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Proportional payoffs in legislative bargaining with weighted voting: A characterization

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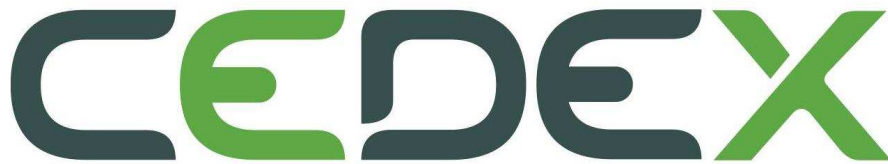
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Proportional payoffs in legislative bargaining with weighted voting: a characterization

Maria Montero*

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Abstract

We examine the relationship between voting weights and expected equilibrium payoffs in legislative bargaining and provide a necessary and sufficient condition for payoffs to be proportional to weights. This condition has a natural interpretation in terms of the supply and demand for coalition partners. An implication of this condition is that Snyder et al.'s (2005) result, that payoffs are proportional to weights in large replicated games, does not necessarily extend to the smaller games that arise in applications. Departures from proportionality may be substantial and may arise even in well-behaved (homogeneous) games.

J.E.L. codes: C78, D72.

Keywords: legislative bargaining, weighted voting, proportional payoffs.

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1 Introduction

Many important collective bodies make decisions by weighted majority voting. Examples are the Electoral College in the United States, the International Monetary Fund, the European Union Council of Ministers and any legislature with disciplined political parties. An important question in this setting is how the distribution of votes affects payoffs. Power indices such as the Shapley-Shubik index coincide with voting weights only rarely. In contrast, Snyder, Ting and Ansolabehere (2005, p. 982) argue that

Elementary microeconomic theory teaches that in competitive situations perfect substitutes have the same price. In a political setting in which votes might be traded or transferred in the formation of coalitions, one might expect the same logic to apply. If a player has k votes, then that player should command a price for those votes equal to the total price of k players that each have one vote.

Snyder et al. (2005) use a noncooperative bargaining game based on the Baron-Ferejohn (1989) model to show that a voter's expected payoff is proportional to its voting weight. They mention two difficulties in proving this result: corner solutions created by equal recognition probabilities, and nonhomogeneity of the game. Equal recognition probabilities may lead to low-weight voters having disproportionately high payoffs due to proposing power, whereas nonhomogeneous games create a difficulty in that players may be substitutes in some minimal winning coalitions but not in others, and it is not immediately obvious what the competitive price for their votes should be. They address these difficulties by making recognition probabilities proportional to voting weights and by replicating the game a finite (though potentially large) number of times (see Proposition 2 in Snyder et al. (2005)).

Since the proof in Snyder et al. (2005) only covers replicated games, how far this result extends to the legislatures with only a few parties that arise in applications is

an open question. The present paper provides a necessary and sufficient condition for proportional equilibrium payoffs. This necessary and sufficient condition is relevant for any weighted voting game; the only assumption needed is that recognition probabilities are proportional to the voting weights. We give an interpretation of this condition in economic terms: there is no excess supply or demand of any player type. An implication of this condition is that, even in the intuitively most favorable case (i.e. uniquely defined homogeneous weights and recognition probabilities proportional to those weights) the equilibrium of the game is not necessarily competitive. It may be possible for larger players to get a disproportionate payoff even if cheaper perfect substitutes appear to be available.

In order to get a rough idea of how often proportional payoffs are predicted in applications, the condition is used to calculate the frequency with which the model actually predicts proportional payoffs in Snyder et al.'s dataset of coalition governments in 14 countries from 1946 to 2001. Proportional payoffs are predicted for about 69% of the legislatures; this proportion varies between countries and can be as high as 100% (for Australia and Austria) or as low as 28% (for Italy). Deviations from proportionality may be substantial, both quantitatively and qualitatively, and this is illustrated with some examples from the dataset. Perhaps the most important qualitative deviation is that it is possible for asymmetric parties to have the same equilibrium expected payoff, even though one of the parties is a more desirable coalition partner and has a greater probability of being proposer. Also, minimal winning coalitions are not necessarily proposed in equilibrium (surplus coalitions in which the proposer is the only nonpivotal player are also possible).

The remainder of the paper is organized as follows. Section 2 presents the model and some simple examples, section 3 contains the characterization result, section 4 turns to the predictions of the model for the dataset, and section 5 concludes.

2 Preliminaries

2.1 Weighted voting games

$N = \{1, \dots, n\}$ is the set of players, $S \subseteq N$ represents a generic coalition and X is the set of alternatives. In the legislative bargaining model under consideration, there is a budget of size 1 to be divided and $X = \{x | x_i \geq 0 \text{ for all } i \text{ and } \sum_{i \in N} x_i \leq 1\}$ is the set of all possible allocations. Player i 's preferences are described by the utility function $u_i(x) = x_i$.

The voting game is described by a set of *winning coalitions* W , where a winning coalition is a coalition that can enforce any alternative in X . A voting game is *proper* if a coalition S and its complement $N \setminus S$ cannot both be winning. A voting game is *strong* if ties are not possible, i.e., S and $N \setminus S$ cannot both be losing. We assume henceforth that the voting game is proper, but not necessarily strong. A *minimal winning coalition* (MWC) S is a coalition that is just large enough to win, that is, S is winning but no $T \subsetneq S$ is winning.

The voting game is *weighted* if it is possible to assign a number of votes (weight) $w_i \geq 0$ to each player and to set a threshold q such that S is winning if and only if $\sum_{i \in S} w_i \geq q$. The combination $[q; w_1, \dots, w_n]$ is a *representation* of the voting game. There are many representations $[q; w_1, \dots, w_n]$ that are equivalent in that they produce the same set of winning coalitions. A representation $[q; w_1, \dots, w_n]$ is called *homogeneous* if all minimal winning coalitions have the same total weight. For example, $[5; 4, 3, 2]$ is not homogeneous because coalition $\{1, 2\}$ has a weight of 7, whereas coalition $\{2, 3\}$ has a weight of 5. A homogeneous representation of the same game is $[2; 1, 1, 1]$. Homogeneous representations are preferred because they give a more accurate description of the situation: $[2; 1, 1, 1]$ reflects the fact that all three players are in a symmetric position (i.e., they are perfect substitutes) since any two of them can form a winning coalition. A game that admits a homogeneous representation is a *homogeneous game*.

Homogeneous voting weights are not necessarily unique. For example, $[5; 3, 2, 2, 1]$ and $[7; 4, 3, 3, 1]$ are homogeneous representations of the same game. A possible approach to deal with this indeterminacy is to use minimal integer weights (MIWs); this approach has been taken in the empirical literature¹ (Ansolabehere et al. (2003), Snyder et al. (2005), Cutler et al. (2014)). A representation has minimal integer weights if all w_i 's are integer numbers and there is no representation with smaller weights (see Ostmann (1987), Freixas and Molinero (2009)).

2.2 The noncooperative model

The noncooperative model is the Baron-Ferejohn (1989) model with weighted voting. Given a set of players N and an associated set of winning coalitions W , bargaining proceeds as follows. Nature randomly selects one of the players to be the proposer, according to a vector $\theta := (\theta_1, \dots, \theta_n)$ of *recognition probabilities*, where $\theta_i \geq 0$ for all $i \in N$ and $\sum_{i=1}^n \theta_i = 1$. The proposer then proposes a distribution (x_1, \dots, x_n) of a budget, with $x_j \geq 0$ for all $j = 1, \dots, n$ and $\sum_{j=1}^n x_j \leq 1$. This proposal is then voted upon.² If the set of voters in favor of the proposal is a winning coalition, the proposal is implemented and the game ends; otherwise the game proceeds to the next round in which the process is repeated. Players share a discount factor $\delta \in (0, 1]$.

A (pure) strategy for player i is a sequence $\sigma_i = (\sigma_i^t)_{t=1}^\infty$, where σ_i^t , the t -th round strategy of player i , prescribes:

1. A proposal, denoted by x .

¹A large body of empirical literature is devoted to testing Gamson's law, which states that ministerial portfolios are allocated proportionally to the "raw" seat shares of parties in government irrespective of the voting weights (see Warwick and Druckman (2006)). Cutler et al. (2014) incorporate both seat shares and MIWs in their statistical model, and find that MIWs have a bearing on which parties get into government, whereas portfolio allocation follows Gamson's law.

²Voters are assumed to vote on the proposal sequentially. This assumption can be replaced by simultaneous voting plus the additional equilibrium refinement that voters always vote as if their vote makes a difference (see Baron and Kalai (1993)).

2. A response function assigning "yes" or "no" to all possible proposals by the other players.

Players may condition their actions on the history of play; however the literature focuses on equilibria in which they do not condition on any elements of history other than the current proposal, if any. These equilibria are called *stationary subgame perfect equilibria* (SSPE). Stationarity requires that players follow the same strategy at every round t regardless of past offers and responses to past offers. An SSPE always exists (Banks and Duggan, 2000). For a fixed $\delta < 1$, all SSPE involve immediate agreement (Okada, 1996) and lead to the same expected payoffs (Eraslan and McLennan, 2013). Expected equilibrium payoffs are usually unique even if $\delta = 1$; when they are not, it is possible to obtain a unique prediction by taking the limit when $\delta \rightarrow 1$. As in Snyder et al. (2005), we consider the model with no discounting, i.e., $\delta = 1$. Calculations for particular numerical examples assume $\delta \rightarrow 1$.

The logic of the Baron-Ferejohn model is simple. Take any stationary strategy combination. Because of stationarity, player i 's expected payoff computed after a proposal has (just) been rejected takes a constant value; we denote this expected payoff (continuation value) by v_i . Since $\delta = 1$, this is also i 's expected payoff computed at the start of the game. These expected payoffs act as prices. It is optimal for player i to accept any proposal that guarantees him at least v_i as a responder and to reject all other proposals.³ Given that responders follow these cutoff strategies, it is optimal for player i as a proposer to find the cheapest group of players whose votes are enough to form a winning coalition and to offer each of them exactly v_j . We say that player i *proposes coalition* S if $i \in S$ and the proposed payoff vector x has $x_j = v_j$ for $j \in S \setminus \{i\}$, $x_i = 1 - \sum_{j \in S \setminus \{i\}} v_j$ and $x_j = 0$ for $j \in N \setminus S$. Let $p_i(S)$ be the probability that i proposes S . Any SSPE

³There is little loss of generality in assuming that ties are always solved in favor of acceptance (see Yan (2002), proposition 2; Eraslan and McLennan (2013), Appendix A).

involves a vector of players' acceptance thresholds $(v_i)_{i \in N}$ and a vector of proposal probabilities $(p_i(S))_{S \ni i, i \in N}$ satisfying two conditions (see Okada (1996), theorem 2):

(1) Proposers propose only the cheapest coalitions available given responders' acceptance thresholds, that is, any coalition with $p_i(S) > 0$ must minimize $\sum_{j \in T \setminus \{i\}} v_j$ (or, equivalently, $\sum_{j \in T} v_j$) subject to the constraint that T is a winning coalition with $T \ni i$.

(2) Responders' acceptance thresholds coincide with their continuation values, that is,

$$v_i = \theta_i \sum_{S: S \ni i} p_i(S) \left[1 - \sum_{j \in S \setminus \{i\}} v_j \right] + \left[\sum_{j \in N \setminus \{i\}} \theta_j \sum_{S \ni \{i, j\}} p_j(S) \right] v_i.$$

Intuitively, this bargaining model is competitive because a player with a disproportionately high v_i would be overpriced and get few proposals if any, which would make it difficult for the player to have a high v_i in the first place.

2.3 Replicated games

Given the original weighted majority game $[q; w_1, \dots, w_n]$, the game with r replications has rn players and a quota qn . The weight vector is found by replacing each player i with r copies with weight w_i .

Snyder et al. (2005) make no claims on how large r needs to be in order to obtain proportional payoffs. Anecdotal evidence suggests that r is often surprisingly small. However, as Laver et al. (2011) point out, the link between a replicated game and the original game may be tenuous. For example, $[5; 4, 3, 2]$ is a symmetric game with three interchangeable players but its replicated game with $r = 2$, $[10; 4, 4, 3, 3, 2, 2]$, has three non-interchangeable types of players. Likewise, $[3; 2, 1, 1]$ is a game in which player 1 belongs to all winning coalitions and therefore has veto power, whereas the corresponding game with $r = 2$, $[6; 2, 2, 1, 1, 1, 1]$, has no veto players. Since the properties of replicated games are not always a good guide to the properties of the original game, it is worth examining the original game directly.

2.4 Some simple examples of nonproportionality

In this section we discuss why equilibrium payoffs may deviate from proportionality, using some simple examples.

The simplest examples of deviation from proportionality are games with a veto player, such as $[3; 2, 1, 1]$. The veto player gets everything (Winter, 1996) even though it has only half of the total weight. It is clear that the substitution argument does not bite in this case, since the veto player must be in all coalitions and cannot be replaced by other players.

Games with veto players are special since the veto player cannot be replaced at all, hence we would not expect proportionality to hold.⁴ However, lack of substitutability is not confined to games with veto players, as the following example illustrates.

Consider the weighted voting game $[5; 3, 2, 2, 1]$, discussed in Montero (2000). This is a homogeneous game, and the weights reported are MIWs. There are two types of MWCs: the large party together with one of the medium-size parties, and the three smaller parties together. Let $v_{[3]}$, $v_{[2]}$ and $v_{[1]}$ denote the expected equilibrium payoffs for a player with 3, 2 and 1 votes respectively. Since each medium-size party could form a coalition with either the large party or the two smaller parties, one would expect $v_{[3]} = v_{[2]} + v_{[1]}$. However, there is no particular reason to expect $v_{[2]} = 2v_{[1]}$. A player with 2 votes need not command a price equal to that of two players with 1 vote each, since no two players with 1 vote are available to replace the player with 2 votes.⁵

Example 1 Consider the weighted voting game $[5; 3, 2, 2, 1]$. Let $\theta = (\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8})$. It is easy to see that $v \neq (\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8})$. All SSPE have $v_{[3]} = \frac{5}{14}$, $v_{[2]} = \frac{4}{14}$ and $v_{[1]} = \frac{1}{14}$.

Proof. See Appendix. ■

⁴Indeed, Snyder et al. (2005) exclude games with veto players from their analysis.

⁵Situations where one player cannot be replaced by smaller players in a MWC are known as games with steps (see Ostmann (1987)).

Note that equilibrium payoffs are quite different from the MIWs we started from. In particular, the ratios $v_{[3]}/v_{[1]}$ and $v_{[2]}/v_{[1]}$ are 5 and 4 respectively instead of 3 and 2. Intuitively, there is an excess demand for the medium-size players. There is competition for the medium-size players, since they are needed by both the large and the small player, and there is no competition at all for the small player. As a result, the medium-size players receive too many proposals and the other two players do not receive enough proposals to sustain payoffs proportional to θ .

The ambiguity of the perfect substitutes argument (or, equivalently, the lack of uniqueness of the homogeneous representation) is not the only reason why equilibrium payoffs may differ from the MIWs. The following example illustrates the lack of proportionality of equilibrium payoffs in a particularly surprising setting, where this issue does not arise.

Example 2 *Consider the game $[20; 5, 5, 5, 5, 5, 1, 1, 1, 1, 1, 1, 1, 1]$. There are two types of MWC in this game: four of the large players, or three of the large players together with five of the small players. The game is clearly homogeneous; furthermore, it has a unique homogeneous representation (up to a multiplicative constant). The substitutability argument points in a very clear direction: a large player can be replaced by five small players, and should get five times as much. Suppose players are recognized with probabilities proportional to their voting weight, i.e., a large player is recognized with probability $\frac{5}{34}$ and a small player is recognized with probability $\frac{1}{34}$. It turns out that the equilibrium is such that $v_{[5]} = \frac{50}{331}$ and $v_{[1]} = \frac{9}{331}$. Hence, the large players are getting a disproportionately high payoff since $v_{[5]} > 5v_{[1]}$.*

Proof. In order to show that this is an equilibrium, we need to find strategies that lead to the expected payoffs and are optimal given the expected payoffs. The strategies are as follows: all players propose a coalition of three large players and five small players, and the proposer offers the coalition partners either $\frac{50}{331}$ (for large players) or $\frac{9}{331}$ (for small players). As a responder, a large player votes in favor of any proposal that gives him at least $\frac{50}{331}$, and a small player votes in favor of

any proposal that gives him at least $\frac{9}{331}$. Proposers are acting optimally given the responders' prices: no other winning coalition would be cheaper. Responders are acting optimally provided that expected payoffs are indeed those, so it only remains to check that expected payoffs are as assumed given the strategies:

$$\begin{aligned} v_{[5]} &= \frac{5}{34} \left[1 - 2 \times \frac{50}{331} - 5 \times \frac{9}{331} \right] + \frac{20}{34} \frac{2}{4} \frac{50}{331} + \frac{9}{34} \frac{3}{5} \frac{50}{331} = \frac{50}{331} \\ v_{[1]} &= \frac{1}{34} \left[1 - 3 \times \frac{50}{331} - 4 \times \frac{9}{331} \right] + \frac{25}{34} \frac{5}{9} \frac{9}{331} + \frac{8}{34} \frac{4}{8} \frac{9}{331} = \frac{9}{331}. \end{aligned}$$

■

It is tempting to conclude that the trouble with the previous example is that there are not enough smaller players to replace the large players. All coalitions that form in equilibrium are of type [55511111]. Proposers would rather replace one of the coalition partners of type [5] with five players of type [1], but this is not possible because there are only nine of those and five are already in the coalition. Indeed, adding another small player would lead to proportional payoffs. Interestingly, this is not the whole story: *removing* one of the small players would also lead to proportional payoffs (more on this in the next section).

3 A necessary and sufficient condition for proportionality

Montero (2006) shows that, if θ coincides with the nucleolus (Schmeidler (1969)), v coincides with the nucleolus as well. A sufficient condition automatically follows from this result: if the weights happen to be proportional to the nucleolus, expected payoffs are proportional to the weights.⁶ Peleg (1968) shows that MIWs are pro-

⁶This condition is not necessary. For example, the nucleolus of the game [10; 3, 3, 3, 2, 2, 2] is $(\frac{2}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$, which is not a system of weights at all, and nevertheless expected payoffs would be proportional to the weights. I'm grateful to Peter Sudhölter for pointing out this example, which appears in Kopelowitz (1967).

portional to the nucleolus for all strong homogeneous games, hence the game being in this class is a sufficient condition for the proportionality of expected payoffs.

The arguments in Montero (2006) can be adapted to provide a more general sufficient condition for expected payoffs to be proportional to an arbitrary system of weights. Moreover, this condition turns out to be *necessary* as well as we show below.

Let W^* be the set of winning coalitions with minimum total weight, i.e. $W^* = \arg \min_{S \in W} \sum_{i \in S} w_i$. This set is *weakly balanced* if it is possible to find a collection of weights $(\lambda_S)_{S \in W^*}$ such that $\lambda_S \geq 0$ for all $S \in W^*$ and $\sum_{S \ni i} \lambda_S = 1$ for all $i \in N$.

Proposition 3 *Let $[q; w_1, \dots, w_n]$ be an arbitrary weighted majority game, normalized so that $\sum_{i \in N} w_i = 1$, and let $\theta = w$. There exists an SSPE with $v = w$ if and only if W^* is weakly balanced.*

Proof. Because we haven't imposed any conditions on the weights, it is possible that no coalition has exactly q votes. Let $\min_{S \in W} \sum_{i \in S} w_i := \bar{q}$.⁷

1. Necessity. Suppose we have an SSPE with $v = w$. Expected payoffs are given by

$$v_i = \theta_i \sum_{S: S \ni i} p_i(S) \left[1 - \sum_{j \in S \setminus \{i\}} v_j \right] + r_i v_i$$

where v_i is i 's expected payoff, θ_i is the probability that i is selected to be proposer, $p_i(S)$ is the probability that i proposes S conditional on i being the proposer, and r_i is the probability that i receives a proposal from another player.

Consider first the case in which $w_i > 0$ for all i and each player belongs to at least one coalition in W^* . Then, if expected payoffs coincide with w , the optimal coalitions for the proposer are the coalitions in W^* to which it belongs. Since these coalitions have a total weight of \bar{q} , $\sum_{j \in S \setminus \{i\}} v_j = \bar{q} - w_i$ for all the proposed

⁷For example, if $w = (\frac{5}{13}, \frac{4}{13}, \frac{4}{13})$ and $q = \frac{7}{13}$, there is no coalition with exactly $\frac{7}{13}$ votes and $\bar{q} = \frac{8}{13}$.

coalitions, and $\sum_{S:S \ni i} p_i(S) \left[1 - \sum_{j \in S \setminus \{i\}} v_j\right]$ can be written as $1 - (\bar{q} - w_i)$. Since both v_i and θ_i coincide with w_i for all i , we can write the equation for expected payoffs as

$$w_i = w_i [1 - (\bar{q} - w_i)] + r_i w_i.$$

Dividing by w_i (which we have assumed to be positive), it must be the case that $r_i = \bar{q} - w_i$, i.e. $r_i + w_i = \bar{q}$. Since w_i is also the probability of being proposer, we see that the total probability of being part of the final coalition (the probability of being proposer, w_i , plus the probability of being responder, r_i) must be the same for all players. This implies that, if $p(S)$ is the equilibrium probability of coalition S forming, $\sum_{S \ni i} p(S) = \bar{q}$ for all i . Notice also that only coalitions with \bar{q} votes form in equilibrium (other coalitions are too expensive), so we may write $\sum_{S:S \in W^*, S \ni i} p(S) = \bar{q}$ for all i . If we divide both sides of the equation by \bar{q} and define $\lambda_S := p(S)/\bar{q}$, we obtain $\sum_{S:S \in W^*, S \ni i} \lambda_S = 1$ for all i . In other words, the set of minimal winning coalitions with \bar{q} votes must be weakly balanced.

If there is a player with $w_i > 0$ who does not belong to any of the coalitions with exactly \bar{q} votes, this player needs to buy more than $\bar{q} - w_i$ votes, and its payoff as a proposer is less than $1 - (\bar{q} - w_i)$. We then write

$$v_i < \theta_i [1 - (\bar{q} - w_i)] + r_i v_i.$$

If we replace v_i and θ_i by w_i and divide everything by w_i , collecting terms we find $r_i + w_i > \bar{q}$. Since $\bar{q} > \frac{1}{2}$ we find that $r_i + w_i > \frac{1}{2}$, i.e., player i 's probability of being in the final coalition is above $\frac{1}{2}$. Let S be one of the coalitions with \bar{q} votes. Players in S will never include i in their proposal. But this then implies that player i is in the final coalition with a probability of at most $1 - \bar{q}$, which is less than $\frac{1}{2}$. Hence, there cannot be an equilibrium with $v = w$ in which a player does not belong to any coalition in W^* .

If $v = w$ and $w_i = 0$ for some i , we can still show that the set W^* must be weakly balanced. Note that players with $w_i = 0$ trivially belong to at least one coalition in

W^* . If $w_i = 0$, the coalition that forms can be viewed as including i (since i receives v_i) or excluding i (since i receives 0). Choose a player k with $w_k > 0$ and adopt the arbitrary accounting convention that players with $w_i = 0$ are considered part of the coalition if and only if player k is part of the coalition. It follows from the analysis above that $\sum_{S \in W^*, S \ni i} \lambda_S = 1$ for all j .

2. Sufficiency. Suppose W^* is weakly balanced. As in Montero (2006), we can use the weights λ_S to construct a mixed strategy equilibrium in which $v = w$. As a proposer, player i proposes one of the coalitions in W^* to which it belongs according to the probability distribution $p_i(S) = \lambda_S$ for all S such that $S \in W^*$, $S \ni i$; $p_i(S) = 0$ for all other S . Proposing S means that player i sets $x_j = w_j$ for all $j \in S \setminus \{i\}$, $x_i = 1 - \sum_{j \in S \setminus \{i\}} w_j$ and $x_j = 0$ for all $j \in N \setminus S$. Since by assumption $\sum_{S \ni i} \lambda_S = 1$ for all i , the strategy is well defined. As a responder, player i accepts proposals if and only if $x_i \geq w_i$.

We now show that this strategy combination leads to $v_i = w_i$ for all i . This is trivially the case if $w_i = 0$, since by assumption this player has no chance of being proposer and no other player offers i a positive payoff as a responder. If $w_i > 0$, player i 's expected payoff given this strategy combination equals

$$v_i = \theta_i \sum_{S: S \ni i} p_i(S) \left[1 - \sum_{j \in S \setminus \{i\}} w_j \right] + \left[\sum_{j \in N \setminus \{i\}} \theta_j \sum_{S \supseteq \{i, j\}} p_j(S) \right] w_i.$$

By assumption, $\theta_i = w_i$. Player i 's payoff as a proposer can be written as $1 - (\bar{q} - w_i)$ since player i only proposes coalitions in W^* and by definition these coalitions have a total weight of \bar{q} . The probability of receiving a proposal, $\sum_{j \in N \setminus \{i\}} \theta_j \sum_{S \supseteq \{i, j\}} p_j(S)$, can be rewritten as $\sum_{S \ni i} \sum_{j \in S \setminus \{i\}} \theta_j p_j(S)$. Hence,

$$v_i = w_i [1 - (\bar{q} - w_i)] + \left[\sum_{S \ni i} \sum_{j \in S \setminus \{i\}} \theta_j p_j(S) \right] w_i.$$

Since by construction only coalitions in W^* are proposed and those have $p_j(S) =$

λ_S , we can write

$$\begin{aligned} v_i &= w_i[1 - (\bar{q} - w_i)] + \left[\sum_{S: S \ni i, S \in W^*} \sum_{j \in S \setminus \{i\}} \theta_j \lambda_S \right] w_i = \\ &= w_i[1 - (\bar{q} - w_i)] + \left[\sum_{S: S \ni i, S \in W^*} \lambda_S \sum_{j \in S \setminus \{i\}} \theta_j \right] w_i. \end{aligned}$$

Furthermore, since $\theta_j = w_j$ and $\sum_{j \in S \setminus \{i\}} w_j = \bar{q} - w_i$ for all $S \in W^*$ we have $\sum_{j \in S \setminus \{i\}} \theta_j = \bar{q} - w_i$. We can then write

$$\begin{aligned} v_i &= w_i[1 - (\bar{q} - w_i)] + \left[\sum_{S: S \ni i, S \in W^*} \lambda_S (\bar{q} - w_i) \right] w_i = \\ &= w_i[1 - (\bar{q} - w_i)] + (\bar{q} - w_i) \left[\sum_{S: S \ni i, S \in W^*} \lambda_S \right] w_i = w_i \end{aligned}$$

where the last equality follows from $\sum_{S: S \ni i, S \in W^*} \lambda_S = 1$.

Since $v_i = w_i$ for all i , players are behaving optimally both as proposers and as responders. Proposers propose only the cheapest coalitions available given the responders' acceptance thresholds (i.e., coalitions in W^*) and responders accept proposals if and only if $x_i \geq v_i$. Hence, we have an SSPE. ■

The introduction of a discount factor $\delta \in (0, 1)$ would not affect the necessary and sufficient condition. If the condition is satisfied, expected payoffs are still proportional and the only effect is that all continuation values shrink proportionally and the proposer advantage increases. If the condition is *not* satisfied, expected equilibrium payoffs cannot be proportional for any $\delta > 0$, and their actual value does in general depend on δ .

The condition can be interpreted as a requirement for the set of cheapest coalitions to be sufficiently rich, so that no player is systematically in excess supply or demand. Fix a vector of prices. The cheapest set of coalitions given those prices are the coalitions most likely to form. If we can have a probability distribution over

those coalitions such that all players are equally likely to be in the final coalition, this is a sign that the prices are competitive. Instead, if a player always or never appears in the final coalition, that player must be underpriced or overpriced.

There are no requirements on $[q; w]$ in order for the condition to apply. Note however that the condition has almost no chance to hold if w is the vector of seat shares, since in general not all parties will belong to a coalition with the minimum number of seats.⁸ MIWs on the other hand ensure that all players belong to a winning coalition of minimum total weight, though even in this case the condition does not necessarily hold as we have seen.⁹

Example 1 is a clear case of violation of this condition: player 4 is an inferior player (Napel and Widgrén, 2001) in that it can only be in a MWC when players 2 and 3 are also present; since player 1 also needs either player 2 or player 3, it is impossible for all players to be in the final coalition with equal probability. Example 2 is a more subtle instance of the same problem. There are two types of MWC, [5555] and [55511111]. Even in the most favorable case for the small players, which is when the only coalition type that forms is [55511111], it is still the case that a type [5] player ends up in the coalition with probability $\frac{3}{5}$, whereas a type [1] player only ends up in the coalition with probability $\frac{5}{9} < \frac{3}{5}$. Either adding or removing a player of type [1] would make the equilibrium competitive again. Adding a player

⁸For example, assuming simple majority voting, the German Bundestag as of September 2014 would be associated with the weighted majority game [316; 311, 192, 64, 63]. There are four minimal winning coalitions: $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, $\{2, 3, 4\}$. The coalition with the minimum total number of seats is $\{2, 3, 4\}$, with 319 seats. Party 1 does not belong to any coalition with 319 seats, hence the corresponding W^* would not be balanced.

⁹Even though there are no requirements on $[q; w]$ in order for the condition to apply, there is an important requirement on the game form: recognition probabilities must be proportional to w . As Kalandrakis (2006) has shown, recognition probabilities have a strong influence on equilibrium payoffs. Diermeier and Merlo (2004) found some empirical support for the hypothesis of formateur selection being proportional to seat shares. To the best of my knowledge there has been no empirical testing of selection proportional to voting weights.

would introduce a new type of MWC that favors the small players, [55111111111]. *Removing* a player leaves the two types of MWC unchanged, but it gives individual type [1] players a greater chance of being part of coalition type [55511111].

It is worth noting that the condition does not have a straightforward connection with other properties such as the homogeneity of the game. The condition always holds for strong homogeneous games, but may hold for other games as well. For example, it holds for the game $[30; 14, 14, 12, 4, 4, 4, 4, 1, 1]$, which is neither homogeneous nor strong.¹⁰

4 Deviations from proportionality in applications

4.1 Predicted frequency of the deviations

Because the condition in Proposition 3 is necessary and sufficient we have a characterization, and are able to answer the question of whether payoffs would be proportional to the voting weights in any particular case. Table 1 shows the frequency of proportional equilibrium payoffs for the weight distributions in Snyder et al. (2005)’s dataset.¹¹ This frequency provides some guidance as to how likely the condition is to hold in applications.¹² As a byproduct, it also indicates the proportion of cases in which the predictions tested by Snyder et al. are supported by the equilibrium

¹⁰Putting Proposition 3 together with Proposition 2 in Snyder et al. (2005), it follows that the condition must be satisfied for sufficiently large replica games. Kurz et al. (2014) show a stronger result: weights of large replica games coincide with the nucleolus.

¹¹These calculations use the MIWs provided by Snyder et al. in their supplementary material, available at www.aeaweb.org. All frequencies are computed as a fraction of the total number of observations in the dataset, where each government is an observation.

¹²Another potential application is the EU Council of Ministers. The condition fails to hold for most of the historical weight distributions (see Le Breton et al. (2012)).

of their theoretical model.

Table 1. Frequency of proportional equilibrium prediction

| | Observations | Proportional | Frequency |
|----------------|--------------|--------------|-----------|
| Australia | 26 | 26 | 1 |
| Austria | 23 | 23 | 1 |
| Belgium | 36 | 25 | 0.69 |
| Denmark | 32 | 25 | 0.78 |
| Finland | 44 | 19 | 0.43 |
| Iceland | 22 | 18 | 0.82 |
| Ireland | 22 | 12 | 0.55 |
| Italy | 46 | 13 | 0.28 |
| Luxembourg | 17 | 11 | 0.65 |
| Netherlands | 23 | 10 | 0.43 |
| Norway | 27 | 26 | 0.96 |
| Portugal | 15 | 11 | 0.73 |
| Sweden | 25 | 24 | 0.96 |
| (West) Germany | 20 | 19 | 0.95 |
| All countries | 378 | 262 | 0.69 |

Some of these observations correspond to trivial cases in which either one party has the overall majority or all parties are de facto symmetric. If we excluded trivial cases (defined as cases in which all minimal integer weights are 0 or 1), the overall frequency of the proportional equilibrium prediction would drop to about 59%.

4.2 Predicted size of the deviations

The necessary and sufficient condition in Proposition 3 provides a yes/no answer on proportionality: if it fails, expected equilibrium payoffs cannot be proportional. But how far are they from being proportional? The following tables compare equilibrium payoffs and weights for all games in the dataset with at most 7 players that

fail to satisfy the condition (excluding games with a veto player, of which there are two in the database). For each game, the table shows w_i (the MIWs), v_i (expected equilibrium payoffs), and two quantitative measures of how far v is from being proportional to w . One such measure is $\frac{v_i}{w_i / \sum_{j \in N} w_j}$, the ratio of payoffs to weights, where weights are normalized so that they add up to 1. This ratio measures how much of a player's weight is translated into expected equilibrium payoffs; if expected equilibrium payoffs were proportional to weights it would always be 1. Another measure is the relative payoffs v_i/v_n , i.e. the exchange rate between players according to equilibrium predictions. If expected equilibrium payoffs were proportional, this exchange rate would always be equal to w_i/w_n (in particular, if $w_n = 1$, this ratio would replicate the MIWs).

Expected payoffs for individual players can be substantially different from weight shares, and this is very often true for the smallest player type, who may get as little as 43% of its weight share. As a result, ratios between a player's payoff and the payoff of the smallest player are very different from w_i/w_n . For example, in the game $[9; 5, 4, 4, 1, 1, 1]$, a player with 5 votes does not get 5 times as much as a player with 1 vote, but about 12 times as much. Nevertheless, if we focus on the ratio of expected payoffs to weights, we see that many players get an expected payoff that is close to their voting weight.

Table 2. Homogeneous games with up to 6 players

| | | | | | | |
|------------------|----------|----------|----------|----------|----------|----------|
| Weights | 7 | 5 | 5 | 2 | 2 | 1 |
| Payoffs | 0.323 | 0.226 | 0.226 | 0.097 | 0.097 | 0.032 |
| Payoffs/weights | 1.014 | 0.993 | 0.993 | 1.067 | 1.067 | 0.699 |
| Relative payoffs | 10.16 | 7.10 | 7.10 | 3.05 | 3.05 | 1 |
| Weights | 5 | 4 | 4 | 1 | 1 | 1 |
| Payoffs | 0.324 | 0.297 | 0.297 | 0.027 | 0.027 | 0.027 |
| Payoffs/weights | 1.038 | 1.190 | 1.190 | 0.430 | 0.430 | 0.430 |
| Relative payoffs | 12.06 | 11.06 | 11.06 | 1 | 1 | 1 |
| Weights | 5 | 3 | 3 | 2 | 1 | |
| Payoffs | 0.376 | 0.208 | 0.208 | 0.168 | 0.040 | |
| Payoffs/weights | 1.053 | 0.970 | 0.970 | 1.178 | 0.556 | |
| Relative payoffs | 9.47 | 5.24 | 5.24 | 4.24 | 1 | |
| Weights | 5 | 2 | 2 | 2 | 1 | |
| Payoffs | 0.412 | 0.176 | 0.176 | 0.176 | 0.059 | |
| Payoffs/weights | 0.988 | 1.059 | 1.059 | 1.059 | 0.706 | |
| Relative payoffs | 7 | 3 | 3 | 3 | 1 | |
| Weights | 4 | 3 | 3 | 1 | 1 | |
| Payoffs | 0.333 | 0.295 | 0.295 | 0.038 | 0.038 | |
| Payoffs/weights | 1.000 | 1.181 | 1.181 | 0.456 | 0.456 | |
| Relative payoffs | 8.77 | 7.77 | 7.77 | 1 | 1 | |
| Weights | 3 | 2 | 2 | 1 | | |
| Payoffs | 0.357 | 0.286 | 0.286 | 0.071 | | |
| Payoffs/weights | 0.952 | 1.143 | 1.143 | 0.571 | | |
| Relative payoffs | 5 | 4 | 4 | 1 | | |

Table 3. Homogeneous games with 7 players

| | | | | | | | |
|------------------|----------|----------|----------|----------|----------|----------|----------|
| Weights | 9 | 7 | 7 | 2 | 2 | 2 | 1 |
| Payoffs | 0.302 | 0.233 | 0.233 | 0.069 | 0.069 | 0.069 | 0.026 |
| Payoffs/weights | 1.006 | 0.998 | 0.998 | 1.035 | 1.035 | 1.035 | 0.771 |
| Relative payoffs | 11.74 | 9.06 | 9.06 | 2.69 | 2.69 | 2.69 | 1 |
| Weights | 9 | 6 | 6 | 3 | 2 | 1 | 1 |
| Payoffs | 0.325 | 0.217 | 0.217 | 0.108 | 0.085 | 0.024 | 0.024 |
| Payoffs/weights | 1.013 | 1.013 | 1.013 | 1.013 | 1.188 | 0.661 | 0.661 |
| Relative payoffs | 13.79 | 9.19 | 9.19 | 4.60 | 3.60 | 1 | 1 |
| Weights | 9 | 3 | 3 | 3 | 2 | 1 | 1 |
| Payoffs | 0.416 | 0.139 | 0.139 | 0.139 | 0.109 | 0.023 | 0.023 |
| Payoffs/weights | 1.017 | 1.017 | 1.017 | 1.017 | 1.200 | 0.650 | 0.650 |
| Relative payoffs | 14.08 | 4.69 | 4.69 | 4.69 | 3.69 | 1 | 1 |

Table 4. Nonhomogeneous games with up to 6 players

| | | | | | | |
|------------------|----------|----------|----------|----------|----------|----------|
| Weights | 9 | 5 | 5 | 3 | 2 | 2 |
| Payoffs | 0.364 | 0.182 | 0.182 | 0.091 | 0.091 | 0.091 |
| Payoffs/weights | 1.051 | 0.945 | 0.945 | 0.788 | 1.182 | 1.182 |
| Relative payoffs | 4 | 2 | 2 | 1 | 1 | 1 |
| Weights | 8 | 6 | 5 | 3 | 3 | 1 |
| Payoffs | 0.320 | 0.227 | 0.206 | 0.113 | 0.113 | 0.020 |
| Payoffs/weights | 1.039 | 0.983 | 1.073 | 0.983 | 0.983 | 0.528 |
| Relative payoffs | 15.75 | 11.16 | 10.16 | 5.58 | 5.58 | 1 |
| Weights | 7 | 6 | 4 | 3 | 3 | 1 |
| Payoffs | 0.280 | 0.280 | 0.140 | 0.140 | 0.140 | 0.018 |
| Payoffs/weights | 0.962 | 1.122 | 0.841 | 1.122 | 1.122 | 0.439 |
| Relative payoffs | 15.33 | 15.33 | 7.67 | 7.67 | 7.67 | 1 |
| Weights | 5 | 4 | 3 | 2 | 2 | |
| Payoffs | 0.290 | 0.280 | 0.150 | 0.140 | 0.140 | |
| Payoffs/weights | 0.928 | 1.119 | 0.801 | 1.119 | 1.119 | |
| Relative payoffs | 2.07 | 2 | 1.07 | 1 | 1 | |

Table 5. Nonhomogeneous games with 7 players

| | | | | | | | |
|------------------|-----------|-----------|----------|----------|----------|----------|----------|
| Weights | 13 | 11 | 9 | 6 | 5 | 4 | 2 |
| Payoffs | 0.261 | 0.218 | 0.174 | 0.130 | 0.088 | 0.087 | 0.043 |
| Payoffs/weights | 1.003 | 0.989 | 0.968 | 1.082 | 0.877 | 1.082 | 1.082 |
| Relative payoffs | 6.03 | 5.03 | 4.03 | 3 | 2.03 | 2 | 1 |
| Weights | 13 | 10 | 9 | 6 | 6 | 3 | 1 |
| Payoffs | 0.264 | 0.198 | 0.198 | 0.132 | 0.132 | 0.066 | 0.010 |
| Payoffs/weights | 0.975 | 0.951 | 1.056 | 1.056 | 1.056 | 1.056 | 0.472 |
| Relative payoffs | 26.86 | 20.14 | 20.14 | 13.43 | 13.43 | 6.71 | 1 |
| Weights | 12 | 10 | 7 | 5 | 4 | 3 | 1 |
| Payoffs | 0.288 | 0.237 | 0.170 | 0.119 | 0.102 | 0.068 | 0.017 |
| Payoffs/weights | 1.009 | 0.996 | 1.017 | 0.997 | 1.070 | 0.947 | 0.704 |
| Relative payoffs | 17.20 | 14.16 | 10.12 | 7.08 | 6.08 | 4.04 | 1 |
| Weights | 11 | 8 | 7 | 4 | 4 | 1 | 1 |
| Payoffs | 0.320 | 0.222 | 0.209 | 0.111 | 0.111 | 0.014 | 0.014 |
| Payoffs/weights | 1.046 | 1.000 | 1.073 | 1.000 | 1.000 | 0.491 | 0.491 |
| Relative payoffs | 23.43 | 16.28 | 15.28 | 8.14 | 8.14 | 1 | 1 |
| Weights | 10 | 9 | 7 | 3 | 3 | 3 | 1 |
| Payoffs | 0.269 | 0.269 | 0.179 | 0.090 | 0.090 | 0.090 | 0.013 |
| Payoffs/weights | 0.969 | 1.077 | 0.923 | 1.077 | 1.077 | 1.077 | 0.462 |
| Relative payoffs | 21 | 21 | 14 | 7 | 7 | 7 | 1 |
| Weights | 10 | 3 | 3 | 3 | 2 | 2 | 1 |
| Payoffs | 0.426 | 0.120 | 0.120 | 0.120 | 0.093 | 0.093 | 0.027 |
| Payoffs/weights | 1.023 | 0.961 | 0.961 | 0.961 | 1.117 | 1.117 | 0.649 |
| Relative payoffs | 15.76 | 4.44 | 4.44 | 4.44 | 3.44 | 3.44 | 1 |
| Weights | 9 | 8 | 5 | 4 | 4 | 1 | 1 |
| Payoffs | 0.278 | 0.278 | 0.139 | 0.139 | 0.139 | 0.014 | 0.014 |
| Payoffs/weights | 0.988 | 1.111 | 0.889 | 1.111 | 1.111 | 0.444 | 0.444 |
| Relative payoffs | 20 | 20 | 10 | 10 | 10 | 1 | 1 |

4.3 Qualitative equilibrium phenomena

An alternative way of looking at the importance of deviations is to focus not on their size, but on the presence of equilibrium phenomena that would be ruled out if

payoffs were proportional to the MIWs. We discuss three such phenomena: players that are not interchangeable may have the same expected equilibrium payoffs, some players may be too expensive to receive any proposals, and surplus coalitions may form. All three phenomena can be illustrated using one of the weighted majority games in the dataset, $[13; 7, 6, 4, 3, 3, 1]$, corresponding to Belgium in 1972.

Example 4 *Consider the weighted majority game $[13; 7, 6, 4, 3, 3, 1]$. Note that types $[7]$ and $[6]$ are genuinely asymmetric: coalition $[733]$ is winning but coalition $[633]$ is losing. Likewise, types $[4]$ and $[3]$ are genuinely asymmetric because $[643]$ is winning but $[633]$ is losing. Let $\theta = (\frac{7}{24}, \frac{6}{24}, \frac{4}{24}, \frac{3}{24}, \frac{3}{24}, \frac{1}{24})$. All SSPE have $v_{[7]} = v_{[6]} = \frac{46}{164}$, $v_{[4]} = v_{[3]} = \frac{23}{164}$ and $v_{[1]} = \frac{3}{164}$.*

Proof. See Appendix. ■

4.3.1 Asymmetric players may have the same payoff

Example 4 shows that it is possible for two players to have the same equilibrium expected payoffs, even though one of the players is more valuable as a coalition partner and has the additional advantage of a higher recognition probability.

Some intuition for this result can be obtained by inspecting the list of MWCs. There are seven MWCs of five types: $[76]$, $[743]$, $[733]$, $[643]$, $[6331]$. All MWCs have exactly 13 votes except for the two coalitions of type $[743]$. If $v_{[7]} > v_{[6]}$ and $v_{[4]} > v_{[3]}$, coalition $[743]$ would be too expensive to be proposed by any player type, because the alternative coalitions $[643]$ and $[733]$ would be cheaper. This leaves four coalition types that could potentially be proposed, $[76]$, $[733]$, $[643]$ and $[6331]$. Given this list, player $[1]$ would need the cooperation of player $[6]$ and both players of type $[3]$, whereas player $[4]$ would need player $[6]$ and one of the players of type $[3]$. Likewise, player $[7]$ would need either player $[6]$ or both players of type $[3]$. Types $[6]$ and $[3]$ would be more in demand than types $[7]$ and $[4]$, and it would not be possible to have $v_{[7]} > v_{[6]}$ and $v_{[4]} > v_{[3]}$. It turns out that, in equilibrium, $v_{[7]} = v_{[6]}$ and $v_{[4]} = v_{[3]}$.

4.3.2 Some players may be too expensive to receive proposals

The equalities $v_{[7]} = v_{[6]}$ and $v_{[4]} = v_{[3]}$ have two implications. On the one hand, coalition type [743] becomes relevant, because it is just as expensive as [643] and [733] despite having one more vote. On the other hand, coalition type [6331] becomes too expensive for anybody other than player [1] even though it only has 13 votes. Players [6] and [3] would rather propose [643] than [6331], since its total cost is $v_{[6]} + v_{[4]} + v_{[3]} = v_{[6]} + 2v_{[3]} < v_{[6]} + 2v_{[3]} + v_{[1]}$.

Note that the substitutability logic applies to this example, but in a somewhat perverse way. Instead of applying to the MWCs with 13 votes (coalition types [76], [733], [643], and [6331]), it applies to coalition types [76], [733], [643] and [743].

Looking at the SSPE payoffs, player [1] appears underpriced since it only expects about 0.02 even though its weight share is about 0.04. Types [6] and [3] are getting a disproportionately high payoff compared to their weight share but this does not result in their exclusion; instead, it is type [1] that is excluded. Indeed, given that $v_{[7]} = v_{[6]}$ and $v_{[4]} = v_{[3]}$, player type [1] would be perceived as too expensive *for any positive value of $v_{[1]}$* .

4.3.3 Surplus coalitions may form

The original Baron-Ferejohn model with symmetric players always leads to minimal winning coalitions, since the proposer could otherwise drop one of the responders and still have a winning coalition. With asymmetric players, it is still true that all coalition partners must be pivotal, but the proposer is not necessarily pivotal. In the previous example, type [1] finds it optimal to propose surplus coalitions such as [7331] or [7431], since they are as expensive as the minimal winning coalition [6331] given that $v_{[7]} = v_{[6]}$ and $v_{[4]} = v_{[3]}$. Hence, surplus coalitions are not ruled out in equilibrium under weighted voting, though the only type of surplus coalition that may form is one in which the proposer is the only member of the coalition who is not pivotal.

5 Concluding remarks

This paper provides a necessary and sufficient condition for equilibrium payoffs to be proportional in the Baron-Ferejohn model with weighted voting. When the condition is satisfied, the set of available coalitions is sufficiently rich so that none of the parties appears systematically too often (excess demand) or too seldom (excess supply) in the final coalition. The condition is relatively easy to check in applications since all equations involved are linear. Using the condition, it is found that the frequency of legislatures in the field with proportional equilibrium payoffs is about 69%, though there is a lot of variation across countries. This frequency may be viewed as sufficiently high to support empirical work, specially in the countries where it is highest.

Most counterexamples are not a result of the competitive bargaining logic failing, but rather of its working in unexpected ways. However, in these cases the deviations from proportionality may be substantial, both quantitatively and qualitatively, as the examples provided illustrate.

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6 Appendix

6.1 Calculations for [5;3,2,2,1]

By contradiction, suppose $v = (\frac{3}{8}, \frac{2}{8}, \frac{2}{8}, \frac{1}{8})$. What would be the optimal proposer behavior given v ? The player with 3 votes needs to buy 2 votes, hence it always offers $v_{[2]}$ to one of the players with 2 votes (the player with 1 vote is of no use to this player, regardless of the value of $v_{[1]}$). The player with 1 vote needs to buy 4 votes, and will buy them from the two players that control 2 votes each. A player with 2 votes needs to buy 3 votes, and would be indifferent between buying them from the large player or from the other two players since $v_{[3]} = v_{[2]} + v_{[1]}$. Let p be the probability that a player with 2 votes proposes to the player with 3 votes (conditional on a player with 2 votes being selected as proposer). Expected payoffs for types [3] and [1] must satisfy the following equations:

$$\begin{aligned}\frac{3}{8} &= \frac{3}{8} \left[1 - \frac{2}{8}\right] + \frac{4}{8} p \frac{3}{8} \\ \frac{1}{8} &= \frac{1}{8} \left[1 - \frac{4}{8}\right] + \frac{4}{8} (1-p) \frac{1}{8}\end{aligned}$$

From the second equation we find $p = 0$. This means that in order to sustain a payoff of $\frac{1}{8}$ for type [1], type [2] must always propose a coalition of type [221]. However, $p = 0$ does not solve the first equation: in order to sustain a payoff of $\frac{3}{8}$ for type [3], p must be $\frac{1}{2}$.

Interestingly, the equilibrium is still competitive in the sense that $v_{[3]} = v_{[2]} + v_{[1]}$. Below we construct an equilibrium strategy profile. Let player [3] propose to each of the two players of type [2] with probability $\frac{1}{2}$, and let each of the players of type [2] propose to player [3] with probability p . The equilibrium values of $v_{[3]}$, $v_{[2]}$, $v_{[1]}$

and p can be found from the following system:

$$\begin{aligned}
v_{[3]} &= \frac{3}{8} [1 - v_{[2]}] + \frac{4}{8} p v_{[3]} \\
v_{[2]} &= \frac{2}{8} [p(1 - v_{[3]}) + (1 - p)(1 - v_{[2]} - v_{[1]})] + \frac{3}{8} \frac{1}{2} v_{[2]} + \frac{2}{8} (1 - p) v_{[2]} + \frac{1}{8} v_{[2]} \\
v_{[1]} &= \frac{1}{8} [1 - 2v_{[2]}] + \frac{4}{8} (1 - p) v_{[1]} \\
v_{[3]} &= v_{[2]} + v_{[1]}
\end{aligned}$$

The solution to this system is $v_{[3]} = \frac{5}{14}$, $v_{[2]} = \frac{4}{14}$, $v_{[1]} = \frac{1}{14}$ and $p = \frac{1}{2}$. This is an equilibrium since players are behaving optimally both as proposers and as responders. Because of the uniqueness result of Eraslan and McLennan (2013), all SSPE must have the same payoff vector.

6.2 Calculations for [13;7,6,4,3,3,1]

There are seven MWCs of five types: [76], [743], [733], [643], [6331]. If expected equilibrium payoffs were proportional, only types [76], [733], [643] and [6331] could be proposed in equilibrium. It can be checked that the necessary and sufficient condition for proportionality does not hold: for any probability distribution over those coalitions, type [6] and/or type [3] would appear in the final coalition disproportionately often.

It turns out that, even though there are five player types, SSPE payoffs divide the players in only three groups, which we denote as L , M and S . We now construct an equilibrium with $v_{[7]} = v_{[6]} := v_L$, $v_{[4]} = v_{[3]} := v_M$, $v_{[1]} := v_S$ and $v_L = 2v_M$. In this situation, player [7] is indifferent between proposing to the other large player and paying v_L , and proposing to two medium players, paying v_M to each (v_L in total). Type [6] is also indifferent between buying votes from the large player or from two medium players (except that, when buying votes from a medium player, one of the two medium players has to be of type [4] because otherwise the coalition would be losing). Coalition [6331] would be too expensive, since on top of $2v_M$ one needs to pay $v_S > 0$. Type [4] may propose [743] or [643]; in both cases it needs to

pay $v_L + v_M$. Likewise, type [3] has three coalition types that are equally optimal: [743], [733] and [643]; coalition [6331] is too expensive. Player [1] has [6331] as its only MWC; given the prices, it could replace [6] with [7] or/and [3] with [4] at no extra cost, hence the surplus coalitions [7331] and [7431] would also be optimal for type [1] (we return to this point below).

We now construct a profile of SSPE strategies. The following table introduces a notation for the strategies. The rows in the table are player types and the columns are coalition types. Each entry in the table represents the probability that the player type in the corresponding row proposes the coalition type in the corresponding column. It is assumed that all players of the same type follow the same strategy and each coalition of the same type is proposed with equal probability.

| | [76] | [743] | [733] | [643] | [6331] |
|-----|----------|---------|----------------------|------------------|--------|
| [7] | α | β | $1 - \alpha - \beta$ | — | — |
| [6] | γ | — | — | $1 - \gamma$ | 0 |
| [4] | — | μ | — | $1 - \mu$ | — |
| [3] | — | π | ρ | $1 - \pi - \rho$ | 0 |
| [1] | — | — | — | — | 1 |

Equilibrium strategies and payoffs solve the following system of equations¹³

$$\begin{aligned}
v_L &= \frac{7}{24}[1 - v_L] + \left[\frac{6}{24}\gamma + \frac{4}{24}\mu + \frac{6}{24}(\pi + \rho) \right] v_L \\
v_L &= \frac{6}{24}[1 - v_L] + \left[\frac{7}{24}\alpha + \frac{4}{24}(1 - \mu) + \frac{6}{24}(1 - \pi - \rho) + \frac{1}{24} \right] v_L \\
v_M &= \frac{4}{24}[1 - v_L - v_M] + \left[\frac{7}{24}\beta + \frac{6}{24}(1 - \gamma) + \frac{6}{24}(1 - \rho) \right] v_M \\
v_M &= \frac{3}{24}[1 - v_L - v_M] + \left[\frac{7}{24}\left(\frac{\beta}{2} + 1 - \alpha - \beta\right) + \frac{6}{24}\frac{1 - \gamma}{2} + \frac{4}{24}\frac{1}{2} + \frac{3}{24}\rho + \frac{1}{24} \right] v_M \\
v_S &= \frac{1}{24}[1 - v_L - 2v_M] \\
v_L &= 2v_M
\end{aligned}$$

There are many solutions to this system, all with $v_L = \frac{46}{164}$, $v_M = \frac{23}{164}$ and $v_S = \frac{3}{164}$. The mixed strategies are not uniquely determined. A possible solution is $\alpha = \mu = \pi = 0$, $\beta = \frac{5}{23}$, $\gamma = \frac{14}{23}$, $\rho = \frac{55}{138}$. These strategies constitute an SSPE since players are behaving optimally both as proposers and as responders: only optimal coalitions are proposed given the acceptance thresholds (v_L , v_M and v_S), and the acceptance thresholds equal the continuation values given the strategies. Due to the uniqueness result of Eraslan and McLennan (2013), all SSPE must have the same v -values.

There are also equilibria in which surplus coalitions are proposed with positive probability. For example, if type [1] proposes [6331] with probability $\frac{1}{2}$ and [7431] with probability $\frac{1}{2}$, the system of equations can be amended accordingly and a new solution for the equilibrium strategies would be $\alpha = \mu = \pi = 0$, $\beta = \frac{12}{161}$, $\gamma = \frac{14}{23}$, $\rho = \frac{29}{92}$ (the v -values are of course unaffected).

¹³Note that we are simplifying the first five equations by using the sixth one (i.e., all coalitions proposed with positive probability in equilibrium must give the same payoff to the proposer). For example, player [6]'s proposer payoff is written as $1 - v_L$ rather than $\gamma[1 - v_L] + (1 - \gamma)[1 - 2v_M]$.