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Working Paper

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Economics Working Paper, No. 2017-05

Provided in Cooperation with:

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Suggested Citation: Lux, Thomas (2017) : On the distribution of links in financial networks: Structural heterogeneity and functional form, Economics Working Paper, No. 2017-05, Kiel University, Department of Economics, Kiel

This Version is available at:

<https://hdl.handle.net/10419/162397>

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On the Distribution of Links in Financial Networks: Structural Heterogeneity and Functional Form

by Thomas Lux



On the Distribution of Links in Financial Networks: Structural Heterogeneity and Functional Form

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June 9, 2017

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Acknowledgements: Part of this work has been conducted in the author's capacity of Bank of Spain Professor in Computational Economics at University Jaume I in Castellon. The almost final version has been completed during a research stay at the University of Kyoto and University of Hyogo in spring 2017 which has been partially supported by a MEXT scholarship within the project 'Exploratory Challenges on Post-K Computer'. The Japanese data has been made available by courtesy of Nikkei Media Marketing Inc. and the consortium of EU FP7 project no.255987 (FOC-II). The excellent hospitality of my hosts, Hideaki Aoyama and Yoshi Fujiwara is most gratefully acknowledged. I am also very grateful to them for many stimulating discussions on the topic of this research and the structure of the Japanese data set that forms part of its empirical basis. My thanks for stimulating discussions also extend to Abhijit Chakraborty, Lutz Honvehlmann, Hiroyasu Inoue, Eliza Lungu and Hazem Krichene. Particular thanks go to Lutz Honvehlmann for his extremely able research assistance.

Keywords: Financial networks, interbank market, degree distribution, credit network.

JEL Classification: G21, G01, E42

Abstract

We investigate the distribution of links in three large data-sets, one of these covering interbank loans in the electronic trading platform e-MID, the other two covering a large part of the loans of banks to non-financial companies in the Spanish and Japanese economies, respectively. In contrast to all previous literature, we do not assume homogeneity of the link distribution over time and across different categories of agents (banks, firms) but apply our hypothesized distributions as regression models. As it turns out, many of the tested sources of heterogeneity turn out to be significant regressors. For instance, we find pervasive time heterogeneity of link formation in all three data sets, and also heterogeneity for different categories of banks/firms that can be identified in the data. Across all networks, the Negative Binominal model always outperforms all alternative models confirming its good performance as a model of economic count data in many previous applications.

I Introduction

Different major classes of networks are routinely associated with their implied distributions of *degrees*, i.e. the number of links of nodes they generate. The major prototypes are Erdős-Renyi and scale-free networks. The former are random networks that are characterized by a constant probability of existence of a link which obviously leads to a Binomial distribution of links that converges toward the Poisson distribution for large networks. Scale-free networks somehow mark the opposite end of the spectrum in that they generate a very broad distribution of links via some kind of amplification mechanism (like preferential attachment of new nodes to those that already possess a large number of connections). As a result, the degree distribution emerging from such a generating mechanism is of a very heterogeneous nature and its scale-free behaviour corresponds to a power-law decay of the distribution of links over its entire range or at least in the upper tail region. Almost all of the related literature focusses on these two possibilities. However, the Poisson and power-law distributions do certainly not constitute an exhaustive list of candidate distributions for the number of links in a network setting. Indeed classes of networks exist which focus on properties other than the degree distribution and for which no general results for the distribution of links are available. Examples are ‘small world’ networks which are defined by a small average distance between nodes (Watts and Strogatz, 1998) or ‘core-periphery’ networks that are defined by a dichotomic classification of nodes into a core group and its periphery (Borgatti and Everett, 1999). Both of these classes might contain members that also share the property of an (asymptotically) power-law like distribution of links or not. In how much these different categorisations overlap or exclude each other, seems to be completely unknown and has not received any attention so far. However, the existence of such alternative categorisations of classes of networks and their pertinent generating mechanisms makes it likely that for some empirical networks, other distributions than the Poisson and scale-free could better describe the data.

This should also apply to *financial networks*, for which the asserted scale-free behaviour had already been disputed in certain cases (cf. Fricke and Lux, 2015). Due to the dominance of the Erdős-Renyi and scale-free paradigms, theoretical modelling has typically made use only of these two classes of models (Nier et al., 2008; Haldane and May, 2011; Anand et al., 2013; Krause and Giansante, 2013). When generating the link structure of a theoretical model in this way, any inference on the stability of the network and its susceptibility to contagion effects after shocks would be determined to a large extent by the (known) properties of the pertinent class of models. Hence, the extent of contagious cascade effects might be underestimated or overestimated because of deviations of these theoretical benchmarks from the empirical structure. It, thus, appears worthwhile to expand the range of candidate distributions and generating mechanisms beyond these classical ones as it appears likely that often the distribution of links is located somewhere between these extremes. A better and hopefully robust characterisation of the degree distribution should, therefore, be valuable input to inform the mushrooming literature on network contagion studies of the banking sector.

Continuing the line of research initiated by Fricke and Lux (2015), this paper will look at some intermediate distributions from the large class of compound Poisson distributions (Karlis and Xekalaki, 2005) that have been found appropriate for modelling discrete events in various fields but have seemingly not been applied to the discrete variables defined by the counts of the number of links within a network so far. We will focus here on the Poisson-Gamma and Poisson-Pareto distributions along with the original Poisson and discrete Pareto (aka power law or scale-free) distribution, and will compare the performance of these four alternatives for three important data sets, one covering interbank credit connections, the other two capturing the network structure of bank-firm loans. As another novel feature within the financial network literature, we will also apply most of the mentioned distributions within a regression framework. In this way, we can identify the influence of certain characteristics of the nodes on their propensity to form links.

We will estimate these models for three large data sets of financial linkages due to loan contracts: interbank loans contracted via the electronic trading platform e-MID, and loans of financial institutions to non-financial firms in the Spanish and Japanese economies. All data sets are available over at least one decade. The e-MID data contains daily recordings of all interbank loans while the other two data sets have yearly granularity. As it will turn out, heterogeneity is pervasive in all three data sets along various dimensions: there is both a change over time of the shape of the estimated distributions as well as a highly significant influence of whether banks/firms belong to some basic classes of agents that can be distinguished in the data. These exogenous effects are mostly very robust as they appear in a qualitatively similar way in all distributions under consideration. Irrespective of inclusion of exogenous effects or not, in almost all cases, the Negative Binominal exhibits the best fit, and dominates all alternatives at any standard level of significance.

The rest of the paper is structured as follows: sec. II introduces the various distributions under investigation and their use as regression models. Sec. III describes our data, and sec. IV provides the empirical results. Sec. V concludes.

II Statistical Models

Since degree distributions are by definition distributions of discrete variables, the present paper confines itself to comparing the performance of discrete distributions. The simplest benchmark is the Poisson distribution given by:

$$P(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad (1)$$

where x is the number of links (the degree of a node), and λ the unique parameter of this distribution function. We note that empirical degree distributions are typically truncated at zero, simply because pertinent data are only collected for entities that are at least minimally

connected to the network under investigation. Hence, in such applications we would have to use the *truncated* Poisson distribution which is given by:

$$P_T(x) = \frac{e^{-\lambda} \lambda^x}{x!(1 - e^{-\lambda})}, \quad (2)$$

where the additional term in the denominator adjusts for the “missing” zero of the empirical data (note that $P(0) = e^{-\lambda}$ in the original Poisson distribution).

The Poisson distribution approximates the exact Binomial distribution of degrees in Erdős-Renyi networks with a high degree of accuracy if the networks are not too small. Since all our applications would be based on at least three-digit numbers of nodes, the Poisson estimates should be virtually identical to estimates for a Binomial distribution.

The power-law characterizing scale-free networks is usually described and estimated in its continuous version, i.e. $p(x) \sim x^{-\alpha}$. However, this, of course neglects the discrete nature of the data. The discrete counterpart of the continuous Pareto distribution is also known as the Zipf or Zeta distribution, and it is given by the probability mass function:

$$P_a(x) = \frac{x^{-\alpha}}{\zeta(\alpha)}, \quad (3)$$

where $\zeta(\alpha)$ is the zeta function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. No adjustment for the lack of zeros is needed in this case as the support of the discrete Pareto covers only positive integers. Besides the elementary Poisson and the discrete Pareto, the most frequently encountered classes of discrete distribution functions are compound Poisson distributions. Two of these are used in this paper: The first is the negative Binomial which results if the parameter λ of the original Poisson distribution (1) is drawn from a Gamma distribution. Note that this amounts to drawing the realizations from a family of Poisson distributions with heterogeneous mean values, hence can be seen as a reflection of heterogeneity of the statistical features of the nodes in a network. We adopt here the following functional form of the negative Binomial:

$$N(x) = \frac{\Gamma(\theta + x) \tau^\theta (1 - \tau)^x}{\Gamma(1 + x) \Gamma(\theta)} \quad \text{with} \quad \tau = \frac{\theta}{\theta + \lambda} \quad (4)$$

with $\Gamma(\cdot)$ the gamma function, $\Gamma(n) = (n-1)!$, and θ and λ the two parameters for the shape of the distribution. Alternative functional forms can be found in Greene (2008). The one of eq. (4) is preferred in the present context as it can be easily related to the Poisson distribution, since the mean value is in both cases identical to the pertinent parameter λ and the negative Binomial converges to the Poisson for $\theta \rightarrow \infty$. The negative Binomial has become hugely popular in many applications featuring discrete data as it is able to capture the widespread phenomenon of *overdispersion*, i.e. the variance exceeding the mean. Namely, while it is well-

known that the variance of the Poisson distribution is $Var_P(x) = \lambda$, for the negative Binomial we obtain $Var_N(x) = \lambda(1 + \frac{\lambda}{\theta}) > \lambda$. For applications without zero counts, we also need to adjust the negative Binomial in an appropriate way to obtain its truncated version:

$$N_T(x) = \frac{\Gamma(\theta + \lambda)\tau^\theta(1 - \tau)^x}{\Gamma(1 + x)\Gamma(\theta)(1 - \tau^\theta)}. \quad (5)$$

The negative Binomial enjoys an almost legendary reputation in marketing as the most versatile tool for fitting purchase frequencies of consumer goods. This literature has been initiated by Ehrenberg (1959) and surveyed by Schmittlein et al. (1985).

The last candidate to be considered in this paper is the Pareto-Poisson mixture. This compound model had been studied before in the actuarial literature (cf. Albrecht, 1984) and has been proposed by Lux (2016) as a model for the degree distribution of credit networks. The justification for this functional form was the plausible observation that the number of credit links of both banks and non-financial firms is increasing with their balance sheet size (de Masi and Gallegati, 2012; de Masi et al., 2011). Taking the size of the underlying entity as a latent variable in a compound Poisson model and taking into account the well-known empirical fact that firm size distributions follow a power or Pareto law then leads to a formalization in which the shape parameter of the Poisson distribution is drawn from a Pareto law:

$$PP(x) = \int_{\underline{\lambda}}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} \alpha \frac{\underline{\lambda}^\alpha}{\lambda^{\alpha+1}} d\lambda \quad (6)$$

which defines a family of distributions with two parameters, α and $\underline{\lambda}$. A closed-form solution for the integral in eq.(6) is not available, so that the probability mass function can only be solved via numerical integration. In eq. (6), α is the usual shape parameter of the Pareto distribution (note that since the latent variable size is a continuous variable, we can adopt here the standard Pareto law), and $\underline{\lambda} > 0$ is a lower boundary for the latent variable which is necessary to guarantee convergence of the integral. Again, we need the zero-truncated counterpart of eq. (6) which formally we obtain by setting:

$$PP_T(x) = \frac{PP(x)}{1 - PP(0)} \quad (7)$$

which again is obtained by numerical integration. It is worthwhile to add that most applications (e.g. in marketing) use the Poisson and negative Binomial as *regression models* (cf. Hilbe, 2007), i.e. modeling the dependency of variables obeying such distributions on exogenous variables. While network data have to the best of my knowledge only be described via unconditional distributions so far, such a perspective would be most informative if additional information on the characteristics of the nodes were available. The Poisson and negative Binomial model could

be embedded into a regression framework by setting:

$$\lambda_i = \exp(\beta_0 + \mathbf{y}_i' \beta) \quad (8)$$

where \mathbf{y}_i is a vector of covariates and $i = 1, \dots, N$ is the sample of nodes of the network. This adds node-specific heterogeneity even in the Poisson model and, in the case of the negative Binomial, could be interpreted as a combination of both observable and unobservable heterogeneity, the later being represented by the Gamma mixing distribution.

I am not aware of any previous use of the Poisson-Pareto model within a regression framework. Nevertheless, this family can also easily be cast into such a format. It can be shown that the mean of eq.(6) is $E[x] = \frac{\alpha}{\alpha-1} \underline{\lambda}$ and so it seems most natural to allow exogenous effects to enter via $\underline{\lambda}$:

$$\underline{\lambda} = \exp(\beta_0 + \mathbf{y}_i' \beta) \quad (9)$$

While eq.(9) is motivated by the Poisson regression framework, it also allows inference on the influence of exogenous factors if the mean actually does not exist, i.e. if $\alpha \leq 1$ holds.

In contrast, no straight-forward way suggests itself to add a regression framework to the discrete Pareto distribution, and so we just apply this alternative in its unconditional format. Since not too much knowledge is available in our data set on the characteristics of individual nodes, the regression framework model is used to allow for fixed effects of different years, as well as different categories of actors the nodes belong to and so we can investigate whether this categorisation is of relevance for the number of their links.

III The data

We consider three large data sets of credit links: The first covers all transactions in the interbank money market conducted within the electronic trading platform e-MID over the years 1999 to 2014.

The second data set is a comprehensive data base of credit extended from banks to non-financial firms in Spain which has been extracted from the SABI (Sistema de Análisis de Balances Ibéricos) archive based on the public commercial registry in Spain. This complete list of bank connections of all publicly registered companies is available for each year from 1997 to 2008 comprising more than 500,000 links between individual banks and their borrowers. Our third data set is a similarly large record of credit links between banks and non-financial firms in Japan collected by Nikkei Media Marketing, Inc. from financial statements of the firms included. These data are available for us over the period from 1979 through 2011.

All three sources have been used in other studies before: The SABI database has been used

by Illueca et al. (2014) who study the effects of the regional expansion of Spanish saving banks during the real-estate boom of the years after the introduction of the Euro. The Japanese data has been investigated from a network perspective by Marotta et al. (2015). The e-MID data features prominently in quite a number of contributions to financial network theory (e.g. de Masi et al., 2006; Fricke and Lux, 2015) as it is the only commercially available data set in this area.

IV Empirical Results

We now move on to the results of estimating various discrete models for the degree distributions computed for these data sets.

IV.1 Interbank Loans from e-MID Platform

We first turn to the *interbank credit data* from the e-MID trading platform. These data have been relatively intensely scrutinized in various previous papers. Among those, de Masi et al. (2006) have reported power-law exponents between 2 and 3 for the distribution of degrees. Fricke and Lux (2015) have questioned this result showing that the histograms of the degree distribution do hardly resemble a power law. Fitting a variety of both continuous and discrete distributions they find that the power law is dominated by many other distributions in terms of proximity to the empirical distribution (evaluated via the Kolmogorov-Smirnov statistic). Which distribution gets closest to the data, varies with the level of time aggregation and across subsamples of the data.

Here we complement this analysis in various ways: First, we use tests based upon likelihood comparisons. Second, we use a larger sample for comparison, namely all banks that have been operating in the money market within the e-MID platform (while Fricke and Lux have confined their analysis to Italian banks which constitute the majority of e-MID users). Third, we do not only estimate the parameters of unconditional distributions, but also apply the Poisson, negative Binomial and Poisson-Pareto mixture as regression models which also allows us a certain assessment of the value added of including exogenous variables to explain the distributions of degrees. Since the implementation of a regression framework is less straightforward in the case of the discrete Pareto distribution, we estimate only the one shape parameter of this family. To be precise, the underlying data here use the set union of the degree distributions extracted from the 64 available quarters 1/1999 to 4/2014. Following Finger et al. (2013) we use such a large level of time aggregation, since at the high-frequency end (e.g. for daily data), the resulting networks would be very sparse. Presumably, over a short time horizon, only a small fraction of all existing links (credit lines) get activated and, thus, high frequency data would provide us with a very small sample of what we want to measure: namely, existing contacts between banks that could be used to obtain a loan in the money market if the need arises. This view is also supported by the finding that many network statistics (such as density, reciprocity etc)

are very volatile at the high-frequency end and become more stable at around the monthly to quarterly level of aggregation. Hence, we define a link between two banks to exist if they have been trading at least once with each other within a quarter and merge the 64 distributions of degrees obtained in this way into a single one.

The later step could be considered problematic as our time span of 16 years covers very different periods: an expansive phase after the launch of this market in which transaction volume and number of market participants had been increasing sharply, the reduction of activity after the outbreak of the financial turmoil in 2007/08 and the subsequent operation of the exchange at a reduced level of turnover. Since this exchange is operated by a company based in Milan, Italy, it has always been predominantly used by Italian banks. However, the fraction of non-Italian banks has been sharply increasing prior to the crisis and collapsed again during the aftermath of the financial turmoil. Finger et al. (2013) and Fricke and Lux (2015) observe that both Italian and non-Italian banks have been mostly trading with counterparts from their own group, so that the network consisted of two largely distinct clusters. Given this outline of the history of trading within the e-MID electronic market, we might hypothesize that one might expect both time heterogeneity of the distribution of degrees as well as an influence of the geographic location of the banks using this system. Because of this clustering, Fricke and Lux (2015) neglect the non-Italian participants and focus their analysis on the majority of market participants operating under Italian law. In our regression framework, we can allow for differences by including country-specific fixed effects. Since except for Italy, other countries are hardly ever represented by more than a handful of banks, we restrict ourselves to using a dummy for the non-Italian origin. In order to account for temporal variation, we additionally include 15 yearly dummies (β_1 to β_{15}) for the years 2000 to 2014.

Table 1 exhibit the results of the estimation of the distributions presented in sec. 2 for this data set together with the factors entering as determinants of their mean. We find the best fit for the Negative Binomial, followed by the Poisson-Pareto mixture and the Discrete Pareto distribution (without exogenous factors). Here and in all other applications, the Poisson distribution provides a definitely much worse fit than all other alternatives. The yearly dummies and the dummy for non-Italian banks are all highly significant according to their t-statistics. The coefficients assume virtually the same values under both the Poisson and Negative Binomial distributions and behave qualitatively similarly under the Poisson-Pareto mixture. Essentially, the coefficients depict an almost monotonic decline of the mean degree which is first caused by mergers and acquisitions and the resulting reduction of the number of market participants and later by the strong decline of interbank trading during the financial crisis. Coefficients are, in fact, almost identical for the years 2002-2007, and the years 2009-2014, respectively, so that one can recognize the well-known phases in the development of this market. The dummy for non-Italian banks is almost exactly -1 for all three models. The effect has to be seen in relation to the other parameter estimates. For instance, for the Negative Binomial distribution, this would imply an expected degree of 72.89 for Italian banks in 1999 (from $\beta_0=4.289$) while

non-Italian banks would be expected to only have 25.76 links on average. This expectation does not exist in the case of the Poisson-Pareto as the estimated tail index of the Pareto is $\alpha=0.81$. In contrast, the Discrete Pareto distribution would indicate existence of the first moment. It seems remarkable that despite the high level of overdispersion, the fat-tailed Poisson-Pareto and Discrete Pareto distributions are both inferior to the Negative Binomial.

As it is also indicated in Table 1, likelihood ratio tests clearly reject restricted models without all dummies for all these distributions that have been used in these regressions. Table 2 shows results of a number of additional tests: The Poisson which is nested in the Negative Binomial, is rejected at all traditional levels of significance. Further, a sequence of Vuong tests (Vuong, 1989) shows that the Negative Binomial significantly outperforms the Poisson-Pareto and Discrete Pareto, and the same is obtained for the Poisson-Pareto against the Discrete Pareto. When adjusting the Vuong test for the difference in estimated parameters¹ (18 in the case of the Negative Binomial and Poisson-Pareto against only one for the simple Discrete Pareto), only the advantage of the Negative Binomial remains, while the parsimonious Discrete Pareto would be preferred under this criterion to the Poisson-Pareto with exogenous factors. Fourth, we have finally compared the Negative Binomial and Poisson-Pareto without exogenous factors to the Discrete Pareto (keeping only parameters θ and β_0 or α and β_0 , respectively) and find the Negative Binomial and the Poisson-Pareto to appear superior to the Discrete Pareto. Since here, the first two alternatives still enjoy the advantage of one more parameter, also the adjusted version of the Vuong test can be applied, which leaves the pattern of dominance unchanged.

IV.2 Spanish bank-firm credit network

We now move on to the analysis of the degree distributions extracted from the bank-firm credit network for the Spanish economy over the years 1997-2008. Since this is a bipartite network, it allows us to investigate degree distributions under different perspectives: Tables 3 and 4 depict the results for the degree distribution of banks within the bipartite network, Tables 5 and 6 exhibit the results obtained from the so-called *one-mode projection* of the original data set. This is the projection of the original adjacency matrix of links, say A , of the bipartite data onto a symmetric binary matrix for banks only that identifies whether two banks have at least one lender in common, or do not have any overlap within their group of lenders. This matrix, say B , is obtained as $B = A^T A$. Tables 7 and 8 present results for the degree distributions of firms from the original bipartite adjacency matrix. The one-mode projection for firms is less interesting as the number of joint lenders assumes very small values throughout. Here, we use the original yearly records as basic input which we merge all into one data set allowing, however, for both differences in the mean in each year and differences in mean due to the category of banks in our data set. The later categories are: commercial banks, saving banks, and credit cooperatives. Particularly the later are typically very small, local institutions which only provide credit to

¹As proposed by Vuong (1989) one then subtracts $0.5(K_1 - K_2)\log N$ from the differences of the likelihoods (K_1 and K_2 the number of parameters of both alternative models, and N the number of observations).

very few borrowers. Not accounting for their different behavior would certainly introduce an element of misspecification into any statistical model of network links. We take commercial banks as the default case, and introduce dummies with coefficients γ_1 and γ_2 for saving banks and credit cooperatives, respectively.

Starting with the degree distribution of *credit relationships of banks to non-financial firms*, Table 3 shows that this distribution is characterized by an even larger degree of overdispersion ($\sim 13,900$) than the interbank network. All dummies except for the first year are significant under the Poisson and Negative Binomial, and pertinent coefficients are again very similar for these two distributions. This data set shows an increase of activity over time which squares with the deregulation of the Spanish banking sector and particularly the regional expansion of activity of saving banks. The dummies for bank categories show a slightly negative effect for savings banks and a much stronger negative effect for credit cooperatives. To get a feeling of the relevance of the coefficients, note that the average degree of commercial banks in the first year, 1997, would have been 550 under the Negative Binomial while the dummy coefficients of -3.16 reduces this number to only ~ 23 for credit cooperatives. Results for the coefficients of exogenous variables differ under the Poisson-Pareto model and are not significant in a number of cases. In particular, while roughly a positive time trend is found, the coefficients for the two categories are positive rather than negative which particularly for credit cooperatives contradicts the basic features of the data. My conjecture is that due to the pronounced fat-tailedness of the estimated mixture distribution ($\alpha=0.35$), the non-stationarity of the resulting model makes identification of exogenous effects very hard as with such a tail index the realisations of the process would be expected to show immense variation anyway. While again the likelihood ratio tests indicate that the dummies are jointly significant for all candidate distributions, the improvement in the fit of the Poisson-Pareto obtained by inclusion of covariates appears quite meagre in absolute terms compared to the other cases. This underscores that many of these dummies do not contribute much in this particular case.

The Discrete Pareto in contrast would again indicate finiteness of the mean, and its estimated shape parameter $\alpha=1.21$ is very close to the one obtained in Table 1 for the degree distribution of interbank credit data. We also find in this case that the Discrete Pareto is closer to the best model, the Negative Binomial, than the Poisson-Pareto. While under the comparison of the likelihood values without adjustment, the Vuong test in Table 4 indicates superiority of the Negative Binomial, adjustment for the number of parameters turns the comparison upside down in favor of the Discrete Pareto. This holds also, if the dummies are discarded and only the two shape parameters of the Negative Binomial are used. The Discrete Pareto also dominates over the Poisson-Pareto under all perspectives. The result of the comparison of the adjusted Vuong test between the Negative Binomial and Discrete Pareto might seem cumbersome as it suggests to neglect known heterogeneity within our data set. This shows that strong heterogeneity within a data set could, in principle, lead to a spurious fit of a homogeneous power law distribution.

Table 5 turns to the results for bank's 'co-lending' degrees from the one-mode projection.

One surprising finding here is that the time dummies are almost never significant under all three regression models. Hence, despite the structural changes of the banking sector and its credit relationships to non-financial firms in this period, banks seemed not to have become generally more connected via joint exposures to the same borrowers. In contrast, the dummies for bank categories are both significant and have identical signs under all three regression models: while saving banks are more connected with other banks via joint borrowers, cooperatives are much less connected. The former finding squares with the observation that saving banks often came in as additional providers of credit to certain firms during the time of their regional expansion (cf. Illueca et al., 2014). Again, we find the Negative Binomial to perform best, followed by the Poisson-Pareto, the one-parameter Discrete Pareto, and finally the Poisson regression. Again the first ranks are relatively close so that the choice between Negative Binomial and Discrete Pareto depends on whether one uses the adjusted version of the Vuong test or not (cf. Table 6). Using only the two shape parameters, the decision is always in favor of the Negative Binomial (which seems plausible given that most of the regression parameters are not significant). Similarly, the Poisson-Pareto dominates over the Discrete Pareto under the non-adjusted Vuong test, but this result changes under adjustment for the number of parameters. Also in this case, the two-parameter Poisson-Pareto without exogenous factors clearly dominates. These patterns are also preserved if we include the dummies for the type of financial institution (indicated by the addition “4 params” in Table 6). As for parameter estimates, it is interesting to note that the tail indices of both the Poisson-Pareto and Discrete Pareto are extremely close to their counterparts of Table 1 (while being different between both models).

Tables 7 and 8 exhibit the results for the degree distribution of firms from the original adjacency matrices of the bank-firm credit network. One notes that these data are characterized by a variance smaller than the mean, i.e. *underdispersion*. Hence, there would not necessarily be a reason to turn to fat-tailed alternatives to the Poisson distribution. Still, we find the Negative Binomial significantly better than the Poisson but the other alternatives perform worse than both of them. Here we only have used time dummies as the bank categories obviously cannot be brought in directly (one could, however, use them to test whether the type of creditor banks would make a difference). In all three regression models, time dummies show a monotonic decrease, i.e., firms have decreased their average number of creditors (from 1.65 in 1997 to 1.16 in 2008 according to the results of the Poisson model). This seems at first view a surprising result as the geographical expansion of savings banks has often led to additional lenders coming in for single firms. One reason for the overall negative trend could be that the general increase of the number of registered firms over this period of strong growth of the Spanish economy has brought many new firms into the data base that initially started out with a single lender and, thus, had a dampening effect on the average. In the absence of overdispersion, in fact, the present models basically capture time variation of a narrow distribution of relatively small entries, so that skipping the regression terms leaves the Negative Binomial even inferior to the Discrete Pareto.

IV.3 Japanese bank-firm credit network

Results from our third data set, the network of bank-firm credit in the Japanese economy, are provided in Tables 9 through 14. Since this record covers a span of more than thirty years, we have abstained from using annual dummies. Instead, we distinguish between three historical episodes as potential candidates for fixed effects: the time up to the climax of the Japanese bubble, the more stagnative period afterwards, and the recent crisis period. Hence, we impose dummies for the years 1990 to 2007, and 2008 to 2018, respectively. In addition, similarly like in the Spanish data set, we can distinguish between different categories of financial institutions. We take as the default category large private banks (labelled ‘city banks’ in our data set), and define as a second category those of regional banks (those designated explicitly as only regionally active banks in the data set as well as those identified as *Shinkin* banks, which are regionally operating credit cooperatives). As a third category, we define insurance companies active in the lending market together with so-called *long-term credit banks* that are both identified as different classes in our data base, since both of them should be more active as long-term lenders pursuing business models different from those of ‘city banks’¹. In contrast to the Spanish data set, we also have more detailed information on firms (including balance sheet information) which here we only use for a binary classification: Namely, in the Japanese case, our selection of non-financial firms as recipients of loans covers those companies only that are listed on an official exchange. While the range of companies covered in this way has been relatively constant from 1979 through 1995, the establishment of the new market and its index JASDAQ has greatly expanded the scope of the data base as of 1996. It would be very questionable whether the firms operating in the New Market would share the structure of loan relationships of the old industries, and so it appears sensible to distinguish between both groups. To do so, we impose a dummy for firms listed in JASDAQ as well as in its later replacement called *Hercules*. Again we apply the same chain of model estimations and specification tests as for the other data. Tables 9 and 10 provide results for the degrees of banks in the original bipartite networks, Tables 11 and 12 those for the one-mode projection for banks’ co-lending relationships and Tables 13 and 14 those for the degrees of firms.

Starting with the degrees of banks in the bipartite network, we find similar results for the fixed effects across models: In all specifications a slight increase of links in 1990 to 2007 and a smaller positive effect (against the benchmark of 1979 to 1989) thereafter are observed. Regionally active banks have distinctly fewer links leading to a significant negative dummy for this category throughout. The dummy for insurances/long-term credit banks also indicates that they have somewhat smaller degrees than city banks, but this effect is not significant for the Poisson-Pareto model. In terms of shape parameters, we find that the Poisson-Pareto again has a very small tail index α close to 0.5 which would indicate non-existence of the theoretical mean while the Discrete Pareto again turns out a value of about 1.2 (both numbers are close to the pertinent results for the Spanish credit network). As we see in Table 10, the dominating model

¹The overall number of cases of these two categories is too small to consider them separately

is again the NBD model which is preferred over all others independently of whether dummies are included or not. The Poisson-Pareto is only preferred over the Discrete Pareto if dummies are included and the Vuong test is adjusted for the different number of parameters. In all other cases, one would reject the discrete Pareto model in favor of the Poisson-Pareto.

In the one-mode projection of the banking sector (Table 11), we find a clearly significant negative effect for the period 2008-2011, and a smaller, but also significant tendency of reduced co-lending in 1990 to 2007 compared to the years before. The regional banks are found to be less connected than city banks in all specifications, while there is no significant effect of the third category of insurance companies and long-term credit banks in any of the models. The ranking of models is the by now ‘usual’ one with NBD dominating over the Poisson-Pareto and Discrete Pareto and all three of them clearly outperforming the simple Poisson model. Specification tests in Table 12 show that the differences in likelihoods are reflected in a preference for the ‘better’ performing model according to the Vuong test at all traditional confidence levels. The tail indices are again remarkably close to their counterparts in the one-mode projection of the Spanish credit network.

Finally, Tables 13 and 14 provide results for the degree distribution of non-financial firms receiving loans from the Japanese banking sector. We first note that the mean degree of 9.47 is much higher than the pertinent value for the Spanish firms. The reason is likely that the restriction to publicly listed entities in the Japanese companies leads to a selection of relatively large firms. In contrast, the Spanish data set is based on the Spanish commercial register and, thus, provides a much broader, nearly comprehensive sample of the population of firms operating in the Spanish economy. In contrast to the Spanish case, the degree distribution is also characterized by overdispersion justifying the estimation of our various fat-tailed alternatives to the elementary Poisson distribution. In all models, the dummies for the stagnation and crisis period as well as the one for firms listed in the new market are significantly negative. Hence, in line with the results for banks (Table 11) the number of credit links per firm has decreased first after the burst of the Japanese bubble and even more so after the onset of the worldwide financial crisis. For instance, under the estimated parameters for the Negative Binomial, the average number of links has decreased from 13.5 in the first period via 9.3 during the stagnation to 6.5 in the post-crisis years. At the same time, an average firm from the new market that started during the second period, would have received credit simultaneously from 5.9 banks.

In the specification tests, we see a somewhat unusual outcome as the Poisson-Pareto dominates over the Negative Binomial. While the margin is small, the difference is significant at all traditional confidence levels. However, the estimated parameters for the Poisson-Pareto are also unusual as with an estimated $\alpha = 1.996$, its tail is much higher than the estimates for the banks’ degree distributions that all were below unity. In comparison, $\alpha = 18.190$ has been obtained for Spanish firms. This later estimate is so high that it would not indicate any visible tail fatness (which indeed is absent given the Spanish firm degrees do not exhibit overdispersion). In the case of Japanese firms, the estimate of about 2 still indicates some mild degree

of tail fatness which among our selection of models is best captured by the Poisson-Pareto. The Discrete Pareto has more typical parameter estimate ($\alpha = 1.430$) but is dominated under all perspectives by both the Poisson-Pareto and the Negative Binomial irrespective of whether fixed effects are included as net.

V Conclusion

Our analysis has demonstrated that heterogeneity is pervasive in the degree distributions extracted from credit networks of different origin. Almost all time dummies and fixed effects for different categories of actors that we have included on our estimations turned out to be highly significant. Hence, we can safely conclude that the structure of network formation in the markets under consideration has changed over time and that different types of actors behave in different ways in these markets. It would therefore be misleading to model the degree distribution of a financial network with any specific unconditional distribution without taking into account the heterogeneity of the data. When accounting for such known sources of heterogeneity, we find in five cases (banks' degrees in the interbank market as well as in the Spanish and Japanese loan markets and their co-lending degrees in the same markets) a clear dominance of the Negative Binomial model. The same applies to firms' degree distribution in the Spanish loan market whereas the Japanese firm degree distribution provided the only case of a dominance of the Poisson-Pareto distribution (which in all other cases was inferior to the Negative Binomial).

If we neglect heterogeneity, we often find the Negative Binomial and the Poisson Pareto in the vicinity of the one-parameter Discrete Pareto distribution. What is more, the Discrete Pareto turns out estimates across all our seven empirical degree distributions that are within a very narrow range (all between 1.2 and 1.4 with the only exception of the Japanese firm degrees). The apparently good fit of power laws that had been reported in previous publications might, then, be an artifact of lumping together different categories of nodes. If a sample contains different classes of agents with different orders of magnitude of links, it might erroneously lead to the impression of a very fat-tailed unconditional distribution. Taking the heterogeneity documented in this paper into account in network studies of contagious defaults should also improve our assessment of the risk of systemic disturbances in loan markets.

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Table 1: E-MID interbank credit data, 1999-2014, time dummies and dummies for Italian/non-Italian origin

Observations: 9304 Mean: 44.36, Variance: 1410.745	Poisson	Negative Binomial	Poisson Pareto	Discrete Pareto
β_0	4.287 (0.001)	4.289 (0.025)	2.640 (0.014)	- -
β_1	-0.060 (0.001)	-0.070 (0.038)	0.110 (0.020)	- -
β_2	-0.110 (0.001)	-0.114 (0.040)	0.128 (0.020)	- -
β_3	-0.215 (0.001)	-0.210 (0.041)	0.076 (0.021)	- -
β_4	-0.240 (0.001)	-0.204 (0.041)	-0.051 (0.021)	- -
β_5	-0.273 (0.001)	-0.213 (0.040)	-0.121 (0.021)	- -
β_6	-0.278 (0.001)	-0.188 (0.040)	-0.096 (0.021)	- -
β_7	-0.277 (0.002)	-0.173 (0.041)	-0.127 (0.021)	- -
β_8	-0.262 (0.002)	-0.172 (0.041)	-0.111 (0.021)	- -
β_9	-0.511 (0.002)	-0.503 (0.042)	-0.283 (0.024)	- -
β_{10}	-0.890 (0.003)	-0.975 (0.043)	-0.623 (0.001)	- -
β_{11}	-0.820 (0.003)	-0.898 (0.045)	-0.677 (0.031)	- -
β_{12}	-0.870 (0.003)	-0.947 (0.045)	-0.624 (0.032)	- -
β_{13}	-1.126 (0.004)	-1.168 (0.049)	-0.884 (0.038)	- -
β_{14}	-1.200 (0.004)	-1.240 (0.050)	-1.032 (0.040)	- -
β_{15}	-1.086 (0.004)	-1.125 (0.053)	-0.696 (0.035)	- -
non-Ital.	-0.918 (0.002)	-1.040 (0.021)	-1.029 (0.018)	- -
α	- -	- -	0.805 (0.019)	1.258 (0.003)
θ	- -	1.992 (0.030)	- -	- -
logL	-114993.37	-42518.11	-46,794.15	-53118.76
Lkl ratio (all dummies=0)	87,202.13 (0.000)	17,828.14 (0.000)	11,242.29 (0.000)	- -
Lkl ratio (non-Ital.=0)	35,296.60 (0.000)	2,032.66 (0.000)	1,966.47 (0.000)	- -

Note: The table shows estimates of the parameters of the various models for pooled degree data extracted from the e-MID electronic platform for interbank credit, 1999-2014. Pooling here refers to the degree distributions of the 64 quarters of this sample. The Poisson, Negative Binomial and Poisson-Pareto mixture models allow for fixed effects depending on the year of the data while the discrete Pareto distribution has been estimated without any covariates. Parentheses contain standard errors.

Table 2: Specification tests for e-MID data

Test	Value	p-value
Poisson vs Negative Binomial (lkl)	144,950.52	(0.000)
Negative Binomial vs Poisson-Pareto	48.47	(0.000)
Negative Binomial vs Discrete Pareto	102.43	(0.000)
Negative Binomial vs Discrete Pareto (adj.)	24.75	(0.000)
Negative Binomial (2 params) vs Discrete Pareto	16.44	(0.000)
Negative Binomial (2 params) vs Discrete Pareto (adj.)	11.87	(0.000)
Discrete Pareto vs Poisson Pareto	-41.34	(1.000)
Discrete Pareto vs Poisson Pareto (adj.)	36.33	(0.000)
Discrete Pareto vs Poisson Pareto (2 params)	-6.10	(1.000)
Discrete Pareto vs Poisson Pareto (2 params, adj.)	-1.53	(0.937)

Note: Specification tests use the Vuong test for non-nested alternatives with the null hypothesis that both models are equally good descriptions of the data against the alternative hypothesis that one model fits better. The test statistics of the Vuong test is based upon the log likelihood differences of both models and, thus, a positive value indicates a better fit of the first model, a negative value a better fit for the second model. The resulting test statistic is asymptotically normally distributed and its p-value is recorded in the last column. The adjusted Vuong tests includes an additional correction factor for the difference in parameters. For the comparison between the Poisson and NBD model, the nestedness of the former in the later allows to conduct a standard likelihood ratio test. Parentheses contain probabilities of acceptance of the null hypothesis.

Table 3: Spanish Bank-Firm Credit, 1997-2007, degrees of banks with dummies for years and type of banks

Observations: 2343 Mean: 915.40, Variance: 12,732,367	Poisson	Negative Binomial	Poisson Pareto	Discrete Pareto
β_0	6.416 (0.003)	6.310 (0.072)	0.749 (0.079)	- -
β_1	0.133 (0.004)	0.128 (0.098)	0.070 (0.096)	- -
β_2	0.177 (0.004)	0.171 (0.095)	-0.082 (0.089)	- -
β_3	0.323 (0.004)	-0.342 (0.094)	-0.035 (0.087)	- -
β_4	0.344 (0.004)	0.373 (0.095)	-0.035 (0.089)	- -
β_5	0.759 (0.004)	0.875 (0.099)	0.254 (0.084)	- -
β_6	0.865 (0.004)	1.006 (0.099)	0.293 (0.084)	- -
β_7	0.981 (0.004)	1.138 (0.100)	0.275 (0.084)	- -
β_8	1.460 (0.004)	1.589 (0.109)	0.211 (0.087)	- -
β_9	1.532 (0.004)	1.688 (0.107)	0.155 (0.088)	- -
β_{10}	1.525 (0.004)	1.696 (0.108)	0.240 (0.007)	- -
γ_1	-0.049 (0.002)	-0.144 (0.078)	1.919 (0.049)	- -
γ_2	-2.933 (0.004)	-3.160 (0.049)	0.147 (0.035)	- -
θ	- -	0.347 (0.011)	- -	- -
α	- -	- -	0.345 (0.012)	1.211 (0.004)
logL	-3,192,707.70	-14,964.63	-22,605.91	-15,818.144
Lkl ratio (beta=0)	2,089,449.39 (0.000)	1,689.87 (0.000)	445.636 (0.000)	- -

Note: The table shows estimates of the parameters of the various models for pooled degree data of the yearly bank-firm credit networks in Spain, 1997-2007. Pooling here refers to the degree distributions of the 11 years. The Poisson, Negative Binomial and Poisson-Pareto mixture models allow for fixed effects depending on the year of the data while the discrete Pareto distribution has been estimated without any covariates. Parentheses contain standard errors. β_1 to β_{10} are dummies for the years 1998 through 2007, while γ_1 and γ_2 are dummies for saving banks and credit cooperatives, respectively.

Table 4: Specification tests for fitted degree distributions of banks in Spanish credit network

Test	Value	p-value
Poisson vs Negative Binomial (lkl)	6,355,486.13	(0.000)
Negative Binomial vs Poisson-Pareto	8.364	(0.000)
Negative Binomial vs Discrete Pareto	16.373	(0.000)
Negative Binomial vs Discrete Pareto (adj.)	-34.061	(1.000)
Negative Binomial (2 params) vs Discrete Pareto	0.155	(0.438)
Negative Binomial (2 params) vs Discrete Pareto (adj.)	-3.724	(0.999)
Discrete Pareto vs Poisson Pareto	7.378	(0.000)
Discrete Pareto vs Poisson Pareto (adj.)	57.812	(0.000)
Discrete Pareto vs Poisson Pareto (2 params)	7.626	(0.000)
Discrete Pareto vs Poisson Pareto (2 params, adj.)	11.506	(0.000)

Note: See Table 2 for details on the specification tests.

Table 5: Spanish Bank-Firm Credit, 1997-2007, degrees of banks from one-mode projection

Observations: 2,310 Mean: 35.003, Variance: 1415.951	Poisson	Negative Binomial	Poisson Pareto	Discrete Pareto
β_0	3.812 (0.011)	3.774 (0.054)	1.723 (0.045)	-
β_1	0.031 (0.015)	0.031 (0.077)	0.053 (0.057)	-
β_2	-0.017 (0.016)	-0.020 (0.077)	-0.033 (0.059)	-
β_3	-0.084 (0.016)	-0.078 (0.078)	-0.102 (0.060)	-
β_4	-0.115 (0.016)	-0.101 (0.076)	-0.126 (0.057)	-
β_5	-0.022 (0.016)	-0.014 (0.079)	0.024 (0.055)	-
β_6	0.009 (0.016)	0.055 (0.079)	0.037 (0.055)	-
β_7	-0.011 (0.016)	0.043 (0.080)	0.022 (0.055)	-
β_8	-0.015 (0.016)	0.057 (0.082)	0.018 (0.056)	-
β_9	-0.007 (0.016)	0.084 (0.081)	0.067 (0.056)	-
β_{10}	-0.005 (0.016)	0.089 (0.081)	0.132 (0.003)	-
γ_1	0.172 (0.008)	0.168 (0.057)	1.210 (0.029)	-
γ_2	-1.202 (0.011)	-1.245 (0.041)	-0.409 (0.001)	-
θ	-	1.430 (0.044)	-	-
α	-	-	0.832 (0.031)	1.286 (0.006)
logL	-32,949.1872	-10,083.1489	-10,558.0512	-12,088.441
Lkl ratio (beta=0)	21,069.027 (0.000)	1,201.556 (0.000)	904.784 (0.000)	-

Note: The table shows estimates of the parameters of the various models for pooled degree data of the yearly bank-firm credit networks in Spain, 1997-2007. Pooling here refers to the degree distributions of the 11 years. The Poisson, Negative Binomial and Poisson-Pareto mixture models allow for fixed effects depending on the year of the data while the discrete Pareto distribution has been estimated without any covariates. Parentheses contain standard errors. β_1 to β_{10} are dummies for the years 1998 through 2007, while γ_1 and γ_2 are dummies for saving banks and credit cooperatives, respectively.

Table 6: Specification tests for the degree distributions of banks ‘co-lending’ degree obtained from the one-mode projection of the bipartite Spanish firm network

Test	Value	p-value
Poisson vs Negative Binomial (lkl)	45,732.077	(0.000)
Negative Binomial vs. Poisson-Pareto	12.718	(0.000)
Negative Binomial (4 params) vs Poisson-Pareto (4 params)	12.963	(0.000)
Negative Binomial vs Discrete Pareto	43.076	(0.000)
Negative Binomial vs Discrete Pareto (adj.)	-7.267	(1.000)
Negative Binomial (2 params) vs Discrete Pareto	32.482	(0.000)
Negative Binomial (2 params) vs Discrete Pareto (adj.)	28.609	(0.000)
Negative Binomial (4 params) vs Discrete Pareto	42.945	(0.000)
Negative Binomial (4 params) vs Discrete Pareto (adj.)	31.328	(0.000)
Discrete Pareto vs Poisson Pareto	-28.697	(1.000)
Discrete Pareto vs Poisson Pareto (adj.)	21.645	(0.000)
Discrete Pareto vs Poisson Pareto (2 params)	-23.722	(1.000)
Discrete Pareto vs Poisson Pareto (2 params, adj.)	-19.850	(1.000)
Discrete Pareto vs Poisson Pareto (4 params)	-28.679	(1.000)
Discrete Pareto vs Poisson Pareto (4 params, adj.)	-17.061	(1.000)

Note: see Table 2 for details on the specification tests.

Table 7: Pooled Spanish Bank-Firm Credit, 1997-2007, degrees of firms

Observations:1,195,432 Mean: 1.794, Variance: 1.295	Poisson	Negative Binomial	Poisson Pareto	Discrete Pareto
β_0	0.502 (0.003)	0.008 (0.008)	1.192 (0.003)	- -
β_1	0.006 (0.004)	0.012 (0.009)	0.001 (0.004)	- -
β_2	-0.028 (0.005)	-0.023 (0.009)	-0.004 (0.005)	- -
β_3	-0.076 (0.005)	-0.078 (0.009)	-0.011 (0.005)	- -
β_4	-0.066 (0.004)	-0.070 (0.009)	-0.009 (0.004)	- -
β_5	-0.218 (0.004)	-0.247 (0.008)	-0.028 (0.004)	- -
β_6	-0.219 (0.004)	-0.249 (0.008)	-0.028 (0.004)	- -
β_7	-0.253 (0.004)	-0.288 (0.008)	-0.032 (0.004)	- -
β_8	-0.326 (0.004)	-0.373 (0.008)	-0.039 (0.004)	- -
β_9	-0.349 (0.004)	-0.399 (0.008)	-0.041 (0.004)	- -
β_{10}	-0.357 (0.004)	-0.409 (0.008)	-0.042 (0.000)	- -
θ	- -	0.943 (0.010)	- -	- -
α	- -	- -	18.190 (0.203)	2.188 (0.001)
logL	-1,492,588.73	-1,466,966.96	-2,088,965.80	-1,614,078.53
Lkl ratio ($\beta = 0$)	18.614.33 (0.000)	362,722.02 (0.000)	69,734.62 (0.000)	- -

Note: The table shows estimates of the parameters of the various models for pooled degree data of the yearly bank-firm credit networks in Spain, 1997-2007. Pooling here refers to the degree distributions of the 11 years. The Poisson, Negative Binomial and Poisson-Pareto mixture models allow for fixed effects depending on the year of the data while the discrete Pareto distribution has been estimated without any covariates. Parentheses contain standard errors. β_0 to β_{10} are dummies for the years 1998 through 2007.

Table 8: Specification tests for fitted degree distributions of firms in Spanish credit network

Test	Value	p-value
Poisson vs Negative Binomial (lkl)	51243.519	(0.000)
Negative Binomial vs Poisson-Pareto	538.215	(0.000)
Negative Binomial vs Discrete Pareto	378.873	(0.000)
Negative Binomial vs Discrete Pareto (adj.)	301.472	(0.000)
Negative Binomial (2 params) vs Discrete Pareto	-52.653	(1.000)
Negative Binomial (2 params) vs Discrete Pareto (adj.)	-59.650	(1.000)
Discrete Pareto vs Poisson Pareto	325.179	(0.000)
Discrete Pareto vs Poisson Pareto (adj.)	402.146	(0.000)
Discrete Pareto vs Poisson Pareto (2 params)	757.484	(0.000)
Discrete Pareto vs Poisson Pareto (2 params, adj.)	764.481	(0.000)

Note: See Table 2 for details on the specification tests.

Table 9: Japanese Bank-Firm Credit, 1979 - 2011, degrees of banks

Observations: 4688 Mean: 128.97, Variance: 55,927.56	Truncated Poisson	Truncated NBD	Truncated Poisson Pareto	Discrete Pareto
β_0	6.484 (0.000)	6.515 (0.131)	4.175 (0.029)	- -
β_1	0.265 (0.000)	0.189 (0.028)	0.140 (0.027)	- -
β_2	0.101 (0.000)	0.089 (0.049)	0.091 (0.040)	- -
γ_1	-2.677 (0.000)	-2.682 (0.130)	-2.682 (0.023)	- -
γ_2	-0.420 (0.000)	-0.406 (0.201)	0.120 (0.016)	- -
θ	- -	0.977 (0.021)	- -	- -
α	- -	- -	0.554 (0.017)	1.237 (0.003)
logL	-165,136.77	-24,732.83	-26,721.45	-28,680.82
Lkl ratio (all dummies=0)	882,806.41 (0.000)	3390.85 (0.000)	1789.64 (0.000)	- -

Note: The time dummies β_1 and β_2 are for the years 1990 - 2007 and 2008 - 2011. Categorical dummies γ_1 and γ_2 are imposed on the groups of regional banks and insurance companies/ long-term credit banks, respectively.

Table 10: Specification tests for the fitted degree distributions of banks from the Japanese credit network

Test	Value	p-value
Poisson vs NBD (lkl)	280,807.87	(0.000)
NBD vs Poisson-Pareto	16.71	(0.000)
NBD vs Discrete Pareto	58.27	(0.000)
NBD vs Discrete Pareto (adj.)	37.14	(0.000)
NBD (2 params) vs Discrete Pareto	39.30	(0.000)
NBD (2 params) vs Discrete Pareto (adj.)	35.08	(0.000)
Discrete Pareto vs Poisson Pareto	-16.51	(1.000)
Discrete Pareto vs Poisson Pareto (adj.)	4.62	(0.000)
Discrete Pareto vs Poisson Pareto (2 params)	-10.86	(1.000)
Discrete Pareto vs Poisson Pareto (2 params, adj.)	-6.64	(1.000)

Note: see Table 2 for details on the specification tests.

Table 11: Japanese Bank-Firm Credit, 1979 - 2011, degrees of banks from one-mode projection

Observations: 4688 Mean: 118.33, Variance: 932.55	Truncated Poisson	Truncated NBD	Truncated Poisson Pareto	Discrete Pareto
β_0	5.006 (0.005)	5.005 (0.059)	4.816 (0.011)	-
β_1	-0.040 (0.001)	-0.038 (0.012)	-0.072 (0.005)	-
β_2	-0.241 (0.002)	-0.240 (0.018)	-0.482 (0.009)	-
γ_1	-0.221 (0.005)	-0.220 (0.058)	-0.796 (0.010)	-
γ_2	-0.004 (0.006)	-0.004 (0.082)	0.003 (0.013)	-
θ	-	12.022 (0.177)	-	-
α	-	-	1.430 (0.052)	1.190 (0.003)
logL	-35,054.67	-23,402.17	-27,538.15	-34,582.49
Lkl ratio (all dummies=0)	5918.29 (0.000)	508.02 (0.000)	1328.29 (0.000)	-

Note: The time dummies β_1 and β_2 are for the years 1990 - 2007 and 2008 - 2011. Categorical dummies γ_1 and γ_2 are imposed on the groups of regional banks and insurance companies/ long-term credit banks, respectively.

Table 12: Specification tests for banks' 'co-lending' degrees obtained from the one-mode projection of the bipartite Japanese credit network

Test	Value	p-value
Poisson vs NBD (lkl)	23,305.02	(0.000)
NBD vs Poisson-Pareto	34.00	(0.000)
NBD vs Discrete Pareto	85.28	(0.000)
NBD vs Discrete Pareto (adj.)	64.14	(0.000)
NBD (2 params) vs Discrete Pareto	87.50	(0.000)
NBD (2 params) vs Discrete Pareto (adj.)	83.27	(0.000)
Discrete Pareto vs Poisson Pareto	-31.88	(1.000)
Discrete Pareto vs Poisson Pareto (adj.)	-10.75	(1.000)
Discrete Pareto vs Poisson Pareto (2 params)	-30.21	(0.000)
Discrete Pareto vs Poisson Pareto (2 params, adj.)	-25.98	(0.000)

Note: see Table 2 for details on the specification tests.

Table 13: Japanese Bank-Firm Credit, 1979 - 2011, degrees of firms

Observations: 66,860 Mean: 9.47, Variance: 66.94	Truncated Poisson	Truncated NBD	Truncated Poisson Pareto	Discrete Pareto
β_0	2.600 (0.001)	2.594 (0.005)	1.906 (0.004)	-
β_1	-0.354 (0.001)	-0.365 (0.005)	-0.327 (0.005)	-
β_2	-0.699 (0.002)	-0.718 (0.008)	-0.555 (0.008)	-
γ	-0.435 (0.003)	-0.452 (0.008)	-0.287 (0.007)	-
θ	-	2.994 (0.018)	-	-
α	-	-	1.996 (0.013)	1.403 (0.002)
logL	-277,219.84	-204,656.73	-203,382.91	-260,337.93
Lkl ratio (all dummies=0)	49,524.32 (0.000)	10,990.09 (0.000)	9,757.78 (0.000)	-

Note: The time dummies β_1 and β_2 are for the years 1990 - 2007 and 2008-2011. γ is a dummy for firms listed in the new market indices JASDAQ and Hercules.

Table 14: Specification tests for the fitted degree distributions of firms in the Japanese credit network

Test	Value	p-value
Poisson vs NBD (lkl)	145,126.20	(0.000)
NBD vs Poisson-Pareto	-7.96	(1.000)
NBD vs Discrete Pareto	203.06	(0.000)
NBD vs Discrete Pareto (adj.)	180.84	(0.000)
NBD (2 params) vs Discrete Pareto	191.87	(0.000)
NBD (2 params) vs Discrete Pareto (adj.)	186.31	(0.000)
Discrete Pareto vs Poisson Pareto	-207.44	(1.000)
Discrete Pareto vs Poisson Pareto (adj.)	-185.22	(1.000)
Discrete Pareto vs Poisson Pareto (2 params)	-201.58	(1.000)
Discrete Pareto vs Poisson Pareto (2 params, adj.)	-196.02	(1.000)

Note: see Table 2 for details on the specification tests.