

Breunig, Christoph

**Working Paper**

## Testing missing at random using instrumental variables

SFB 649 Discussion Paper, No. 2017-007

**Provided in Cooperation with:**

Collaborative Research Center 649: Economic Risk, Humboldt University Berlin

*Suggested Citation:* Breunig, Christoph (2017) : Testing missing at random using instrumental variables, SFB 649 Discussion Paper, No. 2017-007, Humboldt University of Berlin, Collaborative Research Center 649 - Economic Risk, Berlin

This Version is available at:

<https://hdl.handle.net/10419/162508>

**Standard-Nutzungsbedingungen:**

Die Dokumente auf EconStor dürfen zu eigenen wissenschaftlichen Zwecken und zum Privatgebrauch gespeichert und kopiert werden.

Sie dürfen die Dokumente nicht für öffentliche oder kommerzielle Zwecke vervielfältigen, öffentlich ausstellen, öffentlich zugänglich machen, vertreiben oder anderweitig nutzen.

Sofern die Verfasser die Dokumente unter Open-Content-Lizenzen (insbesondere CC-Lizenzen) zur Verfügung gestellt haben sollten, gelten abweichend von diesen Nutzungsbedingungen die in der dort genannten Lizenz gewährten Nutzungsrechte.

**Terms of use:**

*Documents in EconStor may be saved and copied for your personal and scholarly purposes.*

*You are not to copy documents for public or commercial purposes, to exhibit the documents publicly, to make them publicly available on the internet, or to distribute or otherwise use the documents in public.*

*If the documents have been made available under an Open Content Licence (especially Creative Commons Licences), you may exercise further usage rights as specified in the indicated licence.*

SFB 649 Discussion Paper 2017-007

# Testing Missing at Random using Instrumental Variables

Christoph Breunig\*



\*Humboldt-Universität zu Berlin, Germany

This research was supported by the Deutsche  
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

<http://sfb649.wiwi.hu-berlin.de>  
ISSN 1860-5664

SFB 649, Humboldt-Universität zu Berlin  
Spandauer Straße 1, D-10178 Berlin



SFB 649 ECONOMIC RISK BERLIN

# Testing Missing at Random using Instrumental Variables \*

CHRISTOPH BREUNIG \*

*Humboldt-Universität zu Berlin*

February 27, 2017

This paper proposes a test for *missing at random* (MAR). The MAR assumption is shown to be testable given instrumental variables which are independent of response given potential outcomes. A nonparametric testing procedure based on integrated squared distance is proposed. The statistic's asymptotic distribution under the MAR hypothesis is derived. In particular, our results can be applied to testing *missing completely at random* (MCAR). A Monte Carlo study examines finite sample performance of our test statistic. An empirical illustration analyzes the nonresponse mechanism in labor income questions.

*Keywords:* Incomplete data, missing-data mechanism, selection model, nonparametric hypothesis testing, consistent testing, instrumental variable, series estimation.

## 1. Introduction

When confronted with data sets with missing values it is often assumed in applied research that observations are *missing at random* (MAR) in the sense

---

\*The author would like to thank the joint editor Rajeev Dehejia and two anonymous referees for their comments and suggestions that improved the paper. In addition, the paper benefited from discussions with Timothy Armstrong, Xiaohong Chen, Stefan Hoderlein, Arthur Lewbel, Enno Mammen, and Peter Robinson and research assistance by Boryana Ilieva. This research was supported by the DFG postdoctoral fellowship BR 4874/1-1. The author is also grateful for support and hospitality of the Cowles Foundation.

\*Humboldt-Universität zu Berlin, Spandauer Straße 1, 10178 Berlin, Germany, e-mail: [christoph.breunig@hu-berlin.de](mailto:christoph.breunig@hu-berlin.de)

of Rubin [1976]. This condition requires that the probability of observing potential outcomes only depends on observed data. To help to decide whether MAR based techniques could be applied we develop in this paper a test for the MAR assumption. In general, MAR is not refutable without further assumptions and here we rely on instruments that are independent of the response mechanism given potentially observed outcomes. We show that this condition is sufficient to ensure testability of MAR and derive the asymptotic distribution under MAR of a proposed test statistic. We provide an extension of our testing procedure to assess the hypothesis *missing completely at random* (MCAR).

If the missing data mechanism does not follow MAR, a correction of the potential selection bias is necessary to ensure consistency of the estimation procedure. There exist two different instrumental variable approaches to overcome the problem of missing variables. The first approach relies on instruments that determine response but not the outcome and was pioneered by Heckman [1974]. Such instruments, however, are difficult to find when response is directly driven by the outcome. The second approach, also considered in this paper, relies on instruments that are independent of response given potential outcomes. This framework was used in parametric regression analysis by Chen [2001], Liang and Qin [2000], Tang et al. [2003], Ramalho and Smith [2013], and Zhao and Shao [2015]. A nonparametric extension was proposed by D'Haultfoeuille [2010] and Breunig et al. [2015]. While such instrumental variable methods reduce bias in general, if the data are MAR, they unnecessarily increase variance. Indeed, D'Haultfoeuille [2010] showed that estimation of the distribution of the potential outcome leads to a statistical inverse problem that is *ill-posed* in general. This implies that the variance of the estimator becomes arbitrarily large relative to the degree of ill-posedness. We also provide a test for the MCAR assumption which imposes a stronger condition on the response mechanism than MAR. Indeed, MCAR rules out any correlation between response and outcome. When data are MAR but not MCAR various types of correction methods have been suggested so far and include weighted generalized estimating equations (Robins et al. [1994]), nonparametric estimation of the conditional estimating scores (Reilly and Pepe [1995]), and multiple imputation (Rubin [2004], Little and Rubin [2002]). For an overview and further references we refer to Ibrahim et al. [2005]. This literature makes either parametric model assumptions or has difficulties in dealing with continuous data. Using such correction methods reduces bias if MAR holds, under MCAR, however, this unnecessarily increases variance. Thus, it is of interest to examine the observed data for evidence whether the response mechanism satisfies not only MAR but also MCAR.

We show that the MAR hypothesis is equivalent to an identified conditional moment equation and is related to significance testing problems. Based on this moment equation we construct our test statistic using a weighted integrated squared distance. Under the null hypothesis the test statistic converges to a series of independent,  $\chi^2$ -squared distributed random variables. The test

statistic and its critical values can be easily implemented. Also only a slight modification is necessary to obtain a test for MCAR. Under a bounded completeness assumption, our testing procedure is shown to be consistent against fixed alternatives. For significance testing see, for instance, Fan and Li [1996], Lavergne and Vuong [2000], or Delgado and González Manteiga [2001]. In contrast, our test statistic is entirely based on series estimators and weights the generalized Fourier coefficients. We show in this paper, that the new testing procedure has desirable finite sample properties.

Besides a Monte Carlo simulation we demonstrate the finite sample properties in an empirical illustration using data from the German Socio-Economic Panel (SOEP). A common phenomenon in population surveys is that wealth or income questions are typically associated with high rates of item nonresponse. We provide evidence that the item nonresponse for labor income questions is selective, using income information from previous waves as an instrument.

In our instrumental variable framework, a test of MCAR has been proposed by Ramalho and Smith [2013]. Their Hausman type test statistic relies on a parametric model specification with discrete outcomes and differs from our method where no restriction on the marginal distribution of the outcome is imposed. Likelihood ratio tests to verify the hypothesis MCAR have been suggested by Fuchs [1982] and Little [1988], while Chen and Little [1999] considered a Wald-type test and Qu and Song [2002] proposed a generalized score type test based on quadratic inference functions. Kline and Santos [2013] develop a method for assessing the sensitivity of empirical conclusions to departures from MAR based on sharp bounds of conditional quantiles. As far as we know, a consistent test for MAR has not been proposed. We further emphasize that our testing procedure does not require knowledge of the conditional probability of observing potential outcomes up to a finite dimensional parameter.

The remainder of the paper is organized as follows. Section 2 provides sufficient conditions for testability of MAR and MCAR. The asymptotic distributions of the tests are derived and their consistency against fixed alternatives is established. Section 3 examines the finite sample performance of our test in a Monte Carlo simulations study while Section 4 illustrates the usefulness of our procedure in an empirical application.

## **2. The Test Statistic and its Asymptotic Properties**

This section is about testability of missing at random assumptions and the asymptotic behavior of our proposed test statistics. First, we provide sufficient conditions on instruments to ensure testability of MAR and MCAR. Second, we build on identified conditional moment restrictions to construct test statistics. Third, the test statistics' asymptotic distributions under the null

hypotheses are derived and we establish consistency of the tests against fixed alternatives.

## 2.1. Testability

Let  $Y^*$  denote a scalar, partially observed random variable and  $X$  a  $d_x$ -dimensional vector of covariates which are always observed. Further,  $\Delta$  is a binary missing-data indicator for  $Y^*$ , such that  $\Delta = 1$  if a realization of  $Y^*$  is observed and  $\Delta = 0$  otherwise. We write  $Y = \Delta Y^*$ .

First, we consider hypothesis MAR, whether the response mechanism only depends on observed variables  $X$ . In this case, the null hypothesis under consideration is given by

$$\text{MAR} : \mathbb{P}(\Delta = 1|Y^*, X) = \mathbb{P}(\Delta = 1|X) \quad (2.1)$$

and the alternative by  $\mathbb{P}(\mathbb{P}(\Delta = 1|Y^*, X) \neq \mathbb{P}(\Delta = 1|X)) > 0$ .<sup>1</sup>

Second, we consider the MCAR hypothesis whether response is completely at random. As this hypothesis rules out any correlation between response and observed data, MCAR is stronger than MAR. The hypothesis under consideration is

$$\text{MCAR} : \mathbb{P}(\Delta = 1|Y^*, X) = \mathbb{P}(\Delta = 1)$$

and the alternative is  $\mathbb{P}(\mathbb{P}(\Delta = 1|Y^*, X) \neq \mathbb{P}(\Delta = 1)) > 0$ .

We now provide sufficient conditions for testability of the above hypotheses. More precisely, we provide conditions under which the hypotheses MAR and MCAR are equivalent to restrictions of identified conditional moments. We can thus determine these conditional moments exactly given a sufficiently large sample of observed variables only. A key requirement is that an additional vector  $W$ , an instrument, is available which satisfies the following conditions.

*ASSUMPTION 1. For each unit we observe  $\Delta$ ,  $Y$ ,  $X$ , and  $W$ .*

Assumption 1 is satisfied when only observations of  $Y^*$  are missing. In the following, we assume that the random vector  $W$  is independent of the response variable conditional on potentially observed variables  $Y^*$  and covariates  $X$ .

*ASSUMPTION 2. It holds*

$$\Delta \perp\!\!\!\perp W | (Y^*, X).$$

Assumption 2 requires missingness to be primarily determined by  $(Y^*, X)$ . In particular, this exclusion restriction requires any influence of  $W$  on  $\Delta$  to be

---

<sup>1</sup>Since conditional probabilities/expectations are defined only up to equality a.s., all equalities with conditional probabilities/expectations are understood as equalities a.s., even if we do not say so explicitly.

carried solely through  $(Y^*, X)$ . Conditional independence assumptions of this type are quite familiar in the econometrics and statistics literature. Examples are treatment effects (cf. Imbens [2004]) or non-classical measurement error (cf. Hu and Schennach [2008]). For further discussion of Assumption 2 and illustrative examples we refer to Ramalho and Smith [2013] in case of nonresponse and D’Haultfoeuille [2010] in case of counterfactuals. As was shown by D’Haultfoeuille [2010], Assumption 2 is equivalent to a conditional moment restriction (see also Example 2.1 below) and thus testable under a completeness assumption (see Section 2.2 of D’Haultfoeuille [2010]).

**ASSUMPTION 3.** *For all bounded measurable functions  $\phi$ ,  $\mathbb{E}[\phi(Y^*, X)|X, W] = 0$  implies that  $\phi(Y^*, X) = 0$ .*

Assumption 3 is known as bounded completeness. In contrast, to ensure identification in nonparametric instrumental variable models, stronger versions of Assumption 3, such as  $L^2$ -completeness, are required. This type of completeness condition requires Assumption 3 to hold for any measurable function  $\phi$  with  $\mathbb{E}|\phi(Y^*, X)|^2 < \infty$ .  $L^2$ -completeness is also a common assumption in nonparametric hypothesis testing in instrumental variable models, see, for instance, Blundell and Horowitz [2007] or Fève et al.. There are only a few examples in the nonparametric instrumental regression literature where it is sufficient to assume completeness only for bounded functions. One example is estimation of Engel curves as in Blundell et al. [2007] which, by definition, are bounded between zero and one. We emphasize that bounded completeness is much less restrictive than  $L^2$  completeness. Sufficient conditions for bounded completeness have been provided by Mattner [1993] or D’Haultfoeuille [2011] among others. We see below that inference under the considered hypotheses does not require bounded completeness. On the other hand, we need to impose Assumption 3 to ensure consistency against fixed alternatives.

If a valid instrumental variable  $W$  is available then consistent density estimation and regression is possible even if MAR does not hold true. On the other hand, using instrumental variable estimation methods when MAR holds can be inappropriate as the following two examples illustrate.

**EXAMPLE 2.1** (Density Estimation with Selectively Missing Variables). The joint probability density function of  $(Y^*, X)$  satisfies

$$p_{Y^*X}(y, x) = \frac{\mathbb{P}(\Delta = 1, Y^* = y, X = x)}{\mathbb{P}(\Delta = 1|Y^* = y, X = x)}$$

assuming that the conditional probability in the denominator is bounded away from zero. The conditional probability  $\mathbb{P}(\Delta = 1|Y^* = y, X = x)$  is not identified in general. On the other hand, if instrumental variables  $W$  are available, satisfying more restrictive completeness assumptions than Assumption 3, then this probability is identified (see D’Haultfoeuille [2010]) through the

conditional moment restriction

$$\mathbf{E}\left(\frac{\Delta}{\mathbb{P}(\Delta = 1|Y^*, X)} \middle| X, W\right) = 1. \quad (2.2)$$

Estimating  $\mathbb{P}(\Delta = 1|Y^*, X)$  via this equation leads to a large variance relative to the ill-posedness of the underlying inverse problem and the accuracy of this estimator can be very low (see D’Haultfoeuille [2010] and Breunig et al. [2015]). If the data, however, reveals that MAR holds true then  $\mathbb{P}(\Delta = 1|Y^*, X)$  can be directly estimated from the data.  $\square$

**EXAMPLE 2.2** (Regression with Selectively Missing Outcome). Consider estimation of  $\mathbf{E}(\phi(Y^*)|X)$  for some known function  $\phi$  and  $Y^*$  scalar. Either  $\phi$  is the identity function in case of mean regression or  $\phi(Y^*) = \mathbb{1}\{Y^* \leq q\}$  in quantile regression for some quantile  $q \in (0, 1)$ . Let the conditional probability  $\mathbb{P}(\Delta = 1|Y^*, X)$  be bounded away from zero. As in Breunig et al. [2015] (p. 5) it holds

$$\mathbf{E}(\phi(Y^*)|X) = \mathbf{E}\left(\frac{\Delta\phi(Y^*)}{\mathbb{P}(\Delta = 1|Y^*, X)} \middle| X\right)$$

where  $\mathbb{P}(\Delta = 1|Y^*, X)$  can be estimated via the conditional mean restriction (2.2). As shown in Breunig et al. [2015], the first step estimation of  $\mathbb{P}(\Delta = 1|Y^*, X)$  leads to an additional bias term which can reduce accuracy of estimation. Also in this case, imposing MAR is desirable to simplify the estimation procedure and increase estimation precision.  $\square$

**EXAMPLE 2.3** (Relation to Triangular Models). Assumptions 2 and 3 hold true in the triangular model

$$\begin{aligned} \Delta &= \varphi(Y^*, X, \eta) \quad \text{with} \quad \eta \perp\!\!\!\perp (W, \varepsilon) \\ Y^* &= \phi(\psi(X, W) + \varepsilon) \quad \text{with} \quad W \perp\!\!\!\perp \varepsilon \end{aligned}$$

under a large support condition of  $\psi(X, W)$ , regularity assumptions for  $\varepsilon$ , and if the conditional characteristic function of  $\varepsilon$  given  $X$  is infinitely often differentiable and does not vanish on the real line. See D’Haultfoeuille [2011] page 462–463 for further details. Requiring this characteristic function to be nonvanishing is a standard assumption in the deconvolution literature. The normal, Student,  $\chi^2$ , gamma, and double exponential distributions all satisfy this assumption while the uniform and the triangular distributions are the only common distributions to violate this restriction.

In this triangular model, MCAR requires the function  $\varphi$  in the selection equation to be dependent on  $X$  and  $\eta$  only; that is,  $\Delta = \varphi(X, \eta)$ . Under MCAR,  $\varphi$  depends neither on  $Y^*$  nor on  $X$  and hence, the structural equation simplifies to  $\Delta = \varphi(\eta)$ . The triangular model illustrates the difference to Heckman’s approach (cf. its nonparametric version in Das et al. [2003]) where an instrument



enters the selection equation.  $\square$

The following result states that the null hypothesis MAR is testable under the previous conditions. Further, exploiting the properties of the instrument  $W$  shows that MAR is equivalent to an identified conditional moment restriction.

**THEOREM 2.1.** (i) Under Assumptions 1–2, MAR implies  $\mathbf{E}[\Delta - \mathbb{P}(\Delta = 1|X)|X, W] = 0$ . (ii) Under Assumptions 1–3, MAR is equivalent to  $\mathbf{E}[\Delta - \mathbb{P}(\Delta = 1|X)|X, W] = 0$ .

*Proof.* The null hypothesis MAR implies (or is equivalent under Assumption 3) to

$$\mathbf{E} \left[ \mathbb{P}(\Delta = 1|Y^*, X) - \mathbb{P}(\Delta = 1|X) \middle| X, W \right] = 0.$$

By Assumption 2 we have  $\mathbb{P}(\Delta = 1|Y^*, X) = \mathbb{P}(\Delta = 1|Y^*, X, W)$  and thus, the law of iterated expectations yields  $\mathbf{E}[\mathbb{P}(\Delta = 1|Y^*, X)|X, W] = \mathbf{E}[\Delta|X, W]$ . Hence, MAR implies (or is equivalent under Assumption 3) to

$$\mathbf{E} \left[ \Delta - \mathbb{P}(\Delta = 1|X) \middle| X, W \right] = 0$$

where the left hand side is point identified.  $\square$

The following corollary provides a testability result for the hypothesis MCAR. The result follows as in the proof of Theorem 2.1 by replacing  $\mathbb{P}(\Delta = 1|X)$  with  $\mathbb{P}(\Delta = 1)$ .

**COROLLARY 2.2.** (i) Under Assumptions 1–2, MCAR implies  $\mathbf{E}[\Delta - \mathbb{P}(\Delta = 1)|X, W] = 0$ . (ii) Under Assumptions 1–3, MCAR is equivalent to  $\mathbf{E}[\Delta - \mathbb{P}(\Delta = 1)|X, W] = 0$

In the following, we present two examples of possible applications where MAR might be difficult to justify but instrumental variables  $W$  are available that satisfy our exclusion restriction. For further examples we refer to D’Haultfoeuille [2011] in case of counterfactuals in case of schooling data and Zhao and Shao [2015] in an application using health data.

**EXAMPLE 2.4.** Huck et al. [2015] analyze the impact of expectations on stock market returns on the financial investment decisions of households. In their survey, individuals obtain an exogenous treatment which is the historical DAX (Germany’s prime blue chip stock market index) return of a randomly drawn year. Individuals may choose not to respond to questions regarding investment decision, which might be directly related to the latent expectation on the stock market returns. For instance, individuals with very positive expectations on DAX returns could be more likely to respond.

While MAR is difficult to justify here, an exclusion restriction for the instrument can be motivated as follows. Based on the exogenous treatment, an individual updates his latent expectations on future DAX returns. As we control for these expectations it appears reasonable to assume that the treatment has no direct impact on the likelihood of response, i.e.,  $\mathbb{P}(\Delta = 1|Expect^*, Treat) = \mathbb{P}(\Delta = 1|Expect^*)$ . Consequently, we could use such an exogenous treatment

as instrument (given that it is provided prior to an individual's participation decision) to assess the plausibility of MAR.  $\square$

**EXAMPLE 2.5.** In firm level surveys, questions concerning profits are typically associated with high rates of nonresponse. In particular, whether a firm reports its profits might be directly driven by its productivity, e.g., a firm might be less willing to report after weak performance over the fiscal year. On the other hand, the introduction of new technologies such as IT-Outsourcing clearly affects the firm's profits but may not directly influence its response behavior. Breunig et al. [2016] used this variation to identify the firm's response behavior.  $\square$

## 2.2. The Test Statistic

In the previous section, we observed that each null hypothesis is equivalent to a conditional moment restriction

$$\mathbf{E} \left[ r(\Delta, X) | X, W \right] = 0$$

for some bounded function  $r$ . Equivalently, by considering the squared integrated distance we obtain

$$\int \mathbf{E} \left[ r(\Delta, X) | X = x, W = w \right]^2 \pi(x, w) d(x, w) = 0$$

for some weight function  $\pi$  which is strictly positive almost surely (a.s.) on  $\mathcal{X} \times \mathcal{W}$  ( $\mathcal{X}$  and  $\mathcal{W}$  denote the supports of  $X$  and  $W$ , respectively). Let  $p_{XW}$  denote the joint probability density function of  $(X, W)$ . Further, let  $\nu$  be an a.s. strictly positive density function on  $\mathcal{X} \times \mathcal{W}$ . Let us introduce approximating functions  $\{f_j\}_{j \geq 1}$  which are assumed to form an orthonormal basis in the Hilbert space  $L^2_\nu := \left\{ \phi : \int |\phi(x, w)|^2 \nu(x, w) d(x, w) < \infty \right\}$ . Now choosing  $\pi(x, w) = p_{XW}^2(x, w) / \nu(x, w)$  together with Parseval's identity yields

$$\begin{aligned} 0 &= \int \left| \mathbf{E} \left[ r(\Delta, X) | X = x, W = w \right] p_{XW}(x, w) / \nu(x, w) \right|^2 \nu(x, w) d(x, w) \\ &= \sum_{j=1}^{\infty} \left( \mathbf{E} \left[ r(\Delta, X) f_j(X, W) \right] \right)^2. \end{aligned}$$

Given a strictly positive sequence of weights  $(\tau_j)_{j \geq 1}$  the last equation is equivalent to

$$\sum_{j=1}^{\infty} \tau_j \left( \mathbf{E} \left[ r(\Delta, X) f_j(X, W) \right] \right)^2 = 0. \quad (2.3)$$

Our test statistics below are based on an empirical version of the left hand side of (2.3). To do so, we truncate the infinite sum at some finite integer. Below we choose  $(\tau_j)_{j \geq 1}$  to be a strictly decreasing which implies that we reduce weight to those generalized Fourier coefficients as basis functions are becoming more nonlinear. Additional weighting of the testing procedure was also used by Horowitz [2006], Blundell and Horowitz [2007], and Breunig [2015].

Our test statistic is based on an empirical analog of the left hand side of (2.3) given  $(\Delta_1, Y_1, X_1, W_1), \dots, (\Delta_n, Y_n, X_n, W_n)$  of independent and identical distributed (iid.) copies of  $(\Delta, Y, X, W)$  where  $Y = \Delta Y^*$ . Let us introduce the notation for the conditional probability function  $h(\cdot) = \mathbb{P}(\Delta = 1 | X = \cdot)$ . We estimate  $h$  by the series least square estimators

$$\widehat{h}_n(\cdot) = e_{k_n}(\cdot)^t (\mathbf{X}_n^t \mathbf{X}_n)^{-1} \mathbf{X}_n^t \Delta_n$$

where  $\Delta_n = (\Delta_1, \dots, \Delta_n)$ ,  $e_{k_n}(\cdot) := (e_1(\cdot), \dots, e_{k_n}(\cdot))^t$  is a vector of basis functions, and  $\mathbf{X}_n := (e_{k_n}(X_1), \dots, e_{k_n}(X_n))^t$ . In the multivariate case, we consider a tensor-product linear sieve basis, which is the product of univariate linear sieves. The dimension parameter  $k_n$  increases with sample size  $n$ .

Consider the null hypothesis MAR. From the proof of Theorem 2.1, we deduce  $r(\Delta, X) = \Delta - h(X)$ . Replacing  $h$  by the proposed estimator  $\widehat{h}_n$  we obtain the test statistic

$$S_n^{MAR} = \sum_{j=1}^{m_n} \tau_j \left| n^{-1} \sum_{i=1}^n (\Delta_i - \widehat{h}_n(X_i)) f_j(X_i, W_i) \right|^2 \quad (2.4)$$

where  $m_n$  increases with sample size  $n$  and  $(\tau_j)_{j \geq 1}$  is a strictly positive sequence of weights which is nonincreasing. We reject null hypothesis MAR if the test statistic  $S_n^{MAR}$  becomes too large.

For the null hypothesis MCAR, Corollary 2.2 gives  $r(\Delta, X) = \Delta - \mathbb{P}(\Delta = 1)$ . Again, following the derivation of the statistic  $S_n^{MAR}$  we obtain a statistic for MCAR given by

$$S_n^{MCAR} = \sum_{j=1}^{m_n} \tau_j \left| n^{-1} \sum_{i=1}^n (\Delta_i - \widehat{\Delta}_n) f_j(X_i, W_i) \right|^2 \quad (2.5)$$

where  $\widehat{\Delta}_n = n^{-1} \sum_{i=1}^n \Delta_i$ . We reject MCAR if the test statistic  $S_n^{MCAR}$  becomes too large.

### 2.3. Assumptions for Inference

In the following,  $\mathcal{Y}$ ,  $\mathcal{X}$ , and  $\mathcal{W}$  denote the supports of  $Y$ ,  $X$ , and  $W$ , respectively. The usual Euclidean norm is denoted by  $\|\cdot\|$  and  $\|\cdot\|_\infty$  is the supremum norm.

ASSUMPTION 4. (i) We observe a sample  $(\Delta_1, Y_1, X_1, W_1), \dots, (\Delta_n, Y_n, X_n, W_n)$  of independent and identical distributed copies of  $(\Delta, Y, X, W)$  where  $Y = \Delta Y^*$ . (ii) The functions  $\{f_j\}_{j \geq 1}$  form an orthonormal basis in  $L^2_\nu$ . (iii) There exists some constant  $C > 0$  such that  $\sup_{(x,w) \in \mathcal{X} \times \mathcal{W}} \{p_{XW}(x, w)/\nu(x, w)\} \leq C$ .

In our simulations, we used trigonometric basis functions or orthonormalized Hermite polynomials where Assumption 4 (ii) is automatically satisfied if, respectively,  $\nu$  is Lebesgue measure on  $[0, 1]$  or  $\nu$  is the standard normal density. Assumption 4 (iii) is a mild restriction on the density of  $(X, W)$  relative to  $\nu$ . Assumption 4 implies  $\mathbf{E}|f_j(X, W)|^2 \leq C$ . The next assumption involves the linear sieve space  $\mathcal{H}_n := \{\phi : \phi(\cdot) = \beta_n^t e_{k_n}(\cdot) \text{ where } \beta_n \in \mathbb{R}^{k_n}\}$  where the dimension of the sieve space  $k_n$  increases with sample size  $n$ .

ASSUMPTION 5. (i) There exists  $E_{k_n} h \in \mathcal{H}_n$  such that  $\|E_{k_n} h - h\|_\infty^2 = O(1/\gamma_{k_n})$  for some increasing sequence  $(\gamma_l)_{l \geq 1}$ . (ii) It holds  $\sup_{x \in \mathcal{X}} \|e_{k_n}(x)\|^2 = O(k_n)$  such that  $k_n^2 \log(n) = o(n)$ . (iii) The smallest eigenvalue of  $\mathbf{E}[e_{k_n}(X)e_{k_n}(X)^t]$  is bounded away from zero uniformly in  $n$ .

Assumption 5 (i) determines the sieve approximation error for estimating the function  $h$  in the supremum norm and is used to control the bias of the estimator of  $h$ . The sieve approximation error is directly related to the smoothness of the function  $h$ , see also Example 2.6 below for primitive conditions and detailed discussion of the rate requirements. For the relation to  $L^2$  approximation conditions see Belloni et al. [2015] or Chen and Christensen [2015]. An excellent review of approximating properties of different sieve bases is given in Chen [2007]. Assumption 5 (ii) and (iii) restrict the magnitude of the approximating functions  $\{e_j\}_{j \geq 1}$  and impose nonsingularity of their second moment matrix (cf. Newey [1997]). Assumption 5 (ii) is automatically satisfied by trigonometric basis functions, orthonormalized Hermite polynomials, B-splines, or wavelets. Assumption 5 (iii) holds true when  $p_X$  is bounded away from zero and  $e_1, \dots, e_{k_n}$  are orthonormal basis functions. Consistent estimation of critical values requires the following additional assumption, where we use the notation  $\psi_j(\cdot) = \mathbf{E}[f_j(X, W)|X = \cdot]$ .

ASSUMPTION 6. There exists  $E_{k_n} \psi_j \in \mathcal{H}_n$  such that  $\max_{1 \leq j \leq m_n} \|E_{k_n} \psi_j - \psi_j\|_\infty^2 = O(1/\gamma_{k_n})$ .

Assumption 6 ensures that the basis functions  $\{f_j\}_{1 \leq j \leq m_n}$  can be as well approximated as  $h$  by using the sieve space  $\mathcal{H}_n$ . Below we choose for  $f_j$  Hermite polynomials or cosine basis functions which, as analytic functions, can be sufficiently well approximated.

## 2.4. Asymptotic Distribution under MAR

Before establishing the asymptotic distribution of the test statistic  $S_n^{MAR}$  under MAR, we require the following definition. For any realization  $(\delta, x, w)$  of  $(\Delta, X, W)$ , let  $\varepsilon(\delta, x, w)$  be an infinite dimensional vector with  $j$ -th entry given

by

$$\varepsilon_j(\delta, x, w) = \sqrt{\tau_j}(\delta - h(x))(f_j(x, w) - \sum_{l=1}^{\infty} \mathbf{E}[f_j(X, W)e_l(X)]e_l(x)).$$

It holds  $\mathbf{E}[\varepsilon(\Delta, X, W)] = 0$  under MAR. We assume  $\mathbf{E}|\varepsilon_j(\Delta, X, W)|^2 < \infty$ , which is automatically satisfied if  $\{e_l\}_{l \geq 1}$  forms an orthonormal basis. Thereby, under MAR the covariance matrix given by  $\Sigma = \mathbf{E}[\varepsilon(\Delta, X, W)\varepsilon(\Delta, X, W)^t]$  of  $\varepsilon(\Delta, X, W)$  is well defined. The ordered eigenvalues of  $\Sigma$  are denoted by  $(\lambda_j)_{j \geq 1}$ . Furthermore, we introduce a sequence  $\{\chi_{1j}^2\}_{j \geq 1}$  of independent random variables that are distributed as chi-square with one degree of freedom. The proof of the next theorem can be found in the appendix.

**THEOREM 2.3.** *Let Assumptions 1, 2, 4, and 5 hold true. If*

$$\sum_{j=1}^{m_n} \tau_j = O(1), \quad n = o(\gamma_{k_n}), \quad \text{and} \quad m_n^{-1} = o(1) \quad (2.6)$$

then under MAR

$$n S_n^{MAR} \xrightarrow{d} \sum_{j=1}^{\infty} \lambda_j \chi_{1j}^2.$$

The rate  $n = o(\gamma_{k_n})$  ensures that bias for estimating the function  $h$  vanishes sufficiently fast. Below, we show that under classical smoothness assumptions this rate requires an undersmoothed estimator for  $h$ . We also like to emphasize that for the asymptotic result in Theorem 2.3, the bounded completeness condition stated in Assumption 3 is not required. Below we write  $a_n \sim b_n$  when there exist constants  $c, c' > 0$  such that  $cb_n \leq a_n \leq c'b_n$  for all sufficiently large  $n$ .

**EXAMPLE 2.6.** Let  $X$  be continuously distributed. Let  $e_1, \dots, e_{k_n}$  be spline basis functions and  $p$  be the number of continuous derivatives of  $h$ . Then Assumption 5 (i) holds true with  $\gamma_j \sim j^{2p/d_x}$  (see Newey [1997]). Condition  $n = o(\gamma_{k_n})$  and Assumption 5 (ii) is satisfied if  $k_n \sim n^\kappa$  with  $d_x/(2p) < \kappa < 1/(2 + \varepsilon)$  for any small  $\varepsilon > 0$ . Here, the required smoothness of  $h$  is  $p > (2 + \varepsilon)(1 + d_x)/2$ . Hence, the estimator of  $h$  needs to be undersmoothed.  $\square$

**REMARK 2.1 (Estimation of Critical Values).** The asymptotic distribution of our test statistic derived in Theorem 2.3 depends on unknown population quantities. As we see in the following, the critical values can be easily estimated. Let us define

$$\widehat{\varepsilon}_j(\delta, x, w) = \sqrt{\tau_j}(\delta - \widehat{h}_n(x)) \left\{ f_j(x, w) - \sum_{j'=1}^{k_n} \left( \frac{1}{n} \sum_{i=1}^n f_{j'}(X_i, W_i) e_{j'}(X_i) \right) e_{j'}(x) \right\}.$$

We replace  $\Sigma$  by the  $m_n \times m_n$  dimensional matrix

$$\widehat{\Sigma}_n = n^{-1} \sum_{i=1}^n \left( \widehat{\varepsilon}_1(\Delta_i, X_i, W_i), \dots, \widehat{\varepsilon}_{m_n}(\Delta_i, X_i, W_i) \right)^t \left( \widehat{\varepsilon}_1(\Delta_i, X_i, W_i), \dots, \widehat{\varepsilon}_{m_n}(\Delta_i, X_i, W_i) \right).$$

Let  $(\widehat{\lambda}_{jn})_{1 \leq j \leq m_n}$  denote the ordered eigenvalues of  $\widehat{\Sigma}_n$ . To obtain the critical values, we replace  $\sum_{j=1}^{\infty} \lambda_j \chi_{1j}^2$  by the finite sum  $\sum_{j=1}^{m_n} \widehat{\lambda}_{jn} \chi_{1j}^2$ .  $\square$

The following result establishes consistency of the empirical critical values as introduced in the previous remark. In contrast to Theorem 2.3, we have to impose an upper bound on the size of the dimension parameter  $m_n$ .

**PROPOSITION 2.4.** *Let the conditions of Theorem 2.3 and Assumption 6 be satisfied. Assume  $m_n = o(\sqrt{n})$ . Then for all  $z$  it holds*

$$\mathbb{P}\left(\sum_{j=1}^{m_n} \widehat{\lambda}_{jn} \chi_{1j}^2 \leq z\right) - \mathbb{P}\left(\sum_{j=1}^{\infty} \lambda_j \chi_{1j}^2 \leq z\right) = o(1).$$

## 2.5. Asymptotic Distribution under MCAR

We now derive the asymptotic distribution of the statistic for testing  $S_n^{MCAR}$  under the null hypothesis MCAR. For any realization  $(\delta, x, w)$  of  $(\Delta, X, W)$ , let us introduce an infinite dimensional vector  $\nu(\delta, x, w)$  with  $j$ -th entry

$$\nu_j(\delta, x, w) = \sqrt{\tau_j} \left( \delta - \mathbb{P}(\Delta = 1) \right) \left( f_j(x, w) - \mathbb{E}[f_j(X, W)] \right).$$

We have  $\mathbb{E}[\nu_j(\Delta, X, W)] = 0$  under MCAR. Let  $\Sigma_\nu$  be the covariance matrix of  $\nu(\Delta, X, W)$ ; that is,  $\Sigma_\nu = \mathbb{E}[\nu(\Delta, X, W)\nu(\Delta, X, W)^t]$ .

In this subsection, the ordered eigenvalues of  $\Sigma_\nu$  are denoted by  $(\lambda_l)_{l \geq 1}$ . The next result is a direct consequence of Theorem 2.3 and hence, we omit its proof.

**COROLLARY 2.5.** *Let Assumptions 1, 2, and 4 hold true. If*

$$\sum_{j=1}^{m_n} \tau_j = O(1) \quad \text{and} \quad m_n^{-1} = o(1)$$

*then under MCAR*

$$n S_n^{MCAR} \xrightarrow{d} \sum_{j=1}^{\infty} \lambda_j \chi_{1j}^2.$$

**REMARK 2.2 (Estimation of Critical Values).** Estimation of critical values in

case of Corollary 2.5 follows easily from Remark 2.1. Let us define

$$\widehat{v}_j(\delta, x, w) = \sqrt{\tau_j}(\delta - \widehat{\Delta}_n) \left\{ f_j(x, w) - n^{-1} \sum_{i=1}^n f_j(X_i, W_i) \right\}$$

with  $\widehat{\Delta}_n = n^{-1} \sum_{i=1}^n \Delta_i$ . We replace  $\Sigma_v$  by the  $m_n \times m_n$  dimensional matrix

$$\widehat{\Sigma}_n = n^{-1} \sum_{i=1}^n \left( \widehat{v}_1(\Delta_i, X_i, W_i), \dots, \widehat{v}_{m_n}(\Delta_i, X_i, W_i) \right)^t \left( \widehat{v}_1(\Delta_i, X_i, W_i), \dots, \widehat{v}_{m_n}(\Delta_i, X_i, W_i) \right).$$

Let  $(\widehat{\lambda}_{jn})_{1 \leq j \leq m_n}$  denote the ordered eigenvalues of  $\widehat{\Sigma}_n$ . To obtain empirical critical values, we replace  $\sum_{j=1}^{\infty} \lambda_j \chi_{1j}^2$  by the finite sum  $\sum_{j=1}^{m_n} \widehat{\lambda}_{jn} \chi_{1j}^2$ . Consistency follows as in Proposition 2.4.  $\square$

## 2.6. Consistency against Fixed Alternatives

Under each null hypothesis, the asymptotic distribution results remain valid if  $(Y^*, X)$  is not bounded complete for  $(X, W)$ ; that is, Assumption 3 does not hold true. On the other hand, we show that, under bounded completeness, consistency of our tests against fixed alternatives can be obtained. To establish this property we require the following additional assumption.

**ASSUMPTION 7.** *The function  $p_{XW}/v$  is uniformly bounded away from zero.*

If MAR fails, Assumption 7 together with Assumption 3 ensures that the generalized Fourier coefficients  $\mathbf{E}[r(\Delta, X)f_j(X, W)]$  are non-zero for some integer  $j \geq 1$ . Instead of Assumption 7, we may also assume that  $(KTr)(X, W) \neq 0$  and that  $p_{XW}/v$  is uniformly bounded away from zero on the support of  $(KTr)(X, W)$ , where  $T$  denotes the conditional expectation operator defined by  $T\phi = \mathbf{E}[\phi(\Delta, X)|Z]$  and  $K$  is a smoothing operator with eigenvalue decomposition  $\{\sqrt{\tau_j}, f_j\}_{j \geq 1}$ . The following proposition shows that our test has the ability to reject a false null hypothesis with probability 1 as the sample size grows to infinity. For the next results, let us introduce a sequence  $(a_n)_{n \geq 1}$  satisfying  $a_n = o(n)$ . The proof of the next proposition can be found in the appendix.

**PROPOSITION 2.6.** *Assume that MAR does not hold. Let Assumptions 1–7 be satisfied. Then*

$$\mathbb{P}\left(n S_n^{MAR} > a_n\right) = 1 + o(1).$$

The rate  $(a_n)_{n \geq 1}$  is arbitrarily close to the parametric rate  $n^{-1}$  which is due the weighting sequence  $(\tau_j)_{j \geq 1}$  with  $\sum_{j=1}^{m_n} \tau_j = O(1)$ . The next result is a direct consequence of Proposition 2.6 and hence, its proof is omitted.

**COROLLARY 2.7.** *Assume that MCAR does not hold. Let Assumptions 1–4 and 7 be*

satisfied. Then

$$\mathbb{P}\left(n S_n^{\text{MCAR}} > a_n\right) = 1 + o(1).$$

### 3. Monte Carlo Simulations

In this section, we study the finite-sample performance of our test by presenting the results of a Monte Carlo simulation. The experiments use a sample size of 500 and there are 1000 Monte Carlo replications in each experiment. Results are presented for the nominal level 0.05.

As basis functions  $\{f_j\}_{j \geq 1}$  used to construct our test statistic, we use throughout the experiments orthonormalized Hermite polynomials. Hermite polynomials form an orthonormal basis of  $L_\omega^2$  with a weighting function being the density of the standard normal distribution; that is,  $\omega(x) = \exp(-x^2) / \sqrt{2\pi}$ . They can be obtained by applying the Gram–Schmidt procedure to the polynomial series  $1, x, x^2, \dots$  under the inner product  $\langle \phi, \psi \rangle_\omega = (2\pi)^{-1/2} \int \phi(x)\psi(x) \exp(-x^2) dx$ . That is,  $H_1(x) = 1$  and for all  $j = 2, 3, \dots$

$$H_j(x) = \frac{x^{j-1} - \sum_{k=1}^{j-1} \langle id^{j-1}, p_j \rangle_\omega p_k(x)}{\int \left(x^{j-1} - \sum_{k=1}^{j-1} \langle id^{j-1}, p_j \rangle_\omega p_k(x)\right) \omega(x) dx}. \quad (3.1)$$

Our testing procedure is now build up on the basis functions

$$f_j(\cdot) = \frac{H_{j+1}(\cdot)}{\sqrt{\langle H_j, H_j \rangle_\omega}}$$

for all  $j = 1, 2, \dots$ . If the support of the instrument  $W$  or its transformation lies in the interval  $[0, 1]$  then one could also use, for instance, cosine basis functions

$$f_j(x) = \sqrt{2} \cos(\pi j x)$$

for  $j = 1, 2, \dots$ . We also implemented our test statistic with these cosine functions in the settings studied below. But as the results are very similar to the ones with Hermite polynomials presented below we do not report them here. Throughout our simulation study, the number of orthonormalized Hermite polynomials is 10. In the multivariate case, we consider a tensor product of those weighted Hermite polynomials. Due to the weighting sequence  $(\tau_j)_{j \geq 1}$ , results not too sensitive to the number of Hermite polynomials. In contrast, results might be more sensitive to the choice of basis functions  $k_n$  used to estimate  $h$ . Below we use cross validation to choose the appropriate number of basis functions for this function.



**Testing MCAR** Realizations of  $(Y^*, W)$  were generated by  $W \sim \mathcal{N}(0, 1)$  and  $Y^* \sim \rho W + \sqrt{1 - \rho^2} \varepsilon$  where  $\varepsilon \sim \mathcal{N}(0, 1)$ . The constants  $\rho$  characterizes the "strength" of the instrument  $W$  and is varied in the experiments. For a random variable  $V$ , introduce the function  $\phi_2(V) = \mathbb{1}\{V \geq q\} + 0.1 * \mathbb{1}\{V \leq q\}$  where  $q$  is the 0.2 quantile of the empirical distribution of  $V$ . In each experiment,

Model		Empirical Rejection probability of $nS_n$ with			Little's test
$\rho$	$\nu$	$\tau_j = j^{-2}$	$\tau_j = j^{-3}$	$\tau_j = j^{-4}$	
0.2	0.0	0.055	0.057	0.056	0.062
	0.3	0.148	0.159	0.162	0.155
	0.5	0.290	0.297	0.304	0.338
	0.7	0.505	0.529	0.530	0.568
0.3	0.0	0.055	0.057	0.056	0.062
	0.3	0.253	0.265	0.268	0.273
	0.5	0.559	0.588	0.592	0.611
	0.7	0.839	0.855	0.857	0.853
0.4	0.0	0.055	0.057	0.056	0.062
	0.3	0.387	0.393	0.398	0.430
	0.5	0.813	0.831	0.840	0.843
	0.7	0.985	0.988	0.986	0.988

Table 1: Empirical Rejection probabilities for Testing MCAR

realizations of the response variable  $\Delta$  were generated by

$$\Delta \sim \text{Bin}(1, \phi_2(\nu Y^* + \sqrt{1 - \nu^2} \xi))$$

for some constant  $0 \leq \nu \leq 1$  and where  $\xi \sim \mathcal{N}(0, 1)$ . If  $\nu = 0$  then response  $\Delta$  does not depend on  $Y^*$  and hence the null hypothesis MCAR holds true.

The critical values are estimated as in Remark 2.2. For  $m = 100$  we observed that the estimated eigenvalues  $\hat{\lambda}_j$  are sufficiently close to zero for all  $j \geq m$ . To provide a basis for judging whether the power of our test is high or low, we also provide the empirical rejection probabilities when using a test of MCAR for normal data proposed by Little [1988].

The empirical rejection probabilities of test statistic  $S_n^{MCAR}$  using different weightings and Little's test are depicted in Table 1. First, we observe, not surprisingly, that the power of all tests increase as the correlation between  $Y^*$  and  $W$  (measured by  $\rho$ ) becomes larger. Second, power also increases with constant  $\nu$ . From Table 1 we also see that our tests with different weighting sequences have similar power properties and our tests behave similar as Little's test, which, as we want to emphasize, relies on the knowledge of the underlying distribution up to a finite dimensional parameter.

**Testing MAR** Realizations of  $(Y^*, X, W)$  were generated by  $W \sim \mathcal{N}(0, 1)$ ,  $X \sim 0.2W + \sqrt{1 - 0.2^2} \xi$  and  $Y^* \sim \rho W + \sqrt{1 - \rho^2} \xi + \varepsilon$  where  $\xi \sim \mathcal{N}(0, 1)$  and  $\varepsilon \sim \mathcal{N}(0, 0.25)$ . The constant  $\rho$  is varied in the experiments. The critical values are estimated as described in Remark 2.1.

In each experiment, realizations of response  $\Delta$  were generated by

$$\Delta \sim \text{Binomial}\left(1, \phi_{.2}\left(\nu Y^* + \sqrt{1 - \nu^2} X\right)\right)$$

for some constant  $0 \leq \nu \leq 1$ . Clearly, if  $\nu = 0$  then the null hypothesis MAR holds true. We estimate the function  $h$  using B-splines. The number of

Model		Empirical Rejection probability of $nS_n$ with			Delgado and
$\rho$	$\nu$	$\tau_j = j^{-2}$	$\tau_j = j^{-3}$	$\tau_j = j^{-4}$	Manteiga's Test $C_n^{**}$
0.3	0.0	0.045	0.049	0.050	0.049
	0.3	0.091	0.090	0.093	0.201
	0.5	0.105	0.109	0.113	0.301
	0.7	0.147	0.151	0.153	0.399
0.5	0.0	0.045	0.049	0.050	0.051
	0.3	0.354	0.359	0.358	0.230
	0.5	0.595	0.584	0.582	0.386
	0.7	0.759	0.750	0.747	0.492
0.7	0.0	0.045	0.049	0.050	0.050
	0.3	0.733	0.717	0.709	0.323
	0.5	0.943	0.938	0.936	0.508
	0.7	0.986	0.980	0.979	0.626

Table 2: Empirical Rejection probabilities for Testing MAR

knots and orders is chosen via cross validation. Computational procedures were implemented using the statistical software R using the crs Package Hayfield and Racine [2007]. In our experiments, cross validation tended to undersmooth the estimator of  $h$  which implied a sufficiently small bias of this estimator. On the other hand, to obtain appropriate undersmoothing one could also use a data driven choice of basis functions suggested by Picard and Tribouley [2000]. Critical values are estimated as described in Remark 2.1.

We also compare our testing procedure to the bootstrap significance test proposed by Delgado and González Manteiga [2001]. Their test statistic is based on the empirical version of  $\mathbf{E}[T^2(X, W)]$  with  $T(x, w) = \mathbf{E}[p_X(X)(\Delta - h(X)) \mathbb{1}_{\{X \leq x\}} \mathbb{1}_{\{W \leq w\}}]$ , where  $p_X$  denotes the marginal probability density function of  $X$ . Delgado and González Manteiga [2001]'s statistic builds on a kernel estimator of  $h$ . We follow their implementation of their Cramér-von Mises type statistic. Only we use the Gaussian kernel and the bandwidth is chosen with smaller constant, i.e.,  $h = 0.004 * n^{-1/3}$ . This choice of a smaller constant

for the bandwidth ensures that the test has accurate finite sample coverage. The bootstrap version of the test is based on bootstrap innovations as suggested by Mammen [1993], i.e.,  $(1 - \sqrt{5})/2$  with probability  $(1 + \sqrt{5})/(2\sqrt{5})$  and  $(1 + \sqrt{5})/2$  with probability  $1 - (1 + \sqrt{5})/(2\sqrt{5})$ . The implemented test statistic corresponds to  $C_n^{**}$  of Delgado and González Manteiga [2001]. The authors also propose other bootstrap testing procedures but, as they point out, the statistic  $C_n^{**}$  has a slightly better finite sample performance than their other statistics.

Table 2 depicts the empirical rejection probabilities of the tests  $S_n^{MAR}$  when using different weightings and  $C_n^{**}$ . From Table 2, we see that both tests,  $S_n^{MAR}$  and  $C_n^{**}$  have accurate finite sample coverage for varying values of  $\rho$  and  $\nu$ . For  $\rho = 0.3$  the test statistic  $C_n^{**}$  has larger empirical rejection probabilities in the alternative model, while for all other values of  $\rho$  the test  $S_n^{MAR}$  appears to have larger finite sample power. In contrast to our proposed statistic  $S_n^{MAR}$ , the testing procedure of Delgado and González Manteiga [2001] is surprisingly insensitive with respect to the choice of the strength of the instruments captured by  $\rho$ . We also implemented the test of Delgado and González Manteiga [2001] with  $\rho = 0$  and the power properties are similar to the case where  $\rho = 0.3$ . This indicates that, in our setting, the finite sample power of their test rather derives from specific sample properties than the strength of the instrument.

## 4. Empirical Illustration

We now apply our testing procedure to analyze response mechanisms in a data set from the German Socio-Economic Panel (SOEP).

### 4.1. Missing Income Data in the SOEP

As mentioned above, a common phenomenon in population surveys is that wealth and income questions are typically associated with high rates of item nonresponse which, in addition, has been found to be selective (see Watson and Wooden [2009]). Regarding labor income questions in the SOEP, this response behavior was also emphasized by Schräpler [2004] and Frick et al. [2007].

We apply our methodology to a dataset which is a subsample of the SOEP longitudinal survey. The current subsample includes 454 full-time employed male respondents aged between 26–63 in the year 2013 who have not switched jobs in the past year.<sup>2</sup> Further, all individuals are German and have obtained Abitur (comparable to a high school graduation certificate). It has been assured that all subjects in the subsample participated in the SOEP survey

---

<sup>2</sup>In Germany, a typical age for a graduate student to take up a full-time job is 26 and the average retirement age is close to 63.

and were successfully interviewed both in 2013 and in one years of 2000–2012. Missingness in the gross labor income variable is thus due to item nonresponse in the survey questionnaire and our sample contains no individuals that were added to the SOEP (as a replacement for individuals missing due to sample attrition for e.g.) only in 2013. Moreover, we ensure that all individuals in our sample participate in the labor market in 2013. In our subsample, 8.6% of participants do not respond to the question on current gross labor income. In the following, we assess the sensitivity of nonresponse regarding the latent labor income level.

Despite the missing labor income information in 2013, nearly all individuals have reported their gross labor income in one of the previous waves. This possibly indicates that individuals become more sensitive in reporting their labor income during their working career. As instrumental variable we use the latest gross labor income information of the individuals available in the years 2000–2012.<sup>3</sup> In this case, the exclusion restriction translates to the assumption that past income affects the current response behavior only through current income. This pins down to the behavioral assumption that the disutility of reporting current income is driven by the latent current income but not by the observed income of the previous years. In other applications, the exclusion restriction, when using past information as an instrument, might be less evident and has to be justified carefully (for a further discussion we also refer to Hirano et al. [2001] in the case of refreshment samples). Nevertheless, we provide an empirical assessment of its plausibility at the end of this section. In the following, the instrumental variable  $W$  denotes the logarithm of an individual's most recent gross labor income from previous waves. From the rich SOEP data set we pick two covariates that have a significant impact on the response mechanism. We consider an individual's time with a firm and hours worked per week. In the following,  $X_1$  denotes the demeaned logarithm of an individual's length of time with firm and  $X_2$  denotes the demeaned logarithm of individual's actual work time per week. Summary statistics are provided in the following:

	Min.	1st Qu.	Med.	3rd Qu.	Max.	St. Dev.
$W$	-2.47	-0.30	0.02	0.32	2.62	0.54
$X_1$	-2.35	-0.39	0.16	0.55	1.27	0.76
$X_2$	-1.61	-0.12	0.00	0.10	0.57	0.20

<sup>3</sup>There are only 11 individuals who do never report their gross labor income level and for those we use average gross labor income for males of the respective profession, in the respective working sector, available on Eurostat for the year 2010. Similarly, we replace 9 missing values for actual work time by the average work time with respect to job classification according to ISCO-8 for German male workers in the year 2013.

## 4.2. Results and Details of Implementation

**Testing MCAR and MAR** The test statistics are constructed as described in the previous section using Hermite functions. We choose the dimension parameter  $m_n$  such that  $m_n$  is approximately  $\sqrt{n}$ . Again, we emphasize that the results are not sensitive regarding  $m_n$  due to the additional weighting of the coefficients. For testing MAR, we choose the dimension parameter  $k_n$  by cross validation. Finally, we provide results below for varying weights. In practice, we recommend to choose the weighting sequence to maximize the ratio of the value of the test statistic and the associated empirical critical values.

Table 3 depicts the values of our test statistics for testing MCAR and MAR using different control variables. As we see from this table, our test statistic rejects the hypothesis MCAR at the 0.05 nominal level. Our test also rejects MAR if we only control for the length of time with firm but fails to reject MAR if we control for actual working time. As such, the information on length of time with firm alone is not sufficient for explaining the individual response behavior. This indicates that one has to be careful about the choice of covariates for MAR and how the proposed test can be useful in practice.

		$\tau_j = j^{-2}$	$\tau_j = j^{-3}$	$\tau_j = j^{-4}$
MCAR: $\mathbb{P}(\Delta = 1 Y^*) = \mathbb{P}(\Delta = 1)$	Value of $nS_n^{MCAR}$	0.049	0.039	<b>0.036</b>
	Critical Values	0.041	0.033	<b>0.030</b>
MCAR: $\mathbb{P}(\Delta = 1 Y^*, X_1) = \mathbb{P}(\Delta = 1)$	Value of $nS_n^{MCAR}$	0.072	0.061	<b>0.058</b>
	Critical Values	0.055	0.047	<b>0.043</b>
MAR: $\mathbb{P}(\Delta = 1 Y^*, X_1) = \mathbb{P}(\Delta = 1 X_1)$	Value of $nS_n^{MAR}$	0.063	0.053	<b>0.050</b>
	Critical Values	0.054	0.045	<b>0.042</b>
MAR: $\mathbb{P}(\Delta = 1 Y^*, X_2) = \mathbb{P}(\Delta = 1 X_2)$	Value of $nS_n^{MAR}$	<b>0.013</b>	0.011	0.010
	Critical Values	<b>0.019</b>	0.016	0.015

Table 3: Values of  $nS_n^{MCAR}$  and  $nS_n^{MAR}$  for different covariates together with their empirical critical values at 0.05 nominal level.

The values of the test statistic and the critical values vary slightly with the degree of weighting, and so does the fraction of them. For instance, in the second and third rows of Table 3, the fraction of  $nS_n^{MCAR}$  and its critical value is 1.182 if  $\tau_j = j^{-2}$ , 1.191 if  $\tau_j = j^{-3}$ , and 1.199 if  $\tau_j = j^{-4}$ . In Table 3, we depict those values of tests and their empirical values in bold, for which their fraction is maximized.

**Further analysis of the missingness mechanism** In Figure 1, we estimate the conditional probability  $\mathbb{P}(\Delta = 1|Y^*)$ . This conditional probability is identified through the conditional mean equation  $\mathbb{E}[\Delta/\mathbb{P}(\Delta = 1|Y^*)|W] = 1$  (cf. D’Haultfoeuille [2010]). We use a sieve minimum distance estimator based on B-splines as in Breunig et al. [2015] to estimate  $\mathbb{P}(\Delta = 1|Y^*)$ . From Figure 1 we see that the estimator of  $\mathbb{P}(\Delta = 1|Y^*)$  is not constant as required by MCAR. In particular, Figure 1 indicates that individuals with low gross labor income are less likely to report it. This pattern is also shown when we estimate the average of observed income. Identification of  $\mathbb{P}(\Delta = 1|Y^*)$  yields identification of the population mean of the potential income  $Y^*$  since

$$\mathbb{E}[Y^*] = \mathbb{E}[Y^*\Delta/\mathbb{P}(\Delta = 1|Y^*)],$$

see also Breunig et al. [2015]. The value of the empirical version of this inverse probability weighted mean is given by 4834.54 Euros. In contrast, the empirical mean of observed income is 5231.42 Euros. Consequently, without correcting for selective nonresponse we overestimate the mean of the true income.

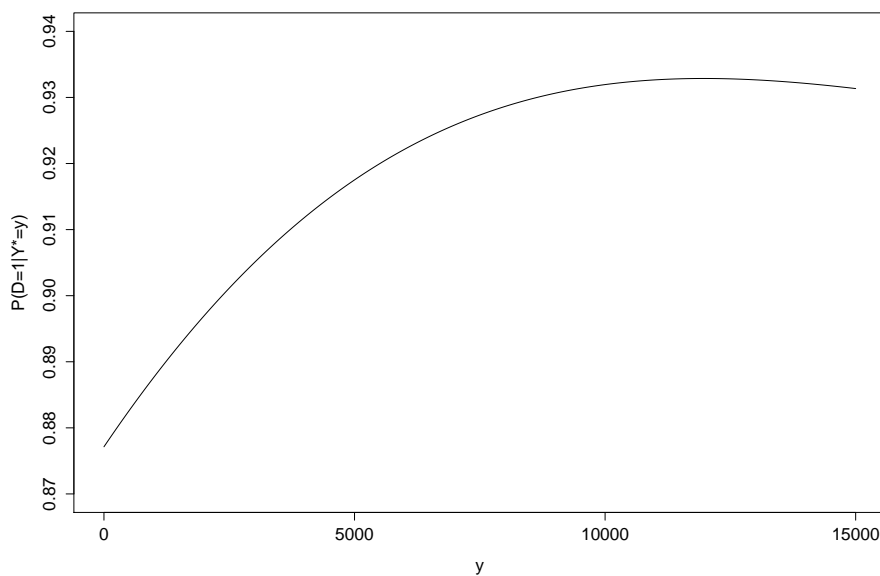


Figure 1: Graph of  $\mathbb{P}(\Delta = 1|Y^* = \cdot)$

**Testing the exclusion restriction for the instrument** Whether or not we observe gross labor income  $Y^*$  is assumed to be independent of past labor income  $W$  conditional on  $Y^*$ . Moreover, we directly perform a test of assumption  $\Delta \perp\!\!\!\perp W|Y^*$  based on Breunig et al. [2015] which builds on Theorem 2.4 of D’Haultfoeuille [2010]. Namely, given that  $\mathbb{P}(\Delta = 1|Y^*)$  is strictly positive

on the support of  $Y^*$  and a slight modification of our completeness assumption as maintained hypotheses, conditional mean independence  $\Delta \perp\!\!\!\perp W|Y^*$  is equivalent to the existence of a function  $\phi(\cdot) \geq 1$  solving  $\mathbf{E}[\Delta\phi(Y^*)|W] = 1$ . Note that the completeness assumption ensures that if such a function exists it holds that  $\phi(Y^*) = 1/\mathbf{P}(\Delta = 1|Y^*)$ . While the completeness assumption is not directly testable, the rejection of the MCAR hypothesis in Table 3 indicates that the instrument  $W$  is sufficiently strong to explain variations in  $Y^*$ . The test is based on checking whether there exists a smooth function  $\phi$  with  $\phi(\cdot) \geq 1$  satisfying  $\mathbf{E}[\Delta\phi(Y^*)|W] = 1$ . The value of the  $L^2$  test statistic proposed by Breunig et al. [2015] is  $-0.626$  (or  $0.783$  when additionally controlling for  $X_1$ ) with p-value  $0.734$  (or  $0.217$  when controlling for  $X_1$ ). At the  $0.05$  nominal level, we thus fail to reject the exclusion assumption on the instrument  $W$ .

## A. Appendix

Throughout the Appendix, let  $C > 0$  denote a generic constant that may be different in different uses. For ease of notation let  $\sum_i = \sum_{i=1}^n$  and  $\sum_{i' < i} = \sum_{i=1}^n \sum_{i'=1}^{i-1}$ . Further, to keep notation simple we define  $Z := (X', W)'$ . In the following,  $\underline{\varepsilon}_{m_n}(\delta, x, w)$  and  $\underline{\varepsilon}_{m_n}^h(\delta, x)$  denote  $m_n$ -dimensional vectors with  $j$ -th entries given by  $\varepsilon_j(\delta, x, w)$  and  $\varepsilon_j^h(\delta, x) := \sqrt{\tau_j}(\delta - h(x)) \sum_{l=1}^{\infty} \mathbf{E}[f_j(Z)e_l(X)]e_l(x)$ , respectively. In the appendix,  $\underline{f}_{m_n}^\tau$  denotes a  $m_n$  dimensional vector with entries  $\sqrt{\tau_j}f_j$  for  $1 \leq j \leq m_n$ .

**PROOF OF THEOREM 2.3.** The proof is based on the decomposition

$$\begin{aligned} n^{-1/2} \sum_i (\Delta_i - \widehat{h}_n(X_i)) \underline{f}_{m_n}^\tau(Z_i) &= n^{-1/2} \sum_i \underline{\varepsilon}_{m_n}(\Delta_i, Z_i) \\ &+ n^{-1/2} \sum_i \left( (h(X_i) - \widehat{h}_n(X_i)) \underline{f}_{m_n}^\tau(Z_i) + \underline{\varepsilon}_{m_n}^h(\Delta_i, X_i) \right) \\ &= I_n + II_n \quad (\text{say}). \end{aligned}$$

Consider  $I_n$ . Consider some fixed integer  $m \geq 1$ . Using Cramer Wold device it is easily seen that

$$n^{-1/2} \sum_i \underline{\varepsilon}_m(\Delta_i, Z_i) \xrightarrow{d} \mathcal{N}(0, \Sigma_m)$$

where  $\Sigma_m$  is the upper  $m \times m$  submatrix of  $\Sigma$ . Hence, we have

$$\sum_{j=1}^m \left| n^{-1/2} \sum_i \varepsilon_j(\Delta_i, Z_i) \right|^2 \xrightarrow{d} \sum_{j=1}^m \lambda_j \chi_{1j}^2$$

with  $\lambda_j$ ,  $1 \leq j \leq m$ , being eigenvalues of  $\Sigma_m$ . On the other hand, observe

$$\sum_{j>m} \mathbf{E} \left| n^{-1/2} \sum_i \varepsilon_j(\Delta_i, Z_i) \right|^2 = \sum_{j>m} \text{Var} \left( n^{-1/2} \sum_i \varepsilon_j(\Delta_i, Z_i) \right) = \sum_{j>m} \mathbf{E} \varepsilon_j^2(\Delta, Z)$$

which becomes sufficiently small for large  $m$  as  $\mathbf{E} \varepsilon_j^2(\Delta, Z)/\tau_j \leq C$  for all  $j \geq 1$ .

Hence, from page 199 in Serfling [1981] we infer that  $\|I_n\|^2 \xrightarrow{d} \sum_{j \geq 1} \lambda_j \chi_{1j}^2$ .

Consider  $II_n$ . We have

$$\begin{aligned} \|II_n\|^2 &\leq 2 \sum_{j=1}^{m_n} \left| \tau_j^{1/2} n^{-1/2} \sum_i \left( \widehat{h}_n(X_i) - (E_{k_n} h)(X_i) \right) f_j(Z_i) - \varepsilon_j^h(\Delta_i, X_i) \right|^2 \\ &\quad + 2 \sum_{j=1}^{m_n} \tau_j \left| n^{-1/2} \sum_i (E_{k_n} h - h)(X_i) f_j(Z_i) \right|^2 \\ &=: A_{n1} + A_{n2}. \end{aligned}$$

Consider  $A_{n1}$ . In the following, we denote  $Q_n := n^{-1} \sum_i e_{k_n}(X_i) e_{k_n}(X_i)^t$ . It holds  $\widehat{h}_n(\cdot) = e_{k_n}(\cdot)^t (n Q_n)^{-1} \sum_i \Delta_i e_{k_n}(X_i)$ . By Assumption 5, the eigenvalues of  $\mathbf{E}[e_{k_n}(X) e_{k_n}(X)^t]$  are bounded away from zero and hence, it may be assumed that  $\mathbf{E}[e_{k_n}(X) e_{k_n}(X)^t] = \mathbf{I}_{k_n}$  where  $\mathbf{I}_{k_n}$  is the  $k_n$  dimensional identity matrix (cf. Newey [1997], p. 161). We observe

$$\begin{aligned} A_{n1} &\leq 2 \sum_{j=1}^{m_n} \left| \tau_j^{1/2} \sum_{l=1}^{k_n} \mathbf{E}[f_j(Z) e_l(X)] Q_n^{-1} n^{-1/2} \sum_i (\Delta_i - E_{k_n} h(X_i)) e_{k_n}(X_i) - \varepsilon_j^h(\Delta_i, X_i) \right|^2 \\ &\quad + 2 \|E_{k_n} h - \widehat{h}_n\|_X^2 \sum_{j=1}^{m_n} \tau_j \sum_{l=1}^{k_n} \left| n^{-1/2} \sum_i e_l(X_i) f_j(Z_i) - \mathbf{E}[e_l(X) f_j(Z)] \right|^2 \\ &=: 2B_{n1} + 2B_{n2} \quad (\text{say}). \quad (\text{A.1}) \end{aligned}$$

For  $B_{n1}$  we evaluate due to the relation  $Q_n^{-1} = \mathbf{I}_{k_n} - Q_n^{-1}(Q_n - \mathbf{I}_{k_n})$  that

$$\begin{aligned} B_{n1} &\leq 2 \left\| \mathbf{E}[f_{m_n}^\tau(Z) e_{k_n}(X)^t] n^{-1/2} \sum_i (\Delta_i - E_{k_n} h(X_i)) e_{k_n}(X_i) - \varepsilon_{m_n}^h(\Delta_i, X_i) \right\|^2 \\ &\quad + 2 \left\| \mathbf{E}[f_{m_n}^\tau(Z) e_{k_n}(X)^t] \right\|^2 \|Q_n - \mathbf{I}_{k_n}\|^2 \|Q_n^{-1}\|^2 \|n^{-1/2} \sum_i (\Delta_i - E_{k_n} h(X_i)) e_{k_n}(X_i)\|^2 \\ &=: 2C_{n1} + 2C_{n2} \quad (\text{say}). \end{aligned}$$

Further, from  $\mathbf{E}[(\Delta - E_{k_n} h(X)) e_{k_n}(X)] = 0$ ,  $\mathbf{E}[(h - E_{k_n} h)(X) e_{k_n}(X)] = 0$ , and



$\mathbf{E}[\varepsilon_{m_n}(\Delta, Z)] = 0$  we deduce

$$\begin{aligned}
C_{n1} &\leq 2 \sum_{j=1}^{m_n} \tau_j \mathbf{E} \left| \sum_{l>k_n} \mathbf{E}[f_j(Z)e_l(X)](h(X) - \Delta)e_l(X) \right|^2 \\
&\quad + 2 \sum_{j=1}^{m_n} \tau_j \mathbf{E} \left| \sum_{l=1}^{k_n} \mathbf{E}[f_j(Z)e_l(X)](E_{k_n}h - h)(X)e_l(X) \right|^2 \\
&\leq 2 \sum_{j=1}^{m_n} \tau_j \mathbf{E} \left| \sum_{l>k_n} \mathbf{E}[f_j(Z)e_l(X)]e_l(X) \right|^2 \\
&\quad + C k_n \|E_{k_n}h - h\|_X^2 \sum_{j=1}^{m_n} \tau_j \sum_{l=1}^{k_n} \mathbf{E}[f_j(Z)e_l(X)]^2 \\
&= o(1)
\end{aligned}$$

using that  $\mathbf{E}[(h(X) - \Delta)^2|X]$  is bounded,  $\sum_{j=1}^{m_n} \tau_j \sum_{l=1}^{k_n} \mathbf{E}[f_j(Z)e_l(X)]^2 = O(1)$ , and by assumption

$$k_n^2 \|E_{k_n}h - h\|_X^2 = O(k_n^2/\gamma_{k_n}) = O(k_n^2/n) = o(1).$$

Consider  $C_{n2}$ . Further, by Rudelson's matrix inequality (see Rudelson [1999] and also Lemma 6.2 of Belloni et al. [2015]) it holds

$$\|Q_n - \mathbf{I}_{k_n}\|^2 = O_p(n^{-1} \log(n) k_n).$$

Moreover, since the difference of eigenvalues of  $Q_n$  and  $\mathbf{I}_{k_n}$  is bounded by  $\|Q_n - \mathbf{I}_{k_n}\|$ , the smallest eigenvalue of  $Q_n$  converges in probability to one and hence,  $\|Q_n^{-1}\|^2 = 1 + o_p(1)$ . Further,

$$\sum_{l=1}^{k_n} \mathbf{E} \left| n^{-1/2} \sum_i (\Delta_i - E_{k_n}h(X_i))e_l(X_i) \right|^2 = \sum_{l=1}^{k_n} \mathbf{E} \left| (\Delta - E_{k_n}h(X))e_l(X) \right|^2 = O(k_n)$$

and hence  $C_{n2} = O_p(n^{-1} \log(n) k_n^2) = o_p(1)$ . Consequently,  $B_{n1} = o_p(1)$ . Consider  $B_{n2}$ . It holds

$$\sum_{l=1}^{k_n} \left| n^{-1} \sum_i e_l(X_i) f_j(Z_i) - \mathbf{E}[e_l(X) f_j(Z)] \right|^2 = O_p(k_n/n).$$

Since  $\|E_{k_n}h - \widehat{h}_n\|_X^2 = O_p(k_n/n)$  (cf. Theorem 1 of Newey [1997]) and  $k_n^2/n = o(1)$  it follows that  $B_{n2} = o(1)$ . Thus, we conclude  $A_{n1} = o_p(1)$ . For  $A_{n2}$  we observe

that

$$\begin{aligned} \mathbf{E} A_{n2} &\leq n \sum_{j=1}^{m_n} \tau_j \mathbf{E} |(E_{k_n} h - h)(X) f_j(Z)|^2 \\ &\leq n \|E_{k_n} h - h\|_\infty^2 \sum_{j=1}^{m_n} \tau_j \mathbf{E} f_j^2(Z) = O(n/\gamma_{k_n}) = o(1). \end{aligned}$$

Consequently, we have  $II_n = o_p(1)$  which completes the proof.  $\square$

For the next proof we introduce the following notations. With  $\gamma_{jn} = \mathbf{E}[f_j(Z)e_{k_n}(X)]$  and  $\widehat{\gamma}_{jn}$  its empirical version, we define  $\psi_{jn}(x, w) = f_j(x, w) - \gamma_{jn}^t e_{k_n}(x)$  and  $\widehat{\psi}_{jn}(x, w) = f_j(x, w) - \widehat{\gamma}_{jn}^t e_{k_n}(x)$ . Moreover, let  $\widetilde{\Sigma}_n$  denote a  $m_n \times m_n$  matrix with  $j, l$ -th entry  $n^{-1} \sum_{i=1}^n \sqrt{\tau_j \tau_l} (\Delta_i - E_{k_n} h_n(X_i))^2 \psi_{jn}(Z_i) \psi_{ln}(Z_i)$ . Let  $\widehat{\beta}_n$  and  $\beta_n$  be such that  $\widehat{h}_n(\cdot) = \widehat{\beta}_n' e_{k_n}(\cdot)$  and  $E_{k_n} h(\cdot) = \beta_n' e_{k_n}(\cdot)$ .

**PROOF OF PROPOSITION 2.4.** We first observe that  $\sum_{j \geq m_n} \lambda_j \chi_{1j}^2 = o_p(1)$  which follows from the following relation of the sum of non-negative eigenvalues of  $\Sigma$ . Namely that the trace the matrix  $\Sigma$  is equal to the sum of its eigenvalues

$$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j &= \sum_{j=1}^{\infty} \tau_j \mathbf{E} \left[ r^2(\Delta, X) \left( f_j(Z) - \sum_{l=1}^{\infty} \mathbf{E}[f_j(Z)e_l(X)] e_l(X) \right)^2 \right] \\ &= O(1). \end{aligned}$$

Consequently, it is sufficient to consider

$$\sum_{j=1}^{m_n} \widehat{\lambda}_{jn} \chi_{1j}^2 = \sum_{j=1}^{\infty} \lambda_j \chi_{1j}^2 + \sum_{j=1}^{m_n} (\widehat{\lambda}_{jn} - \lambda_j) \chi_{1j}^2 + o(1).$$

Let  $\chi_{m_n}^2$  denote the chi-square distribution with  $m_n$  degrees of freedom. We have

$$\sum_{j=1}^{m_n} (\widehat{\lambda}_{jn} - \lambda_j) \chi_{1j}^2 \leq \chi_{m_n}^2 \max_{1 \leq j \leq m_n} |\widehat{\lambda}_{jn} - \lambda_j|.$$

It holds  $\chi_{m_n}^2 = O_p(m_n)$  and it thus remains to show  $\max_{1 \leq j \leq m_n} |\widehat{\lambda}_{jn} - \lambda_j| = O_p(k_n^2/n)$ . Let  $\Sigma_{m_n}$  denote the upper  $m_n \times m_n$  matrix of  $\Sigma$ . We have

$$\max_{1 \leq j \leq m_n} |\widehat{\lambda}_{jn} - \lambda_j| \leq \|\widehat{\Sigma}_n - \Sigma_{m_n}\| \leq \|\widehat{\Sigma}_n - \widetilde{\Sigma}_n\| + \|\widetilde{\Sigma}_n - \Sigma_{m_n}\|,$$

where for the first summand on the right hand side we conclude

$$\begin{aligned}
& \|\widehat{\Sigma}_n - \widetilde{\Sigma}_n\|^2 \\
& \leq \sum_{j,l=1}^{m_n} \tau_j \tau_l \left| \frac{1}{n} \sum_i \left( 2(\Delta_i - E_{k_n} h(X_i))(\widehat{h}_n(X_i) - E_{k_n} h(X_i)) + (\widehat{h}_n(X_i) - E_{k_n} h(X_i))^2 \right) \psi_{jn}(Z_i) \psi_{ln}(Z_i) \right. \\
& \quad \left. + (\Delta_i - \widehat{h}_n(X_i))^2 \left( 2(\widehat{\psi}_{jn} - \psi_{jn})(Z_i) \psi_{ln}(Z_i) + (\widehat{\psi}_{jn} - \psi_{jn})(Z_i) (\widehat{\psi}_{ln} - \psi_{ln})(Z_i) \right) \right|^2 \\
& \leq C \sum_{j,l=1}^{m_n} \tau_j \tau_l \left\{ \left| \mathbf{E}[(\Delta - E_{k_n} h(X)) \psi_{jn}(Z) \psi_{ln}(Z) e_{\underline{k}_n}(X)^t] (\widehat{\beta}_n - \beta_n) \right|^2 \right. \\
& \quad \left. + \left| (\widehat{\beta}_n - \beta_n)^t \mathbf{E}[e_{\underline{k}_n}(X) \psi_{jn}(Z) \psi_{ln}(Z) e_{\underline{k}_n}(X)^t] (\widehat{\beta}_n - \beta_n) \right|^2 \right. \\
& \quad \left. + \left| \mathbf{E}[(\Delta - E_{k_n} h(X))^2 \psi_{ln}(Z) e_{\underline{k}_n}(X)^t] (\widehat{\gamma}_{jn} - \gamma_{jn}) \right|^2 \right. \\
& \quad \left. + \left| (\widehat{\gamma}_{jn} - \gamma_{jn})^t \mathbf{E}[(\Delta - E_{k_n} h(X))^2 e_{\underline{k}_n}(X) e_{\underline{k}_n}(X)^t] (\widehat{\gamma}_{ln} - \gamma_{ln}) \right|^2 \right\} \\
& \quad \quad \quad + O_p(k_n^4/n^2) \\
& \quad \quad \quad = O_p(1/n).
\end{aligned}$$

Finally, it is easily seen that  $\|\widetilde{\Sigma}_n - \Sigma_{m_n}\|^2 = O_p(1/\gamma_{k_n} + 1/n) = O_p(1/n)$ , which proves the result.  $\square$

For the next proof, recall the definition of the smoothing operator  $K$  which is determined by the eigenvalue decomposition  $\{\sqrt{\tau_j}, f_j\}_{j \geq 1}$  and the conditional expectation operator  $T$  defined by  $T\phi = \mathbf{E}[\phi(\Delta, X)|Z]$  for any bounded function  $\phi$ .

**PROOF OF PROPOSITION 2.6.** Since  $p_Z/v$  is uniformly bounded away from zero by some constant  $C > 0$  we obtain

$$\begin{aligned}
S_n &= \sum_{j=1}^{m_n} \tau_j \left| n^{-1} \sum_i r(\Delta_i, X_i) f_j(Z_i) \right|^2 + o_p(1) \\
&= \sum_{j=1}^{m_n} \tau_j \left| \mathbf{E} \left[ r(\Delta, X) f_j(Z) \right] \right|^2 + o_p(1) \\
&= \sum_{j=1}^{\infty} \left| \int_{\mathcal{X} \times \mathcal{W}} \sqrt{\tau_j} \mathbf{E}(r(\Delta, X)|Z = z) \frac{p_Z(z)}{v(z)} f_j(z) v(z) dz \right|^2 + o_p(1) \\
&= \int_{\mathcal{X} \times \mathcal{W}} \left| (KT r)(z) \frac{p_Z(z)}{v(z)} \right|^2 v(z) dz + o_p(1) \\
&\geq C \mathbf{E} \left| (KT r)(Z) \right|^2 + o_p(1).
\end{aligned}$$

Since  $K$  is nonsingular by construction it follows from the proof of Theorem 2.1 that  $\mathbf{E} \left| (KT r)(Z) \right|^2 > 0$ .  $\square$

## References

- A. Belloni, V. Chernozhukov, D. Chetverikov, and K. Kato. Some new asymptotic theory for least squares series: Pointwise and uniform results. *Journal of Econometrics*, 186(2):345–366, 2015.
- R. Blundell and J. Horowitz. A nonparametric test of exogeneity. *Review of Economic Studies*, 74(4):1035–1058, Oct 2007.
- R. Blundell, X. Chen, and D. Kristensen. Semi-nonparametric iv estimation of shape-invariant engel curves. *Econometrica*, 75(6):1613–1669, 2007.
- C. Breunig. Goodness-of-fit tests based on series estimators in nonparametric instrumental regression. *Journal of Econometrics*, 184(2):328–346, 2015.
- C. Breunig, E. Mammen, A. Simoni, et al. Nonparametric estimation in case of endogenous selection. Technical report, Sonderforschungsbereich 649, Humboldt University, Berlin, Germany, 2015.
- C. Breunig, M. Kummer, J. Ohnemus, and S. Viète. Does it outsourcing increase firm success? Technical report, 2016.
- H. Y. Chen and R. Little. A test of missing completely at random for generalised estimating equations with missing data. *Biometrika*, 86(1):1–13, 1999.
- K. Chen. Parametric models for response-biased sampling. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 63(4):775–789, 2001.
- X. Chen. Large sample sieve estimation of semi-nonparametric models. *Handbook of Econometrics*, 2007.
- X. Chen and T. M. Christensen. Optimal uniform convergence rates and asymptotic normality for series estimators under weak dependence and weak conditions. *Journal of Econometrics*, 2015.
- M. Das, W. K. Newey, and F. Vella. Nonparametric estimation of sample selection models. *The Review of Economic Studies*, 70(1):33–58, 2003.
- M. A. Delgado and W. González Manteiga. Significance testing in nonparametric regression based on the bootstrap. *Annals of Statistics*, pages 1469–1507, 2001.
- X. D’Haultfoeuille. A new instrumental method for dealing with endogenous selection. *Journal of Econometrics*, 154(1):1–15, 2010.
- X. D’Haultfoeuille. On the completeness condition in nonparametric instrumental problems. *Econometric Theory*, 27(03):460–471, 2011.

- Y. Fan and Q. Li. Consistent model specification tests: omitted variables and semiparametric functional forms. *Econometrica: Journal of the econometric society*, pages 865–890, 1996.
- F. Fève, J.-P. Florens, and I. Van Keilegom. Estimation of conditional ranks and tests of exogeneity in nonparametric nonseparable models. *Journal of Business & Economic Statistics*.
- J. Frick, M. Grabka, and J. Marcus. Editing and multiple imputation of item-non-response in the 2002 wealth module of the german socio-economic panel (soep). Technical report, DIW Berlin, The German Socio-Economic Panel (SOEP), 2007.
- C. Fuchs. Maximum likelihood estimation and model selection in contingency tables with missing data. *Journal of the American Statistical Association*, 77(378):270–278, 1982.
- T. Hayfield and J. S. Racine. np: Nonparametric kernel smoothing methods for mixed datatypes. *R package version 0.13-1*, 2007.
- J. Heckman. Shadow prices, market wages, and labor supply. *Econometrica: Journal of the Econometric Society*, pages 679–694, 1974.
- K. Hirano, G. W. Imbens, G. Ridder, and D. B. Rubin. Combining panel data sets with attrition and refreshment samples. *Econometrica*, 69(6):1645–1659, 2001.
- J. L. Horowitz. Testing a parametric model against a nonparametric alternative with identification through instrumental variables. *Econometrica*, 74(2):521–538, 2006.
- Y. Hu and S. M. Schennach. Instrumental variable treatment of nonclassical measurement error models. *Econometrica*, 76(1):195–216, 2008.
- S. Huck, T. Schmidt, and G. Weizsacker. The standard portfolio choice problem in germany. Technical report, 2015.
- J. G. Ibrahim, M.-H. Chen, S. R. Lipsitz, and A. H. Herring. Missing-data methods for generalized linear models: A comparative review. *Journal of the American Statistical Association*, 100(469):332–346, 2005.
- G. W. Imbens. Nonparametric estimation of average treatment effects under exogeneity: A review. *Review of Economics and statistics*, 86(1):4–29, 2004.
- P. Kline and A. Santos. Sensitivity to missing data assumptions: Theory and an evaluation of the us wage structure. *Quantitative Economics*, 4(2):231–267, 2013.

- P. Lavergne and Q. Vuong. Nonparametric significance testing. *Econometric Theory*, 16(04):576–601, 2000.
- K.-Y. Liang and J. Qin. Regression analysis under non-standard situations: a pairwise pseudolikelihood approach. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 62(4):773–786, 2000.
- R. J. Little. A test of missing completely at random for multivariate data with missing values. *Journal of the American Statistical Association*, 83(404):1198–1202, 1988.
- R. J. Little and D. B. Rubin. *Statistical analysis with missing data*. 2002.
- E. Mammen. Bootstrap and wild bootstrap for high dimensional linear models. *The annals of statistics*, pages 255–285, 1993.
- L. Mattner. Some incomplete but boundedly complete location families. *The Annals of Statistics*, 21(4):2158–2162, 1993.
- W. K. Newey. Convergence rates and asymptotic normality for series estimators. *Journal of Econometrics*, 79(1):147 – 168, 1997.
- D. Picard and K. Tribouley. Adaptive confidence interval for pointwise curve estimation. *Annals of statistics*, pages 298–335, 2000.
- A. Qu and P. X.-K. Song. Testing ignorable missingness in estimating equation approaches for longitudinal data. *Biometrika*, 89(4):841–850, 2002.
- E. A. Ramalho and R. J. Smith. Discrete choice non-response. *The Review of Economic Studies*, 80(1):343–364, 2013.
- M. Reilly and M. S. Pepe. A mean score method for missing and auxiliary covariate data in regression models. *Biometrika*, 82(2):299–314, 1995.
- J. M. Robins, A. Rotnitzky, and L. P. Zhao. Estimation of regression coefficients when some regressors are not always observed. *Journal of the American Statistical Association*, 89(427):846–866, 1994.
- D. B. Rubin. Inference and missing data. *Biometrika*, 63(3):581–592, 1976.
- D. B. Rubin. *Multiple imputation for nonresponse in surveys*, volume 81. John Wiley & Sons, 2004.
- M. Rudelson. Random vectors in the isotropic position. *Journal of Functional Analysis*, 164(1):60–72, 1999.
- J.-P. Schräpler. Respondent behavior in panel studies a case study for income nonresponse by means of the german socio-economic panel (soep). *Sociological Methods & Research*, 33(1):118–156, 2004.

- R. J. Serfling. *Approximation theorems of mathematical statistics*. Wiley Series in Probability and Statistics. Wiley, Hoboken, NJ, 1981.
- G. Tang, R. J. Little, and T. E. Raghunathan. Analysis of multivariate missing data with nonignorable nonresponse. *Biometrika*, 90(4):747–764, 2003.
- N. Watson and M. Wooden. Identifying factors affecting longitudinal survey response. *Methodology of longitudinal surveys*, 1:157–182, 2009.
- J. Zhao and J. Shao. Semiparametric pseudo-likelihoods in generalized linear models with nonignorable missing data. *Journal of the American Statistical Association*, 110(512):1577–1590, 2015.

# SFB 649 Discussion Paper Series 2017

For a complete list of Discussion Papers published by the SFB 649, please visit <http://sfb649.wiwi.hu-berlin.de>.

- 001 "Fake Alpha" by Marcel Müller, Tobias Rosenberger and Marliese Uhrig-Homburg, January 2017.
- 002 "Estimating location values of agricultural land" by Georg Helbing, Zhiwei Shen, Martin Odening and Matthias Ritter, January 2017.
- 003 "FRM: a Financial Risk Meter based on penalizing tail events occurrence" by Lining Yu, Wolfgang Karl Härdle, Lukas Borke and Thijs Benschop, January 2017.
- 004 "Tail event driven networks of SIFIs" by Cathy Yi-Hsuan Chen, Wolfgang Karl Härdle and Yarema Okhrin, January 2017.
- 005 "Dynamic Valuation of Weather Derivatives under Default Risk" by Wolfgang Karl Härdle and Maria Osipenko, February 2017.
- 006 "RiskAnalytics: an R package for real time processing of Nasdaq and Yahoo finance data and parallelized quantile lasso regression methods" by Lukas Borke, February 2017.
- 007 "Testing Missing at Random using Instrumental Variables" by Christoph Breunig, February 2017.

**SFB 649, Spandauer Straße 1, D-10178 Berlin**  
**<http://sfb649.wiwi.hu-berlin.de>**

This research was supported by the Deutsche  
Forschungsgemeinschaft through the SFB 649 "Economic Risk".

