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Aggregate Fluctuations in Adaptive Production Networks

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Aggregate Fluctuations in Adaptive Production Networks[☆]

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Abstract

We study production networks where firms' products can be described by a set of input and output characteristics, and links are formed only if the output characteristics of a seller match the input characteristics of a customer. We introduce a fully endogenous network formation model with monopolistically competitive firms, in which firms exit due to exogenous shocks, or the propagation of shocks through the network. Firms can replace suppliers they have lost due to exit subject to switching costs and search frictions. This enables us to study the impact of shocks on aggregate production in an adaptive network, and we show that depending on the nature of the shocks, adaptivity can make the network more or less stable.

Key words: production networks, shocks, supply chains, resilience, aggregate fluctuations
JEL: D85, L24, O33

1. Introduction

Production networks have played a crucial role in determining the robustness of the output and production of an economy during crises such as the dot-com bubble or the recent financial crisis. However, the formation of these networks at the level of firms and their behavior in the presence of shocks is only barely understood.[1] In this paper we develop a simple model of a monopolistically competitive economy where firms use the intermediate goods of other firms as inputs for production while their output, in turn, is used as intermediate input by other firms, and for final good consumption. Firms' production recipes can be described by a set of input and output characteristics, and input-output links are formed only if the output characteristics of a seller match the input characteristics of a buyer [cf. Hausmann and Hidalgo, 2011]. Our model builds on Acemoglu et al. [2006], and is similar to the input-output economy considered in Chaney [2013]. We show that sales (and

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[1] For example Carvalho [2014] describes the area of progress that needs to be made in our understanding of production networks as follows: "...Relative to sectors, progress on firm-level production networks needs to deal with three added complications: First, on the theory side, it is more difficult to brush aside the complexities of *market structure* (as I have done here by appealing to identical, perfectly competitive firms inside each sector). Second, at this level of disaggregation it is clear that we have to distinguish between easily substitutable inputs and crucial, hard-to-substitute, inputs where firms are locked-in and *switching costs* are large. Third, relative to sector-level data, *input-output information at the firm-level* is in very short supply."

profits) of a firm are determined by the Bonacich centrality of the firm in the network of technological complementarities between intermediate goods [cf. Acemoglu et al., 2012; Ballester et al., 2006]. For certain parameter choices we can further simplify the model and show that profits are proportional to the number of buyers of a firm.

We then introduce a dynamic process for the formation of the production network with entry and exit, where the network dynamically adjusts to exogenous (small and large) shocks. Firms are hit by shocks which may lead them to exit, and these shocks can propagate to their customers, also leading them to exit. Firms can replace exiting suppliers with new suppliers subject to search frictions or switching costs. The existing literature, such as Acemoglu et al. [2012], does not investigate the capacity of the network to adjust to shocks, neither do they study the role of switching costs, and thus might not fully capture the impact of shocks within an adaptive production network. Our framework allows us to investigate the region in between the two extreme cases of no adaptivity and full adaptivity.

We further show that a mix of random small and large shocks with rewiring can give rise to "self-organized criticality" [cf. Bak, 1996; Bak et al., 1993; Krugman, 1996; Scheinkman and Woodford, 1994], with frequent crashes and recoveries of the supply chain network [cf. Horvath, 1998]. The basic mechanism is one in which the resilience of large firms to small shocks and rewiring leads large firms to accumulate many links. However, when rare large shocks hit these large firms then the entire network breaks down. This gives rise to an endogenous business cycle [Matsuyama, 1999].

Finally, our model enables us to study the impact of shocks on aggregate production in an adaptive network, and we show that the relationship between network structure and aggregate fluctuations is not simple or monotonic. Moreover, we show that, contrary to conventional wisdom, greater network adaptivity does not necessarily make the network more stable nor resilient.

Relation to the Literature. Acemoglu et al. [2012] analyze the propagation of shocks at the sectoral level within an exogenously given input-output network between sectors. Differently to these authors, we fully endogenize the production network and analyze its formation over time with firm entry and exit. Carvalho [2009] also analyzes the propagation of shocks in a network between sectors, but assumes that links between sectors are drawn at random and sectors which are associated with general purpose technologies have a higher probability to supply to other sectors. Differently to Carvalho [2009], we study the network at the firm level, and do not consider a random graph for this network, but a dynamic process with entry and exit.

Atalay et al. [2011] have studied the formation of production networks, but in a mechanistic way, without providing a micro-foundation for why links are formed, but assuming that links are formed at random or through preferential attachment. Differently to Atalay et al. [2011] in our model firms come in different types, depending on their input and output characteristics, and links are formed only if the output characteristics of a seller match the input characteristics of a customer. While in Atalay et al. [2011] the accumulation of links is mainly an age effect, in our model it is a technology effect. Moreover, we allow for the propagation of shocks, where firms that have lost their suppliers are themselves more likely to exit, while Atalay et al. [2011] do not consider such an effect. Furthermore, Atalay et al. [2011] assume that firms exit at random, while here we assume that less profitable firms are more likely to exit the market. We show that this gives rise to self organized critically and frequent crashes and recoveries of the economy [cf. Bak et al., 1993; Horvath, 1998; Scheinkman and Woodford, 1994].

In a related paper, Hausmann and Hidalgo [2011] introduce heterogeneity in terms of firms'

product characteristics, and how the match between input and output characteristics form an input-output network structure. However, the authors do not analyze the evolution of the network over time, and how it reacts to exogenous shocks.

Oberfield [2011] develops a model for production networks at the firm level with bilateral two-part pricing for intermediate input goods. Similarly, we study an input-output model of production with monopolistically competitive firms based on Acemoglu et al. [2006] and Chaney [2013]. However, differently to Oberfield [2011], where the production network is static, here we study the dynamic formation of the network and the adjustment of the network to firm-specific shocks.

Recently, Carvalho and Voigtländer [2014] investigate the formation of input-output networks based on the characteristics of the different varieties produced by firms and their suppliers, similar to the models studied in Jackson and Rogers [2007]; König [2011]. However, the authors do not investigate the adaptivity of the network against shocks, and their analysis is based on simulation studies, while here we provide an analytically tractable framework.

Lim [2016] analyzes an endogenous production network formation model and applies it to a similar dataset as we do here. However, in his model the formation of a buyer-supplier link depends only on the sizes of the two firms involved in the link, and he does not incorporate firm exit, nor the replacement of suppliers or customers lost due to exit. Moreover, differently to Lim [2016] we evaluate the welfare loss due to exit of a firm in an adaptive production network.

We investigate the impact of profit shocks (which can be interpreted as a fixed cost or credit constraints) on the network structure, and the feedback of structural changes in the network on the output of the economy. The impact of shocks is also at the center of the analysis in Bigio and Jennifer [2013] and Kelly et al. [2012]. However, in the first the network is exogenous, while in the latter, micro-foundations for why links are formed are missing, and results are based on numerical simulations instead of an analytic treatment as we do here. Finally, our model is related to previous studies in the mathematics, computer science and physics literature, such as Juher and Saldaña [2011]; Karrer and Ghoshal [2008]; Kong and Roychowdhury [2008]; Kong et al. [2008]; Moore et al. [2006]; Saldaña [2007],[2] who study random network growth processes with random node decay. However, these works do not provide proper micro-foundations for the formation of links, or the removal of nodes. Instead, here we derive this process from an economic model of production in input-output networks with monopolistically competitive firms based on Acemoglu et al. [2006].

Barrot and Sauvagnat [2016] study the propagation of shocks due to natural disasters on the customers of firms in a production network of the U.S. economy using similar data as we do here. They find a significant effect of a shock to a supplier of a firm. Differently to Barrot and Sauvagnat [2016] we develop a theoretical model that allows for the network to respond to shocks and thus we make the production network endogenous.

[2]In particular, Kong and Roychowdhury [2008]; Kong et al. [2008] study undirected networks and do not allow for the rewiring of links. They also do not allow for "shock propagation", i.e. the fact that firms' who have lost a supplier are more likely to exit themselves in future periods. However, they do not allow firms to replace lost suppliers (rewiring), as we do here.

2. Firms' Profits and Production

Similar to Acemoglu et al. [2006] we consider a production function of the following form[3,4]

$$x_i = \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} z \sum_{j=1}^{n} l_{ji}^{1-\alpha} x_{ji}^{\alpha}, \tag{2.1}$$

with $\alpha \in (0,1)$, productivity z, labor input, l_{ji}, for intermediate input j, and the quantity of intermediate inputs, x_{ji}, produced by firm j and sold to firm i. We assume that the input specific labor input can be written as $l_{ji} = l_i w_{ji}$, where the total labor force of firm i is given by l_i and $\sum_{j=1}^{n} w_{ji} = 1$ [cf. Acemoglu et al., 2012]. We can write $w_{ji} = a_{ji}/d_i^-$ with $a_{ji} \in \{0,1\}$ indicating a possible supply link from j to i in the production network,[5] and $d_i^- = \sum_{j=1}^{n} a_{ji}$ is the in-degree of firm i. For the remainder of this section we normalize the productivity to one, i.e. we set $z = 1$ in the production function in Equation (2.1). We then can state the following proposition regarding the firms' profits and total output in equilibrium for any given network structure G.

Proposition 1. *Consider the production function of Equation (2.1) for a given network $G \in \mathcal{G}^n$ representing the input-output relationships between firms, let the wage be $h > 0$ and define $\phi \equiv \frac{1}{h^{2-\alpha}} \alpha^{\frac{4-3\alpha}{1-\alpha}}$.*

(i) The Nash equilibrium profits are given by $\pi_i = \frac{(1-\alpha)h}{c} b_i(\mathbf{W}, \phi)$, where $\mathbf{b}(\mathbf{W}, \phi) \equiv (\mathbf{I}_n - \phi \mathbf{W})^{-1}\mathbf{u}$ is the Bonacich centrality of $\mathbf{W}$ with parameter ϕ and total output is given by $Y = \frac{h}{c} \sum_{i=1}^{n} b_i(\mathbf{W}, \phi)$.

(ii) In the limit of small ϕ profits of firm i are a linear function of the number of customers of i, i.e. $\pi_i = \frac{(1-\alpha)h}{c} \left(1 + \phi d_i^+\right) + O\left(\phi^2\right)$, and output is proportional to the number of firms, i.e. $Y = \frac{n(1+\phi)h}{c} + O\left(\phi^2\right)$.

In the following sections we will assume that ϕ is small, and that the first-order Taylor approximation in part (ii) of Proposition 1 holds. The main implication is that the profit of firm i is a linearly increasing function of the number d_i^+ of customers of i. Similarly, Klette and Kortum [2004] assume that the profit of a firm is proportional to the number of products it is selling.

3. Network Formation and Entry-Exit Dynamics

In the following we introduce two alternative models that govern the formation of the buyer-supplier network G and the entry and exit of firms. In Section 3.1 we introduce a model with ex ante

[3] See also Chaney [2013] and Oberfield [2011].

[4] Instead of Equation (2.1) we can also consider a more general CES production function. However, with the specific form of Equation (2.1) we can greatly simplify the computation of equilibrium prices, and thus make the model more tractable.

[5] The possibility of a supply relationship from j to i hinges on the technological constraints imposed by the output characteristics of firm j and the input requirements of firm i, so that $a_{ji} \in \{0,1\}$ is an indicator function for whether the product of firm j matches the input requirements of firm i. This will be discussed in more detail in Section 3.2.

homogeneous firms, while in Section 3.2 we analyze a model with ex ante heterogeneous firms, where the heterogeneity is represented by varying input and output characteristics across firms.

3.1. Homogeneous Firms

We first determine the exit probability of a firm. Assume that firm i is hit by a cost shock ζ. The probability of firm i to exit after the cost shock ζ is given by

$$\begin{aligned}\mathbb{P}\left(\pi_i - \zeta < 0\right) &= \mathbb{P}\left(\pi_i - \bar{c} - \varepsilon < 0\right) \\ &= 1 - \mathbb{P}\left(\varepsilon < \pi_i - \bar{c}\right) \\ &= 1 - F_\varepsilon\left(\pi_i - \bar{c}\right) \\ &= \left(\frac{\pi_i - \bar{c}}{\bar{\varepsilon}}\right)^{-\chi},\end{aligned}$$

where we have assumed that the cost ζ has deterministic component $\bar{c}$ and a random component ε which is Pareto distributed with parameter χ, lower bound $\bar{\varepsilon}$ and cdf $F_\varepsilon(x) = 1 - \left(\frac{x}{\bar{\varepsilon}}\right)^{-\chi}$. Using the fact that (cf. Proposition 1) $\pi_i \approx \frac{(1-\alpha)h}{c}\left(1 + \phi d_i^+\right)$, normalizing $\frac{(1-\alpha)h}{c} = 1$ gives $\pi_i - \bar{c} \approx (1 - \bar{c}) + \phi d_i^+$. Further, setting $\bar{\varepsilon} = \phi$ and $\bar{c} = 1 - \phi$ so that $\frac{1-\bar{c}}{\phi} = 1$ and assuming that $\chi = 1$, we can simplify the exit probability of firm i to

$$\mathbb{P}\left(\pi_i - \zeta < 0\right) = \frac{1}{1 + d_i^+}. \tag{3.1}$$

As larger firms have a lower probability to exit, we call this *preferential survival* [cf. Kong et al., 2008]. This is consistent with the empirical evidence [cf. e.g. Evans, 1987].

After having defined the shocks, we are now able to introduce a Markov chain that describes the formation of the production network. Firms that are hit by a small shock exit with a probability that depends on their customer base (or profits, respectively), while if they are hit by a large shock they exit independently of size.

Definition 2. *We consider a Markov chain $(G_t)_{t=0}^{\infty}$ comprising a sequence of networks $G_0, G_1, \ldots$ where $G_t = (\mathcal{N}_t, \mathcal{E}_t)$ with $\mathcal{N}_t$ being the set of firms and $\mathcal{E}_t$ the set of buyer-supplier relationships (edges/links) between them. Starting from an initial state G_0, in every step from t to $t+1$ the following events happen:*

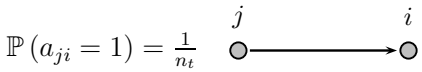
Entry *At every period $t = 1, 2, \ldots$, a new firm i is born and selects a randomly chosen firm j among the incumbents as a supplier.*

Exit *(i) Large Shocks: With probability $\delta \in [0, 1]$ a randomly selected firm exits.*

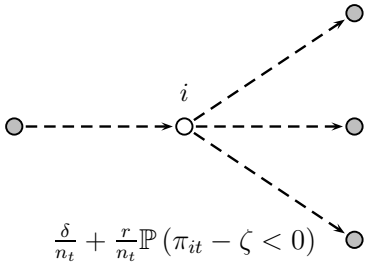
(ii) Small Shocks: With probability $r \in [0, 1]$ a randomly selected firm is hit by an additive shock (fixed cost) ζ following a Pareto distribution, and if the firm's after shock profit is negative, it exits.

(iii) Shock Propagation: With probability $\rho \in [0, 1]$ a firm which does not have a supplier exits.

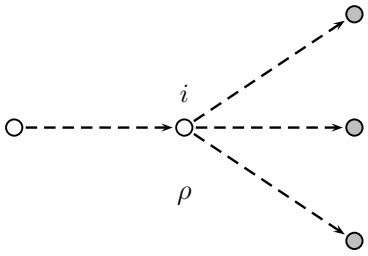
Rewiring *If a firm looses a supplier due to exit, with probability $\gamma \in [0, 1]$ it attempts to replace it with a firm drawn uniformly at random among the incumbents.*



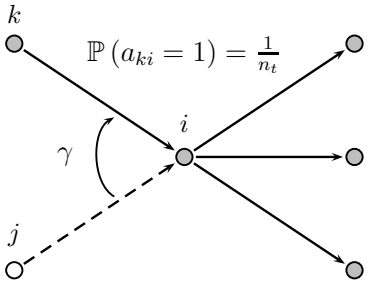
(a) Entry of a firm i and uniform attachment to an incumbent firm j with probability $\mathbb{P}(a_{ji} = 1) = \frac{1}{n_t}$.



(b) The exit of a firm i due to a large shock with probability $\frac{\delta}{n_t}$ or small shock with probability $\frac{r}{n_t}\mathbb{P}(\pi_{it} - \zeta < 0)$.



(c) Shock propagation leading to the exit of a firm i with in-degree zero with probability ρ.



(d) Replacement of a supplier j that has exited with probability γ. Then a new firm k becomes the supplier to i with probability $\mathbb{P}(a_{ki} = 1) = \frac{1}{n_t}$.

Figure 1: The different events that happen during the time evolution of the Markov chain introduced in Definition 2: (a) entry, (b) exit due to a large or small shock, (c) shock propagation and (d) replacement of a supplier after exit. Filled circles indicate firms that have not exited, while empty circles indicate firms that have exited. All shocks are assumed to be Zipf distributed (i.e. $\chi = 1$).

The different events that happen during the time evolution of the Markov chain introduced in Definition 2 are illustrated in Figure 1.

The parameter γ is a measure of the *adaptivity* of the network. It is similar to macroeconomic models of price stickiness, where the opportunity for firms to reset their prices in any particular period is a random event, and the probability that they are unable to do so is known as the "Calvo probability" [cf. Calvo, 1983].

The following proposition characterizes the out-degree distribution, that is, the distribution over the number of customers across firms, together with the firms' age distribution, assuming that $\gamma = 0$, so that there is no rewiring (i.e. replacement of lost suppliers due to exit).

Proposition 3. *Assume that $\gamma = 0$ (without rewiring), let τ be the asymptotic fraction of firms with in-degree zero and κ denote the average shifted inverse out-degree.*

(i) Assume that $\rho > 0$. Then the expected number of firms is given by

$$n_t = \frac{\left(1 - e^{-t\rho\tau}\right)(1 - r\kappa - \delta)}{\rho\tau}, \tag{3.2}$$

with the limit

$$\lim_{t\to\infty} n_t = \frac{1 - \delta - r\kappa}{\rho\tau}.$$

(ii) Let $a \equiv \frac{\rho\tau}{1-\delta-r\kappa}$, $b \equiv a\,(\delta + r\tilde{\kappa})$, then under the mean field approximation the distribution of the firm's lifetime, T, is given by

$$\mathbb{P}(T > t) = e^{-(br + a\delta + \rho)t}. \tag{3.3}$$

(iii) Under the assumption of weak degree correlations, the asymptotic out-degree distribution is given by

$$P^+(k) = \left(1 + (2 + r(1-\kappa))\Gamma(C^+)\Gamma(C^-)\sum_{k=1}^{\infty}\frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^+ + k)\Gamma(1+C^- + k)}\right)^{-1}$$
$$\times \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}(2 + r(1-\kappa))\Gamma(C^+)\Gamma(C^-)}{\Gamma(1+C^+ + k)\Gamma(1+C^- + k)}, \tag{3.4}$$

where

$$C^{\pm} \equiv \frac{2 + \delta \pm \sqrt{(2-\delta-r\kappa)^2 - 2r(2\delta + \kappa(2-\delta-r\kappa)) + (\kappa-4)\kappa r^2}}{2(\delta + \kappa r)}.$$

with $P^+(k) \sim \left(\frac{1}{\delta+\kappa r}\right)^{k+1}\left(\frac{e}{k}\right)^k k^{\frac{1}{2}-C^+-C^-}$ for large k, while κ is determined by $\kappa = \sum_{k=0}^{\infty}\frac{1}{1+k}P^+(k)$.

The next proposition characterizes the out-degree distribution, that is, the distribution over the number of customers across firms, together with the firms' age distribution, assuming that $\gamma > 0$, allowing for rewiring and replacement of lost suppliers due to exit.

Proposition 4. *Assume that $\gamma > 0$ (with rewiring), let τ be the asymptotic fraction of firms with in-degree zero and κ denote the average shifted inverse out-degree.*

(i) Assume that $\rho > 0$. Then the expected number of firms is given by

$$n_t = \frac{\left(1 - e^{-t\rho\tau}\right)(1 - r\kappa - \delta)}{\rho\tau}, \tag{3.5}$$

with the limit

$$\lim_{t\to\infty} n_t = \frac{1 - \delta - r\kappa}{\rho\tau}.$$

(ii) Denote by $a \equiv \frac{\rho\tau}{1-\delta-r\kappa}\left(1 + \gamma(1 + \bar{k})\right)$, $b \equiv \frac{\rho\tau}{1-\delta-r\kappa}\left(\delta + r\tilde{\kappa}\right)$, $c \equiv \frac{\rho\tau}{1-\delta-r\kappa}$. Then under the mean field approximation the firm's lifetime T distribution is given by

$$\mathbb{P}(T > t) = \left(\frac{b}{ae^{bt} - a + b}\right)^r \exp\left(-\frac{\rho e^{-c(1-\gamma)t}}{c(1-\gamma)} - \rho\left(-\frac{1}{c(1-\gamma)} + t\right) - c\delta t\right), \tag{3.6}$$

and $\mathbb{P}(T > t) \sim e^{-(br + c\delta + \rho)t}$ as t becomes large.

(iii) Under the assumption of weak degree correlations, the asymptotic out-degree distribution is given by

$$P^+(k) = \left(1 + \frac{\left(1 + \gamma\left(1 + r\tilde{\kappa} - r\kappa\right)\bar{k}\right)\Gamma(C^+)\Gamma(C^-)}{\delta + r\kappa}\sum_{k=1}^{\infty}\frac{\Gamma(k+2)\left(\frac{1+\gamma\bar{k}(1-r\kappa+r\tilde{\kappa})}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1 + C^+ + k)\Gamma(1 + C^- + k)}\right)^{-1}$$
$$\times \frac{\Gamma(k+2)\left(\frac{1+\gamma\bar{k}(1-r\kappa+r\tilde{\kappa})}{\delta+\kappa r}\right)^{k+1}(2 - r\kappa + r + \gamma\bar{k}(1 - r\kappa + r\tilde{\kappa}))\Gamma(C^+)\Gamma(C^-)}{(\delta + r\kappa)\Gamma(1 + C^+ + k)\Gamma(1 + C^- + k)}, \tag{3.7}$$

where

$$C^{\pm} \equiv \frac{1}{2(\delta + \kappa r)}\Big[2 + \delta + \gamma\bar{k}(1 - r\kappa + r\tilde{\kappa}) \pm \Big(\left(2 + 3\delta + 2\kappa r + \gamma\bar{k}\left(1 + r\tilde{\kappa} - r\kappa\right)\right)^2$$
$$-4(\delta + \kappa r)\left(4 + r + 2\delta + 2\gamma\bar{k}\left(1 + r\tilde{\kappa} - r\kappa\right)\right)\Big)^{1/2}\Big],$$

with $P^+(k) \sim \left(\frac{1}{\delta+\kappa r}\right)^{k+1}\left(\frac{e}{k}\right)^k k^{\frac{1}{2}-C^+-C^-}$ for large k, while κ is determined by $\kappa = \sum_{k=0}^{\infty}\frac{1}{1+k}P^+(k)$, $\tilde{\kappa}$ is determined by $\tilde{\kappa} = \frac{1}{k}\sum_{k=0}^{\infty}\frac{k}{1+k}P^+(k)$, and $\bar{k}$ is determined by $\bar{k} = \sum_{k=0}^{\infty} kP^+(k)$.

The left panel in Figure 2 shows the out-degree distribution, $P^+(k)$, from a simulation of the stochastic process in Definition 2 and the predictions of Equation (2). The right panel in Figure 2 shows the relative output loss, ΔY (respectively, the number of surviving firms), compared to the case with $\gamma = 1$ (no shock propagation) for varying values of $\gamma \in [0, 1]$. We observe that high switching costs, respectively, low rewiring probabilities γ, are associated with considerable output losses. The out-degree distribution, $P^+(k)$, in Equations (3.4) and (3.7), respectively, is a power-law distribution.[6] The power-law degree distribution is consistent with earlier works on production

[6]Further, note that the out-degree distribution $P^+(k)$ can be used to compute the coefficient of variation, as it

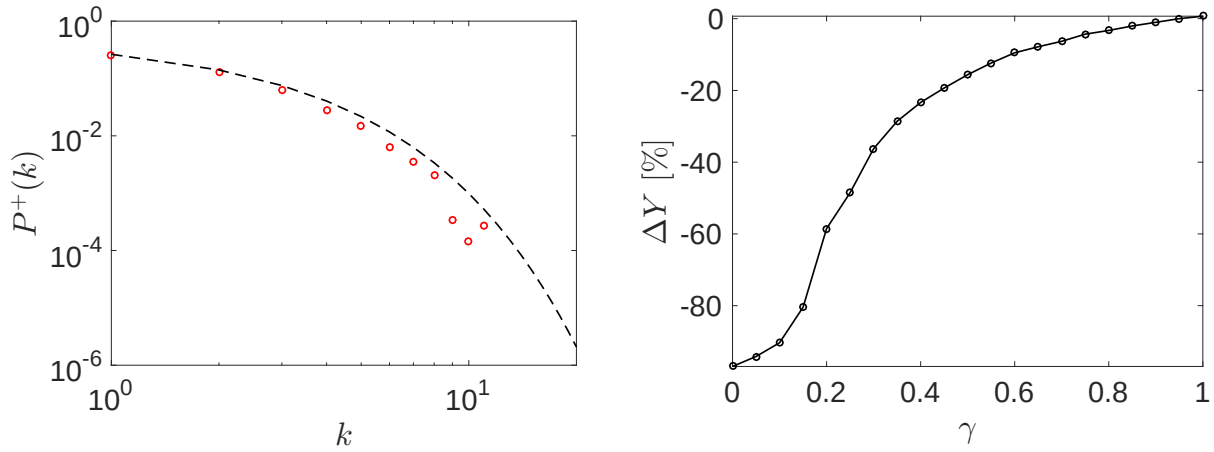


Figure 2: (Left panel) The out-degree distribution, $P^+(k)$, from a simulation of the stochastic process in Definition 2 and the predictions of Equation (3.7). (Right panel) The relative output loss ΔY (respectively, the number of surviving firms) compared to the case with $\gamma = 1$ (no shock propagation) for varying values of $\gamma \in [0, 1]$.

networks [cf. Atalay et al., 2011], and implies a Pareto firm size distribution as we find it in the data [Gabaix, 2009]. The lifetime distribution, $\mathbb{P}(T > t)$, in Equations (3.3) and (3.6), respectively, is asymptotically exponential, confirming previous empirical studies [cf. Coad, 2010].

3.2. Heterogeneous Input-Output Characteristics

Similar to Hausmann and Hidalgo [2011],[7] we assume that each firm is characterized by a tuple $\mathbf{h}_i = (\mathbf{h}_i^-, \mathbf{h}_i^+)$ of a zero-one vector of input characteristics $\mathbf{h}_i^- \in \{0,1\}^N$ and output characteristics $\mathbf{h}_i^+ \in \{0,1\}^N$ of length N.[8] An illustration of these input and output characteristics is shown in Figure 3.

Further, following Hausmann and Hidalgo [2011] we assume that each input characteristic h_{ik}^- is one with probability $p \in [0,1]$ and zero otherwise. Similarly, each output characteristic h_{ik}^+ is one with probability $q \in [0,1]$ and zero otherwise for all $k = 1, \dots, N$. Under this assumption of independently drawn input/output characteristics, each stock of input characteristics $|\mathsf{S}(\mathbf{h}_i^-)|$ follows a Binomial distribution with success probability p (i.e. $\mathbb{P}(h_{ik}^- = 1) = p$ for all $i = 1, \dots, n$ and $k = 1, \dots, N$), and each stock of output characteristics $|\mathsf{S}(\mathbf{h}_i^+)|$ follows a Binomial distribution with success probability q (i.e. $\mathbb{P}(h_{ik}^+ = 1) = q$ for all $i = 1, \dots, n$ and $k = 1, \dots, N$).[9]

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uniquely defines the first and second moments.

[7] See also Hidalgo and Hausmann [2009]; Hidalgo et al. [2007]. Note, however, that differently to the current setup these authors study a static network at the country level.

[8] Note that the output characteristics of a firm can be obtained from the primary and secondary SIC codes in the Compustat database, and the input characteristics from the information on suppliers and their primary and secondary product characteristics in the Compustat Segments database.

[9] A firm i with a large stock of output characteristics $|\mathsf{S}(\mathbf{h}_i^+)|$ produces a good that can be used by many firms as input. This refers to so called "general purpose technologies" [cf. Bresnahan and Trajtenberg, 1995; Helpman, 1998;

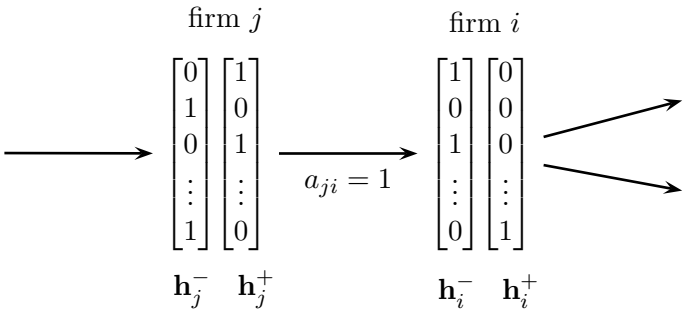


Figure 3: Illustration of the vectors of input and output characteristics of firm i being a customer of firm j.

After having defined the propensities of firms to become suppliers, we are now able to introduce a Markov chain that describes the formation of the production network.

Definition 5. *We consider a Markov chain $(G_t)_{t=0}^{\infty}$ comprising a sequence of networks $G_0, G_1, \ldots$ where $G_t = (\mathcal{N}_t, \mathcal{E}_t)$ with $\mathcal{N}_t$ being the set of firms and $\mathcal{E}_t$ the set of buyer-supplier relationships (edges/links) between them. Starting from an initial state G_0, in every step from t to $t+1$ the following events happen:*

Entry *At every period $t = 1, 2, \ldots$, a new firm is born and its input and output characteristics are drawn independently with probabilities $p \in [0,1]$ and $q \in [0,1]$, respectively. Then an incumbent firm whose output characteristics match the entrant's input characteristics becomes the supplier to the entrant.*

Exit *(i) Large Shocks: With probability $\delta \in [0,1]$ a randomly selected firm exits.*

(ii) Small Shocks: With probability $r \in [0,1]$ a randomly selected firm is hit by an additive shock (fixed cost) ζ following a Pareto distribution, and if the firm's after shock profit is negative, it exits.

(iii) Shock Propagation: With probability $\rho \in [0,1]$ a firm which does not have a supplier exits.

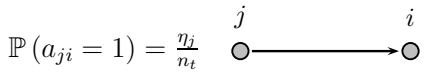
Rewiring *If a firm looses a supplier due to exit, with probability $\gamma \in [0,1]$ it attempts to replace it with a firm whose output characteristics match the firm's input characteristics.*

An illustration of the different events taking place during the evolution of the Markov chain is given in Figure 4.

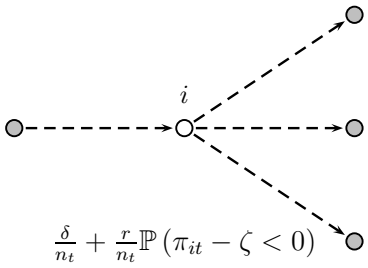
The following proposition characterizes the out-degree distribution, that is, the distribution over the number of customers across firms, and a firm's lifetime distribution.

Proposition 6. *Assume that each input characteristic is drawn independently with probability $p \in [0,1]$ and each output characteristic is drawn independently with probability $q \in [0,1]$. Further, denote*

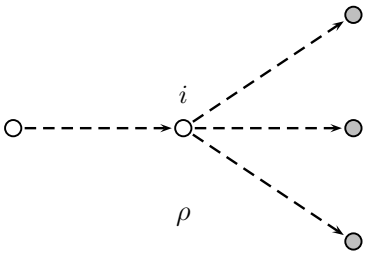
Jovanovic and Rousseau, 2005].



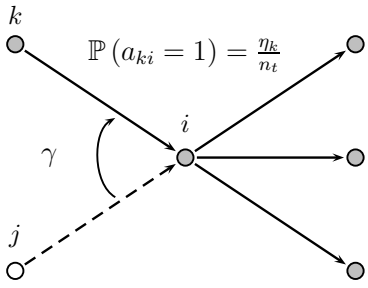
(a) Entry of a firm i and uniform attachment to an incumbent firm j with probability $\mathbb{P}(a_{ji} = 1) = \frac{\eta_j}{n_t}$.



(b) The exit of a firm i due to a large shock with probability $\frac{\delta}{n_t}$ or small shock with probability $\frac{r}{n_t}\mathbb{P}(\pi_{it} - \zeta < 0)$.



(c) Shock propagation leading to the exit of a firm i with in-degree zero with probability ρ.



(d) Replacement of a supplier j that has exited with probability γ. Then a new firm k becomes the supplier to i with probability $\mathbb{P}(a_{ki} = 1) = \frac{\eta_k}{n_t}$.

Figure 4: The different events that happen during the time evolution of the Markov chain introduced in Definition 5: (a) entry, (b) exit due to a large or small shock, (c) shock propagation and (e) replacement of a supplier after exit. Filled circles indicate firms that have not exited, while empty circles indicate firms that have exited.

by $a \equiv \left(1+\gamma\bar{k}(\delta+r\kappa+\rho\tau)\right)\frac{\rho\tau}{1-\delta-r\kappa}$, $b \equiv \frac{\rho\tau(\delta+r\kappa)}{1-\delta-r\kappa}$ *and*

$$\eta_{\max} \equiv (1-p)^N\left(1+\frac{p}{q(1-p)}\right)^N - 1 = \left(1-p\left(1-\frac{1}{q}\right)\right)^N - 1, \tag{3.8}$$

with $\tilde{\sigma} \equiv \sqrt{Nq(1-q)}\ln\left(1+\frac{p}{q(1-p)}\right)$ *and* $\tilde{\mu} \equiv Nq\ln\left(1+\frac{p}{q(1-p)}\right)+N\ln(1-p)$.

(i) Assume that $\rho > 0$. *Then the expected number of firms is given by*

$$n_t = \frac{\left(1-e^{-t\rho\tau}\right)(1-r\kappa-\delta)}{\rho\tau}, \tag{3.9}$$

with the limit

$$\lim_{t\to\infty} n_t = \frac{1-\delta-r\kappa}{\rho\tau}.$$

(ii) The firm's lifetime T *distribution is given by*

$$\mathbb{P}(T>t) = \int_0^{\eta_{\max}} e^{-\left(\rho+\frac{b}{\delta+r\kappa}\left(\delta+\frac{rb}{a\eta}\right)\right)t}d\eta. \tag{3.10}$$

(iii) The out-degree distribution is given by

$$P_t^+(k) = \int_0^{\eta_{\max}} P_t^+(\eta,k)\frac{1}{\sqrt{2\pi\tilde{\sigma}^2}}\frac{1}{\eta}e^{-\frac{(\ln\eta-\tilde{\mu})^2}{2\tilde{\sigma}^2}}d\eta. \tag{3.11}$$

where

$$P_t^+(\eta,k) = \frac{b}{(\delta+r\kappa)(b-a\eta)}\left(\frac{a\eta_i-bk}{a\eta_i-b}\right)^{\frac{\rho}{b}-1+\frac{1}{\delta+r\kappa}\left(\delta+\frac{rb}{a\eta_i}\right)}.$$

4. Adaptivity, Network Concentration and Aggregate Volatility

In this section we analyze the relationship between network concentration measured by the out-degree coefficient of variation, CV_k^+, and aggregate volatility, σ_n, defined as the standard deviation of the number of firms n_t over a predefined time window w given by

$$\sigma_n = \sqrt{\sum_{s=t-w}^{t}\left(g_s - \frac{1}{w}\sum_{s'=t-w}^{t} g_{s'}\right)^2},$$

where the growth rate in the number of firms is given by

$$g_t = \frac{n_t - n_{t-1}}{n_{t-1}}.$$

Figure 5 shows the number of firms, n_t, and the volatility, σ_n over time t with large shocks without rewiring $\gamma = 0$ (left panels), weak adaptivity $\gamma = 0.5$ (middle panels) and strong adaptivity $\gamma = 1$

(right panels). It is assumed that the small shocks are negligible, i.e. we have set $r = 0$, and there is no replacement of suppliers that have exited (after a shock). Note that in this case we observe that with increasing γ the average number of surviving firms is reduced, while the volatility and the coefficient of variation in the out-degree are increased. The mechanism underlying this relationship is that due to the firms replacing their suppliers with larger firms (before a shock), the network becomes increasingly concentrated on large firms. However, when a large firm is hit by a shock and forced to exit in such a concentrated network, this triggers a cascade of exits of many firms linked to it. Figure 7 shows the volatility, σ_n, the out-degree coefficient of variation, CV_k^+ and the volatility versus the out-degree coefficient of variation across different values of the rewiring probability γ. Both, σ_n and CV_k^+ are increasing with ν.

In contrast, Figure 6 shows the same quantities as in Figure 5, but with non-negligible small shocks, $r = 0.15$, which can be better absorbed by large firms, and the replacement of suppliers that have exited at a cost that can be better absorbed by large firms ($\chi = 2$). In this case we observe that with increasing γ the average number of surviving firms is increased, while the volatility declines and the coefficient of variation of the out-degree is increasing. We thus observe the opposite relationship of Figure 5. This is due to the fact that now large firms can absorb shocks (direct or indirect through the exit of their suppliers), and hence, an increasingly concentrated network relying on these firms becomes more resilient and reduces volatility. Figure 8 shows for the same parameter values that σ_n is decreasing with ν, CV_k^+ are increasing with γ.

Figures 7 and 8 show the volatility, σ_n, the out-degree coefficient of variation, CV_k^+ and the volatility versus the out-degree coefficient of variation across different values of the rewiring probability γ. In the case of $r = 0$ (no small shocks, and no preferential survival) both, σ_n and CV_k^+ are increasing with γ. In contrast, in the case of $r = 0.15$ (with small shocks and preferential survival) we have that σ_n is decreasing with γ while CV_k^+ is increasing with γ. This shows that in the presence of small shocks and preferential survival the relationship between network concentration and aggregate volatility can be reversed. This is because when larger firms are better able to absorb shocks then a network that is more concentrated can become more resilient.

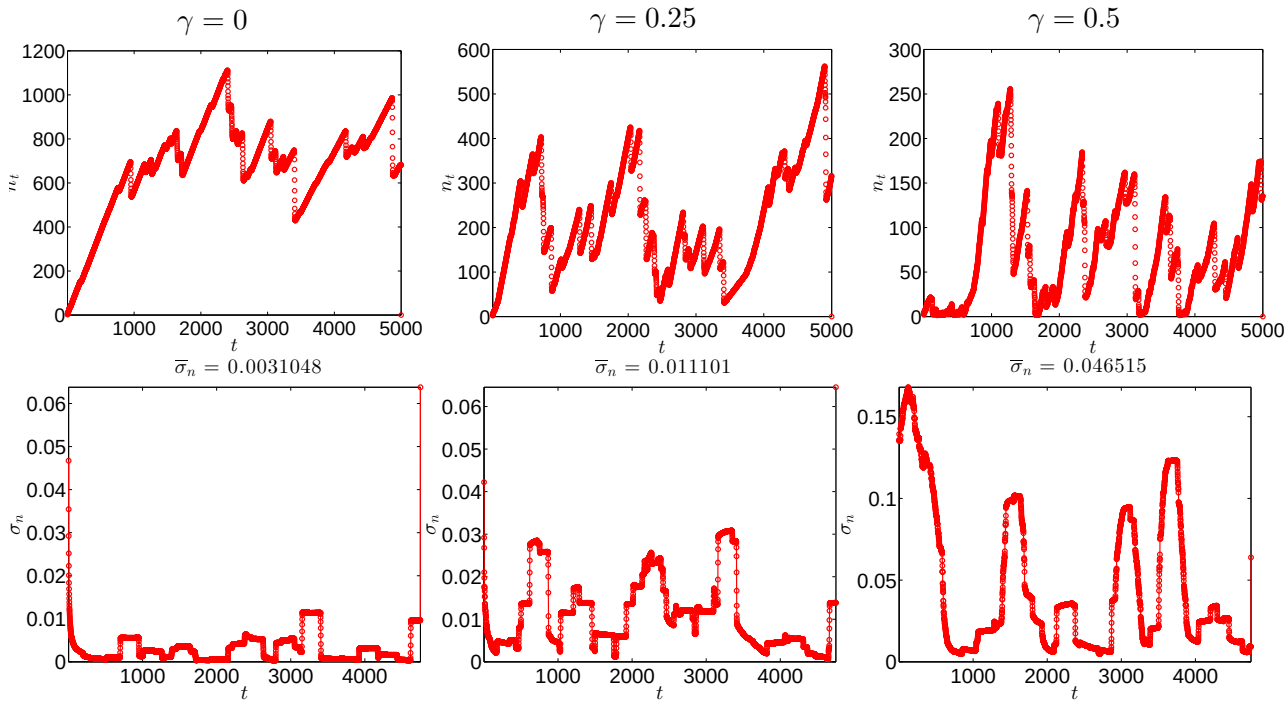


Figure 5: The number of firms, n_t, (top panels) and the volatility, σ_n (bottom panels) over time t with large shocks without rewiring $\gamma = 0$ (left panels), weak adaptivity $\gamma = 0.5$ (middle panels) and strong adaptivity $\gamma = 1$ (right panels). The remaining parameters are $T = 5000$, $\chi = 1$, $\rho = 0.5$, $r = 0$ and $\delta = 0.01$. Simulations are averaged over 5 independent realizations. Note that we have assumed that the small shocks are negligible, i.e. we have set $r = 0$, and there is no advantage of large firms to escape the propagation of shocks, $\gamma = 0$. In this case we observe that with increasing ν the average number of surviving firms is reduced, while the volatility and the coefficient of variation in the out-degree are increased.

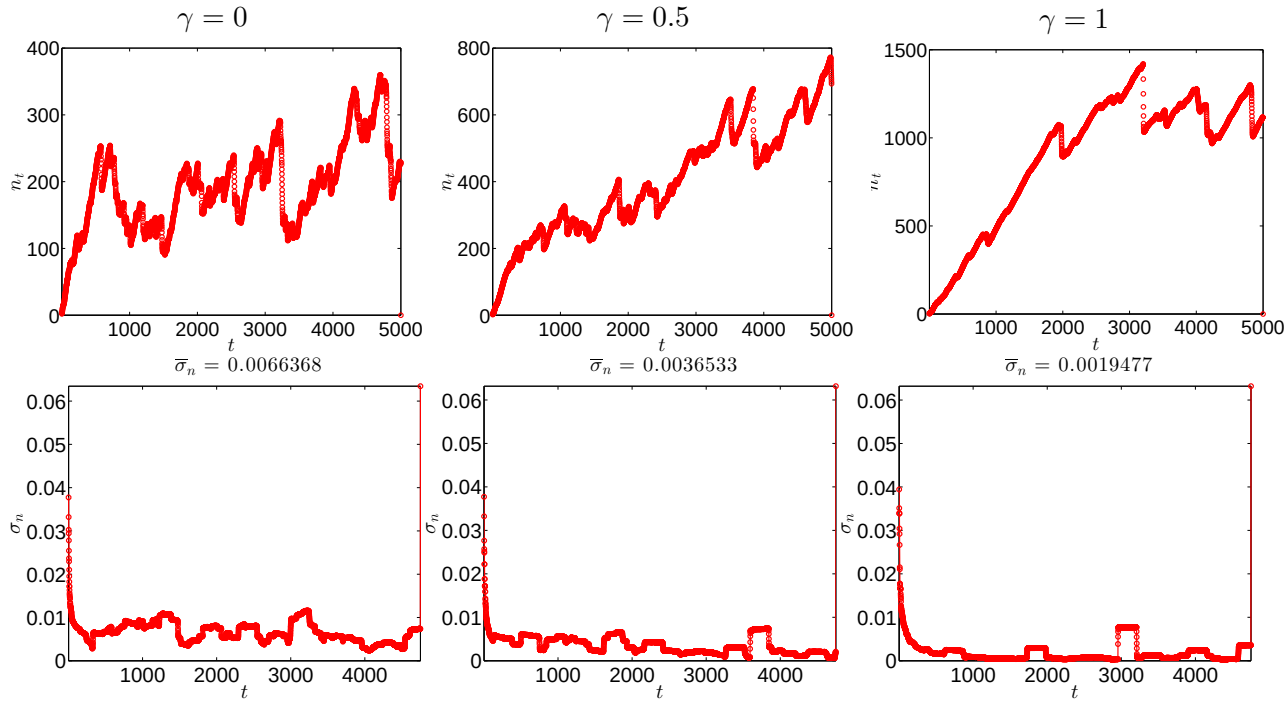


Figure 6: The number of firms, n_t, (top panels) and the volatility, σ_n, (bottom panels) over time t with large shocks without rewiring $\gamma = 0$ (left panels), weak rewiring $\gamma = 0.5$ (middle panels) and strong rewiring $\gamma = 1$ (right panels). The remaining parameters are $T = 5000$, $\rho = 0.5$, $r = 0.15$ and $\delta = 0.03$. Simulations are averaged over 5 independent realizations. Note that we have assumed non-negligible small shocks, $r = 0.15$, which can be better absorbed by large firms, and preferential rewiring of large firms due to the propagation of shocks. In this case we observe that with increasing γ the average number of surviving firms is increased, while the volatility declines and the coefficient of variation in the out-degree is increasing.

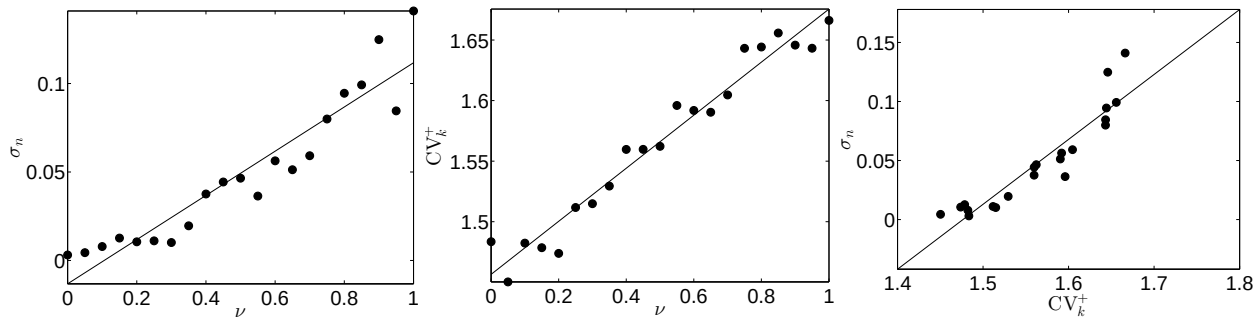


Figure 7: The volatility, σ_n (left panel), the out-degree coefficient of variation, CV_k^+ (middle panel) and the volatility versus the out-degree coefficient of variation across different values of the rewiring probability ν. Simulations are averaged over 5 independent realizations. The parameters are $T = 5000$, $\chi = 1$, $\rho = 0.5$, $r = 0$, and $\delta = 0.01$. Both, σ_n and CV_k^+ are increasing with ν.

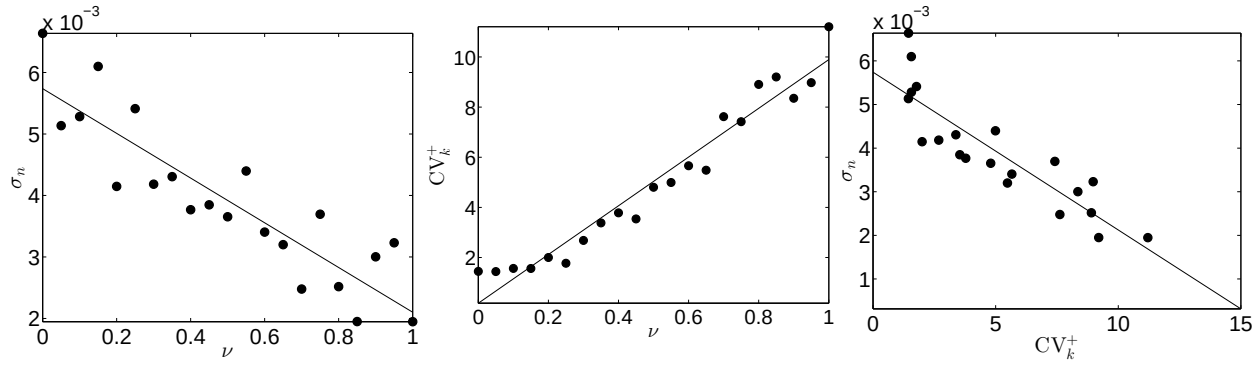


Figure 8: The volatility, σ_n (left panel), the out-degree coefficient of variation, CV_k^+ (middle panel) and the volatility versus the out-degree coefficient of variation across different values of the rewiring probability ν. Simulations are averaged over 5 independent realizations. The parameters are $T = 5000$, $\chi = 2$, $\rho = 0.5$, $r = 0.15$ and $\delta = 0.03$. While σ_n is decreasing with ν, CV_k^+ is increasing with ν.

5. Conclusion

In this paper we study the formation of production networks with endogenous entry and exit. We introduce a fully endogenous model of production networks with monopolistically competitive firms, in which firms exit due to exogenous shocks. Firms can replace suppliers they have lost due to exit subject to switching costs and search frictions. This enables us to study the impact of shocks on aggregate production in an adaptive network, and we show that the relationship between network structure and aggregate fluctuations is not simple or monotonic. Moreover, we show that, contrary to conventional wisdom, greater network adaptivity does not necessarily make the network more stable.

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Appendix

A. Proofs

Proof of Proposition 1. We first consider part (i) of the proposition. In the following we solve the problem of the firm in two stages. In the first stage, firms choose the optimal input quantities to minimize the input cost subject to producing a certain level of output and taking prices of the intermediate input goods as given. In the second stage, given these optimal input choices, firms set the price of their output for each buyer separately.

The firm minimizes the cost of inputs, labor l_i and intermediate inputs $(x_{ji})_{j=1}^n$, subject to producing a given level of output $\bar{x}$. The firm's problem can thus be written as follows

$$\begin{aligned} &\min_{l_i,\{x_{ji}\}_{j=1}^n} \left\{ h l_i + \sum_{j=1}^n a_{ji} x_{ji} p_{ji} \right\} \\ &\text{s.t. } x_i = \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} l_i^{1-\alpha} \sum_{j=1}^n w_{ji}^{1-\alpha} x_{ji}^\alpha = \bar{x}, \end{aligned} \tag{A.1}$$

where h is the exogenous wage rate. The Lagrangian is given by

$$\mathscr{L}_i = h l_i + \sum_{j=1}^n a_{ji} x_{ji} p_{ji} - \lambda_i \left(\frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} l_i^{1-\alpha} \sum_{j=1}^n w_{ji}^{1-\alpha} x_{ji}^\alpha - \bar{x} \right).$$

The FOC wrt. x_{ji} is given by

$$\frac{\partial \mathscr{L}_i}{\partial x_{ji}} = a_{ji} p_{ji} - \lambda_i \frac{1}{\alpha^\alpha (1-\alpha)^{1-\alpha}} l_i^{1-\alpha} \alpha w_{ji}^{1-\alpha} x_{ji}^{\alpha-1} = 0.$$

This can be rearranged as

$$x_i = \frac{1}{\alpha} l_i \lambda^{\frac{\alpha}{1-\alpha}} \sum_{j=1}^n w_{ji} p_{ji}^{\frac{\alpha}{\alpha-1}}.$$

The FOC wrt. l_i is given by

$$\frac{\partial \mathscr{L}_i}{\partial l_i} = h - \lambda_i (1-\alpha) \frac{x_i}{l_i} = 0,$$

so that

$$l_i = \lambda_i (1-\alpha) \frac{x_i}{h}$$

Inserting yields

$$\lambda_i = h^{1-\alpha} \left(\sum_{j=1}^n w_{ji} p_{ji}^{\frac{\alpha}{\alpha-1}} \right)^{\alpha-1} \equiv P_i,$$

where P_i is the Dixit-Stiglitz price index. From the FOC wrt. x_{ji} we further find that

$$x_{ji} = x_{ki}\frac{w_{ji}}{w_{ki}}\left(\frac{p_{ki}}{p_{ji}}\right)^{\frac{1}{1-\alpha}}.$$

Inserting into the production function yields

$$x_i = \frac{1}{\alpha^\alpha(1-\alpha)^{1-\alpha}} l_i^{1-\alpha} p_{ki}^{\frac{\alpha}{1-\alpha}} w_{ki}^{-\alpha} x_{ki}^{\alpha} \sum_{j=1}^{n} w_{ji} p_{ji}^{\frac{\alpha}{\alpha-1}}.$$

Further, using the fact that

$$\sum_{j=1}^{n} w_{ji} p_{ji}^{\frac{\alpha}{\alpha-1}} = h P_i^{\frac{1}{\alpha-1}}$$

and

$$l_i = \frac{(1-\alpha)P_i x_i}{h}$$

we can write

$$x_{ki} = \frac{\alpha}{h} P_i^{\frac{2-\alpha}{1-\alpha}} w_{ki} p_{ki}^{\frac{1}{\alpha-1}} x_i.$$

Total expenditures of firm i for intermediate inputs can then be written as

$$\begin{aligned} C_i &= \sum_{j=1}^{n} a_{ji} p_{ji} x_{ji} + h l_i \\ &= \frac{\alpha}{h} P_i^{\frac{2-\alpha}{1-\alpha}} x_i \sum_{j=1}^{n} w_{ji} p_{ji}^{\frac{\alpha}{\alpha-1}} + h l_i \\ &= \alpha P_i x_i + (1-\alpha) P_i x_i \\ &= P_i x_i, \end{aligned}$$

so that the marginal cost of production is given by the price index

$$MC_i \equiv \frac{C_i}{x_i} = P_i.$$

Firms choose the price to maximize profits. The profit function of the firm can then be written as follows

$$\pi_i = \sum_{j=1}^{n} \left(p_{ij} x_{ij} - P_i x_{ij}\right).$$

Using the fact that

$$x_{ij} = \frac{\alpha}{h} w_{ij} P_j^{\frac{2-\alpha}{1-\alpha}} p_{ij}^{-\frac{1}{1-\alpha}} x_j,$$

we can write

$$\pi_i = \sum_{j=1}^{n} \left(p_{ij}^{1-\frac{1}{1-\alpha}} - P_i p_{ij}^{-\frac{1}{1-\alpha}}\right) \frac{\alpha}{h} w_{ij} P_j^{\frac{2-\alpha}{1-\alpha}} x_j.$$

The FOC wrt. p_{ij} is given by

$$\frac{\partial \pi}{\partial p_{ij}} = \left(-\frac{\alpha}{1-\alpha} p_{ij}^{-\frac{1}{1-\alpha}} + \frac{1}{1-\alpha} p_{ij}^{-\frac{1}{1-\alpha}-1} \right) \frac{\alpha}{h} w_{ij} P_j^{\frac{2-\alpha}{1-\alpha}} x_j = 0,$$

from which we find that the firm charges a constant markup, $\frac{1}{\alpha}$, over marginal cost

$$p_{ij} = \frac{1}{\alpha} P_i \equiv p_i,$$

and hence we have that

$$p_i = \frac{1}{\alpha} h^{1-\alpha} \left(\sum_{j=1}^{n} w_{ji} p_j^{\frac{\alpha}{\alpha-1}} \right)^{\alpha-1}. \tag{A.2}$$

The profit of firm i is then given by

$$\pi_i = p_i x_i - P_i x_i = (1-\alpha) p_i x_i = (1-\alpha) s_i,$$

and so the profit of firm i is proportional to its sales $s_i = p_i x_i$. Next, denoting by $\tilde{p}_i \equiv p_i^{\frac{1}{\alpha-1}}$, we can write Equation (A.2) as follows

$$\tilde{p}_i = \frac{1}{\alpha} h^{1-\alpha} \sum_{j=1}^{n} w_{ji} \tilde{p}_j^{\alpha}.$$

This is a generalized linear system similar to the one studied in Allen et al. [2014], where nonlinear versions of the Perron-Frobenius Theorem apply to show existence and uniqueness of the solution.[10] Making the Ansatz $\tilde{p}_i = \tilde{p}$ we get

$$\tilde{p} = \frac{1}{\alpha} h^{1-\alpha} \tilde{p}^{\alpha} \sum_{j=1}^{n} w_{ji}.$$

By assumption, we have that $\sum_{j=1}^{n} w_{ji} = 1$ so that we obtain

$$\tilde{p} = \alpha^{\frac{1}{\alpha-1}} h,$$

and the price is given by

$$p_i = \tilde{p}^{\alpha-1} = \alpha h^{\alpha-1}.$$

Hence, by solving the price equilibrium in Equation (A.2) we find that all firms charge the same price $p_i = p = \alpha h^{-(\alpha-1)}$ for all $i = 1, \ldots, n$. A similar price homogeneity is obtained in Ciccone [2002] and

[10] See also Krause [1986]; Nussbaum and Lunel [1999] and Lecture 5 in the trade notes by Treb Allen (https://sites.google.com/site/treballen/teaching/econ-460).

the recent model by Carvalho and Voigtländer [2014].[11] The market clearing condition implies that

$$x_i = \sum_{j=1}^{n} x_{ij} + q_i,$$

where q_i is the consumption of good i. The utility of the consumer representing Cobb–Douglas preferences is given by

$$u_i = A \prod_{j=1}^{c} q_j^{\frac{1}{c}},$$

with the budget constraint $\sum_{i=1}^{c} p_i q_i = h \sum_{i=1}^{c} l_i = h$ and $1 \leq c \leq n$ being the number of goods consumed.[12] The optimal consumption level is given by $q_i = \frac{h}{p_i c}$. Inserting into the market clearing condition then gives

$$x_i = \sum_{j=1}^{n} a_{ij} x_{ij} + \frac{h}{p_i c},$$

Multiplying with p_i and inserting optimal input levels yields

$$s_i = p_i x_i = \frac{h}{c} + \frac{1}{h} \alpha^{\frac{3-2\alpha}{1-\alpha}} \sum_{j=1}^{n} w_{ij} \frac{p_i^{\frac{\alpha}{\alpha-1}}}{p_j^{\frac{1}{\alpha-1}}} s_j.$$

Inserting the price then yields

$$s_i = \frac{h}{c} + \underbrace{\frac{1}{h^{2-\alpha}} \alpha^{\frac{4-3\alpha}{1-\alpha}}}_{\equiv \phi} \sum_{j=1}^{n} w_{ij} s_j,$$

where we have denoted by $\phi \equiv \frac{1}{h^{2-\alpha}} \alpha^{\frac{4-3\alpha}{1-\alpha}}$. We have that $0 \leq \phi$ and $\lim_{\alpha \to 1} \phi = \frac{1}{e}$. An illustration of the function $\phi \equiv \frac{1}{h^{2-\alpha}} \alpha^{\frac{4-3\alpha}{1-\alpha}}$ for $h = 1$ and varying values of $\alpha \in (0, 1)$ can be seen in Figure A.1. It follows that the vector of sales $\mathbf{s} = (p_1 x_1, p_2 x_2, \ldots, p_n x_n)^\top$ must satisfy[13]

[11] In particular, Carvalho and Voigtländer [2014] state that "...Thus, in expectation our model features a symmetric equilibrium with all new varieties facing the same marginal cost and therefore charging the same price. Note, however, that variety producers use different sets of inputs. Thus, the out-degree may be asymmetric if some varieties are more popular suppliers than others. Nevertheless, the total demand for an input affects neither its pricing nor its own adoption of inputs. Consequently, in our setup, symmetry of prices is compatible with asymmetry in the number of forward linkages."

[12] We may assume that the consumer makes a uniform random draw of $1 \leq c \leq n$ goods from a basket with a total of n available goods.

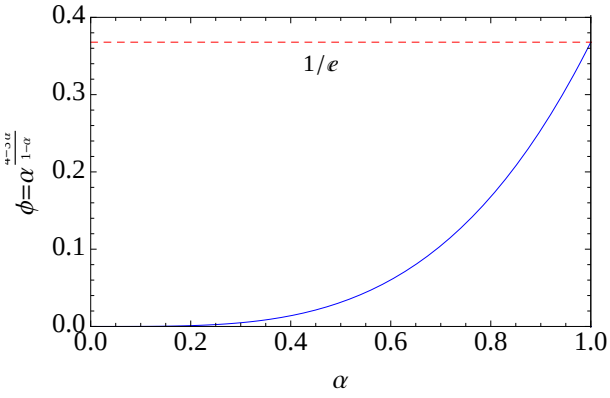


Figure A.1: The function $\phi \equiv \frac{1}{h^{2-\alpha}}\alpha^{\frac{4-3\alpha}{1-\alpha}}$ for $h = 1$ and varying values of $\alpha \in (0, 1)$.

$$\begin{aligned} \mathbf{s} &= \frac{h}{c}\mathbf{u} + \phi\mathbf{W}\mathbf{s} \\ \Leftrightarrow\ (\mathbf{I}_n - \phi\mathbf{W})\mathbf{s} &= \frac{h}{m}\mathbf{u} \\ \Leftrightarrow\ \mathbf{s} &= \frac{h}{c}\underbrace{(\mathbf{I}_n - \phi\mathbf{W})^{-1}\mathbf{u}}_{\mathbf{b}(\mathbf{W},\phi)}, \end{aligned}$$

where $\mathbf{b}(\mathbf{W}, \phi) = (\mathbf{I}_n - \phi\mathbf{W})^{-1}\mathbf{u}$ is the Bonacich centrality of $\mathbf{W}$ with parameter ϕ [Ballester et al., 2006; Bonacich, 1987].[14] The profit of the firm can then be written as follows[15]

$$\pi_i = \frac{(1-\alpha)h}{c} b_i(\mathbf{W}, \phi). \tag{A.3}$$

Next, we consider part (ii) of the proposition. Note that we can write the Bonacich centrality

[13] From the expression of $\mathbf{s} = \frac{h}{c}\mathbf{u} + \phi\mathbf{W}\mathbf{s}$ we see that the sales vector $\mathbf{s}$ is a linear combination of the homogeneous term $\frac{h}{c}\mathbf{u}$ and the interaction term $\phi\mathbf{W}\mathbf{s}$. When the second term dominates (for small h), we have that $\mathbf{s} \approx \phi\mathbf{W}\mathbf{s}$, and the sales vector $\mathbf{s}$ is the eigenvector of the matrix $\mathbf{W}$.

[14] The matrix $\mathbf{I}_n - \phi\mathbf{W}$ is invertible if $|\phi| < \lambda_{\text{PF}}(\mathbf{W}) = 1$, where the last equality follows from the fact that $\mathbf{W}$ is a row stochastic matrix and so its principal eigenvalue is one.

[15] Total sales and total profits are thus proportional to the sum of the Bonacich centralities $\sum_{i=1}^{n} b_i(\mathbf{W}, \phi)$, and hence, the firm which reduces total profits and total sales the most is determined by $i^* = \arg\max_{i=1,\ldots,n} b_i(G, \alpha)/N_i(G, \phi)$ with $b_i(G, \phi)$ being the Bonacich centrality of firm i in G and $N_i(G, \phi)$ being the generating function of the number of closed walks that start and terminate at node i [Ballester et al., 2006].

as[16]

$$\mathbf{b}(\mathbf{W},\phi)=(\mathbf{I}_n-\phi\mathbf{A}\mathbf{D}^{-1})^{-1}\mathbf{u}=\sum_{k=0}^{\infty}\phi^k(\mathbf{A}\mathbf{D}^{-1})^k\mathbf{u}=\mathbf{u}+\phi\mathbf{A}\mathbf{D}^{-1}\mathbf{u}+O\left(\phi^2\right).$$

With

$$[\mathbf{A}\mathbf{D}^{-1}\mathbf{u}]_i=\left[\mathbf{A}\begin{pmatrix}\frac{1}{d_1^-}&0&\cdots\\0&\frac{1}{d_2^-}&\cdots\\0&0&\ddots\end{pmatrix}\begin{pmatrix}1\\1\\\vdots\end{pmatrix}\right]_i=\left[\mathbf{A}\begin{pmatrix}\frac{1}{d_1^-}\\\frac{1}{d_2^-}\\\vdots\end{pmatrix}\right]_i=\sum_{j=1}^{n}a_{ij}\frac{1}{d_j^-},$$

we then can write the Bonacich centrality of firm i as

$$b_i(\mathbf{W},\phi)=1+\phi\sum_{j=1}^{n}a_{ij}\frac{1}{d_j^-}+O\left(\phi^2\right),$$

and sales of firm i are given by

$$s_i=\frac{h}{c}b_i(\mathbf{W},\phi)=\frac{h}{c}\left(1+\phi\sum_{j=1}^{n}a_{ij}\frac{1}{d_j^-}\right)+O\left(\phi^2\right).$$

Hence, we can write the profit of firm i as follows

$$\pi_i=\frac{(1-\alpha)h}{c}\left(1+\phi\sum_{j=1}^{n}a_{ij}\frac{1}{d_j^-}\right)+O\left(\phi^2\right). \tag{A.4}$$

Moreover, total profits can be written as

$$\Pi=\sum_{i=1}^{n}\pi_i=\frac{(1-\alpha)h}{c}\left(n+\phi\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\frac{1}{d_j^-}\right)+O\left(\phi^2\right)=\frac{n(1+\phi)(1-\alpha)h}{c}+O\left(\phi^2\right), \tag{A.5}$$

where we have used the fact that $\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}\frac{1}{d_j^-}=\sum_{j=1}^{n}\frac{1}{d_j^-}\sum_{i=1}^{n}a_{ij}=\sum_{j=1}^{n}\frac{1}{d_j^-}d_j^-=n$. Similarly, total output is given by

$$Y=\sum_{i=1}^{n}s_i=\frac{n(1+\phi)h}{c}+O\left(\phi^2\right). \tag{A.6}$$

Hence, up to leading order terms in ϕ, both, total profits, Π, and total output, Y, are increasing with the number of firms, n. Our measure of welfare will thus be proportional to the number of firms in

[16] Observe that Google's Page rank centrality satisfies the following equation: $x_i=1+\phi\sum_{k=1}^{n}a_{ik}\frac{1}{d_k^+}x_k$. In contrast, the Bonacich centrality satisfies $x_i=1+\phi\sum_{k=1}^{n}a_{ik}x_k$, and the eigenvector centrality satisfies $x_i=\phi\sum_{k=1}^{n}a_{ik}x_k$ with $\phi=1/\lambda_{\text{PF}}$. Denoting by $\mathbf{D}$ the diagonal matrix with diagonal entries given the out-degrees of the nodes, we can write $\mathbf{x}=\mathbf{u}+\phi\mathbf{A}\mathbf{D}^{-1}\mathbf{x}$, and we can write the Page rank centrality vector as follows $\mathbf{x}=(\mathbf{I}_n-\phi\mathbf{A}\mathbf{D}^{-1})^{-1}\mathbf{u}=\mathbf{D}(\mathbf{D}-\phi\mathbf{A})^{-1}\mathbf{u}$.

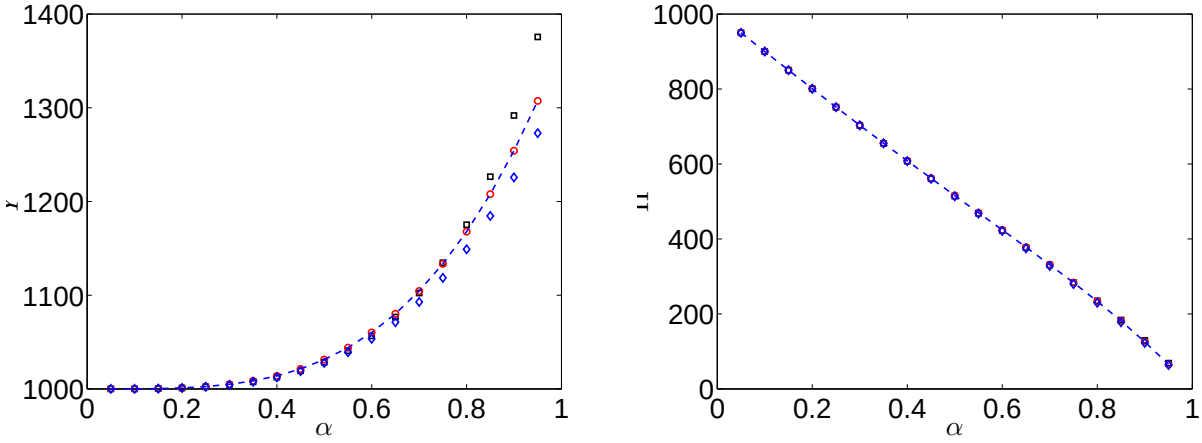


Figure A.2: Total output, Y (left panel), and total profits, $\Pi = \sum_{i=1}^{n} \pi_i$ (right panel), using Equation (A.3) and the approximation in Equations (A.6) and (A.5), respectively, indicated with a dashed line for varying values of $\alpha \in (0, 1)$ with $h = c = 1$ with parameters $n = 1000$.

the economy. Similarly, volatility in output is equivalent to volatility in the number of firms.

Finally, note that due to price equalization, a firm is indifferent from having only a single supplier or multiple suppliers. In particular, assuming that the establishment of a buyer-supplier relationship comes at a (small) fixed cost, a firm's optimal sourcing decision is to buy from a single supplier only [cf. Oberfield, 2011]. The profit of firm i is then given by

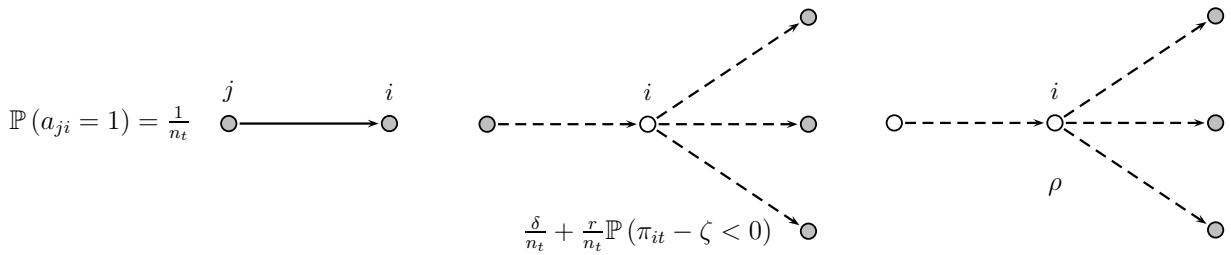
$$\pi_i = \frac{(1-\alpha)h}{c}\left(1 + \phi \sum_{j=1}^{n} a_{ij} \frac{1}{d_j^-}\right) + O\left(\phi^2\right) = \frac{(1-\alpha)h}{c}\left(1 + \phi d_i^+\right) + O\left(\phi^2\right).$$

This concludes the proof of the proposition. □

Proof of Proposition 3. We assume that $\gamma = 0$ (without rewiring). An illustration of the different events taking place during the evolution of the Markov chain is given in Figure A.3. We first give a proof of part (i) of the proposition. Let N_t be the number of firms at time t. Then we have that

$$\begin{aligned} N_{t+1} = N_t + 1 &- \sum_{i=1}^{N_t} \mathbb{1}_{\{\text{firm } i \text{ exits due to a large shock}\}} \\ &- \sum_{i=1}^{N_t} \mathbb{1}_{\{\text{firm } i \text{ exits due to a small shock}\}} \\ &- \sum_{i=1}^{N_t} \mathbb{1}_{\{\text{firm } i \text{ exits because } d_{it}^- = 0\}} \mathbb{1}_{\{d_{it}^- = 0\}}. \end{aligned}$$

Taking the expectation conditional on the filtration $\mathcal{F}_t$ (everything that has happened up to time t)



(a) Entry of a firm i and uniform attachment to an incumbent firm j with probability $\mathbb{P}(a_{ji}=1)=\frac{1}{n_t}$.

(b) The exit of a firm i due to a large shock with probability $\frac{\delta}{n_t}$ or small shock with probability $\frac{r}{n_t}\mathbb{P}(\pi_{it}-\zeta<0)$.

(c) Shock propagation leading to the exit of a firm i with in-degree zero with probability ρ.

Figure A.3: The different events that happen during the time evolution of the Markov chain introduced in Definition 2: (a) entry, (b) exit due to a large or small shock, and (c) shock propagation. Filled circles indicate firms that have not exited, while empty circles indicate firms that have exited. All shocks are assumed to be Zipf distributed (i.e. $\chi=1$).

yields

$$\begin{aligned}\mathbb{E}(N_{t+1}|\mathcal{F}_t)-N_t = 1 &- \sum_{i=1}^{N_t}\mathbb{P}(\{\text{firm } i \text{ exits due to a large shock}\})\\ &- \sum_{i=1}^{N_t}\mathbb{P}(\{\text{firm } i \text{ exits due to a small shock}\})\\ &- \sum_{i=1}^{N_t}\mathbb{P}\big(\{\text{firm } i \text{ exits because } d_{it}^-=0\}\big)\,\mathbb{1}_{\{d_{it}^-=0\}}.\end{aligned}$$

Using the fact that

$$\begin{aligned}\mathbb{P}(\{\text{firm } i \text{ exits due to a large shock}\}) &= \frac{\delta}{N_t},\\ \mathbb{P}(\{\text{firm } i \text{ exits due to a small shock}\}) &= \frac{r}{N_t}\frac{1}{1+d_{it}^+}=\frac{r}{N_t}\frac{1}{\tilde{d}_{it}^+},\\ \mathbb{P}\big(\{\text{firm } i \text{ exits because } d_{it}^-=0\}\big) &= \rho,\end{aligned}$$

where we have introduced the shifted out-degree $\tilde{d}_{it}^+\equiv d_{it}^++1$, this can be written as

$$\begin{aligned}\mathbb{E}(N_{t+1}|\mathcal{F}_t)-N_t &= 1-\delta-\frac{r}{N_t}\sum_{i=1}^{N_t}\frac{1}{\tilde{d}_{it}^+}-\rho N_t\frac{1}{N_t}\sum_{i=1}^{N_t}\mathbb{1}_{\{d_{it}^-=0\}}\\ &= 1-\delta-r\kappa-\rho\tau N_t,\end{aligned}\tag{A.7}$$

where[17]

$$\tau = \frac{1}{N_t}\sum_{i=1}^{N_t} \mathbb{1}_{\{d_{it}^- = 0\}}$$

denotes the fraction of firms with in-degree zero, and we have denoted by

$$\kappa \equiv \frac{1}{N_t}\sum_{i=1}^{N_t} \frac{1}{\tilde{d}_{it}^+}. \tag{A.8}$$

Taking the (unconditional) expectation on both sides of Equation (A.21), and denoting by $n_t = \mathbb{E}\left(N_{t+1}\right) = \mathbb{E}\left(\mathbb{E}\left(N_{t+1}|\mathcal{F}_t\right)\right)$, gives [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\frac{dn_t}{dt} = 1 - \delta - r\kappa - \rho\tau n_t. \tag{A.9}$$

The solution to this differential equation is given by

$$n_t = \left(\rho\tau n_0 + (1-\delta-r\kappa)(e^{\rho\tau t}-1)\right)\frac{e^{-\rho\tau t}}{\rho\tau}, \tag{A.10}$$

with the limit

$$\lim_{t\to\infty} n_t = \frac{1-\delta-r\kappa}{\rho\tau}. \tag{A.11}$$

With the initial condition, $n_0 = 0$, we get

$$n_t = \frac{\left(1-e^{-t\rho\tau}\right)(1-r\kappa-\delta)}{\rho\tau}.$$

Next we prove part (ii) of the proposition. Let d_{it}^- be the in-degree of firm i at time t, conditional on not having exited before time t. Then we have that

$$\begin{aligned} d_{i,t+1}^- = d_{it}^- &- \sum_{j=1}^{N_t} a_{ji,t}\mathbb{1}_{\{j \text{ exits because of a large shock}\}} \\ &- \sum_{j=1}^{N_t} a_{ji,t}\mathbb{1}_{\{j \text{ exits because of a small shock}\}} \\ &- \sum_{j=1}^{N_t} a_{ji,t}\mathbb{1}_{\{j \text{ exits because } d_{jt}^- = 0\}}\mathbb{1}_{\{d_{jt}^- = 0\}}, \end{aligned}$$

for all $t > i$ and the initial condition (at the date of entry) $d_{ii}^- = 1$. Taking the expectation conditional

[17] Note that τ converges to $\mathbb{E}(\mathbb{1}_{\{d_{it}^- = 0\}}) = \mathbb{P}(\{d_{it}^- = 0\}) = 1 - \mathbb{P}(\{d_{it}^- = 1\}) = 1 - \mathbb{E}(d_{it}^-) = 1 - k_{it}^-$.

on $\mathcal{F}_t$ gives

$$\begin{aligned}\mathbb{E}\left(d^-_{i,t+1}|\mathcal{F}_t\right) - d^-_{it} = &-\sum_{j=1}^{N_t} a_{ji,t}\mathbb{P}\left(\{j \text{ exits because of a large shock}\}\right)\\ &-\sum_{j=1}^{N_t} a_{ji,t}\mathbb{P}\left(\{j \text{ exits because of a small shock}\}\right)\\ &-\sum_{j=1}^{N_t} a_{ji,t}\mathbb{P}\left(\{j \text{ exits because } d^-_{jt}=0\}\right)\mathbb{1}_{\{d^-_{jt}=0\}}.\end{aligned}$$

Using the fact that $\mathbb{P}\left(\pi_{it}-\zeta<0\right)=\frac{1}{\tilde{d}^+_{it}}$, and that

$$\begin{aligned}\mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right) &= \frac{\delta}{N_t},\\ \mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right) &= \frac{r}{N_t}\frac{1}{\tilde{d}^+_{it}},\\ \mathbb{P}\left(\{\text{firm } i \text{ exits because } d^-_{it}=0\}\right) &= \rho,\end{aligned}$$

we then get

$$\mathbb{E}\left(d^-_{i,t+1}|\mathcal{F}_t\right) - d^-_{it} = -\sum_{j=1}^{N_t} a_{ji,t}\frac{\delta}{N_t} - \sum_{j=1}^{N_t} a_{ji,t}\frac{r}{N_t}\frac{1}{\tilde{d}^+_{jt}} - \sum_{j=1}^{N_t} a_{ji,t}\rho\tau.$$

We can write this as follows

$$\mathbb{E}\left(d^-_{i,t+1}|\mathcal{F}_t\right) - d^-_{it} = -\frac{1}{N_t}\sum_{j=1}^{N_t} a_{ji,t}\left(\delta + \frac{r}{\tilde{d}^+_{jt}} + \rho\tau N_t\right). \tag{A.12}$$

Taking the (unconditional) expectation on both sides of Equation (A.26), denoting by $k^-_{i,t+1} = \mathbb{E}\left(d^-_{i,t+1}\right) = \mathbb{E}\left(\mathbb{E}\left(d^-_{i,t+1}|\mathcal{F}_t\right)\right)$ and $\bar{k}_t \equiv \frac{1}{N_t}\sum_{j=1}^{N_t}\tilde{d}^+_{jt}$, gives [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\frac{dk^-_{it}}{dt} = -\frac{1}{n_t}k^-_{it}\left(\delta + \frac{r}{1+\bar{k}_t} + \rho\tau n_t\right).$$

Inserting the asymptotic number of firms from Equation (A.25) gives

$$\begin{aligned}\frac{dk_{it}^-}{dt} &= -\frac{\rho\tau}{1-\delta-r\kappa}k_{it}^-\left(\delta+\frac{r}{1+\bar{k}_t}+\rho\tau\frac{1-\delta-r\kappa}{\rho\tau}\right)\\ &= -\frac{\rho\tau}{1-\delta-r\kappa}k_{it}^-\left(1+r\left(\frac{1}{1+\bar{k}_t}-\kappa\right)\right)\\ &= -\frac{\rho\tau}{1-\delta-r\kappa}k_{it}^-\left(1+r\left(\frac{1}{1+\bar{k}_t}-\kappa\right)\right).\end{aligned}$$

When $(1+\bar{k}_t)^{-1}\approx\kappa=\lim_{t\to\infty}\frac{1}{N_t}\sum_{i=1}^{N_t}\frac{1}{1+d_{it}^+}$ this can be written as

$$\frac{dk_{it}^-}{dt} = -\frac{\rho\tau}{1-\delta-r\kappa}k_{it}^-, \tag{A.13}$$

with the initial condition $k_{ii}^-=1$, and the solution

$$k_{it}^- = e^{-\frac{(t-i)\rho\tau}{1-\delta-r\kappa}}. \tag{A.14}$$

This shows that the in-degree of firm i is exponentially decaying.

Next, let $\tilde{d}_{it}^+$ be the shifted out-degree of firm i at time t, conditional on not having exited before time t. Then we have that

$$\begin{aligned}\tilde{d}_{i,t+1}^+ = \tilde{d}_{it}^+ + \mathbb{1}_{\{i\to t\}} &- \sum_{j=1}^{N_t} a_{ij,t}\mathbb{1}_{\{j \text{ exits due to a large shock}\}}\\ &- \sum_{j=1}^{n_t} a_{ij,t}\mathbb{1}_{\{j \text{ exits due to a small shock}\}},\end{aligned}$$

for all $t>i+1$ and the initial condition $\tilde{d}_{ii}^+=1$. Taking the expectation conditional on $\mathcal{F}_t$ gives

$$\begin{aligned}\mathbb{E}\left(\tilde{d}_{i,t+1}^+|\mathcal{F}_t\right)-\tilde{d}_{it}^+ = \mathbb{P}\left(\{i\to t\}\right) &- \sum_{j=1}^{N_t} a_{ij,t}\mathbb{P}\left(\{j \text{ exits due to a large shock}\}\right)\\ &- \sum_{j=1}^{N_t} a_{ij,t}\mathbb{P}\left(\{j \text{ exits due to a small shock}\}\right).\end{aligned}$$

Using the fact that

$$\begin{aligned}\mathbb{P}\left(\{i\to t\}\right) &= \frac{1}{N_t},\\ \mathbb{P}\left(\{\text{firm } j \text{ exits due to a large shock}\}\right) &= \frac{\delta}{N_t},\\ \mathbb{P}\left(\{\text{firm } j \text{ exits due to a small shock}\}\right) &= \frac{r}{N_t}\frac{1}{\tilde{d}_{jt}^+},\end{aligned}$$

we can write this as follows

$$\mathbb{E}\left(\tilde{d}^+_{i,t+1}|\mathcal{F}_t\right) - \tilde{d}^+_{it} = \frac{1}{N_t} - \sum_{j=1}^{N_t} a_{ij,t}\frac{\delta}{N_t} - \sum_{j=1}^{N_t} a_{ij,t}\frac{r}{N_t}\frac{1}{\tilde{d}^+_{jt}}.$$

This can be written as

$$\mathbb{E}\left(\tilde{d}^+_{i,t+1}|\mathcal{F}_t\right) - \tilde{d}^+_{it} = \frac{1}{N_t} - \sum_{j=1}^{N_t} a_{ij,t}\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{\tilde{d}^+_{jt}}\right).$$

Taking the expectation on both sides of the equation and denoting by $k^+_{it} = \mathbb{E}\left(\tilde{d}^+_{it}\right)$ yields [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\frac{dk^+_{it}}{dt} = \frac{1}{n_t} - \left(\delta + \frac{r}{1+\bar{k}_t}\right)\frac{k^+_{it}}{n_t}.$$

In the stationary state we have that $\bar{k}_t = \bar{k}$, $\lim_{t\to\infty} n_t = \frac{1-\delta-r\kappa}{\rho\tau}$, so that we can write this as follows

$$\frac{dk^+_{it}}{dt} = a - bk^+_{it},$$

where we have denoted by

$$\begin{aligned} a &\equiv \frac{\rho\tau}{1-\delta-r\kappa}, \\ b &\equiv \frac{\rho\tau}{1-\delta-r\kappa}\left(\delta + r\tilde{\kappa}\right) = a\left(\delta + r\tilde{\kappa}\right). \end{aligned}$$

With the initial condition $k^+_{ii} = 1$ the solution is given by

$$k^+_{it} = \frac{a + (b-a)e^{-b(t-i)}}{b}, \tag{A.15}$$

with the limit

$$\lim_{t\to\infty} k^+_{it} = \frac{a}{b} = \left(\delta + r\tilde{\kappa}\right)^{-1}.$$

Next, let S_{it} be the indicator variable indicating whether firm i is still alive at time t. Then we have that

$$S_{i,t+1} = S_{it}\left(1 - \mathbb{1}_{\{i \text{ exits due to a large shock}\}} - \mathbb{1}_{\{i \text{ exits due to a small shock}\}} - \mathbb{1}_{\{i \text{ exits because } d^-_{it}=0\}}\mathbb{1}_{\{d^-_{it}=0\}}\right).$$

Taking the expectation with respect to $\mathcal{F}_t$ gives

$$\begin{aligned} \mathbb{E}\left(S_{i,t+1}|\mathcal{F}_t\right) = S_{it}\big(1 &- \mathbb{P}\left(\{i \text{ exits due to a large shock}\}\right) - \mathbb{P}\left(\{i \text{ exits due to a small shock}\}\right) \\ &- \mathbb{P}\left(\{i \text{ exits because } d^-_{it}=0\}\right)\mathbb{1}_{\{d^-_{it}=0\}}\big). \end{aligned}$$

Using the fact that

$$\begin{aligned}\mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right) &= \frac{\delta}{N_t},\\ \mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right) &= \frac{r}{N_t}\frac{1}{\tilde{d}_{it}^{+}},\\ \mathbb{P}\left(\{\text{firm } i \text{ exits because } d_{it}^{-}=0\}\right) &= \rho,\end{aligned}$$

gives

$$\mathbb{E}\left(S_{i,t+1}|\mathcal{F}_t\right) = S_{it}\left(1-\frac{\delta}{N_t}-\frac{r}{N_t}\frac{1}{\tilde{d}_{it}^{+}}-\rho\mathbb{1}_{\{d_{it}^{-}=0\}}\right),$$

or equivalently

$$\mathbb{E}\left(S_{i,t+1}|\mathcal{F}_t\right) - S_{it} = -S_{it}\left(\frac{\delta}{N_t}+\frac{r}{N_t}\frac{1}{\tilde{d}_{it}^{+}}+\rho\mathbb{1}_{\{d_{it}^{-}=0\}}\right).$$

Taking the expectation and denoting by $s_{it}=\mathbb{E}\left(S_{it}\right)$ and $k_{it}^{+}=\mathbb{E}\left(\tilde{d}_{it}^{+}\right)$ then gives [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\frac{ds_{it}}{dt} = -s_{it}\left(\frac{\delta}{n_t}+\frac{r}{n_t}\frac{1}{k_{it}^{+}}+\rho\mathbb{P}\left(\{d_{it}^{-}=0\}\right)\right), \tag{A.16}$$

with the initial condition $s_{ii}=1$. Note that by assumption d_{it}^{-} can be either zero or one, so that

$$\begin{aligned}\mathbb{P}\left(\{d_{it}^{-}=0\}\right) &= 1-\mathbb{P}\left(\{d_{it}^{-}=1\}\right)\\ &= 1-\mathbb{E}\left(d_{it}^{-}\right)\\ &= 1-k_{it}^{-}.\end{aligned} \tag{A.17}$$

Inserting Equation (A.32) into Equation (A.31) gives

$$\frac{ds_{it}}{dt} = -s_{it}\left(\frac{\delta}{n_t}+\frac{r}{n_t}\frac{1}{k_{it}^{+}}+\rho(1-k_{it}^{-})\right), \tag{A.18}$$

With k_{it}^{-} from Equation (A.14) and k_{it}^{+} from Equation (A.15) we then get

$$\frac{ds_{it}}{dt} = -s_{it}\left(a\delta+\frac{arb}{a+(b-a)e^{-b(t-i)}}+\rho\left(1-e^{-a(t-i)}\right)\right),$$

or equivalently

$$\frac{d\ln s_{it}}{dt} = -\left(a\delta+\frac{arb}{a+(b-a)e^{-b(t-i)}}+\rho\left(1-e^{-a(t-i)}\right)\right). \tag{A.19}$$

With the initial condition $s_{ii} = 1$ the solution is given by

$$s_{it} = \left(\frac{b}{b - a\left(1 - e^{b(t-i)}\right)}\right)^r \times \exp\left(\frac{\rho}{b}\left(e^{-a(t-i)}\left(\frac{ae^{b(t-i)}}{a-b}\right)^{a/b}\left(B_{\frac{ae^{b(t-i)}}{a-b}}\left(-\frac{a}{b},0\right) - B_{\frac{ae^{b(t-i)}}{a-b}}\left(1-\frac{a}{b},0\right)\right)\right.\right.$$
$$\left.\left. + \left(\frac{a}{a-b}\right)^{a/b}\left(B_{\frac{a}{a-b}}\left(1-\frac{a}{b},0\right) - B_{\frac{a}{a-b}}\left(-\frac{a}{b},0\right)\right)\right) - (t-i)(a\delta+\rho)\right).$$

When $a + (b-a)e^{-b(t-i)} \approx a$ and $1 - e^{-a(t-i)} \approx 1$ for large t we can simplify Equation (A.34) to

$$\frac{d\ln s_{it}}{dt} = -\left(a\delta + rb + \rho\right),$$

and the solution is given by

$$s_{it} = e^{-(br+a\delta+\rho)(t-i)}, \tag{A.20}$$

which is an exponentially decaying function.

We next prove part (iii) of the proposition. Let $N_t^+(k)$ denote the number of firms with out-degree k at time t. Taking the expectation with respect to $\mathcal{F}_t$ gives

$$\mathbb{E}\left(N_{t+1}^+(k)|\mathcal{F}_t\right) - N_t^+(k) = \frac{1}{N_t}N_t^+(k-1) - \frac{1}{N_t}N_t^+(k) - \frac{\delta}{N_t}N_t^+(k) - \frac{r}{N_t}\frac{1}{1+k}N_t^+(k) - \rho N_t^{-+}(0,k)$$
$$- kN_t^+(k)\left(\frac{\delta}{N_t} + \frac{r}{N_t}\sum_{l=0}^{\infty}P_t^+(l|k)\frac{1}{l+1},\right),$$

where $N_t^{-+}(0,k)$ denotes the number of firms with in-degree 0 and out-degree k and $P_t^+(l|k) \equiv \mathbb{P}(d_{j,t}^+ = l|d_{i,t} = k, a_{ij,t} = 1)$. Assuming that there are only weak degree correlations we then can write

$$\begin{aligned} N_t^{-+}(0,k) &= \mathbb{P}\left(d_{it}^+ = k, d_{it}^- = 0\right)N_t \\ &= \left(\mathbb{P}\left(d_{it}^+ = k\right)\mathbb{P}\left(d_{it}^- = 0\right) + \mathrm{Cov}\left(\mathbb{1}_{\{d_{it}^-=0\}}, \mathbb{1}_{\{d_{it}^+=k\}}\right)\right)N_t \\ &\approx \mathbb{P}\left(d_{it}^+ = k\right)\mathbb{P}\left(d_{it}^- = 0\right)N_t \\ &= \frac{N_t^+(k)N_t^-(0)}{N_t}, \end{aligned}$$

where we have assumed that $\mathrm{Cov}\left(\mathbb{1}_{\{d_{it}^-=0\}}, \mathbb{1}_{\{d_{it}^+=k\}}\right) \approx 0$, and

$$P_t^+(l|k) \equiv \mathbb{P}(d_{j,t}^+ = l|d_{i,t} = k, a_{ij,t} = 1) \approx \mathbb{P}(d_{j,t}^+ = l) = \frac{N_t^+(l)}{N_t}.$$

Inserting gives

$$\begin{aligned}\mathbb{E}\left(N_{t+1}^{+}(k)|\mathcal{F}_t\right) - N_t^{+}(k) &= \frac{1}{N_t}N_t^{+}(k-1) - \frac{1}{N_t}N_t^{+}(k) - \frac{\delta}{N_t}N_t^{+}(k) - \frac{r}{N_t}\frac{1}{1+k}N_t^{+}(k) - \rho\tau N_t^{+}(k) \\ &\quad - kN_t^{+}(k)\left(\frac{\delta}{N_t} + \frac{r}{N_t}\sum_{l=0}^{\infty}\frac{N_t^{+}(l)}{N_t}\frac{1}{l+1}\right) \\ &= \frac{1}{N_t}N_t^{+}(k-1) - \frac{1}{N_t}N_t^{+}(k) - \frac{\delta}{N_t}N_t^{+}(k) - \frac{r}{N_t}\frac{1}{1+k}N_t^{+}(k) - \rho\tau N_t^{+}(k) \\ &\quad - k\left(\frac{\delta}{N_t} + \frac{r}{N_t}\kappa\right)N_t^{+}(k),\end{aligned}$$

where $\tau = \frac{N_t^{-}(0)}{N_t}$ denotes the asymptotic fraction of firms with in-degree zero and $\kappa \equiv \sum_{k=0}^{\infty}\frac{N_t^{+}(k)}{N_t}\frac{1}{k+1} = \sum_{k=0}^{\infty}P_t^{+}(k)\frac{1}{k+1}$. We then can write [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\frac{dN_t^{+}(k)}{dt} = \frac{1}{N_t}\left(N_t^{+}(k-1) - \left(1+\delta+\frac{r}{1+k}+\rho N_t^{-}(0)+k(\delta+r\kappa)\right)N_t^{+}(k)\right).$$

In the stationary state, where $\frac{dN_t^{+}(k)}{dt} = 0$, we then have that

$$N^{+}(k) = \frac{1}{1+\delta+\frac{r}{1+k}+\rho N^{-}(0)+k(\delta+r\kappa)}N^{+}(k-1),$$

from which we get

$$\begin{aligned}N^{+}(k) &= N^{+}(0)\prod_{l=1}^{k}\frac{1}{1+\delta+\frac{r}{1+l}+\rho N^{-}(0)+(\delta+r\kappa)l} \\ &= N^{+}(0)\frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}(1+\delta+N^{-}(0)\rho+r)\Gamma(C^{+})\Gamma(C^{-})}{\Gamma(1+C^{+}+k)\Gamma(1+C^{-}+k)},\end{aligned}$$

where

$$C^{\pm} \equiv \frac{1+\kappa r+2\delta+N^{-}(0)\rho \pm \sqrt{(N^{-}(0)\rho+1)^2 - 2r(2\delta+\kappa+\kappa N^{-}(0)\rho)+(\kappa-4)\kappa r^2}}{2(\delta+\kappa r)}.$$

Note that $N^{+}(0) = N - \sum_{k=1}^{\infty}N^{+}(k)$, and consequently

$$N^{+}(0) = N\left(1+(1+\delta+N^{-}(0)\rho+r)\Gamma(C^{+})\Gamma(C^{-})\sum_{k=1}^{\infty}\frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^{+}+k)\Gamma(1+C^{-}+k)}\right)^{-1}.$$

The out-degree distribution $P^+(k) = \frac{N^+(k)}{N}$ is then given by

$$P^+(k) = \left(1 + (1+\delta+N^-(0)\rho+r)\Gamma(C^+)\Gamma(C^-) \sum_{k=1}^{\infty} \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^++k)\Gamma(1+C^-+k)} \right)^{-1}$$
$$\times \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}(1+\delta+N^-(0)\rho+r)\Gamma(C^+)\Gamma(C^-)}{\Gamma(1+C^++k)\Gamma(1+C^-+k)}.$$

For large k we have that

$$\begin{aligned}\frac{\Gamma(k+2)}{\Gamma(C^++k+1)\Gamma(C^-+k+1)} &\sim \frac{k\Gamma(k+1)}{k^{C^+}k^{C^-}\Gamma(k+1)\Gamma(k+1)} \\ &\sim \frac{k^{-C^+-C^-+1}}{\Gamma(k+1)} \\ &\sim \frac{e^k k^{-C^+-C^--k+\frac{1}{2}}}{\sqrt{2\pi}},\end{aligned}$$

where we have used the fact that for large k, $\lim_{k\to\infty}\frac{\Gamma(k+\alpha)}{\Gamma(k)k^\alpha} = 1, \alpha \in \mathbb{R}$, and Stirling's formula, $\Gamma(k+1) \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$, as $k \to \infty$. The out-degree distribution is then asymptotically given by

$$P^+(k) \sim \left(\frac{1}{\delta+\kappa r}\right)^{k+1}\left(\frac{e}{k}\right)^k k^{\frac{1}{2}-C^+-C^-}.$$

Next, recall that the number of firms, N_t, evolves according to

$$\frac{dN_t}{dt} = 1 - \delta - r\kappa_t - \rho N_t^-(0),$$

with $\kappa_t = \sum_{k=0}^{\infty}\frac{1}{1+k}P_t^+(k)$, so that in the stationary state we obtain

$$N^-(0) = \frac{1-\delta-r\kappa}{\rho}.$$

Inserting yields

$$P^+(k) = \left(1 + (2+r(1-\kappa))\Gamma(C^+)\Gamma(C^-) \sum_{k=1}^{\infty} \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^++k)\Gamma(1+C^-+k)} \right)^{-1}$$
$$\times \frac{\Gamma(k+2)\left(\frac{1}{\delta+\kappa r}\right)^{k+1}(2+r(1-\kappa))\Gamma(C^+)\Gamma(C^-)}{\Gamma(1+C^++k)\Gamma(1+C^-+k)},$$

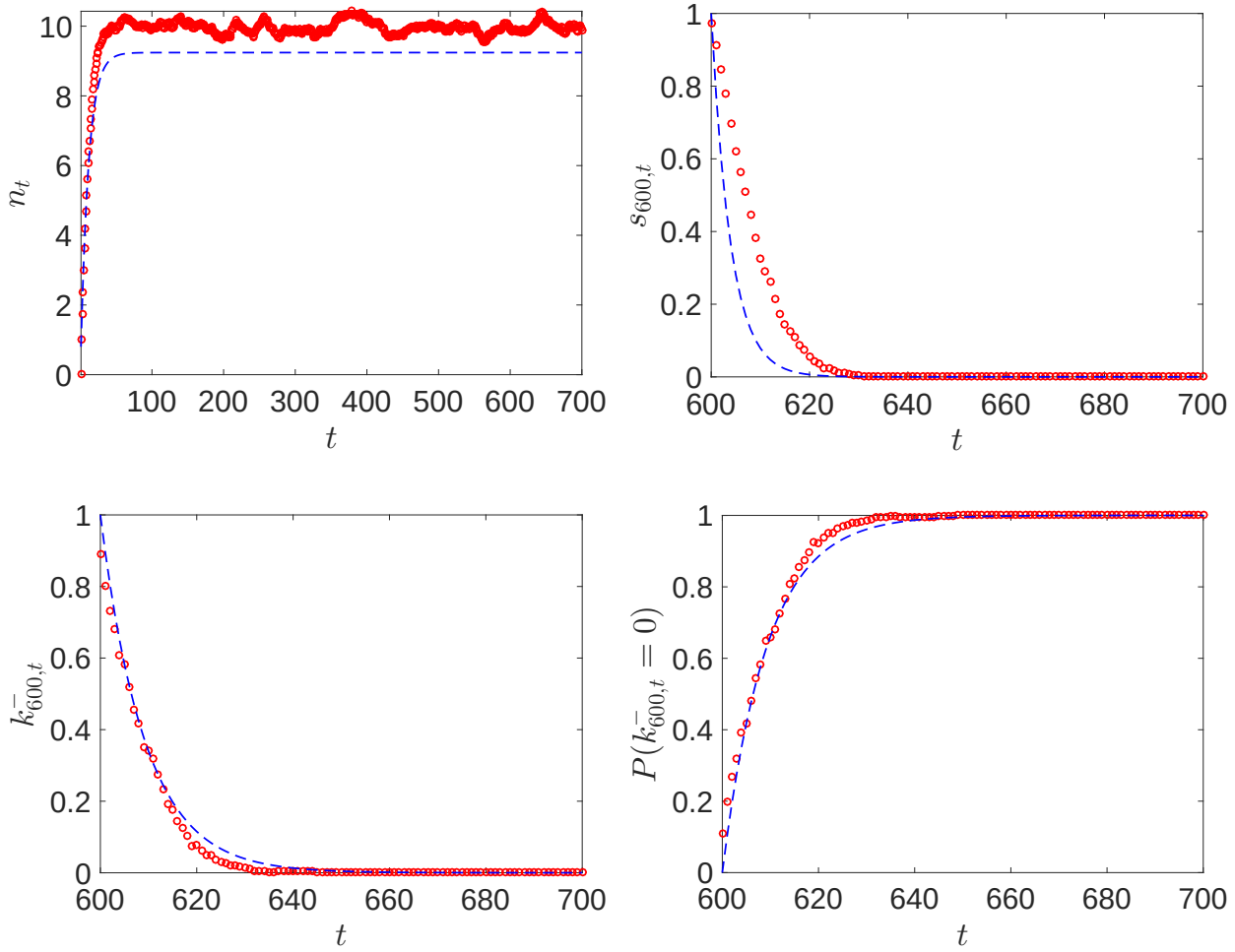


Figure A.4: Simulation results (∘) and theoretical predictions (−−) of the model without rewiring ($\gamma = 0$). (Top left panel) The number of firms n_t over time t. (Top right panel) The survival probability s_{it} of a specific firm $i = 600$ over time t. (Bottom left panel) The in-degree k^-_{it} of a specific firm $i = 600$ over time t. (Bottom right panel) The probability $\mathbb{P}\left(k^-_{it} = 0\right)$ that the in-degree of a specific firm $i = 600$ is zero over time t. The parameters used are $\delta = 0.01$, $\rho = 0.5$, $r = 0.2$, $\rho = 0.25$ and $T = 700$ averaged over 500 independent Monte Carlo simulations.

where

$$C^{\pm} \equiv \frac{2 + \delta \pm \sqrt{(2 - \delta - r\kappa)^2 - 2r(2\delta + \kappa(2 - \delta - r\kappa)) + (\kappa - 4)\kappa r^2}}{2(\delta + \kappa r)}.$$

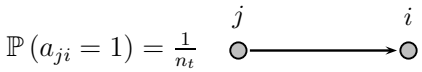
while κ is determined by

$$\kappa = \sum_{k=0}^{\infty} \frac{1}{1+k} P^+(k).$$

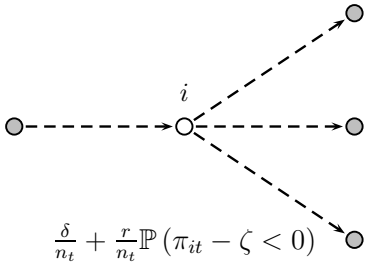
This nonlinear equation can be solved for κ numerically using for example an iterated fixed point algorithm.

□

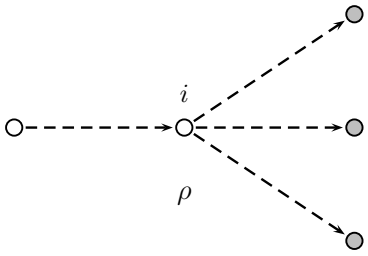
Proof of Proposition 4. We assume that $\gamma > 0$ (with rewiring). An illustration of the different events taking place during the evolution of the Markov chain is given in Figure A.5. We start with



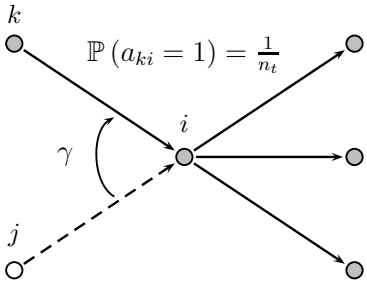
(a) Entry of a firm i and uniform attachment to an incumbent firm j with probability $\mathbb{P}(a_{ji} = 1) = \frac{1}{n_t}$.



(b) The exit of a firm i due to a large shock with probability $\frac{\delta}{n_t}$ or small shock with probability $\frac{r}{n_t}\mathbb{P}(\pi_{it} - \zeta < 0)$.



(c) Shock propagation leading to the exit of a firm i with in-degree zero with probability ρ.



(d) Replacement of a supplier j that has exited with probability γ. Then a new firm k becomes the supplier to i with probability $\mathbb{P}(a_{ki} = 1) = \frac{1}{n_t}$.

Figure A.5: The different events that happen during the time evolution of the Markov chain introduced in Definition 2: (a) entry, (b) exit due to a large or small shock, (c) shock propagation and (d) replacement of a supplier after exit. Filled circles indicate firms that have not exited, while empty circles indicate firms that have exited. All shocks are assumed to be Zipf distributed (i.e. $\chi = 1$).

the proof of part (i) of the proposition. Let N_t be the number of firms at time t. Then we have that

$$\begin{aligned} N_{t+1} = N_t + 1 &- \sum_{i=1}^{N_t} \mathbb{1}_{\{\text{firm } i \text{ exits due to a large shock}\}} \\ &- \sum_{i=1}^{N_t} \mathbb{1}_{\{\text{firm } i \text{ exits due to a small shock}\}} \\ &- \sum_{i=1}^{N_t} \mathbb{1}_{\{\text{firm } i \text{ exits because } d_{it}^- = 0\}} \mathbb{1}_{\{d_{it}^- = 0\}}. \end{aligned}$$

Taking the expectation conditional on the filtration $\mathcal{F}_t$ (everything that has happened up to time t) yields

$$\begin{aligned} \mathbb{E}\left(N_{t+1}|\mathcal{F}_t\right) - N_t = 1 &- \sum_{i=1}^{N_t} \mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right) \\ &- \sum_{i=1}^{N_t} \mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right) \\ &- \sum_{i=1}^{N_t} \mathbb{P}\left(\{\text{firm } i \text{ exits because } d_{it}^- = 0\}\right) \mathbb{1}_{\{d_{it}^- = 0\}}. \end{aligned}$$

Using the fact that

$$\begin{aligned} \mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right) &= \frac{\delta}{N_t}, \\ \mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right) &= \frac{r}{N_t}\frac{1}{\tilde{d}_{it}^+}, \\ \mathbb{P}\left(\{\text{firm } i \text{ exits because } d_{it}^- = 0\}\right) &= \rho, \end{aligned}$$

this can be written as

$$\begin{aligned} \mathbb{E}\left(N_{t+1}|\mathcal{F}_t\right) - N_t &= 1 - \delta - \frac{r}{N_t}\sum_{i=1}^{N_t}\frac{1}{\tilde{d}_{it}^+} - \rho\tau N_t \\ &= 1 - \delta - r\kappa - \rho\tau N_t, \end{aligned} \tag{A.21}$$

where

$$\tau = \frac{1}{N_t}\sum_{i=1}^{N_t} \mathbb{1}_{\{d_{it}^- = 0\}}$$

denotes the fraction of firms with in-degree zero, and we have denoted by

$$\kappa \equiv \frac{1}{N_t}\sum_{i=1}^{N_t}\frac{1}{\tilde{d}_{it}^+}. \tag{A.22}$$

Taking the (unconditional) expectation on both sides of Equation (A.21), and denoting by $n_t = \mathbb{E}(N_{t+1}) = \mathbb{E}(\mathbb{E}(N_{t+1}|\mathcal{F}_t))$, gives [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\frac{dn_t}{dt} = 1 - \delta - r\kappa - \rho\tau n_t. \tag{A.23}$$

The solution to this differential equation is given by

$$n_t = \left(\rho\tau n_0 + (1 - \delta - r\kappa)(e^{\rho\tau t} - 1)\right) \frac{e^{-\rho\tau t}}{\rho\tau}, \tag{A.24}$$

with the limit

$$\lim_{t\to\infty} n_t = \frac{1 - \delta - r\kappa}{\rho\tau}. \tag{A.25}$$

With the initial condition, $n_0 = 0$, we obtain

$$n_t = \frac{\left(1 - e^{-t\rho\tau}\right)(1 - r\kappa - \delta)}{\rho\tau}.$$

Next, we consider part (ii) of the proposition. Let d_{it}^- be the in-degree of firm i at time t, conditional on not having exited before time t. Then we have that

$$\begin{aligned} d_{i,t+1}^- = d_{it}^- &- \sum_{j=1}^{N_t} a_{ji,t} \mathbb{1}_{\{j \text{ exits because of a large shock}\}} \mathbb{1}_{\{i \text{ fails to replace } j\}} \\ &- \sum_{j=1}^{N_t} a_{ji,t} \mathbb{1}_{\{j \text{ exits because of a small shock}\}} \mathbb{1}_{\{i \text{ fails to replace } j\}} \\ &- \sum_{j=1}^{N_t} a_{ji,t} \mathbb{1}_{\{j \text{ exits because } d_{jt}^- = 0\}} \mathbb{1}_{\{d_{jt}^- = 0\}} \mathbb{1}_{\{i \text{ fails to replace } j\}}, \end{aligned}$$

for all $t > i$ and the initial condition (at the date of entry) $d_{ii}^- = 1$. Taking the expectation conditional on $\mathcal{F}_t$ gives

$$\begin{aligned} \mathbb{E}\left(d_{i,t+1}^- | \mathcal{F}_t\right) - d_{it}^- = &- \sum_{j=1}^{N_t} a_{ji,t} \mathbb{P}\left(\{j \text{ exits because of a large shock}\}\right) \mathbb{P}\left(\{i \text{ fails to replace } j\}\right) \\ &- \sum_{j=1}^{N_t} a_{ji,t} \mathbb{P}\left(\{j \text{ exits because of a small shock}\}\right) \mathbb{P}\left(\{i \text{ fails to replace } j\}\right) \\ &- \sum_{j=1}^{N_t} a_{ji,t} \mathbb{P}\left(\{j \text{ exits because } d_{jt}^- = 0\}\right) \mathbb{1}_{\{d_{jt}^- = 0\}} \mathbb{P}\left(\{i \text{ fails to replace } j\}\right). \end{aligned}$$

Using the fact that $\mathbb{P}\left(\pi_{it}-\zeta<0\right)=\frac{1}{\tilde{d}_{it}^{+}}$, and that

$$\begin{aligned}\mathbb{P}\left(i \text{ fails to replace } j\right)&=1-\gamma,\\ \mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right)&=\frac{\delta}{N_t},\\ \mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right)&=\frac{r}{N_t}\frac{1}{\tilde{d}_{it}^{+}},\\ \mathbb{P}\left(\{\text{firm } i \text{ exits because } d_{it}^{-}=0\}\right)&=\rho,\end{aligned}$$

we then get

$$\mathbb{E}\left(d_{i,t+1}^{-}|\mathcal{F}_t\right)-d_{it}^{-}=-\sum_{j=1}^{N_t}a_{ji,t}\frac{\delta}{N_t}\left(1-\gamma\right)-\sum_{j=1}^{N_t}a_{ji,t}\frac{r}{N_t}\frac{1}{\tilde{d}_{jt}^{+}}\left(1-\gamma\right)-\sum_{j=1}^{N_t}a_{ji,t}\rho\tau\left(1-\gamma\right).$$

We can write this as follows

$$\mathbb{E}\left(d_{i,t+1}^{-}|\mathcal{F}_t\right)-d_{it}^{-}=-\frac{1}{N_t}\left(1-\gamma\right)\sum_{j=1}^{N_t}a_{ji,t}\left(\delta+\frac{r}{\tilde{d}_{jt}^{+}}+\rho\tau N_t\right). \tag{A.26}$$

Taking the (unconditional) expectation on both sides of Equation (A.26), and denoting by $k_{i,t+1}^{-}=\mathbb{E}\left(d_{i,t+1}^{-}\right)=\mathbb{E}\left(\mathbb{E}\left(d_{i,t+1}^{-}|\mathcal{F}_t\right)\right)$, gives [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\frac{dk_{it}^{-}}{dt}=-\frac{1}{n_t}\left(1-\gamma\right)k_{it}^{-}\left(\delta+\frac{r}{1+\bar{k}_t}+\rho\tau n_t\right).$$

Inserting the asymptotic number of firms from Equation (A.25) gives

$$\begin{aligned}\frac{dk_{it}^{-}}{dt}&=-\frac{\rho\tau(1-\gamma)}{1-\delta-r\kappa}k_{it}^{-}\left(\delta+\frac{r}{1+\bar{k}_t}+\rho\tau\frac{1-\delta-r\kappa}{\rho\tau}\right)\\ &=-\frac{\rho\tau(1-\gamma)}{1-\delta-r\kappa}k_{it}^{-}\left(1+r\left(\frac{1}{1+\bar{k}_t}-\kappa\right)\right)\\ &=-\frac{\rho\tau(1-\gamma)}{1-\delta-r\kappa}\left(1+r\left(\frac{1}{1+\bar{k}_t}-\kappa\right)\right)k_{it}^{-}.\end{aligned} \tag{A.27}$$

When $(1+\bar{k}_t)^{-1}\approx\kappa$ this can be written as

$$\frac{dk_{it}^{-}}{dt}=-\frac{\rho\tau(1-\gamma)}{1-\delta-r\kappa}k_{it}^{-}. \tag{A.28}$$

With the initial condition $k_{ii}^{-}=1$, the solution is given by

$$k_{it}^{-}=e^{-\frac{\rho\tau(1-\gamma)}{1-\delta-r\kappa}(t-i)}. \tag{A.29}$$

This shows that the in-degree of firm i is exponentially decaying.

Next, let $\tilde{d}^+_{it}$ be the shifted out-degree of firm i at time t, conditional on not having exited before time t. Then we have that

$$\begin{aligned}\tilde{d}^+_{i,t+1} = \tilde{d}^+_{it} + \mathbb{1}_{\{i\to t\}} &- \sum_{j=1}^{N_t} a_{ij,t}\mathbb{1}_{\{j \text{ exits due to a large shock}\}} \\ &- \sum_{j=1}^{n_t} a_{ij,t}\mathbb{1}_{\{j \text{ exits due to a small shock}\}} \\ &+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\mathbb{1}_{\{k \text{ exits due to a large shock}\}}\mathbb{1}_{\{j \text{ rewires the link to } i\}} \\ &+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\mathbb{1}_{\{k \text{ exits due to a small shock}\}}\mathbb{1}_{\{j \text{ rewires the link to } i\}} \\ &+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\mathbb{1}_{\{k \text{ exits because } d^-_{kt}=0\}}\mathbb{1}_{\{j \text{ rewires the link to } i\}}\mathbb{1}_{\{d^-_{kt}=0\}},\end{aligned}$$

for all $t > i+1$ and the initial condition $\tilde{d}^+_{ii} = 1$. Taking the expectation conditional on $\mathcal{F}_t$ gives

$$\begin{aligned}\mathbb{E}\left(\tilde{d}^+_{i,t+1}|\mathcal{F}_t\right) - \tilde{d}^+_{it} = \mathbb{P}\left(\{i\to t\}\right) &- \sum_{j=1}^{N_t} a_{ij,t}\mathbb{P}\left(\{j \text{ exits due to a large shock}\}\right) \\ &- \sum_{j=1}^{N_t} a_{ij,t}\mathbb{P}\left(\{j \text{ exits due to a small shock}\}\right) \\ &+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\mathbb{P}\left(\{k \text{ exits due to a large shock}\}\right)\mathbb{P}\left(\{j \text{ rewires the link to } i\}\right) \\ &+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\mathbb{P}\left(\{k \text{ exits due to a small shock}\}\right)\mathbb{P}\left(\{j \text{ rewires the link to } i\}\right) \\ &+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\mathbb{P}\left(\{k \text{ exits because } d^-_{kt}=0\}\right)\mathbb{P}\left(\{j \text{ rewires the link to } i\}\right)\mathbb{1}_{\{d^-_{kt}=0\}}.\end{aligned}$$

Using the fact that

$$\begin{aligned}
\mathbb{P}\left(\{i \to t\}\right) &= \frac{1}{N_t}, \\
\mathbb{P}\left(\{\text{firm } j \text{ exits due to a large shock}\}\right) &= \frac{\delta}{N_t}, \\
\mathbb{P}\left(\{\text{firm } j \text{ exits due to a small shock}\}\right) &= \frac{r}{N_t}\frac{1}{\tilde{d}_{jt}^{+}}, \\
\mathbb{P}\left(\{\text{firm } k \text{ exits because } d_{kt}^{-}=0\}\right) &= \rho, \\
\mathbb{P}\left(\{j \text{ rewires the link to } i\}\right) &= \gamma\frac{1}{n_t},
\end{aligned}$$

we can write this as follows

$$\begin{aligned}
\mathbb{E}\left(\tilde{d}_{i,t+1}^{+}|\mathcal{F}_t\right) - \tilde{d}_{it}^{+} = \frac{1}{N_t} &- \sum_{j=1}^{N_t} a_{ij,t}\frac{\delta}{N_t} - \sum_{j=1}^{N_t} a_{ij,t}\frac{r}{N_t}\frac{1}{\tilde{d}_{jt}^{+}} \\
&+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\frac{\delta}{N_t}\gamma\frac{1}{n_t} \\
&+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\frac{r}{N_t}\frac{1}{\tilde{d}_{kt}^{+}}\gamma\frac{1}{n_t} \\
&+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\mathbb{1}_{\{d_{kt}^{-}=0\}}\rho\gamma\frac{1}{n_t}.
\end{aligned}$$

This can be written as

$$\begin{aligned}
\mathbb{E}\left(\tilde{d}_{i,t+1}^{+}|\mathcal{F}_t\right) - \tilde{d}_{it}^{+} = \frac{1}{N_t} &- \sum_{j=1}^{N_t} a_{ij,t}\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{\tilde{d}_{jt}^{+}}\right) \\
&+ \gamma\frac{1}{n_t}\sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{\tilde{d}_{kt}^{+}} + \rho\mathbb{1}_{\{d_{kt}^{-}=0\}}\right).
\end{aligned}$$

Taking the expectation on both sides of the equation and denoting by $k_{it}^{+} = \mathbb{E}\left(\tilde{d}_{it}^{+}\right)$ yields [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\begin{aligned}
\frac{dk_{it}^{+}}{dt} &= \frac{1}{n_t} - \frac{k_{it}^{+}}{n_t}\left(\delta + \frac{r}{1+\bar{k}_t}\right) + \gamma\frac{1}{n_t}n_t\frac{1+\bar{k}_t}{n_t}\left(\delta + \frac{r}{1+\bar{k}_t} + \rho\tau n_t\right) \\
&= \frac{1}{n_t}\left(1 + \gamma\left(\delta + \frac{r}{1+\bar{k}_t} + \rho\tau n_t\right)(1+\bar{k}_t)\right) - \frac{1}{n_t}\left(\delta + \frac{r}{1+\bar{k}_t}\right)k_{it}^{+}.
\end{aligned}$$

In the stationary state we have that $\bar{k}_t = \bar{k}$, $\lim_{t\to\infty} n_t = \frac{1-\delta-r\kappa}{\rho\tau}$, so that we can write this as follows

$$\frac{dk_{it}^+}{dt} = a - bk_{it}^+,$$

where we have denoted by

$$\begin{aligned}
a &\equiv \frac{\rho\tau}{1-\delta-r\kappa}\left(1+\gamma\left(\delta+r\tilde{\kappa}+1-\delta-r\kappa\right)\left(1+\bar{k}\right)\right), \\
&= \frac{\rho\tau}{1-\delta-r\kappa}\left(1+\gamma\left(1+r\left(\frac{1}{1+\bar{k}}-\kappa\right)\right)\left(1+\bar{k}\right)\right) \\
&\approx \frac{\rho\tau}{1-\delta-r\kappa}\left(1+\gamma(1+\bar{k})\right) \\
b &\equiv \frac{\rho\tau}{1-\delta-r\kappa}\left(\delta+r\tilde{\kappa}\right).
\end{aligned}$$

With the initial condition $k_{ii}^+ = 0$ the solution is given by

$$k_{it}^+ = \frac{a+(b-a)e^{-b(t-i)}}{b}, \tag{A.30}$$

with the limit

$$\lim_{t\to\infty} k_{it}^+ = \frac{a}{b} = \left(1+\gamma\left(1+r\left(\frac{1}{1+\bar{k}}-\kappa\right)\right)\left(1+\bar{k}\right)\right)\left(\delta+r\tilde{\kappa}\right)^{-1} \approx \left(1+\gamma(1+\bar{k})\right)\left(\delta+r\tilde{\kappa}\right)^{-1}.$$

Next, let S_{it} be the indicator variable indicating whether firm i is still alive at time t. Then we have that

$$S_{i,t+1} = S_{it}\left(1 - \mathbb{1}_{\{i \text{ exits due to a large shock}\}} - \mathbb{1}_{\{i \text{ exits due to a small shock}\}} - \mathbb{1}_{\{i \text{ exits because } d_{it}^- = 0\}}\mathbb{1}_{\{d_{it}^-=0\}}\right).$$

Taking the expectation with respect to $\mathcal{F}_t$ gives

$$\begin{aligned}
\mathbb{E}\left(S_{i,t+1}|\mathcal{F}_t\right) = S_{it}\big(1 &- \mathbb{P}\left(\{i \text{ exits due to a large shock}\}\right) - \mathbb{P}\left(\{i \text{ exits due to a small shock}\}\right) \\
&-\mathbb{P}\left(\{i \text{ exits because } d_{it}^- = 0\}\right)\mathbb{1}_{\{d_{it}^-=0\}}\Big).
\end{aligned}$$

Using the fact that

$$\begin{aligned}
\mathbb{P}\left(\{\text{firm } i \text{ exits due to a large shock}\}\right) &= \frac{\delta}{N_t}, \\
\mathbb{P}\left(\{\text{firm } i \text{ exits due to a small shock}\}\right) &= \frac{r}{N_t}\frac{1}{\tilde{d}_{it}^+}, \\
\mathbb{P}\left(\{\text{firm } i \text{ exits because } d_{it}^- = 0\}\right) &= \rho,
\end{aligned}$$

gives

$$\mathbb{E}\left(S_{i,t+1}|\mathcal{F}_t\right) = S_{it}\left(1 - \frac{\delta}{N_t} - \frac{r}{N_t}\frac{1}{\tilde{d}_{it}^+} - \rho\mathbb{1}_{\{d_{it}^-=0\}}\right),$$

or equivalently

$$\mathbb{E}\left(S_{i,t+1}|\mathcal{F}_t\right) - S_{it} = -S_{it}\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{\tilde{d}_{it}^+} + \rho\mathbb{1}_{\{d_{it}^-=0\}}\cdot\right).$$

Taking the expectation and denoting by $s_{it} = \mathbb{E}\left(S_{it}\right)$ and $k_{it}^+ = \mathbb{E}\left(\tilde{d}_{it}^+\right)$ then gives [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\frac{ds_{it}}{dt} = -s_{it}\left(\frac{\delta}{n_t} + \frac{r}{n_t}\frac{1}{k_{it}^+} + \rho\mathbb{P}\left(\{d_{it}^- = 0\}\right)\right), \tag{A.31}$$

with the initial condition $s_{ii} = 1$. Note that by assumption d_{it}^- can be either zero or one, so that

$$\begin{aligned}\mathbb{P}\left(\{d_{it}^- = 0\}\right) &= 1 - \mathbb{P}\left(\{d_{it}^- = 1\}\right)\\ &= 1 - \mathbb{E}\left(d_{it}^-\right)\\ &= 1 - k_{it}^-.\end{aligned} \tag{A.32}$$

Inserting Equation (A.32) into Equation (A.31) gives

$$\frac{ds_{it}}{dt} = -s_{it}\left(\frac{\delta}{n_t} + \frac{r}{n_t}\frac{1}{k_{it}^+} + \rho(1 - k_{it}^-)\right), \tag{A.33}$$

With k_{it}^- from Equation (A.14), k_{it}^+ from Equation (A.15) and denoting by $c = 1/\lim_{t\to\infty} n_t = \frac{\rho\tau}{1-\delta-r\kappa}$, we then get

$$\frac{ds_{it}}{dt} = -s_{it}\left(c\delta + \frac{arb}{a + (b-a)e^{-b(t-i)}} + \rho\left(1 - e^{-c(1-\gamma)(t-i)}\right)\right),$$

or equivalently

$$\frac{d\ln s_{it}}{dt} = -\left(c\delta + \frac{arb}{a + (b-a)e^{-b(t-i)}} + \rho\left(1 - e^{-c(1-\gamma)(t-i)}\right)\right). \tag{A.34}$$

With the initial condition $s_{ii} = 1$ the solution is given by

$$s_{it} = \left(\frac{b}{ae^{b(t-i)} - a + b}\right)^r \exp\left(-\frac{\rho e^{-c(1-\gamma)(t-i)}}{c(1-\gamma)} - \rho\left(-\frac{1}{c(1-\gamma)} - i + t\right) - c\delta(t-i)\right).$$

When $a+(b-a)e^{-b(t-i)} \approx a$ and $1-e^{-c(1-\gamma)(t-i)} \approx 1$ for large t we can simplify Equation (A.34) to

$$\frac{d\ln s_{it}}{dt} = -\left(c\delta + rb + \rho\right),$$

and the solution is given by

$$s_{it} = e^{-(br+c\delta+\rho)(t-i)}, \tag{A.35}$$

which is an exponentially decaying function.

We next give a proof of part (iii) of the proposition. Let $N_t^+(k)$ denote the number of firms with out-degree k at time t. Then we have that

$$\begin{aligned}
\mathbb{E}\left(N_{t+1}^+(k)|\mathcal{F}_t\right) - N_t^+(k) = {} & \frac{1}{N_t}N_t^+(k-1) - \frac{1}{N_t}N_t^+(k) - \frac{\delta}{N_t}N_t^+(k) - \frac{r}{N_t}\frac{1}{1+k}N_t^+(k) - \rho N_t^{-+}(0,k) \\
& - kN_t^+(k)\left(\frac{\delta}{N_t} + \frac{r}{N_t}\sum_{l=0}^{\infty}P_t^+(l|k)\frac{1}{l+1}\right) \\
& + \frac{\gamma}{N_t}N_t(k-1)\left(\sum_{l=0}^{\infty}N_t^+(l)l\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{l+1}\right) + \sum_{l=0}^{\infty}\rho N_t^{-+}(0,l)l\right) \\
& - \frac{\gamma}{N_t}N_t(k)\left(\sum_{l=0}^{\infty}N_t^+(l)l\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{l+1}\right) + \sum_{l=0}^{\infty}\rho N_t^{-+}(0,l)l\right),
\end{aligned}$$

where $N_t^{-+}(0,k)$ denotes the number of firms with in-degree 0 and out-degree k and $P_t^+(l|k) \equiv \mathbb{P}(d_{j,t}^+ = l|d_{i,t} = k, a_{ij,t} = 1)$. Assuming that there are only weak degree correlations we then can write

$$\begin{aligned}
N_t^{-+}(0,k) &= \mathbb{P}\left(d_{it}^+ = k, d_{it}^- = 0\right)N_t \\
&\approx \mathbb{P}\left(d_{it}^+ = k\right)\mathbb{P}\left(d_{it}^- = 0\right)N_t \\
&= \frac{N_t^+(k)N_t^-(0)}{N_t},
\end{aligned}$$

and

$$P_t^+(l|k) \equiv \mathbb{P}(d_{j,t}^+ = l|d_{i,t} = k, a_{ij,t} = 1) \approx \mathbb{P}(d_{j,t}^+ = l) = \frac{N_t^+(l)}{N_t}.$$

Inserting gives

$$\begin{aligned}
\mathbb{E}\left(N_{t+1}^+(k)|\mathcal{F}_t\right) - N_t^+(k) = {} & \frac{1}{N_t}N_t^+(k-1) - \frac{1}{N_t}N_t^+(k) - \frac{\delta}{N_t}N_t^+(k) - \frac{r}{N_t}\frac{1}{1+k}N_t^+(k) - \rho\frac{N_t^-(0)}{N_t}N_t^+(k) \\
& - kN_t^+(k)\left(\frac{\delta}{N_t} + \frac{r}{N_t}\sum_{l=0}^{\infty}\frac{N_t^+(l)}{N_t}\frac{1}{l+1}\right) \\
& + \frac{\gamma}{N_t}\left(N_t(k-1) - N_t(k)\right)\left((\delta + \rho N_t^-(0))\frac{1}{N_t}\sum_{l=0}^{\infty}lN_t^+(l) + \frac{r}{N_t}\sum_{l=0}^{\infty}\frac{l}{l+1}N_t^+(l)\right).
\end{aligned}$$

Denoting by

$$\bar{k}_t \equiv \frac{1}{N_t}\sum_{l=0}^{\infty} l N_t^+(l),$$

$$\kappa_t \equiv \frac{1}{N_t}\sum_{l=0}^{\infty} \frac{1}{l+1} N_t^+(l),$$

and

$$\tilde{\kappa}_t \equiv \bar{k}_t^{-1}\frac{1}{N_t}\sum_{l=0}^{\infty} \frac{l}{l+1} N_t^+(l),$$

we can write this as

$$\begin{aligned}\mathbb{E}\left(N_{t+1}^+(k)|\mathcal{F}_t\right) - N_t^+(k) = \frac{1}{N_t}N_t^+(k-1) - \frac{1}{N_t}N_t^+(k) - \frac{\delta}{N_t}N_t^+(k) - \frac{r}{N_t}\frac{1}{1+k}N_t^+(k) - \rho\frac{N_t^-(0)}{N_t}N_t^+(k) \\ + \frac{\gamma}{N_t}\left(N_t(k-1) - N_t(k)\right)\bar{k}_t\left(\delta + \rho N_t^-(0) + r\tilde{k}_t\right) - \frac{1}{N_t}k(\delta + r\kappa_t)N_t^+(k).\end{aligned}$$

We then can write [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\begin{aligned}\frac{dN_t^+(k)}{dt} = \frac{1}{N_t}\left(\left(1 + \gamma\bar{k}_t\left(\delta + r\tilde{\kappa}_t + \rho N_t^-(0)\right)\right)N_t^+(k-1)\right. \\ \left. - \left(1 + \delta + \frac{r}{1+k} + \rho N_t^-(0) + \gamma\bar{k}_t\left(\delta + r\tilde{\kappa}_t + \rho N_t^-(0)\right) + k(\delta + r\kappa_t)\right)N_t^+(k)\right).\end{aligned}$$

In the stationary state, where $\frac{dN_t^+(k)}{dt} = 0$, we then have that

$$N^+(k) = \frac{1 + \gamma\left(\delta + r\tilde{\kappa} + \rho N^-(0)\right)\bar{k}}{1 + \delta + \frac{r}{1+k} + \rho N^-(0) + \gamma\left(\delta + r\tilde{\kappa} + \rho N^-(0)\right)\bar{k} + k(\delta + r\kappa)}N^+(k-1),$$

from which we get

$$\begin{aligned}N^+(k) &= N^+(0)\prod_{l=1}^{k}\frac{1 + \gamma\left(\delta + r\tilde{\kappa} + \rho N^-(0)\right)\bar{k}}{1 + \delta + \frac{r}{1+l} + \rho N^-(0) + \gamma\left(\delta + r\tilde{\kappa} + \rho N^-(0)\right)\bar{k} + l(\delta + r\kappa)} \\ &= N^+(0)\frac{\Gamma(k+2)\left(\frac{1+\gamma\bar{k}(\delta+\rho N^-(0)+r\tilde{\kappa})}{\delta+\kappa r}\right)^{k+1}(1 + \delta + N^-(0)\rho + r + \gamma\bar{k}(\delta + \rho N^-(0) + r\tilde{\kappa}))\Gamma(C^+)\Gamma(C^-)}{(\delta + r\kappa)\Gamma(1 + C^+ + k)\Gamma(1 + C^- + k)},\end{aligned}$$

where

$$\begin{aligned}C^{\pm} \equiv \frac{1}{2(\delta + \kappa r)}\left[1 + \kappa r + 2\delta + N^-(0)\rho + \gamma\bar{k}(\delta + \rho N^-(0) + r\tilde{\kappa})\right. \\ \pm\left(\left(1 + 4\delta + N^-(0)\rho + 3\kappa r + \gamma\bar{k}\left(r\tilde{\kappa} + \delta + N^-(0)\rho\right)\right)^2\right. \\ \left.\left.-4(\delta + \kappa r)\left(2 + r + 4\delta + 2N^-(0)\rho + 2\kappa r + 2\gamma\bar{k}\left(r\tilde{\kappa} + \delta + N^-(0)\rho\right)\right)\right)^{1/2}\right].\end{aligned}$$

Note that $N^+(0) = N - \sum_{k=1}^{\infty} N^+(k)$, and consequently

$$N^+(0) = N \left(1 + \frac{(1+\gamma\left(\delta + r\tilde{\kappa} + \rho N^-(0)\right)\bar{k})\Gamma(C^+)\Gamma(C^-)}{\delta + r\kappa} \sum_{k=1}^{\infty} \frac{\Gamma(k+2)\left(\frac{1+\gamma\bar{k}(\delta+\rho N^-(0)+r\tilde{\kappa})}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^++k)\Gamma(1+C^-+k)} \right)^{-1}.$$

The out-degree distribution $P^+(k) = \frac{N^+(k)}{N}$ is then given by

$$P^+(k) = \left(1 + \frac{(1+\gamma\left(\delta + r\tilde{\kappa} + \rho N^-(0)\right)\bar{k})\Gamma(C^+)\Gamma(C^-)}{\delta + r\kappa} \sum_{k=1}^{\infty} \frac{\Gamma(k+2)\left(\frac{1+\gamma\bar{k}(\delta+\rho N^-(0)+r\tilde{\kappa})}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^++k)\Gamma(1+C^-+k)} \right)^{-1}$$
$$\times \frac{\Gamma(k+2)\left(\frac{1+\gamma\bar{k}(\delta+\rho N^-(0)+r\tilde{\kappa})}{\delta+\kappa r}\right)^{k+1}(1+\delta+N^-(0)\rho+r+\gamma\bar{k}(\delta+\rho N^-(0)+r\tilde{\kappa}))\Gamma(C^+)\Gamma(C^-)}{(\delta+r\kappa)\Gamma(1+C^++k)\Gamma(1+C^-+k)}.$$

For large k we have that

$$\begin{aligned}\frac{\Gamma(k+2)}{\Gamma(C^++k+1)\Gamma(C^-+k+1)} &\sim \frac{k\Gamma(k+1)}{k^{C^+}k^{C^-}\Gamma(k+1)\Gamma(k+1)} \\ &\sim \frac{k^{-C^+-C^-+1}}{\Gamma(k+1)} \\ &\sim \frac{e^k k^{-C^+-C^--k+\frac{1}{2}}}{\sqrt{2\pi}},\end{aligned}$$

where we have used the fact that for large k, $\lim_{k\to\infty}\frac{\Gamma(k+\alpha)}{\Gamma(k)k^\alpha} = 1, \alpha \in \mathbb{R}$, and Stirling's formula, $\Gamma(k+1) \sim \sqrt{2\pi k}\left(\frac{k}{e}\right)^k$, as $k \to \infty$. The out-degree distribution is then asymptotically given by

$$P^+(k) \sim \left(\frac{1}{\delta+\kappa r}\right)^{k+1}\left(\frac{e}{k}\right)^k k^{\frac{1}{2}-C^+-C^-}.$$

Next, recall that the number of firms, N_t, evolves according to

$$\frac{dN_t}{dt} = 1 - \delta - r\kappa_t - \rho N_t^-(0),$$

with $\kappa_t = \sum_{k=0}^{\infty}\frac{1}{1+k}P_t^+(k)$, so that in the stationary state we obtain

$$N^-(0) = \frac{1-\delta-r\kappa}{\rho}.$$

Inserting yields

$$P^+(k) = \left(1 + \frac{(1+\gamma\,(1+r\tilde{\kappa}-r\kappa)\,\bar{k})\Gamma(C^+)\Gamma(C^-)}{\delta+r\kappa}\sum_{k=1}^{\infty}\frac{\Gamma(k+2)\left(\frac{1+\gamma\bar{k}(1-r\kappa+r\tilde{\kappa})}{\delta+\kappa r}\right)^{k+1}}{\Gamma(1+C^++k)\Gamma(1+C^-+k)}\right)^{-1}$$
$$\times\frac{\Gamma(k+2)\left(\frac{1+\gamma\bar{k}(1-r\kappa+r\tilde{\kappa})}{\delta+\kappa r}\right)^{k+1}(2-r\kappa+r+\gamma\bar{k}(1-r\kappa+r\tilde{\kappa}))\Gamma(C^+)\Gamma(C^-)}{(\delta+r\kappa)\Gamma(1+C^++k)\Gamma(1+C^-+k)},$$

where

$$C^{\pm} \equiv \frac{1}{2(\delta+\kappa r)}\Big[2+\kappa r+\delta-r\kappa+\gamma\bar{k}(1-r\kappa+r\tilde{\kappa})$$
$$\pm\Big(\big(2+3\delta-r\kappa+3\kappa r+\gamma\bar{k}\,(1+r\tilde{\kappa}-r\kappa)\big)^2$$
$$-4(\delta+\kappa r)\left(2+r+4\delta+2(1-\delta-r\kappa)+2\kappa r+2\gamma\bar{k}\,(1+r\tilde{\kappa}-r\kappa)\right)\Big)^{1/2}\Big],$$

while κ is determined by

$$\kappa = \sum_{k=0}^{\infty}\frac{1}{1+k}P^+(k),$$

$\tilde{\kappa}$ is given by

$$\tilde{\kappa} = \frac{1}{\bar{k}}\sum_{k=0}^{\infty}\frac{k}{1+k}P^+(k),$$

and $\bar{k}$ is determined by

$$\bar{k} = \sum_{k=0}^{\infty}kP^+(k),$$

The above nonlinear equations for κ, $\tilde{\kappa}$ and $\bar{k}$ can be solved numerically using for example an iterated fixed point algorithm. □

Figure A.6 shows simulation results and theoretical predictions for the out-degree distribution, $P^+(k)$, in Equation (3.7) with rewiring.

Proof of Proposition 6. Figure A.7 illustrates the different events that happen during the time evolution of the Markov chain introduced in Definition 5.

Part (i) of the proposition is identical to the proof of Proposition 4.

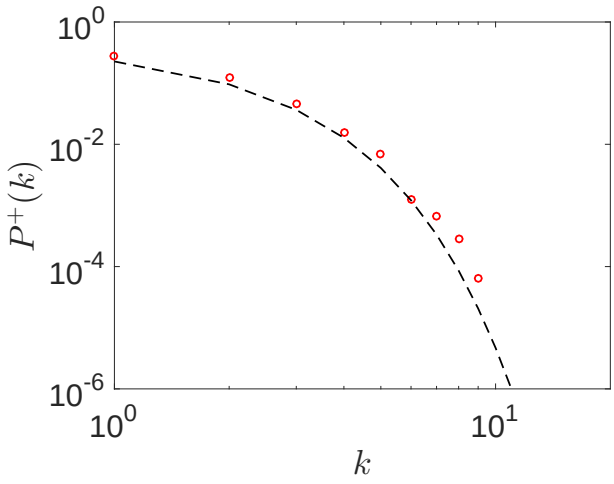
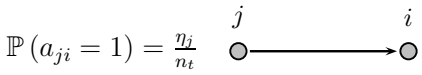
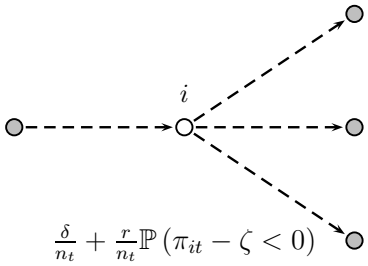


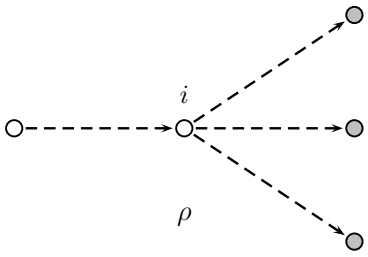
Figure A.6: Simulation results (○) and theoretical predictions (−−) for the out-degree distribution, $P^+(k)$, in Equation (3.7) with rewiring ($\gamma = 0.1$).



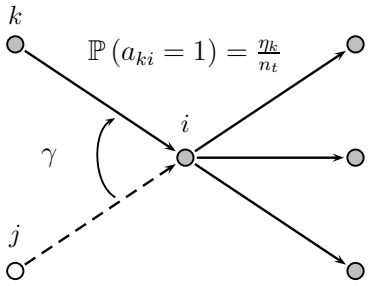
(a) Entry of a firm i and uniform attachment to an incumbent firm j with probability $\mathbb{P}(a_{ji} = 1) = \frac{\eta_j}{n_t}$.



(b) The exit of a firm i due to a large shock with probability $\frac{\delta}{n_t}$ or small shock with probability $\frac{r}{n_t}\mathbb{P}(\pi_{it} - \zeta < 0)$.



(c) Shock propagation leading to the exit of a firm i with in-degree zero with probability ρ.



(d) Replacement of a supplier j that has exited with probability γ. Then a new firm k becomes the supplier to i with probability $\mathbb{P}(a_{ki} = 1) = \frac{\eta_k}{n_t}$.

Figure A.7: The different events that happen during the time evolution of the Markov chain introduced in Definition 5: (a) entry, (b) exit due to a large or small shock, (c) shock propagation and (e) replacement of a supplier after exit. Filled circles indicate firms that have not exited, while empty circles indicate firms that have exited.

For the proof of part (ii) of the proposition, we use the fact that

$$\begin{aligned}
\mathbb{P}\left(i \text{ fails to replace } j\right) &= 1-\gamma,\\
\mathbb{P}\left(\{i \to t\}\right) &= \frac{\eta_i}{N_t},\\
\mathbb{P}\left(\{\text{firm } j \text{ exits due to a large shock}\}\right) &= \frac{\delta}{N_t},\\
\mathbb{P}\left(\{\text{firm } j \text{ exits due to a small shock}\}\right) &= \frac{r}{N_t}\frac{1}{d_{jt}^+},\\
\mathbb{P}\left(\{\text{firm } j \text{ exits because } d_{jt}^- = 0\}\right) &= \rho,\\
\mathbb{P}\left(\{j \text{ rewires the link to } i\}\right) &= \gamma\frac{\eta_i}{N_t},
\end{aligned}$$

where n_t is still given by Equation (A.24), and similarly, the change in the in-degree of firm i is given by Equation (A.28), that is [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\frac{dk_{it}^-}{dt} = -\frac{\rho\tau\left(1-\gamma\right)}{1-\delta-r\kappa}k_{it}^-. \tag{A.36}$$

where we have assumed that $\bar{k}_t^{-1} \approx \kappa$. With the initial condition $k_{ii}^- = 1$, the solution is given by

$$k_{it}^- = e^{-\frac{\rho\tau(1-\gamma)}{1-\delta-r\kappa}(t-i)}. \tag{A.37}$$

This shows that the in-degree of firm i is exponentially decaying.

For the change in the out-degree of firm i we now obtain

$$\begin{aligned}
\mathbb{E}\left(d_{i,t+1}^+|\mathcal{F}_t\right) - d_{it}^+ = \frac{\eta_i}{N_t} &- \sum_{j=1}^{N_t} a_{ij,t}\frac{\delta}{N_t} - \sum_{j=1}^{N_t} a_{ij,t}\frac{r}{N_t}\frac{1}{d_{jt}^+}\\
&+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\frac{\delta}{N_t}\gamma\frac{\eta_i}{N_t}\\
&+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\frac{r}{N_t}\frac{1}{d_{kt}^+}\gamma\frac{\eta_i}{N_t}\\
&+ \sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\mathbb{1}_{\{d_{kt}^-=0\}}\rho\gamma\frac{\eta_i}{N_t}.
\end{aligned}$$

This can be written as

$$\mathbb{E}\left(d_{i,t+1}^{+}|\mathcal{F}_t\right) - d_{it}^{+} = \frac{\eta_i}{N_t} - \sum_{j=1}^{N_t} a_{ij,t}\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{d_{jt}^{+}}\right) + \gamma\frac{\eta_i}{N_t}\sum_{j=1}^{N_t}\sum_{k=1}^{N_t} a_{kj,t}\left(\frac{\delta}{N_t} + \frac{r}{N_t}\frac{1}{d_{kt}^{+}} + \rho\mathbb{1}_{\{d_{kt}^{-}=0\}}\right).$$

Taking the expectation on both sides of the equation and denoting by $k_{it}^{+} = \mathbb{E}\left(d_{it}^{+}\right)$ yields [cf. Darling and Norris, 2008; Kurtz, 1971; Wormald, 1995]

$$\begin{aligned}\frac{dk_{it}^{+}}{dt} &= \frac{\eta_i}{n_t} - \frac{k_{it}^{+}}{n_t}\left(\delta + \frac{r}{\bar{k}_t}\right) + \gamma\frac{\eta_i}{n_t}n_t\frac{\bar{k}_t}{n_t}\left(\delta + \frac{r}{\bar{k}_t} + \rho\tau n_t\right)\\ &= \left(1 + \gamma\left(\delta + \frac{r}{\bar{k}_t} + \rho\tau n_t\right)\bar{k}_t\right)\frac{\eta_i}{n_t} - \left(\delta + \frac{r}{\bar{k}_t}\right)\frac{k_{it}^{+}}{n_t}.\end{aligned}$$

In the stationary state we have that $\bar{k}_t = \bar{k}$, and $\lim_{t\to\infty} n_t = \frac{1-\delta-r\kappa}{\rho\tau}$, so that we can write this as follows

$$\frac{dk_{it}^{+}}{dt} = a\eta_i - bk_{it}^{+}, \tag{A.38}$$

where we have denoted by

$$\begin{aligned}a &\equiv \left(1 + \gamma\left(\delta + r\frac{1}{\bar{k}_t} + \rho\tau\right)\bar{k}_t\right)\frac{\rho\tau}{1-\delta-r\kappa} \approx \left(1 + \gamma\bar{k}(\delta + r\kappa + \rho\tau)\right)\frac{\rho\tau}{1-\delta-r\kappa},\\ b &\equiv \left(\delta + r\frac{1}{\bar{k}_t}\right)\frac{\rho\tau}{1-\delta-r\kappa} \approx \frac{\rho\tau(\delta + r\kappa)}{1-\delta-r\kappa},\end{aligned}$$

and we have assumed that $\frac{1}{k_t} \approx \kappa$. With the initial condition $k_{ii}^{+} = 1$ (for the shifted out-degree) the solution is given by

$$k_{it}^{+} = \frac{a\eta_i + (b - a\eta_i)e^{-b(t-i)}}{b}, \tag{A.39}$$

with the limit

$$\lim_{t\to\infty} k_{it}^{+} = \frac{a\eta_i}{b} \approx \frac{\left(1 + \gamma\bar{k}(\delta + r\kappa + \rho\tau)\right)}{\delta + r\kappa}\eta_i.$$

For the probability that firm i is still alive at time t Equation (A.33) still holds, that is

$$\frac{ds_{it}}{dt} = -s_{it}\left(\frac{\delta}{n_t} + \frac{r}{n_t}\frac{1}{k_{it}^{+}} + \rho(1 - k_{it}^{-})\right).$$

With k_{it}^{-} from Equation (A.37) and k_{it}^{+} from Equation (A.39) we then get

$$\frac{ds_{it}}{dt} = -s_{it}\left(\frac{b\delta}{\delta + r\kappa} + \frac{rb^2}{(\delta + r\kappa)(a\eta_i + (b - a\eta_i)e^{-b(t-i)})} + \rho\left(1 - e^{-\frac{\rho\tau(1-\gamma)}{1-\delta-r\kappa}(t-i)}\right)\right),$$

or equivalently

$$\frac{d\ln s_{it}}{dt} = -\left(\frac{b\delta}{\delta + r\kappa} + \frac{rb^2}{(\delta + r\kappa)(a\eta_i + (b - a\eta_i)e^{-b(t-i)})} + \rho\left(1 - e^{-\frac{\rho\tau(1-\gamma)}{1-\delta-r\kappa}(t-i)}\right)\right),$$

with the initial condition $s_{ii} = 1$. When $a\eta_i + (b - a\eta_i)e^{-b(t-i)} \approx a\eta_i$ and $e^{-\frac{\rho\tau(1-\gamma)}{1-\delta-r\kappa}(t-i)} \approx 0$, we can simplify this to

$$\frac{d\ln s_{it}}{dt} = -\left(\frac{b\delta}{\delta + r\kappa} + \frac{rb^2}{a\eta_i(\delta + r\kappa)} + \rho\right),$$

and the solution is given by

$$s_{it} = e^{-\left(\rho + \frac{b}{\delta + r\kappa}\left(\delta + \frac{rb}{a\eta_i}\right)\right)(t-i)}. \tag{A.40}$$

Next, we give a proof of part (iii) of the proposition. The out-degree distribution is given by

$$P_t^+(k) = \int P_t^+(\eta, k) f(\eta) d\eta,$$

where

$$\begin{aligned} P_t^+(\eta, k) &= \frac{1}{n_t}\sum_{i=1}^{t} s_{it}\mathbb{1}_{\{k_{it}^+ = k\}} \\ &= \frac{1}{n_t}\int_1^t s_{it}\delta(k - k_{it}^+)di \\ &= \frac{1}{n_t}\int_0^\ell s_{it}\delta(k - k_{it}^+)\left|\frac{di}{dk}\right|dk \\ &= \frac{1}{n_t}\left|\frac{dk_{it}^+}{di}\right|^{-1} s_{it}\Big|_{i:k_{it}^+ = k}, \end{aligned}$$

where we have used the inverse function theorem $\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}$ with $b = f(a)$. From Equation (A.39) we have that

$$\frac{dk_{it}^+}{di} = (b - a\eta_i)e^{-b(t-i)}.$$

Further, from setting $k_{it}^+ = k$ in Equation (A.39) and solving for i we obtain

$$i = t + \frac{\ln\left(\frac{a\eta_i - bk}{a\eta_i - b}\right)}{b}.$$

Hence,

$$t - i = -\ln\left(\frac{a\eta_i - bk}{a\eta_i - b}\right)^{\frac{1}{b}}.$$

Inserting the above expression for i into Equation (A.40) we get for the lifetime distribution

$$s_{it} = \left(\frac{a\eta_i - bk}{a\eta_i - b}\right)^{\frac{\rho}{b}+\frac{1}{\delta+r\kappa}\left(\delta+\frac{rb}{a\eta_i}\right)},$$

and

$$\begin{aligned} P_t^+(\eta,k) &= \frac{1}{n_t}\left|\frac{dk_{it}^+}{di}\right|^{-1} s_{it}\Big|_{i:k_{it}^+=k} \\ &= \frac{b}{(\delta+r\kappa)(b-a\eta)} e^{b(t-i)} s_{it}\Big|_{i=t+\frac{\ln\left(\frac{a\eta-bk}{a\eta-b}\right)}{b}} \\ &= \frac{b}{(\delta+r\kappa)(b-a\eta)}\left(\frac{a\eta_i - bk}{a\eta_i - b}\right)^{\frac{\rho}{b}-1+\frac{1}{\delta+r\kappa}\left(\delta+\frac{rb}{a\eta_i}\right)}. \end{aligned}$$

Integrating over η yields the out-degree distribution

$$\begin{aligned} P_t^+(k) &= \int P_t^+(\eta,k) f(\eta) d\eta \\ &= \int_0^{\eta_{\max}} P_t^+(\eta,k) \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \frac{1}{\eta} e^{-\frac{(\ln\eta-\tilde{\mu})^2}{2\tilde{\sigma}^2}} d\eta, \end{aligned} \tag{A.41}$$

where we have denoted by

$$\eta_{\max} \equiv (1-p)^N \left(1+\frac{p}{q(1-p)}\right)^N - 1 = \left(1-p\left(1-\frac{1}{q}\right)\right)^N - 1, \tag{A.42}$$

and

$$\begin{aligned} \tilde{\sigma} &\equiv \sqrt{Nq(1-q)}\ln\left(1+\tfrac{p}{q(1-p)}\right) \\ \tilde{\mu} &\equiv Nq\ln\left(1+\tfrac{p}{q(1-p)}\right) + N\ln(1-p), \end{aligned} \tag{A.43}$$

with the pdf of a log-normally distributed random variable

$$f(\eta) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \frac{1}{\eta} e^{-\frac{(\ln\eta-\tilde{\mu})^2}{2\tilde{\sigma}^2}}.$$

□

Supplement to "Aggregate Fluctuations in Adaptive Production Networks"

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B. Heterogenous Input-Output Characteristics

Assuming that links are drawn according to whether a firm's output characteristics match another firm's input characteristics we can obtain the following proposition for the out-degree distribution.

Proposition 7. *If we denote by* $a \equiv (1-p)^N$ *and* $b \equiv 1 + \frac{p}{q(1-p)}$*, then the out-degree distribution is given by*

$$P^+(k) = \binom{N}{\max\left(0, \min\left(N, \left[\ln\left(1+\frac{k}{a}\right)/\ln(b)\right]\right)\right)} q^{\max\left(0,\min\left(N,\left[\ln\left(1+\frac{k}{a}\right)/\ln(b)\right]\right)\right)} \times (1-q)^{N-\max\left(0,\min\left(N,\left[\ln\left(1+\frac{k}{a}\right)/\ln(b)\right]\right)\right)}, \tag{B.1}$$

while the in-degree distribution is given by $P^-(k) = \delta_{k,1}$*.*

Proof of Proposition 7. Observe that in a large population, there exists at least one firm with any vector of output characteristics. Hence, for any firm i with input characteristics vector $\mathbf{h}_i^-$ there exists at least one firm j such that $\mathbf{h}_i^- \cap \mathbf{h}_j^+ = \mathbf{h}_i^-$. We assume that a firm with input characteristics $\mathbf{h}_i^-$ draws uniformly at random from all firms j which have these output characteristics, i.e. it must hold that if $h_{ik}^- = 1$ then $h_{jk}^+ = 1$ for all $k = 1 \ldots, N$ for a firm j to qualify as a supplier to firm i.[18,19] Let the support of $\mathbf{h}$ be given by $\mathsf{S}(\mathbf{h})$ and its cardinality is given by $|\mathsf{S}(\mathbf{h})| = \langle \mathbf{h}, \mathbf{u} \rangle$, counting the number of nonzero entries in $\mathbf{h}$, and $\langle \cdot, \cdot \rangle$ denoting the scalar product in $\{0,1\}^N$. Consider a vector $\mathbf{s}_i$ of size $|\mathsf{S}(\mathbf{s}_i)| = i$ corresponding to a subset of the output characteristics of a firm with k output characteristics in total, $i \leq k$. Note that there are $|\{\mathbf{s}_i \in \{0,1\}^N : |\mathsf{S}(\mathbf{s}_i)| = i \leq k\}| = \binom{k}{i}$ such subsets of size i of a vector with k characteristics in total. Let $X_{\mathbf{s}_i}^j \in \{0,1\}$ be the indicator variable for a firm j which needs the input characteristics $\mathbf{s}_i$, i.e. $\mathbf{h}_i^- = \mathbf{s}_i$, to receive a link (being a customer) from a firm who possesses the output characteristics $\mathbf{s}_i$. There are $np^i(1-p)^{N-i}$ firms which need exactly the characteristics $\mathbf{s}_i$, and there are nq^i firms which possess the characteristics in $\mathbf{s}_i$ (note that these firms might also possess additional characteristics). The probability that a firm j which needs $\mathbf{s}_i$ forms an incoming link to (buys from) a firm which has $\mathbf{s}_i$ as output characteristics is then given by

$$\mathbb{P}(X_{\mathbf{s}_i}^j = 1) = \frac{1}{nq^i}.$$

[18] Note however, that the converse $h_{jk}^- = 1 \Rightarrow h_{ik}^+ = 1$ is not required to hold. This means that a firm can buy a product which has more characteristics than are actually required for the input of the firm.

[19] In an extension of the current model following Eeckhout and Jovanovic [2002]; Stigler [1961] we could assume that finding a new supplier proceeds by first taking κ independent draws from the population of firms. Let the match of a firm i's input characteristics with the other firms' output characteristics be denoted by $m_{ij} = |\mathsf{S}(\mathbf{h}_i^- \cap \mathbf{h}_j^+)| \leq \min(|\mathsf{S}(\mathbf{h}_i^-)|, |\mathsf{S}(\mathbf{h}_j^+)|)$. For each i let the cumulative distribution of m_{ij} be $F_i(a) = \sum_{b=0}^a P_i(b)$, and $P_i(b)$ the corresponding probability mass function. Then the maximum value $\max_{1 \leq j \leq \kappa}\{m_{ij}\}$ is distributed as $G_i(a) = F_i(a)^\kappa$ and has a probability mass function of $g_i(a) = \kappa F_i(a)^{\kappa-1} P_i(a)$. The parameter κ measures the directedness of search. In the limit of $\kappa \to \infty$, the firm selects the best matching firm, while in the limit of $\kappa \to 1$ the firm selects at random. With this assumption, the expected value of the match is given by $\mathbb{E}\left(\max_{1 \leq j \leq k}\{m_{ij}\}\right) = \sum_{a=0}^{|\mathsf{S}(\mathbf{h}_i^-)|} a\kappa F_i(a)^{\kappa-1} P_i(a)$.

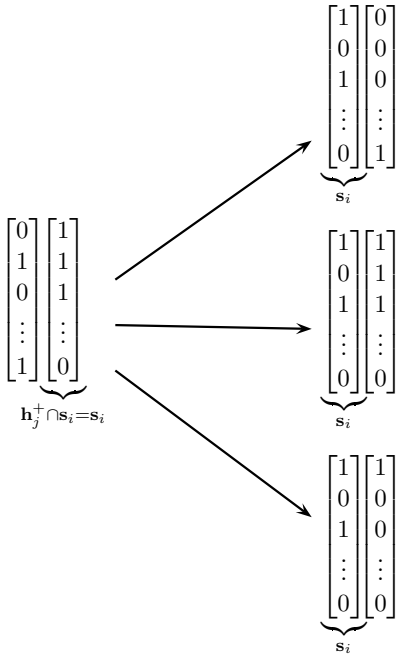

Figure B.1: An illustration of firms with input characteristics $\mathbf{s}_i$ being customers of a firm j which has the output characteristics $\mathbf{h}_j^+$ such that $\mathbf{h}_j^+ \cap \mathbf{s}_i = \mathbf{s}_i$.

This is because the firms which have $\mathbf{s}_i$ as output characteristics are drawn uniformly at random by j, and there are nq^i such firms. Observe that $X_{\mathbf{s}_i}^j$ is a Bernoulli random variable with success probability $\frac{1}{nq^i}$. Let

$$S_{\mathbf{s}_i} = \sum_{j:\mathbf{h}_j^- = \mathbf{s}_i} X_{\mathbf{s}_i}^j$$

count the number of links created to firms with input characteristics $\mathbf{s}_i$ from a firm which has the output characteristics $\mathbf{s}_i$. Note that this is a sum of $np^i(1-p)^{N-i}$ independent random variables. An illustration can be seen in Figure B.1. Further, let

$$S^k = \sum_{\mathbf{s}_i : |\,\mathsf{S}(\mathbf{s}_i)| = i \leq k} S_{\mathbf{s}_i}$$

count the number of links created to firms which require in total i input characteristics for all $i \leq k$. Note that $|\{\mathbf{s}_i \in \{0,1\}^N : |\,\mathsf{S}(\mathbf{s}_i)| = i \leq k\}| = \binom{k}{i}$, so that S_i^k is a sum over over $\binom{k}{i}$ sums, each being a sum of $np^i(1-p)^{N-i}$ independent random variables. To see this note that for a firm j with output characteristics $\mathbf{h}_j$ such that $|\,\mathsf{S}(\mathbf{h}_j^+)| = k \geq i$ there are

$$\binom{k}{1} np(1-p)^{N-1}$$

firms which need exactly one of the characteristics as input, there are

$$\binom{k}{2} np^2(1-p)^{N-2}$$

firms which need exactly two of the output characteristics of firm j as input, ..., and there are

$$\binom{k}{i} np^i(1-p)^{N-i}$$

firms which need exactly i of the k output characteristics of firm j as input.

Since $X^j_{\mathbf{s}_i}$ is a Bernoulli random variable with success probability $\frac{1}{nq^i}$, its generating function is given by

$$G_{X^j_{\mathbf{s}_i}}(z) = 1 - \frac{1}{nq^i} + z\frac{1}{nq^i}.$$

Similarly, $S_{\mathbf{s}_i}$ is the sum of independent random variables, and its generating function is

$$G_{S_{\mathbf{s}_i}}(z) = \prod_{j:\mathbf{h}_j^- = \mathbf{s}_i} G_{X^j_{\mathbf{s}_i}}(z) = \left(1 - \frac{1}{nq^i} + z\frac{1}{nq^i}\right)^{n(1-p)^{N-i}p^i}.$$

The generating function of S^k is then given by

$$G_{S^k}(z) = \prod_{\mathbf{s}_i:|\,\mathsf{S}(\mathbf{s}_i)|=i\leq k} G_{S_{\mathbf{s}_i}}(z) = \prod_{\mathbf{s}_i:|\,\mathsf{S}(\mathbf{s}_i)|=i\leq k}\ \prod_{j:\mathbf{h}_j^- = \mathbf{s}_i} G_{X^j_{\mathbf{s}_i}}(z) = \prod_{i=1}^{k}\left(1 - \frac{1}{nq^i} + z\frac{1}{nq^i}\right)^{\binom{k}{i}n(1-p)^{N-i}p^i}.$$

It then follows that

$$\begin{aligned} G'_{S^k}(z) = &\sum_{j=1}^{k}\prod_{i\neq j}^{k}\left(1 - \frac{1}{nq^i} + z\frac{1}{nq^i}\right)^{\binom{k}{i}n(1-p)^{N-i}p^i} \\ &\times \binom{k}{j}n(1-p)^{N-j}p^j\left(1 - \frac{1}{nq^j} + z\frac{1}{nq^j}\right)^{\binom{k}{j}n(1-p)^{N-j}p^j - 1}\frac{1}{nq^j}, \end{aligned}$$

so that[20]

$$\begin{aligned} \mathbb{E}(\,d_i^+\,\big|\,|\mathsf{S}(\mathbf{h}^+)| = k) &= \mathbb{E}\left(S^k\right) \\ &= G'_{S^k}(1) \\ &= \sum_{j=1}^{k}\binom{k}{j}\frac{(1-p)^{N-j}p^j}{q^j} \\ &= (1-p)^N\left(\left(1 + \frac{p}{q(1-p)}\right)^k - 1\right). \end{aligned}$$

The cumulant generating function of S^k is given by

$$\begin{aligned} R_{S^k}(z) &\equiv \ln G_{S^k}(z) \\ &= \sum_{i=1}^{k}\binom{k}{i}n(1-p)^{N-i}p^i\left(1 - \frac{1}{nq^i} + z\frac{1}{nq^i}\right) \\ &= (1-p)^N\left(n\left(\left(\frac{1}{1-p}\right)^k - 1\right) + (z-1)\left(\left(\frac{p}{q-pq} + 1\right)^k - 1\right)\right). \end{aligned}$$

[20] Note that when $p = q$ we obtain the mean field approximation in Theorem 5 in Anderson [2012], which has been derived in a related context.

Mean and variance can then be computed from $R_{S^k}(0)'$ and $R_{S^k}(0)''$. However, since all second and higher derivatives of the cumulant are zero, we have a constant random variable. Denoting by $a \equiv (1-p)^N$ and $b \equiv 1 + \frac{p}{q(1-p)}$, it follows that the out-degree distribution is given by

$$\begin{aligned}
P^+(k) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{1}_{\{d_i^+=k\}}\right) \\
&= \mathbb{P}(d_i^+ = k) \\
&= \sum_{j=1}^{N} \mathbb{P}(\, d_i^+ = k \big|\, |\mathsf{S}(\mathbf{h}_i^+)| = j)\mathbb{P}(|\mathsf{S}(\mathbf{h}_i^+)| = j) \\
&= \sum_{j=1}^{N} \mathbb{1}_{\{k=\max(0,\min(N,[a(b^j-1)]))\}} \binom{N}{j} q^j (1-q)^{N-j} \\
&= \binom{N}{\max\left(0,\min\left(N,\left[\ln\left(1+\frac{k}{a}\right)/\ln(b)\right]\right)\right)} q^{\max\left(0,\min\left(N,\left[\ln\left(1+\frac{k}{a}\right)/\ln(b)\right]\right)\right)} \\
&\times (1-q)^{N-\max\left(0,\min\left(N,\left[\ln\left(1+\frac{k}{a}\right)/\ln(b)\right]\right)\right)},
\end{aligned}$$

while the in-degree distribution is given by $P^-(k) = \delta_{k,1}$. □

For large k (respectively, for small $a = (1-p)^N$, when N is large), we can make the approximation $\ln\left(1+\frac{k}{a}\right) \approx \ln k - \ln a$. Moreover, we have the following bounds for the binomial coefficient $\left(\frac{n}{k}\right)^k \leq \binom{n}{k} \leq \left(\frac{ne}{k}\right)^k$. We then obtain a lower bound for the out-degree distribution

$$\underline{P}^+(k) = \left(\frac{N}{f(k)}\right)^{f(k)} q^{f(k)}(1-q)^{N-f(k)},$$

and an upper bound is given by

$$\overline{P}^+(k) = \left(\frac{Ne}{f(k)}\right)^{f(k)} q^{f(k)}(1-q)^{N-f(k)},$$

where $f(k) = \max\left(0, \min\left(N, \left[\left(\ln k - \ln a\right)/\ln(b)\right]\right)\right)$. An example for the out-degree distribution can be seen in Figure B.2.[21]

The probability that an incumbent i becomes a supplier to the entrant t with k output charac-

[21] In an extension of the model we can assume that firms produce more than one product. Let a firm produce ℓ products. Then the in-degree of a firm will be ℓ, while the out- degree follows from a similar calculation as above after we have replaced the number of firms n with $n\ell$.

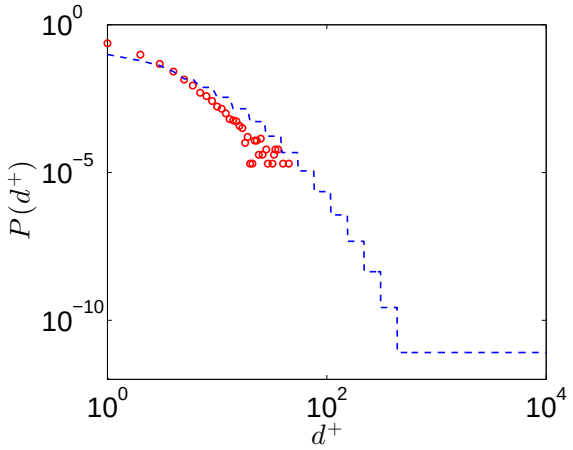

Figure B.2: The out-degree distribution from a simulation of the network (○) and the theoretical prediction (--) of Equation (B.1). The parameters used are $\alpha = 0.75$, $\eta = 0.9$, $p = 0.2$, $q = 0.6$, $N = 50$ and $n = 10000$.

teristics is then given by

$$\begin{aligned}\mathbb{P}(\{i \to t\} || \,\mathsf{S}(\mathbf{h}_i^+)| = k) &= \sum_{j=1}^{k} \binom{k}{j} \frac{p^j (1-p)^{N-j}}{n_t q^j} \\ &= \frac{(1-p)^N}{n_t} \left(\left(1 + \frac{p}{q(1-p)} \right)^k - 1 \right) \\ &\equiv \frac{\eta(k)}{n_t},\end{aligned}$$

while the probability that any incumbent forms a link to the entrant is

$$\begin{aligned}\sum_{i=1}^{n_t} \sum_{k=1}^{N} \mathbb{P}(\{i \to t\} || \,\mathsf{S}(\mathbf{h}_i^+)| = k) \mathbb{P}(|\,\mathsf{S}(\mathbf{h}_i^+)| = k) &= n_t \sum_{k=1}^{N} \binom{N}{k} q^k (1-q)^{N-k} \frac{(1-p)^N}{n_t} \left(\left(1 + \frac{p}{q(1-p)} \right)^k - 1 \right) \\ &= 1 - (1-p)^N \xrightarrow[N \to \infty]{} 1.\end{aligned}$$

The term $(1-p)^N$ corresponds to the probability of the event that an entrant does not require any input to produce. We can write $\eta(\nu) = a(b^\nu - 1)$, where

$$\begin{aligned} a &\equiv (1-p)^N, \\ b &\equiv 1 + \tfrac{p}{q(1-p)}, \end{aligned}$$

and ν (the stock of output characteristics of a firm) follows a binomial distribution, $\mathrm{Bin}(N, q)$. For large N we can approximate $\mathrm{Bin}(N, q)$ with a normal distribution with mean $\mu = Nq$ and standard deviation $\sigma = \sqrt{Nq(1-q)}$, that is, we can write the pdf of ν as follows

$$f_\nu(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi Nq(1-q)}} e^{-\frac{(x-Nq)^2}{2Nq(1-q)}}.$$

It then follows for the cumulative distribution of η

$$\begin{aligned} F_\eta(y) = \mathbb{P}\left(a(b^\nu - 1) \leq y\right) & \\ &= \mathbb{P}\left(b^\nu \leq 1 + \frac{y}{a}\right) \\ &= \mathbb{P}\left(\nu \leq \frac{\ln\left(1+\frac{y}{a}\right)}{\ln b}\right) \\ &= \int_{-\infty}^{\frac{\ln\left(1+\frac{y}{a}\right)}{\ln b}} \frac{1}{\sqrt{2\pi N q(1-q)}} e^{-\frac{(x-Nq)^2}{2Nq(1-q)}} dx. \end{aligned}$$

Differentiating with respect to y of $F_\eta(y)$ gives the corresponding pdf

$$f_\eta(y) = F'_\eta(y) = \frac{1}{\sqrt{2\pi N q(1-q)}} \frac{1}{(a+y)\ln b} e^{-\frac{\left(\frac{\ln\left(1+\frac{y}{a}\right)}{\ln b} - Nq\right)^2}{2Nq(1-q)}}.$$

When the term $a = (1-p)^N$ is small, then we can write this as

$$\begin{aligned} f_\eta(y) = F'_\eta(y) &= \frac{1}{\sqrt{2\pi N q(1-q)}} \frac{1}{y \ln b} e^{-\frac{\left(\frac{\ln y - \ln a}{\ln b} - Nq\right)^2}{2Nq(1-q)}} \\ &= \frac{1}{\sqrt{2\pi}\sqrt{Nq(1-q)}\ln b} \frac{1}{y} e^{-\frac{(\ln y - \ln a - Nq\ln b)^2}{2Nq(1-q)(\ln b)^2}}. \end{aligned}$$

Denoting by

$$\begin{aligned} \tilde{\sigma} &\equiv \sigma \ln b = \sqrt{Nq(1-q)} \ln\left(1 + \tfrac{p}{q(1-p)}\right) \\ \tilde{\mu} &\equiv \mu \ln b + \ln a = Nq \ln\left(1 + \tfrac{p}{q(1-p)}\right) + N\ln(1-p), \end{aligned} \tag{B.2}$$

we obtain the pdf of a log-normally distributed random variable[22]

$$f_\eta(y) = \frac{1}{\sqrt{2\pi\tilde{\sigma}^2}} \frac{1}{y} e^{-\frac{(\ln y - \tilde{\mu})^2}{2\tilde{\sigma}^2}}.$$

[22]The log-normal distribution is given by

$$f_\eta(y) = \frac{1}{\sqrt{2\pi}\tilde{\sigma} y} e^{-\frac{(\ln y - \tilde{\mu})^2}{2\tilde{\sigma}^2}}.$$

Taking logs delivers

$$\ln f_\eta(y) = -\frac{(\ln y)^2}{2\tilde{\sigma}^2} + \left(\frac{\tilde{\mu}}{\tilde{\sigma}^2} - 1\right)\ln y - \log\left(\sqrt{2\pi}\tilde{\sigma}\right) - \frac{\tilde{\mu}^2}{2\tilde{\sigma}^2}.$$

When $\tilde{\sigma}$ becomes large, $\ln f_\eta(y)$ becomes a linear function of $\ln y$, and we get $f_\eta(y) \sim y^{-\left(1-\frac{\tilde{\mu}}{\tilde{\sigma}^2}\right)}$, that is, a power-law with exponent $1 - \frac{\tilde{\mu}}{\tilde{\sigma}^2}$. This approximation is good as long as y is not much larger than $\tilde{\mu}$ [cf. e.g. Mitzenmacher, 2004; Sornette, 2000].