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The Lottery Contest is a Best-Response Potential Game

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Abstract. It is shown that the n -player lottery contest admits a best-response potential (Voorneveld, 2000, *Economics Letters*). This is true also when the contest technology reflects the possibility of a draw. The result implies, in particular, the existence of a nontrivial example of a strictly competitive game that is best-response equivalent to a game with identical payoff functions.

Keywords. Potential games · Tullock contest · Best-response equivalence · Zero-sum games

JEL-Codes. C62, C72, D72

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1. Introduction

Potential games are interesting because they allow conclusions not only regarding existence and uniqueness of a Nash equilibrium, but also regarding the outcome of dynamic and boundedly rational adjustment processes. Since Monderer and Shapley's (1996) seminal contribution, the literature has produced increasingly flexible variants of the initial concepts. One such generalization has led to the notion of a *best-response potential* (Voorneveld, 2000; Kukushkin, 2004; Dubey et al., 2006; Uno, 2007, 2011; Park, 2015). According to the definition, a game with continuous strategy spaces admits a best-response potential if there is a game with identical payoff functions that is *best-response equivalent* (henceforth *BR-equivalent*, etc.) to the original game, i.e., that has the same BR-correspondence, mapping any profile of pure strategies to a set of pure strategy profiles, as the original game.

This paper has two parts. In the first part, we show that the n -player lottery contest admits a BR-potential. This holds true regardless of whether the contest allocates the prize with probability one (Haavelmo, 1954; Tullock, 1975; Bell et al., 1975; Baron, 1994) or there is a probability of a draw (Loury, 1979; Dasgupta and Nti, 1998; Blavatskyy, 2010; Jia, 2012).¹ In the second part of the paper, we exploit the useful strategic equivalence between contests and zero-sum games,² so as to derive a perplexing implication of our result. Specifically, it is shown that a zero-sum game may be BR-equivalent to a game with identical payoff functions.

The lottery contest and its natural generalizations have found widespread application in economics and political theory (cf. Konrad, 2009). It corresponds to a Cournot game with isoelastic inverse demand and constant marginal costs. Deschamps (1975) proved convergence of fictitious play in a two-player Cournot oligopoly with strictly declining BR-functions. Thorlund-Peterson (1990) extended this result to an arbitrary number of firms that hold point beliefs. Even an exact potential is possessed by a Cournot game with linear demand (Slade, 1994). More generally, sufficient conditions for the existence of a BR-potential have been found for aggregative games that allow monotone BR-selections (Huang, 2002; Dubey et al., 2006; Jensen, 2010). However, all these methods do not apply

¹Like this paper, Dasgupta and Nti (1998) allow for both cases.

²By strategic equivalence, we mean here vNM-equivalence (Morris and Ui, 2004), which will be defined later in the paper. The vNM-equivalence between contests and zero-sum games, which is implicit in the early work by Moulin and Vial (1978), was used by Pavlov (2013) to study correlated equilibria in the first-price all-pay auction. More recent applications include network conflicts (Ewerhart and Valkanova, 2016) and decomposable normal-form games (Hwang and Rey-Bellet, 2017).

to the lottery contest whose BR-function is not monotone (cf. Dixit, 1987). Also more recent examples of games that allow a BR-potential (Dragone et al., 2012; Bourlès et al., 2015) do not cover the case of the lottery contest.

2. The lottery contest admits a BR-potential

In a *lottery contest*, with noise parameter $a \geq 0$, common valuation $V > 0$, and $n \geq 2$ players, each player $i \in \{1, \dots, n\}$ simultaneously and independently chooses an effort $x_i \geq 0$, and subsequently receives a payoff of

$$u_i^a(x_1, \dots, x_n) = \begin{cases} \frac{x_i}{a + \bar{x}}V - x_i & \text{if } a + \bar{x} > 0 \\ V/n & \text{if } a + \bar{x} = 0, \end{cases} \quad (1)$$

where $\bar{x} = x_1 + \dots + x_n$ denotes aggregate effort. This game is known to possess a unique pure-strategy Nash equilibrium that is necessarily symmetric (Dasgupta and Nti, 1998).³

An n -person game $G = (X_1, \dots, X_n, u_1, \dots, u_n)$ with strategy spaces X_i and payoff functions $u_i : X \equiv X_1 \times \dots \times X_n \rightarrow \mathbb{R}$ for players $i = 1, \dots, n$ is called a *BR-potential game* (Voorneveld, 2000) if there exists a function $P : X \rightarrow \mathbb{R}$ such that

$$\arg \max_{x_i \in X_i} P(x_i, x_{-i}) = \arg \max_{x_i \in X_i} u_i(x_i, x_{-i}) \quad (2)$$

for any $i = 1, \dots, n$ and any $x_{-i} = (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in X_{-i} \equiv X_1 \times \dots \times X_{i-1} \times X_{i+1} \times \dots \times X_n$.

The following observation has, to the author's knowledge, not been documented in the literature.⁴

Proposition 1. *For any $a \geq 0$, the n -player lottery contest is a BR-potential game.*

Proof. Consider first the case $a > 0$. We claim that, in this case,

$$P^a(x_1, \dots, x_n) = \left\{ a\bar{x} + \sum_{j < k} x_j x_k \right\} V - \frac{1}{3}(a + \bar{x})^3 \quad (3)$$

³For $a = 0$, Dasgupta and Nti (1998) assume a different tie-breaking rule, viz. $u_i^0(0, \dots, 0) = 0$ for $i = 1, \dots, n$. The BR-correspondence is the same, however.

⁴Cf., e.g., the surveys by Chowdhury and Sheremata (2015) and González-Sánchez and Hernández-Lerma (2016).

is a BR-potential for the lottery contest. Indeed, differentiating (3), we find

$$\frac{\partial P^a(x_i, x_{-i})}{\partial x_i} = (a + \bar{x}_{-i})V - (a + \bar{x})^2, \quad (4)$$

where $\bar{x}_{-i} = x_1 + \dots + x_{i-1} + x_{i+1} + \dots + x_n$. Moreover,

$$\frac{\partial^2 P^a(x_i, x_{-i})}{\partial x_i^2} = (-2) \cdot (a + \bar{x}) < 0, \quad (5)$$

i.e., the problem of maximizing $P^a(\cdot, x_{-i})$ subject to $x_i \geq 0$ is strictly concave. The unique solution $x_i^* \equiv x_i^*(x_{-i}, a)$ is given by

$$x_i^* = \begin{cases} \sqrt{(a + \bar{x}_{-i})V} - a - \bar{x}_{-i} & \text{if } \bar{x}_{-i} \leq V - a \\ 0 & \text{if } \bar{x}_{-i} > V - a. \end{cases} \quad (6)$$

But this is just player i 's best-response function in the lottery contest with noise parameter a . Hence,

$$\arg \max_{x_i \geq 0} P^a(x_i, x_{-i}) = \arg \max_{x_i \geq 0} u_i^a(x_i, x_{-i}), \quad (7)$$

as claimed. Consider next the case $a = 0$. Denote by $\pi(x)$ the number of nonzero entries of the vector $x = (x_1, \dots, x_n)$. We claim that, in this case,

$$P^0(x_1, \dots, x_n) = \begin{cases} (\sum_{j < k} x_j x_k)V - \frac{1}{3}\bar{x}^3 & \text{if } \pi(x_1, \dots, x_n) \geq 2 \\ -\frac{1}{3}x_j V^2 & \text{if } \pi(x_1, \dots, x_n) = 1 \text{ and } x_j > 0 \\ -\frac{1}{3}\frac{n-1}{n}V^3 & \text{if } \pi(x_1, \dots, x_n) = 0 \end{cases} \quad (8)$$

is a BR-potential for the lottery contest. To see this, suppose first that x_{-i} has at least two nonzero entries. Then, certainly $\pi(x_1, \dots, x_n) \geq 2$, so that from (8),

$$P^0(x_1, \dots, x_n) = (\sum_{j < k} x_j x_k)V - \frac{1}{3}\bar{x}^3. \quad (9)$$

Moreover, $u_i^0(\cdot, x_{-i})$ is differentiable, so that in straightforward extension of the case $a > 0$ considered above,

$$\arg \max_{x_i \geq 0} P^0(x_i, x_{-i}) = \arg \max_{x_i \geq 0} u_i^0(x_i, x_{-i}). \quad (10)$$

Suppose next that x_{-i} has precisely one nonzero entry $x_j > 0$. Then, $\pi(x_1, \dots, x_n) = 2$ if $x_i > 0$, and $\pi(x_1, \dots, x_n) = 1$ if $x_i = 0$. Hence, using (8) another time,

$$P^0(x_1, \dots, x_n) = \begin{cases} x_i x_j V - \frac{1}{3}(x_i + x_j)^3 & \text{if } x_i > 0 \\ -\frac{1}{3}x_j V^2 & \text{if } x_i = 0. \end{cases} \quad (11)$$

We have to show that the problem of maximizing $P^0(\cdot, x_{-i})$ subject to $x_i \geq 0$ has the unique solution that is given by the best-response function of the lottery contest with noise parameter $a = 0$, i.e., by $x_i^* = \sqrt{x_j V} - x_j$ if $x_j \leq V$ and by $x_i^* = 0$ if $x_j > V$. From the above, it clearly suffices to show that the problem $\max_{x_i \geq 0} P^0(x_i, x_{-i})$ has an interior solution if and only if $x_j < V$. But indeed, using (11), one can easily check that for $x_j > 0$,

$$\lim_{\substack{x_i \rightarrow 0 \\ x_i > 0}} P^0(x_i, x_{-i}) = -\frac{1}{3}x_j^3 > -\frac{1}{3}x_j V^2 = P^0(0, x_{-i}) \quad (12)$$

if and only if $x_j < V$, as illustrated in Figure 1.

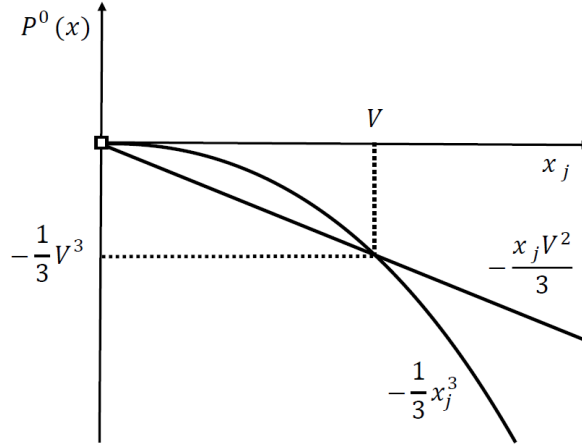


Figure 1. Constructing a BR-potential in the case $a = 0$.

Finally, suppose that all entries of x_{-i} are zero. Then, again from (8), $P^0(x_i, x_{-i}) = -\frac{1}{3}x_i V^2$ if $x_i > 0$, but $P^0(0, x_{-i}) < 0$, so that $\arg \max_{x_i \in X_i} P^0(x_i, x_{-i}) = \emptyset$. Similarly, $u_i^0(x_i, x_{-i}) = V - x_i$ if $x_i > 0$, but $u_i^0(0, x_{-i}) = V/n$, so that $\arg \max_{x_i \in X_i} u_i^0(x_i, x_{-i}) = \emptyset$. This proves that P^0 is indeed a BR-potential for the lottery contest with noise parameter $a = 0$. \square

Park (2015) has argued that, in cases where preferences are complete but the BR-set may be empty, the BR-potential should generate the same preferences over strategies as the original payoff function. To check that this condition holds for the BR-potential (8) constructed above, note that for $x_{-i} = 0$,

$$P^0(x_i, x_{-i}) = \begin{cases} -\frac{1}{3}x_i V^2 & \text{if } x_i > 0 \\ -\frac{1}{3}\frac{n-1}{n}V^3 & \text{if } x_i = 0. \end{cases} \quad (13)$$

Thus, the BR-potential induces a preference for lower strategies among positive strategies $x_i > 0$, and an indifference between $x_i = 0$ and $x_i = \frac{n-1}{n}V$. This is likewise true for

preferences reflecting player i 's original payoffs when $x_{-i} = 0$, which are given by

$$u_i^0(x_i, x_{-i}) = \begin{cases} V - x_i & \text{if } x_i > 0 \\ V/n & \text{if } x_i = 0. \end{cases} \quad (14)$$

Thus, the BR-potential constructed in the proof of Proposition 1 does satisfy Park's condition.

Proposition 1 easily extends to the class of contests in which the contest success function is a strict convex combination of a purely random allocation and the above specification (Haavelmo, 1954; Baron, 1994). The same is true for Amegashie's (2006) contest success function with noise that creates a lottery contest without the possibility of a draw yet minimum efforts. Further generalizations would clearly be interesting.

3. An implication

In this section, we show that a zero-sum game may be BR-equivalent to a game with identical payoff functions.

Note first that this type of non-trivial example is impossible for the stronger notion of *vNM-equivalence* (Morris and Ui, 2004).⁵ Indeed, suppose that a two-person zero-sum game with payoff functions $u_1(x_1, x_2) = -u_2(x_1, x_2)$ is vNM-equivalent to a game in which players have the identical payoff function $u(x_1, x_2)$. Then, there exist constants $\lambda_1 > 0$, $\lambda_2 > 0$, and functions $\mu_1 = \mu_1(x_2)$, $\mu_2 = \mu_2(x_1)$ such that

$$\lambda_1 u_1(x_1, x_2) + \mu_1(x_2) = u(x_1, x_2) = \lambda_2 u_2(x_1, x_2) + \mu_2(x_1). \quad (15)$$

Using the zero-sum property, this implies

$$u_1(x_1, x_2) = \frac{\mu_2(x_1) - \mu_1(x_2)}{\lambda_1 + \lambda_2}. \quad (16)$$

Repeating these arguments for player 2, the game is seen to be vNM-equivalent to two independent decision problems. Consistent with this observation, games with identical payoff functions and zero-sum games have traditionally been considered as polar cases in a broad spectrum of possible forms of strategic interaction.

⁵According to Morris and Ui (2004), two games are *vNM-equivalent* if, for each player, the payoff function in one game is equal to a positive constant times the payoff function in the other game, plus a term that depends only on the opponents' strategies.

The following result shows that this dichotomy must be considered with care.

Proposition 2. *For any $n \geq 2$, there exists a nontrivial n -person zero-sum game that is BR-equivalent to a game with identical payoff functions.*

Proof. It follows from Hwang and Rey-Bellet (2017, Example 2) that the lottery contest with noise parameter $a = 0$ is vNM-equivalent to a zero-sum game.⁶ Since vNM-equivalence implies BR-equivalence, the claim is now immediate from Proposition 1. \square

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⁶For a direct proof, consider the n -player game G with payoffs $u_i(x) = u_i^0(x) - \frac{V}{n} + x_{i+1}$, where $x_{n+1} = x_1$. Since G is zero-sum, this proves the claim.

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