is given by (A.16), and using the definition of $\omega^{\sup}$ given by (21), we obtain:

$$\frac{\upsilon(b^*_{\mathbf{h}};\mathbf{h})}{\omega^{\max}_{\mathbf{h}}} = \gamma^{-1}\delta^*_{\mathbf{h}}(\gamma) - (1 - x_{\mathbf{h}})$$

From (A.1) and (A.2), for $\delta_t \in (a_{\inf}, a^{\sup})$, we then find*:*

$$\delta^*_{\mathbf{h}}(\gamma) = \gamma\left[1 - \beta\chi(\delta_{\mathbf{h}},\mathbf{h}) + \beta\chi\left(\delta^*_{\mathbf{h}}(\gamma),\mathbf{h}\right)\right] \tag{A.23}$$

which allows us to get, using the fact that $\frac{\partial \chi(\delta_{\mathbf{h}},\mathbf{h})}{\partial \delta_{\mathbf{h}}} = 0$:

$$\frac{\partial \delta^*_{\mathbf{h}}(\gamma)}{\partial \mathbf{h}} = \gamma\beta \cdot \left(\frac{\frac{\partial \chi\left(\delta^*_{\mathbf{h}}(\gamma),\mathbf{h}\right)}{\partial \mathbf{h}} - \frac{\partial \chi(\delta_{\mathbf{h}},\mathbf{h})}{\partial \mathbf{h}}}{1 - \gamma\beta \frac{\partial \chi\left(\delta^*_{\mathbf{h}}(\gamma),\mathbf{h}\right)}{\partial \delta^*_{\mathbf{h}}(\gamma)}} \right).$$

Using (A.3), (A.4) and $\Phi(\delta,\mathbf{h}) = \partial\chi(\delta,\mathbf{h})/\partial\delta$, we get:

$$\frac{\partial \delta^*_{\mathbf{h}}(\gamma)}{\partial \mathbf{h}} = \beta\gamma \left(\frac{G\left(\delta^*_{\mathbf{h}}(\gamma)\right) - G\left(\delta_{\mathbf{h}}\right)}{1 - \gamma\beta\Phi\left(\delta^*_{\mathbf{h}}(\gamma),\mathbf{h}\right)} \right)$$

From (A.4), $\Phi\left(\delta^*_{\mathbf{h}}(\gamma),\mathbf{h}\right) < E(1/a)$ and from Assumption 1.1, $\beta E(1/a) < 1$. As we assume $\gamma \leq 1/\beta E\left(\frac{1}{a}\right)$, we therefore have $\gamma\beta\Phi\left(\delta^*_{\mathbf{h}}(\gamma),\mathbf{h}\right) < 1$, and the denominator in the RHS is positive. From Proposition 3, we know that $b^*_{\mathbf{h}}(\gamma) \leq b^{\max}_{\mathbf{h}}$, which implies:

$$\delta^*_{\mathbf{h}}(\gamma) \equiv \frac{b^*_{\mathbf{h}}(\gamma)}{\omega^{\max}_{\mathbf{h}}} < \delta_{\mathbf{h}} \equiv \frac{b^{\max}_{\mathbf{h}}}{\omega^{\max}_{\mathbf{h}}}$$

and the numerator is negative as $G(\cdot)$ is increasing. Hence $\partial\delta^*_{\mathbf{h}}(\gamma)/\partial\mathbf{h}$ is negative for any value of $\mathbf{h}$ and $\psi(\mathbf{h},\gamma)$ is monotonously increasing in $\mathbf{h}$.

By computing $\psi(0,\gamma)$ and $\psi(\gamma,\gamma)$, we get:

$$\psi(0,\gamma) = -\delta^*_0(\gamma) < 0 \quad \text{and} \quad \psi(\gamma,\gamma) = \gamma - \delta^*_\gamma(\gamma) > 0$$

as $\delta^*_\gamma(\gamma) \equiv b^*_\gamma(\gamma)/\omega^{\max}_\gamma < \gamma$ from Proposition 3. Therefore there exists a value $\mathbf{H}_\gamma$ such that $0 < \mathbf{H}_\gamma < \gamma$ and $\psi(\mathbf{H}_\gamma,\gamma) = 0$, or equivalently $\Psi(\mathbf{H}_\gamma,\gamma) = 0$.

From the previous equality, $\mathbf{H}_\gamma$ is a function of γ which we denote by $H(\gamma)$. Thus:

$$\frac{\partial H(\gamma)}{\partial \gamma} = -\frac{\psi_\gamma\left(H(\gamma),\gamma\right)}{\psi_{\mathbf{h}}\left(H(\gamma),\gamma\right)}$$

The denominator is positive as shown above. We observe that:

$$\psi_\gamma\left(H(\gamma),\gamma\right) = -\frac{\partial \delta^*_{\mathbf{h}}(\gamma)}{\partial \gamma} \tag{A.24}$$

From (A.23) we get:

$$\frac{\partial \delta_{\mathbf{h}}^{*}(\gamma)}{\partial \gamma} = \frac{1-\chi\left(\delta_{\mathbf{h}},\mathbf{h}\right)+\chi\left(\delta_{\mathbf{h}}^{*}(\gamma),\mathbf{h}\right)}{1-\gamma\frac{\partial\chi\left(\delta_{\mathbf{h}}^{*}(\gamma),\mathbf{h}\right)}{\partial\delta_{\mathbf{h}}^{*}(\gamma)}} = \frac{1}{\gamma}.\frac{\delta_{\mathbf{h}}}{1-\gamma\beta\Phi\left(\delta_{\mathbf{h}}^{*}(\gamma),\mathbf{h}\right)} > 0.$$

Thus $\partial H(\gamma)/\partial\gamma$ is positive.

Points 1 and 2 of Proposition 5 are direct applications of Proposition 4.