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Inequality Reducing Properties of Progressive Income Tax Schedules: The Case of Endogenous Income^{*}

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Abstract

The case for progressive income taxation is often based on the classic result of Jakobsson (1976) and Fellman (1976), according to which progressive and only progressive income taxes—in the sense of increasing average tax rates on income—ensure a reduction in income inequality. This result has been criticized on the ground that it ignores the possible disincentive effect of taxation on work effort, and the resolution of this critique has been a long-standing problem in public finance. This paper provides a normative rationale for progressivity that takes into account the effect of an income tax on labor supply. It shows that a tax schedule is inequality reducing only if it is progressive—in the sense of increasing *marginal* tax rates on income, and identifies a necessary and sufficient condition on primitives under which progressive and only progressive taxes are inequality reducing.

Keywords: progressive taxation; income inequality; incentive effects of taxation.

JEL classifications: D63, D71.

1 Introduction

A prominent and largely studied normative rationale for progressive income taxation derives from the fundamental result of Jakobsson (1976) and Fellman (1976), which asserts that progressive and only progressive income taxes—in the sense of increasing average tax rates on income—reduce income inequality (regardless of the income distribution they are applied

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to) according to the relative Lorenz dominance criterion.¹ Jakobsson (1976) and Fellman (1976) were early contributors to a vast literature on the redistributive effects of tax systems initiated by Musgrave and Thin (1948).² This literature is for the most part framed in terms of exogenous income. In particular, while the work of Jakobsson (1976) and Fellman (1976) has been extended in various directions,³ treatments that incorporate the disincentive effects of taxation tend to emphasize negative results, such as the existence of non-pathological consumer preferences for which progressive tax schedules increase income inequality. As shown in Ebert and Moyes (2007), the Jakobsson-Fellman result can be extended to the case of endogenous income, but in this case inequality reducing tax schedules are no longer completely characterized by average rate progressivity. Instead, the effect of a tax on gross incomes, in addition to its shape, determines the redistributive effect.⁴ In particular, a progressive tax schedule may well increase income inequality if the associated elasticity of gross income with respect to non-taxed income is large enough. Allingham (1979) and Ebert and Moyes (2003, 2007) provide examples of such progressive tax schedules.

In this paper we recover a version of the Jakobsson-Fellman result that takes into account the incentive effects of taxation.⁵ Our first result states that a piecewise linear tax schedule is inequality reducing (*i.e.*, it reduces income inequality whatever the distribution of abilities) only if it is *marginal-rate progressive* (*i.e.*, it exhibits nondecreasing marginal tax rates on income). A second result asserts that the set of all inequality reducing tax schedules is precisely the set of all marginal-rate progressive tax schedules if and only if the linear tax schedules are inequality reducing.⁶

The main results are formulated within the standard Mirrlees model (see Mirrlees, 1971), which provides a suitable framework for the analysis of nonlinear income taxation with endogenous labor supply, and rest on mild assumptions. First, the set of admissible tax schedules is defined as the set of all continuous, piecewise linear maps from pre-tax incomes to tax liabilities that are nondecreasing and preserve the pre-tax ranking of income (*i.e.*, marginal tax rates are less than unity). Second, consumers require an increasingly large and unbounded compensation for an extra unit of labor as their leisure time tends to zero.⁷

¹A second normative rationale for income tax progressivity is based on the principle of equal sacrifice, which dates back to Samuelson (1947). See Young (1990); Berliant and Gouveia (1993); Ok (1995); Mitra and Ok (1996, 1997). Progressive income taxation has also been studied from a positive perspective as an equilibrium outcome of a voting game (*e.g.*, see Snyder and Kramer, 1988; Marhuenda and Ortuño-Ortín, 1995; Roemer, 1999; Carbonell-Nicolau and Klor, 2003; Carbonell-Nicolau and Ok, 2007).

²See Lambert (2001) for a survey.

³See, *e.g.*, Kakwani (1977); Hemming and Keen (1983); Eichhorn et al. (1984); Liu (1985); Formby et al. (1986); Thon (1987); Latham (1988); Thistle (1988); Moyes (1988, 1994); Le Breton et al. (1996); Ebert and Moyes (2000); Ju and Moreno-Ternero (2008).

⁴See also Onrubia et al. (2005).

⁵See Section 4 for a discussion of the relationship between our results and the Jakobsson-Fellman result.

⁶A sufficient condition for the linear tax schedules to be inequality reducing is that the wage elasticity of labor supply be nondecreasing with respect to productivity and nonincreasing with respect to exogenous income. See Section 4.

⁷In other words, the marginal rate of substitution of labor for consumption tends to infinity as leisure time vanishes.

Finally, consumer preferences satisfy *agent monotonicity*, a condition introduced by Mirrlees (1971, Assumption B, p. 182) and named by Seade (1982). This condition requires that the marginal rate of substitution of gross income for consumption be nonincreasing with productivity, and, as pointed out by Mirrlees (1971), is equivalent to the condition that, in the absence of taxation, consumption increases as the wage rate increases.

The paper is organized as follows. Section 2 lays out the formal setting. The main results, together with sketches of their proofs, are presented in Section 3. Technical proofs are relegated to the Appendix.

2 The model

Consider an economy with a finite number $n \geq 2$ of individuals. The welfare of an individual is measured by a continuous utility function $u : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ defined over consumption-labor pairs $(c, l) \in \mathbb{R}_+ \times [0, 1]$ such that $u(\cdot, l)$ is strictly increasing in c for each $l \in [0, 1]$, and $u(c, \cdot)$ is strictly decreasing in l for each $c \in \mathbb{R}_+$. It is assumed that u is strictly quasiconcave on $\mathbb{R}_{++} \times [0, 1)$ and has continuous partial derivatives on $\mathbb{R}_+ \times [0, 1)$,

$$u_c(c, l) := \frac{\partial u(c, l)}{\partial c} \quad \text{and} \quad u_l(c, l) := \frac{\partial u(c, l)}{\partial l}.$$

For $(c, l) \in \mathbb{R}_+ \times [0, 1)$, let

$$MRS(c, l) := -\frac{u_l(c, l)}{u_c(c, l)}$$

denote the marginal rate of substitution of labor for consumption. (Observe that $MRS(c, l) \geq 0$ for each (c, l) .) We assume that, for each $c > 0$,

$$\lim_{l \rightarrow 1} MRS(c, l) = +\infty \quad \text{and} \quad 0 < \lim_{l \rightarrow 0} MRS(c, l) < +\infty.$$

The second condition is readily acceptable. According to the first condition, the compensation required by an individual for an extra unit of working time tends to infinity as the agent's leisure time approaches zero.⁸

Let \mathcal{U} denote the set of all utility functions satisfying the above conditions.

Prior to formulating the agents' utility maximization problem, we need the formal definition of a tax schedule. Throughout the sequel we confine attention to nondecreasing and order-preserving piecewise linear tax schedules.

Definition 1. Let $(\alpha_0, \mathbf{t}, \bar{\mathbf{y}}) = (\alpha_0, (t_0, \dots, t_K), (\bar{y}_0, \dots, \bar{y}_K))$, where $\alpha_0 \geq 0$, $K \in \mathbb{Z}_+$, $t_k \in [0, 1)$ for each $k \in \{0, \dots, K\}$, $t_k \neq t_{k+1}$ whenever $k \in \{0, \dots, K-1\}$ and $K \geq 1$, and $0 = \bar{y}_0 < \dots < \bar{y}_K$. A **piecewise linear tax schedule** is a real-valued map T on \mathbb{R}_+ uniquely determined by

⁸This condition is used in Lemma 2.

$(\alpha_0, \mathbf{t}, \bar{\mathbf{y}})$ as follows:

$$T(y) := \begin{cases} -\alpha_0 + t_0 y & \text{if } 0 = \bar{y}_0 \leq y \leq \bar{y}_1, \\ -\alpha_0 + t_0 \bar{y}_1 + t_1 (y - \bar{y}_1) & \text{if } \bar{y}_1 < y \leq \bar{y}_2, \\ \vdots & \vdots \\ -\alpha_0 + t_0 \bar{y}_1 + t_1 (\bar{y}_2 - \bar{y}_1) + \cdots + t_{K-1} (\bar{y}_K - \bar{y}_{K-1}) + t_K (y - \bar{y}_K) & \text{if } \bar{y}_K < y. \end{cases}$$

Here $T(y)$ is interpreted as the tax liability for gross income level y .

Letting $\bar{y}_{K+1} := +\infty$, the tax schedule corresponding to the vector $(\alpha_0, \mathbf{t}, \bar{\mathbf{y}})$ can be succinctly expressed as

$$T(y) = t_k y - \alpha_k \quad \text{for } \bar{y}_k \leq y \leq \bar{y}_{k+1}, \quad k \in \{0, \dots, K\},$$

where $\alpha_k := \alpha_{k-1} + (t_k - t_{k-1})\bar{y}_k$ for $k \in \{1, \dots, K\}$.

We write $(\alpha_0, \mathbf{t}, \bar{\mathbf{y}})$ and the associated map T interchangeably. The set of piecewise linear tax schedules is denoted by \mathcal{T} .

Individuals differ in their abilities. An **ability distribution** is a vector $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{R}_{++}^n$ such that $a_1 \leq \dots \leq a_n$ (that is, without loss of generality, individuals are sorted in ascending order according to their ability). The set of all ability distributions is denoted by \mathcal{A} .

An agent of ability $a > 0$ who chooses $l \in [0, 1]$ units of labor and faces a tax schedule $T \in \mathcal{T}$ consumes $c = al - T(al)$ units of the good and obtains a utility of $u(c, l)$. Thus, the agent's problem is

$$\max_{l \in [0, 1]} u(al - T(al), l). \quad (1)$$

Because the members of \mathcal{U} and \mathcal{T} are continuous, for given $u \in \mathcal{U}$, $a > 0$, and $T \in \mathcal{T}$, the optimization problem in (1) has a solution, although it need not be unique. A **solution function** is a map $l^u : \mathbb{R}_{++} \times \mathcal{T} \rightarrow [0, 1]$ such that $l^u(a, T)$ is a solution to (1) for each $(a, T) \in \mathbb{R}_{++} \times \mathcal{T}$. The **pre-tax** and **post-tax income functions** associated to a solution function l^u , denoted by $y^u : \mathbb{R}_{++} \times \mathcal{T} \rightarrow \mathbb{R}_+$ and $x^u : \mathbb{R}_{++} \times \mathcal{T} \rightarrow \mathbb{R}_+$ respectively, are given by

$$y^u(a, T) := al^u(a, T) \quad \text{and} \quad x^u(a, T) := al^u(a, T) - T(al^u(a, T)).$$

The superscript u will sometimes be omitted to lighten notation.

Given $a > 0$, let $U^a : \mathbb{R}_+ \times [0, a] \rightarrow \mathbb{R}$ be defined by $U^a(c, y) := u(c, \frac{y}{a})$. For $(c, y, a) \in \mathbb{R}_+^2 \times \mathbb{R}_{++}$ with $y < a$, define

$$U_c^a(c, y) := \frac{\partial U^a(c, y)}{\partial c}, \quad U_y^a(c, y) := \frac{\partial U^a(c, y)}{\partial y}, \quad \text{and} \quad \eta^a(c, y) := -\frac{U_y^a(c, y)}{U_c^a(c, y)}.$$

Observe that $\eta^a(c, y)$ can be viewed as the marginal rate of substitution of gross income for consumption at (c, y) for an agent with ability a .

The following condition plays an important role in the proofs of our main results. It was introduced by Mirrlees (1971, Assumption B, p. 182) and termed *agent monotonicity* by Seade (1982). Agent monotonicity is a single-crossing condition on the agents' indifference curves in the space of gross income-consumption pairs (y, c) .

Definition 2. A utility function $u \in \mathcal{U}$ satisfies **agent monotonicity** if $\eta^a(c, y) \geq \eta^{a'}(c, y)$ for each $(c, y) \in \mathbb{R}_+^2$ and $0 < a < a'$ with $y < a$.

The set of all the members of \mathcal{U} satisfying agent monotonicity is represented as \mathcal{U}^* .

An **income distribution** is a vector $\mathbf{z} = (z_1, \dots, z_n) \in \mathbb{R}_+^n$ of incomes arranged in increasing order, *i.e.*, $z_1 \leq \dots \leq z_n$; here z_1 denotes the income of the poorest agent, z_2 denotes the income of the second poorest agent, and so on.

In this paper we use the standard relative Lorenz ordering to make inequality comparisons between income distributions. Given two income distributions $\mathbf{z} = (z_1, \dots, z_n)$ and $\mathbf{z}' = (z'_1, \dots, z'_n)$ with $z_1, z'_1 > 0$, we say that \mathbf{z} is at least as equal as \mathbf{z}' if \mathbf{z} **Lorenz dominates** \mathbf{z}' , *i.e.*, if

$$\frac{\sum_{i=1}^k z_i}{\sum_{i=1}^n z_i} \geq \frac{\sum_{i=1}^k z'_i}{\sum_{i=1}^n z'_i}, \quad \text{for all } k \in \{1, \dots, n\}.$$

For $u \in \mathcal{U}^*$, and given pre-tax and post-tax income functions y^u and x^u , an ability distribution $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{A}$ and a tax schedule $T \in \mathcal{T}$ determine a **pre-tax income distribution**

$$\mathbf{y}^u(\mathbf{a}, T) := (y^u(a_1, T), \dots, y^u(a_n, T))$$

and a **post-tax income distribution**

$$\mathbf{x}^u(\mathbf{a}, T) := (x^u(a_1, T), \dots, x^u(a_n, T)).^9$$

In the absence of taxation, *i.e.*, if $T \equiv 0$, one has $\mathbf{y}^u(\mathbf{a}, 0) = \mathbf{x}^u(\mathbf{a}, T)$.

Definition 3. Let $u \in \mathcal{U}$. A tax schedule $T \in \mathcal{T}$ is **income inequality reducing with respect to u** , which we denote as (u -iir), if $\mathbf{x}^u(\mathbf{a}, T)$ Lorenz dominates $\mathbf{y}^u(\mathbf{a}, 0)$ for each ability distribution $\mathbf{a} := (a_1, \dots, a_n) \in \mathcal{A}$ and for each pre-tax and post-tax income functions y^u and x^u .

The functions y^u and x^u are uniquely determined at almost every point of their domain (*cf.* Lemma 1 below). The quantifier for y^u and x^u in Definition 3 deals with the cases when these maps are not uniquely defined.

3 The main results

We begin by defining two important subclasses of the set \mathcal{T} of piecewise linear tax schedules.

Definition 4. A tax schedule $T \in \mathcal{T}$ is **marginal-rate progressive** if it is a convex function.

⁹Under the agent monotonicity condition, in both cases the vector components are arranged in increasing order. See Lemma 1 below.

The set of all marginal-rate progressive tax schedules in \mathcal{T} is denoted by \mathcal{T}_{prog} .

Definition 5. A tax schedule $T \in \mathcal{T}$ is **linear** if $T(y) = -\alpha_0 + t_0 y$ for all $y \in \mathbb{R}_+$ and some $\alpha_0 \geq 0$ and $t_0 \in [0, 1)$.

The set of all linear tax schedules in \mathcal{T} is denoted by \mathcal{T}_{lin} .

We now present the two main results of the paper. Theorem 1 states that only marginal-rate progressive tax schedules can be inequality reducing. Theorem 2 asserts that if the linear members of \mathcal{T} are inequality reducing, then the set of all inequality reducing tax schedules coincides with the set of marginal-rate progressive tax schedules.

Theorem 1. *Given $u \in \mathcal{U}^*$, a tax schedule in \mathcal{T} is u -iir only if it is marginal-rate progressive.*

Theorem 2. *Given $u \in \mathcal{U}^*$, the set of all u -iir tax schedules in \mathcal{T} equals \mathcal{T}_{prog} if and only if the members of \mathcal{T}_{lin} are u -iir.*

Remark. The reader may wish to consider tax schedules that are inequality reducing not only for a *fixed* utility function $u \in \mathcal{U}^*$ but rather for *all* utility functions u in some subdomain \mathcal{U}^{**} of \mathcal{U} . Given $\mathcal{U}^{**} \subseteq \mathcal{U}$, call a tax schedule $T \in \mathcal{T}$ **income inequality reducing with respect to \mathcal{U}^{**} (\mathcal{U}^{**} -iir)** if $(x^u(a_1, T), \dots, x^u(a_n, T))$ Lorenz dominates $(y^u(a_1, 0), \dots, y^u(a_n, 0))$ for each pre-tax and post-tax income functions y^u and x^u , each $(a_1, \dots, a_n) \in \mathcal{A}$, and every $u \in \mathcal{U}^{**}$. Theorems 1 and 2 immediately give the following variants in terms of this strengthening of Definition 3.

Corollary 1 (to Theorem 1). *Given $\mathcal{U}^{**} \subseteq \mathcal{U}$, a tax schedule in \mathcal{T} is \mathcal{U}^{**} -iir only if it is marginal-rate progressive.*

Corollary 2 (to Theorem 2). *Given $\mathcal{U}^{**} \subseteq \mathcal{U}$, the set of all \mathcal{U}^{**} -iir tax schedules in \mathcal{T} equals \mathcal{T}_{prog} if and only if the members of \mathcal{T}_{lin} are \mathcal{U}^{**} -iir.¹⁰*

3.1 Proofs of the main results

In this subsection, we present the main arguments for the proofs of Theorems 1 and 2 and relegate technical details to the Appendix. We begin with three preparatory lemmas. The first lemma says that pre-tax and post-tax income functions are monotone in a , and that the solution to the agents' maximization problem (1) is almost always unique. This result is well-known (see Mirrlees, 1971, Theorem 1 and the ensuing discussion on p. 183).

Lemma 1. *Let $u \in \mathcal{U}^*$, $T \in \mathcal{T}$. For every pre-tax and post-tax income functions x^u and y^u , the maps $a \mapsto x^u(a, T)$ and $a \mapsto y^u(a, T)$ are nondecreasing on \mathbb{R}_{++} . Moreover, given $T \in \mathcal{T}$, there is a unique solution to (1) for all $a > 0$, except for a set of measure zero.*

¹⁰Observe that in the particular case when $\mathcal{U}^{**} = \mathcal{U}^*$ (or $\mathcal{U}^{**} = \mathcal{U}$ for that matter), Corollary 2 is vacuous, for we know (as per Proposition 3 in Ebert and Moyes (2007)) that there exist preferences for which no proportional tax is inequality reducing.

Lemma 2. Given $u \in \mathcal{U}$, $(c, y) \in \mathbb{R}_{++}^2$, and $q \in (0, +\infty)$, there exists an $a > y$ such that $\eta^a(c, y) = q$.¹¹

Proof. Observe that

$$\lim_{a \rightarrow \infty} \eta^a(c, y) = \lim_{a \rightarrow \infty} \frac{1}{a} MRS(c, y/a) = 0 \quad \text{and} \quad \lim_{a \downarrow y} \eta^a(c, y) = \lim_{a \downarrow y} \frac{1}{a} MRS(c, y/a) = +\infty.$$

Since the map $a \mapsto \eta^a$ is continuous, the lemma follows from the Intermediate Value Theorem. ■

The third lemma gives an alternative characterization of an inequality reducing tax schedule (recall Definition 3); its proof is relegated to Appendix A.¹²

Lemma 3. Given $u \in \mathcal{U}$, a tax schedule $T \in \mathcal{T}$ is u -iir if and only if for any ability distribution $\mathbf{a} \in \mathcal{A}$ and for any pre-tax and post-tax income functions y^u and x^u ,

$$\frac{x^u(a_i, T)}{y^u(a_i, 0)} \geq \frac{x^u(a_{i+1}, T)}{y^u(a_{i+1}, 0)} \quad \forall i \in \{1, \dots, n-1\}. \quad (2)$$

3.1.1 Proof of Theorem 1

The following lemma plays an essential role in the proof of Theorem 1.

Lemma 4. Let $u \in \mathcal{U}^*$ and $T \in \mathcal{T}$, and let x^u be a post-tax income function. Then the map $a \mapsto x^u(a, T)$ is continuous on \mathbb{R}_{++} if and only if T is marginal-rate progressive.

The formal proof of Lemma 4 is furnished in Appendix B. Here we provide intuition for this result. Consider the agents' budget line in the space of pre-tax and post-tax income pairs (y, x) for a given tax schedule. If agents face a marginal-rate progressive tax schedule T , this budget line is concave, and since preferences satisfy strict quasiconcavity, there is a unique optimal pre-tax and post-tax income pair for each agent; in this case the continuity of the map $a \mapsto x^u(a, T)$ follows from Berge's Maximum Theorem. Under a non-convex tax T , on the other hand, the budget line must be non-concave somewhere, as the black line in Figure 1. In this case there are multiple optimal pre-tax and post-tax income pairs for some ability level, say a^* (points $(\underline{y}, \underline{x})$ and (\bar{y}, \bar{x}) in Figure 1). Given the agent monotonicity condition (recall Definition 2), this multiplicity generates a discontinuity of the map $a \mapsto x^u(a, T)$ at a^* . Thus, continuity of the map $a \mapsto x^u(a, T)$ implies convexity of T .

Given Lemma 4, Theorem 1 can be concisely proven as follows. Take $u \in \mathcal{U}^*$. By Lemma 3, we only need to find, for each $T \in \mathcal{T}$ that is not marginal-rate progressive, $\mathbf{a} \in \mathcal{A}$ and pre-tax and post-tax income functions y^u and x^u violating (2). Note that given y^u and x^u , an ability distribution $\mathbf{a} \in \mathcal{A}$ will violate (2) if the map $a \mapsto x^u(a, T)$ defined on \mathbb{R}_{++} has a

¹¹Feasibility requires $a \geq y$ (recall that $l = \frac{y}{a}$ and $l \leq 1$), and the lemma states that there is an $a > y$ such that $\eta^a(c, y) = q$.

¹²Lemma 3 is analogous to Lemma 1 in Jakobsson (1976), Proposition 2.1 in Moyes (1994), and Lemma 2 in Ebert and Moyes (2007).

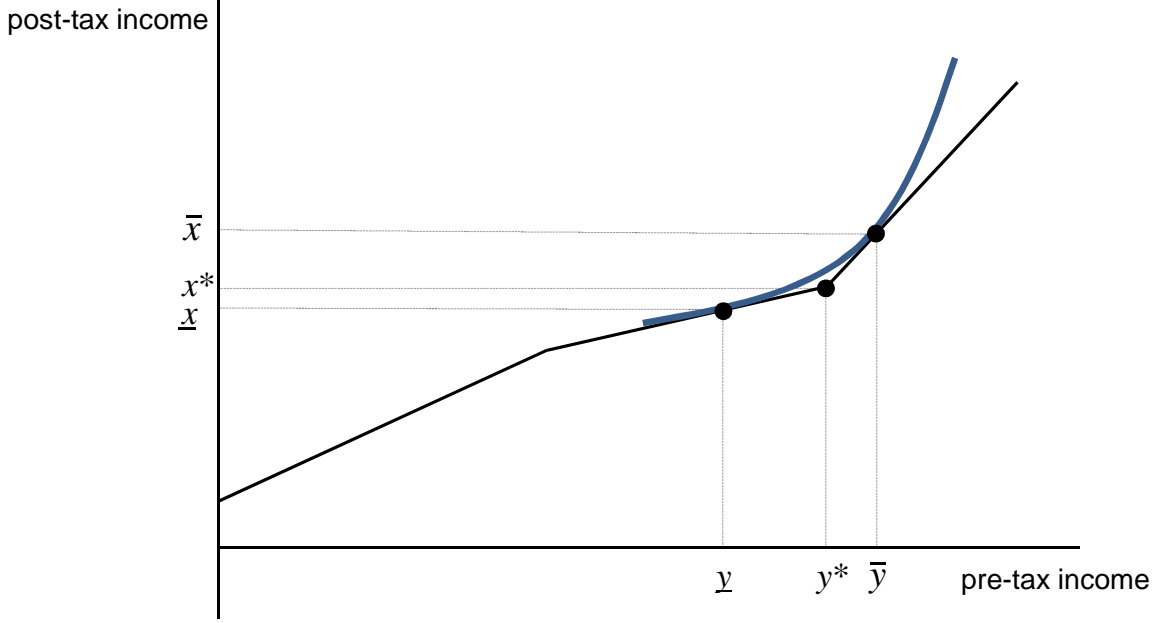


Figure 1: Figure for Theorem 1

discontinuity point and the map $a \mapsto y^u(a, 0)$ defined on \mathbb{R}_{++} is continuous. Indeed, in this case, letting $a^* > 0$ be a discontinuity point for the map $a \mapsto x^u(a, T)$,

$$\lim_{a \uparrow a^*} x^u(a, T) < \lim_{a \downarrow a^*} x^u(a, T) \quad \text{and} \quad \lim_{a \uparrow a^*} y^u(a, 0) = \lim_{a \downarrow a^*} y^u(a, 0),$$

since x^u is nondecreasing (Lemma 1), implying

$$\lim_{a \uparrow a^*} \frac{x^u(a, T)}{y^u(a, 0)} < \lim_{a \downarrow a^*} \frac{x^u(a, T)}{y^u(a, 0)}.$$

Thus, Theorem 1 is a consequence of Lemma 4. (Observe that $x^u(\cdot, T) = y^u(\cdot, 0)$ whenever $T \equiv 0$.)

3.1.2 Proof of Theorem 2

To lighten notation, we will omit the superscript u throughout the proof.

It is clear that if the set of all u -iir tax schedules in \mathcal{T} equals \mathcal{T}_{prog} , then the members of \mathcal{T}_{lin} are u -iir.

Suppose that the members of \mathcal{T}_{lin} are u -iir. By Theorem 1 it follows that the set of all u -iir tax schedules in \mathcal{T} is contained in \mathcal{T}_{prog} . It remains to show the reverse containment. Let $T = (\alpha_0, \mathbf{t}, \bar{\mathbf{y}}) \in \mathcal{T}$ be marginal-rate progressive (recall Definition 1). By Lemma 3, we only need to show that condition (2) holds for any ability distribution $\mathbf{a} \in \mathcal{A}$ and for any pre-tax and post-tax income functions y^u and x^u .

Now, for each income threshold \bar{y}_k of T , define the linear tax schedule $T_k(y) := t_k y - \alpha_k$ for $k \in \{0, \dots, K\}$, where $\alpha_k := \alpha_{k-1} + (t_k - t_{k-1})\bar{y}_k$ for $k \in \{1, \dots, K\}$.

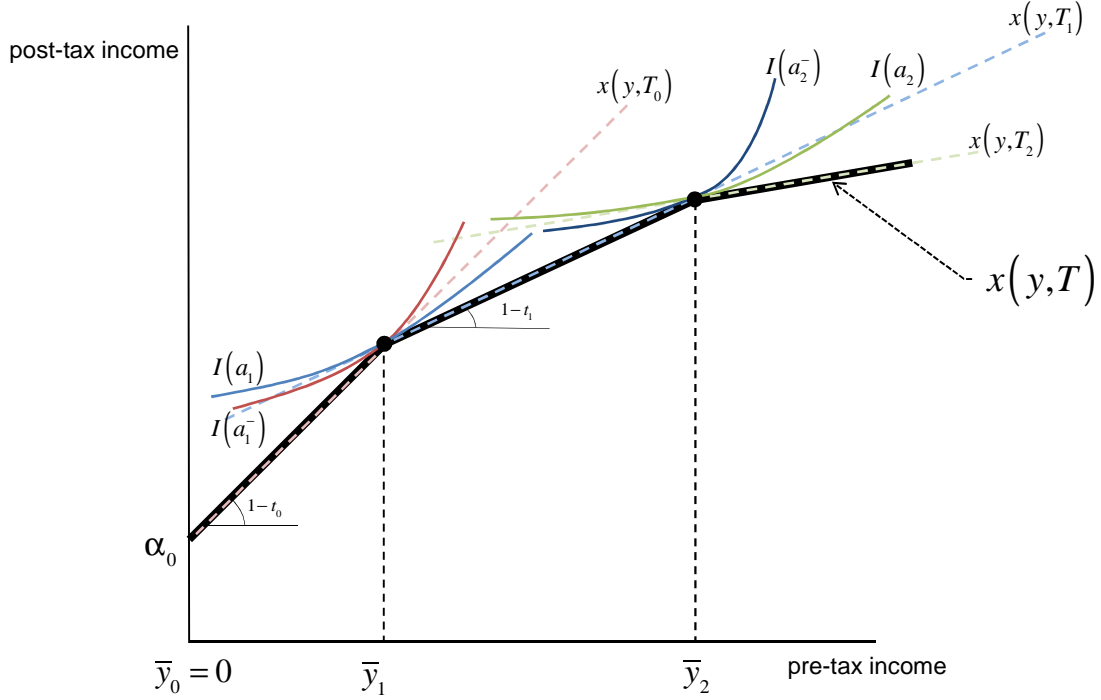


Figure 2: Figure for Theorem 2

Pre-tax and post-tax income functions, y and x , are uniquely defined, since preferences are strictly quasiconcave and the tax function T is convex. For $k \in \{1, \dots, K\}$, define the abilities a_k^- and a_k such that

$$a_k^- := \min \{a : y(a, T_{k-1}) = \bar{y}_k\} \quad \text{and} \quad a_k := \max \{a : y(a, T_k) = \bar{y}_k\}$$

(see Figure 2). Lemma 2 guarantees that a_k^- and a_k exist and are well defined for all $k \in \{1, \dots, K\}$.

Furthermore, since T is marginal-rate progressive (and hence $t_{k-1} \leq t_k$ for all $k \in \{1, \dots, K\}$), agent monotonicity (Definition 2) implies that $a_k^- \leq a_k \leq a_{k+1}^-$.¹³

Next, define the following family of sets covering $(0, +\infty)$:

$$\mathfrak{A} := \left\{ (0, a_1^-], \{[a_k^-, a_k]\}_{k=1}^K, \{[a_k, a_{k+1}^-]\}_{k=1}^{K-1}, [a_K, +\infty) \right\}.$$

We first show that condition (2) is satisfied for ability distributions contained in each element of the family \mathfrak{A} .

(i) Consider first the interval $(0, a_1^-]$. Observe that $y(a, T) = y(a, T_0)$ for all $a \leq a_1^-$. Because T_0 is a linear tax, it is *u-iir*, and so Lemma 3 gives

$$\frac{x(a, T)}{y(a, 0)} = \frac{x(a, T_0)}{y(a, 0)} \geq \frac{x(a', T_0)}{y(a', 0)} = \frac{x(a', T)}{y(a', 0)} \quad \forall a \leq a' \leq a_1^-. \quad (3)$$

¹³Observe that, letting $\bar{x}_k = x(a_k^-, T) = x(a_k, T)$, $\eta^{a_k^-}(\bar{x}_k, \bar{y}_k) = 1 - t_{k-1} > 1 - t_k = \eta^{a_k}(\bar{x}_k, \bar{y}_k)$. On the other hand, $\eta^{a_k}(\bar{x}_{k+1}, \bar{y}_{k+1}) > 1 - t_k = \eta^{a_{k+1}^-}(\bar{x}_{k+1}, \bar{y}_{k+1})$.

(ii) For $[a_K, +\infty)$, a symmetric argument shows that

$$\frac{x(a, T)}{y(a, 0)} \geq \frac{x(a', T)}{y(a', 0)} \quad \forall a_K \leq a \leq a'. \quad (4)$$

(iii) Now consider the interval $[a_k^-, a_k]$ for $k \in \{1, \dots, K\}$. Observe that

$$y(a_k, T) = y(a_k, T_k) = \bar{y}_k = y(a_k^-, T_{k-1}) = y(a_k^-, T).$$

Hence, by monotonicity of the map $a \mapsto y(a, T)$ (Lemma 1), $y(a, T) = \bar{y}_k$ for all $a \in [a_k^-, a_k]$. Therefore, because $y(a', 0) \geq y(a, 0)$ for all $a_k^- \leq a \leq a' \leq a_k$ by Lemma 1,

$$\frac{x(a, T)}{y(a, 0)} = \frac{\bar{y}_k - T(\bar{y}_k)}{y(a, 0)} \geq \frac{\bar{y}_k - T(\bar{y}_k)}{y(a', 0)} = \frac{x(a', T)}{y(a', 0)} \quad \forall a, a' \in [a_k^-, a_k], a \leq a'. \quad (5)$$

(iv) Finally, consider the interval $[a_k, a_{k+1}^-]$ for $k \in \{1, \dots, K-1\}$. By construction, we have $y(a, T) = y(a, T_k)$ for all $a \in [a_k, a_{k+1}^-]$. Therefore, since T_k is a linear (hence *u-iir*) tax, Lemma 3 gives

$$\frac{x(a, T)}{y(a, 0)} = \frac{x(a, T_k)}{y(a, 0)} \geq \frac{x(a', T_k)}{y(a', 0)} = \frac{x(a', T)}{y(a', 0)} \quad \forall a, a' \in [a_k, a_{k+1}^-], a \leq a'. \quad (6)$$

Combining equations (3)-(6) we obtain (2) for every $a \in \mathcal{A}$.

4 Concluding remarks

This paper provides a normative foundation for progressive income taxes. Our work—which goes beyond the classic results of Jakobsson (1976) and Fellman (1976) in that it takes into account the disincentive effects of taxation on work effort—relies on three basic conditions: the agent monotonicity condition, which is standard in the literature on nonlinear taxation with endogenous labor supply; piecewise linearity of admissible tax schedules, an ubiquitous feature of actual statutory tax schedules; and an increasingly large marginal rate of substitution of labor for consumption for vanishingly small amounts of leisure time. The latter condition fails in the Jakobsson-Fellman setting, viewed in the Mirrlees framework as the particular case of costless work effort.

Theorem 1 implies that tax schedules aimed at reducing income inequality must be marginal-rate progressive. In other words, only marginal-rate progressive tax schedules can secure a reduction in consumption inequality compared to a situation with no taxes.¹⁴ Theorem 2 can be combined with the results on linear taxation in Ebert and Moyes (2007) to obtain a necessary and sufficient condition on primitives under which the members of \mathcal{T}_{prog} , and only the members of \mathcal{T}_{prog} , are inequality reducing. In fact, if in the absence of

¹⁴When individual incomes are subjected to different rounds of taxation, an analyst may wish to study the inequality reducing properties of the composition of different tax schedules. Le Breton et al. (1996) address this issue in the context of exogenous income. The extension of their analysis to the case of endogenous income is left for future research.

taxation income is differentiable with respect to productivity, then all linear tax schedules are *u-iir* (for $u \in \mathcal{U}^*$) if the wage elasticity of labor supply is nondecreasing with respect to productivity and nonincreasing with respect to exogenous income (Ebert and Moyes, 2007, Proposition 2). Thus, given $u \in \mathcal{U}^*$, a sufficient condition for the set of all *u-iir* tax schedules to be precisely \mathcal{T}_{prog} is that the wage elasticity of labor supply be nondecreasing with respect to productivity and nonincreasing with respect to exogenous income.¹⁵

We conclude with a philosophical comment. This paper focuses on the reduction of income inequality through the tax system, as does virtually all the related literature. Some authors have suggested that considering instead the *welfare* inequality reducing properties of taxes might be more reasonable. After all, individuals ultimately care about their well-being. While we believe that this idea deserves further investigation, we would like to point out that a meaningful characterization of the welfare inequality reducing properties of progressive tax schedules seems problematic even in the standard context of exogenous incomes. Indeed, consider the marginal-rate progressive (hence income inequality reducing) tax function $T(y) := y - \ln(y + 1)$, together with the utility function $u(x) := \ln(x)$; since the ratio $\frac{u(y-T(y))}{u(y)} = \frac{\ln(\ln(y+1))}{\ln(y)}$ is strictly increasing in y , it follows from Lemma 1 in Jakobsson (1976) that T is welfare inequality *increasing*.¹⁶

Appendix

A Proof of Lemma 3

Lemma 3. *Given $u \in \mathcal{U}$, a tax schedule $T \in \mathcal{T}$ is *u-iir* if and only if for any ability distribution $\mathbf{a} \in \mathcal{A}$ and for any pre-tax and post-tax income functions y^u and x^u ,*

$$\frac{x^u(a_i, T)}{y^u(a_i, 0)} \geq \frac{x^u(a_{i+1}, T)}{y^u(a_{i+1}, 0)} \quad \forall i \in \{1, \dots, n-1\}. \quad (7)$$

Proof. We adapt the proof of Lemma 2 in Ebert and Moyes (2007).¹⁷

(\Leftarrow) A tax schedule is *u-iir* if condition (7) holds for any $\mathbf{a} \in \mathcal{A}$ and any pre-tax and post-tax income functions y^u and x^u . This is a direct consequence of Marshall et al. (2011, B.1.b in chapter 5), since for each $\mathbf{a} \in \mathcal{A}$, $x^u(a_i, T) > 0$ for each $i \in \{1, \dots, n\}$ and $\sum_{i=1}^n y^n(a_i, 0) > 0$.

(\Rightarrow) Suppose that there exists $\bar{\mathbf{a}} \in \mathcal{A}$ and x^u and y^u such that

$$\frac{x^u(\bar{a}_h, T)}{y^u(\bar{a}_h, 0)} < \frac{x^u(\bar{a}_{h+1}, T)}{y^u(\bar{a}_{h+1}, 0)} \quad \text{for some } h \in \{1, \dots, n-1\}. \quad (8)$$

Choose $\mathbf{a}^* := (a_1^*, \dots, a_n^*)$ where $a_1^* := \bar{a}_h$ and $a_i^* := \bar{a}_{h+1}$ for $i \in \{2, \dots, n\}$. By definition, $a_1^* < a_2^* = \dots = a_n^*$. It follows that $x^u(a_1^*, T) = x^u(\bar{a}_h, T)$ and $x^u(a_i^*, T) = x^u(\bar{a}_{h+1}, T)$ for $i \in \{2, \dots, n\}$.

¹⁵For a necessary and sufficient condition, see Ebert and Moyes (2007, p. 669).

¹⁶According to Ebert and Moyes (2007, footnote 19), in the case of exogenous income, the Jakobsson-Fellman result can be stated in terms of welfare inequality if and only if the utility function is isoelastic, *i.e.*, it takes the form $u(x) = \chi x^\xi$, where χ and ξ are constants.

¹⁷Since we do not assume the existence of a unique solution to the agents' maximization problem, (1), Lemma 2 in Ebert and Moyes (2007) does not subsume Lemma 3.

And similarly for y . From (8),

$$\frac{x^u(a_1^*, T)}{y^u(a_1^*, 0)} < \frac{x^u(a_2^*, T)}{y^u(a_2^*, 0)} = \dots = \frac{x^u(a_n^*, T)}{y^u(a_n^*, 0)}.$$

Appealing to Marshall et al. (2011, B.1.b in Chapter 5),

$$\left(\frac{x^u(a_1^*, T)}{\sum_j x^u(a_j^*, T)}, \dots, \frac{x^u(a_n^*, T)}{\sum_j x^u(a_j^*, T)} \right)$$

is majorized by

$$\left(\frac{y^u(a_1^*, 0)}{\sum_j y^u(a_j^*, 0)}, \dots, \frac{y^u(a_n^*, 0)}{\sum_j y^u(a_j^*, 0)} \right).^{18}$$

Therefore,

$$\sum_{i=1}^k \left(\frac{x^u(a_i^*, T)}{\sum_j x^u(a_j^*, T)} \right) \leq \sum_{i=1}^k \left(\frac{y^u(a_i^*, 0)}{\sum_j y^u(a_j^*, 0)} \right) \quad \forall k \in \{1, \dots, n\}.$$

That is, $(y^u(a_1^*, 0), \dots, y^u(a_n^*, 0))$ Lorenz dominates $(x^u(a_1^*, T), \dots, x^u(a_n^*, T))$, and hence T is not u -iir. \blacksquare

B Proof of Lemma 4

Lemma 4. *Let $u \in \mathcal{U}^*$ and $T \in \mathcal{T}$, and let x^u be a post-tax income function. Then the map $a \mapsto x^u(a, T)$ is continuous on \mathbb{R}_{++} if and only if T is marginal-rate progressive.*

Proof. Take $u \in \mathcal{U}^*$ and $T = (\alpha, \mathbf{t}, \bar{\mathbf{y}}) \in \mathcal{T}$. First observe that T is marginal-rate progressive (i.e., convex) if and only if the map $y \mapsto x = y - T(y)$ defined on \mathbb{R}_+ is concave.

(\Leftarrow) Let T be marginal-rate progressive. Because the map $y \mapsto x(y) := y - T(y)$ is concave and $u(c, l)$ is strictly quasiconcave (and strictly increasing (resp. decreasing) in c (resp. l)), for each $a > 0$ the problem

$$\max_{y \in [0, a]} u\left(y - T(y), \frac{y}{a}\right) \quad (9)$$

has a unique solution. Consequently, there is a unique map that assigns to each ability level $a > 0$ the pre-tax income $y(a)$ that solves (9), and by virtue of Berge's Maximum Theorem, this map is continuous. But then the map $a \mapsto y(a) - T(y(a))$ defined on \mathbb{R}_{++} is continuous. In other words, for any post-tax income function x^u , the map $a \mapsto x^u(a, T)$ is continuous on \mathbb{R}_{++} .

(\Rightarrow) To prove the converse assertion we assume that T is not marginal-rate progressive and show that the map $a \mapsto x^u(a, T)$ has a discontinuity point in \mathbb{R}_{++} .

If T is not marginal-rate progressive, the map $x(y)$ is not concave. Define

$$y^* := \inf\{y \in \mathbb{R}_+ : x(y)|_{[0, y^*]} \text{ is concave}\}.$$

¹⁸For $x, y \in \mathbb{R}^n$, in increasing arrangement, we say that x is majorized by y if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ for $k = 1, \dots, n-1$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. (Marshall et al., 2011, p. 8.)

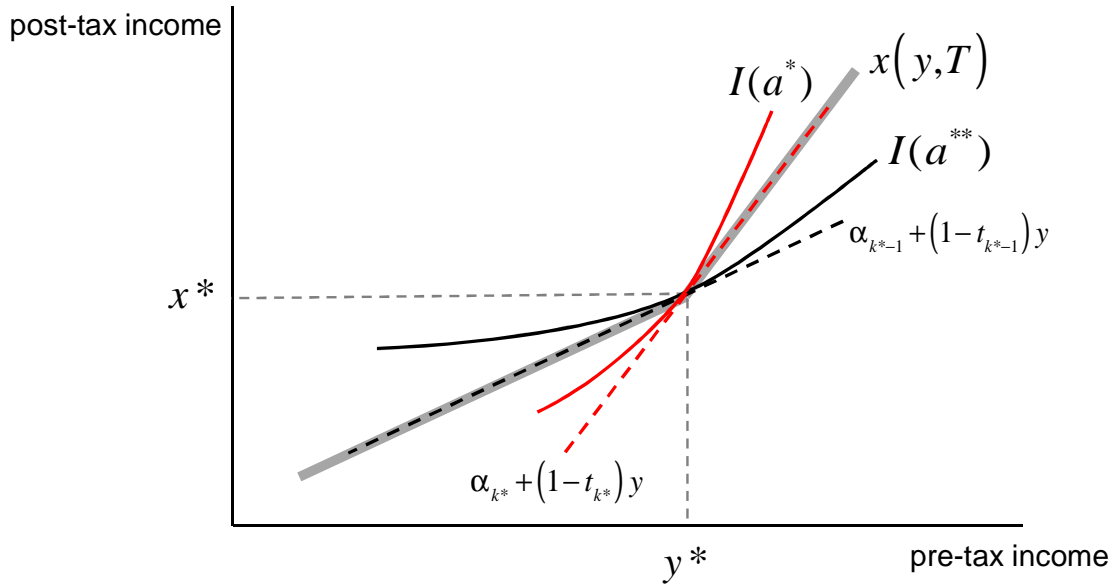


Figure 3: Figure for Lemma 4

It is easy to see that $y^* = \bar{y}_{k^*}$ for some $k^* \in \{1, \dots, K\}$. In addition, the restriction of $x(y)$ to $[\bar{y}_{k^*-1}, \bar{y}_{k^*+1}]$ is convex (here $\bar{y}_{K+1} := +\infty$) and $(x^* := y^* - T(y^*), y^*) \gg 0$ and $0 < 1 - t_{k^*-1} < 1 - t_{k^*}$. Applying Lemma 2 it follows that there exist $0 < a^* < a^{**}$ such that

$$\eta^{a^*}(x^*, y^*) = 1 - t_{k^*} \quad \text{and} \quad \eta^{a^{**}}(x^*, y^*) = 1 - t_{k^*-1},$$

implying $x^u(a^*, T) < x^* < x^u(a^{**}, T)$ and $x^u(a, T) \neq x^*$ for all $a \in (a^*, a^{**})$. (Refer to Figure 3.) If the map $a \mapsto x^u(a, T)$ were continuous on \mathbb{R}_{++} , the Intermediate Value Theorem would give $\alpha \in (a^*, a^{**})$ with $x^u(\alpha, T) = x^*$, a contradiction. We conclude that the map $a \mapsto x^u(a, T)$ has a discontinuity point in \mathbb{R}_{++} . ■

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