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# Macroeconomic and stock market interactions with endogenous aggregate sentiment dynamics

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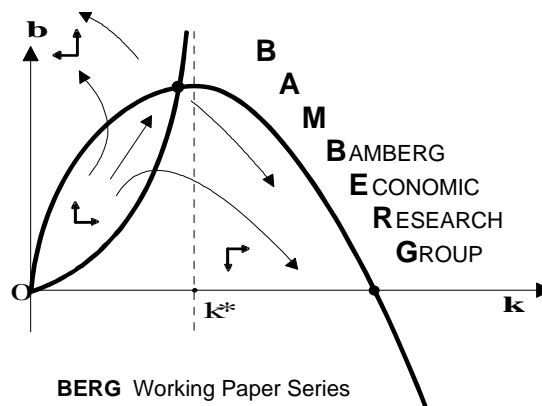
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# Macroeconomic and Stock Market Interactions with Endogenous Aggregate Sentiment Dynamics

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1           **Macroeconomic and Stock Market Interactions with Endogenous**  
2                                   **Aggregate Sentiment Dynamics**

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11                                   May 30, 2017

12                                   **Abstract**

13           This paper studies the implications of heterogeneous capital gain expectations on output and  
14       asset prices. We consider a disequilibrium macroeconomic model where agents' expectations on  
15       future capital gains affect aggregate demand. Agents' beliefs take two forms – fundamentalist  
16       and chartist – and the relative weight of the two types of agents is endogenously determined. We  
17       show that there are two sources of instability arising from the interaction of the financial with the  
18       real part of the economy, and from the heterogeneous opinion dynamics. Two main conclusions  
19       are derived. On the one hand, perhaps surprisingly, the non-linearity embedded in the opinion  
20       dynamics far from the steady state can play a stabilizing role by preventing the economy from  
21       moving towards an explosive path. On the other hand, however, real-financial interactions and  
22       sentiment dynamics do amplify exogenous shocks and tend to generate persistent fluctuations and  
23       the associated welfare losses. We consider alternative policies to mitigate these effects.

24       **Keywords:** Real-financial interactions, heterogeneous expectations, aggregate sentiment dynam-  
25       ics, macro-financial instability

26       **JEL classifications:** E12, E24, E32, E44.

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## 1 Introduction

The way in which the dynamic interaction between stock markets and the macroeconomy has been understood by the economics profession has evolved significantly over the last thirty years. As Shiller (2003) has argued, while the rational representative agent framework and the related Efficient Market Hypothesis represented the dominant theoretical modeling paradigm in financial economics during the 1970s, the behavioral finance approach has gained increasing ground within the economics community over the last two decades. The main reason for this significant paradigm shift is well known: following Shiller (1981) and LeRoy and Porter (1981), a large number of studies have documented various empirical regularities of financial markets – such as the excess volatility of stock prices – which are clearly inconsistent with the Efficient Market Hypothesis, see e.g. Frankel and Froot (1987, 1990), Shiller (1989), Allen and Taylor (1990), and Brock et al. (1992), among many others. During the 1990s several researchers like Day and Huang (1990), Chiarella (1992), Kirman (1993), Lux (1995) and Brock and Hommes (1998) have developed models of financial markets with heterogeneous agents following the seminal work by Beja and Goldman (1980) in order to explain such empirical regularities. Ever since, financial market models with heterogeneous agents using rule-of-thumb strategies have become central in the behavioral finance literature, see e.g. Chiarella and He (2001, 2003), De Grauwe and Grimaldi (2005), Chiarella et al. (2006), and Dieci and Westerhoff (2010).

The importance of different types of heterogeneity (regarding preferences, risk aversion or available information) and boundedly rational behavior at the micro level for the dynamics of the macroeconomy has also been increasingly acknowledged in macroeconomics (Akerlof, 2002, 2007). In this context, a particularly fruitful new strand of the literature has focused on the consequences of heterogeneous boundedly rational expectations for the dynamics of the macroeconomy and the conduct of economic policy, see e.g. Branch and McGough (2010), Branch and Evans (2011), De Grauwe (2011, 2012), Proaño (2011, 2013), among others. In these studies, the Brock and Hommes (1997) (BH) approach has been the preferred specification for the endogenous switch between alternative heuristics. In contrast, the development of macroeconomic models using the Weidlich-Haag-Lux (WHL) approach (see Weidlich and Haag, 1983 and Lux, 1995) is still in a nascent stage, with Franke (2012), Franke and Ghonghadze (2014), Flaschel et al. (2015), Chiarella et al. (2015) and Lojak (2016) as notable exceptions.

While the WHL and the BH approaches are quite similar in spirit – and similarly close to Keynes' (1936) and Simon's (1957) views on expectations under bounded rationality (see also Kahneman and Tversky, 1973 and Kahneman, 2003) – there is a fundamental difference between them: In the BH approach the variation in the share of agents using a particular heuristic depends on a measure of utility, or forecast accuracy, related to that particular rule of thumb which is thought to be relevant at the microeconomic level. In contrast, in the WHL approach the switch between different heuristics or attitudes, such as optimism or pessimism, is determined by an aggregate sentiment index composed

63 e.g. by macroeconomic variables describing the state of the economy in the business cycle, see also  
64 Franke (2014). The WHL approach thus incorporates an additional link from the macroeconomic  
65 environment to microeconomic decision-making based on psychological grounds and on Keynes' notion  
66 that "Knowing that our own individual judgment is worthless, we endeavor to fall back on the judgment  
67 of the rest of the world which is perhaps better informed. That is, we endeavor to conform with the  
68 behavior of the majority or the average. The psychology of a society of individuals each of whom is  
69 endeavoring to copy the others leads to what we may strictly term a *conventional* judgment." (Keynes,  
70 1937, p. 114; his emphasis).<sup>1</sup>

71 In this latter line of research the main contribution of this paper is to study the effects of aggregate  
72 sentiments in stock markets on the real economy using the WHL approach to model the expectations  
73 formation process in stock markets. More specifically, we incorporate aggregate sentiment dynamics  
74 in a stock market populated by heterogeneous agents, and examine the effects of herding and spec-  
75 ulative behavior in combination with real-financial market interactions. We adopt the distinction  
76 between *chartists* and *fundamentalists* which may be a key ingredient to explain bubbles as argued  
77 by Brunnermeier (2008). Ceteris paribus, chartists tend to exert a destabilizing influence on the price  
78 of financial assets, whereas the presence of fundamentalists is stabilizing.

79 In spite of its simplicity, our model features a variety of interesting aspects. The presence of  
80 self-reinforcing mechanisms in the aggregate dynamics allows for the existence of nontrivial multiple  
81 equilibria. In the economy, there are two sources of instability deriving from the feedback effects  
82 between real and financial markets via Tobin's  $q$  (as in Blanchard's 1981 seminal model) *and* from the  
83 endogenous aggregate sentiment dynamics produced by the interaction of heterogeneous agents in the  
84 stock markets. We prove that the dynamical system describing the evolution of the economy always has  
85 either a single steady state (with uniformly distributed agents) or three steady states (the equilibrium  
86 with uniformly distributed agents, one with a dominance of chartists and one where fundamentalists  
87 dominate), but even though various subdynamics of the model can be stable (at either the uniform or  
88 the fundamentalist of the three steady states), the complete system may be repelling around all of its  
89 equilibria. Given the complexity of the 4D nonlinear system, we use numerical simulations to explore  
90 the properties of the economy. Our results show that the dynamical system describing the economy  
91 is generally bounded: all trajectories remain in an economically meaningful subset of the state space.  
92 In this sense, unfettered markets with possibly accelerating real-financial feedback mechanisms may  
93 have some in-built stabilizing mechanism (based on aggregate sentiment dynamics) that prevent the  
94 economy from moving along an infeasible path. Nonetheless, real-financial interactions and sentiment  
95 dynamics *do* amplify exogenous shocks and may generate persistent fluctuations and the associated  
96 welfare losses. Indeed, despite the relatively simple behavior of the subsystem describing the evolution

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<sup>1</sup>Indeed, the central equation of the WHL approach which describes the dynamics of population shares might be provided from game theoretic foundations along the lines of Brock and Durlauf (2001), Blume and Durlauf (2003) and He et al. (2016). We are grateful to Tony He for pointing this link out to us.

97 of output without heterogeneous beliefs, the dynamics of the complete system can exhibit somewhat  
98 irregular fluctuations.

99 Finally, it is worth stressing that, unlike in most of the current macroeconomic literature, our model  
100 is based on a dynamic disequilibrium approach in which the evolution of the variables over time is  
101 described by gradual adjustment processes, and no equilibrium condition is imposed a priori. This  
102 dynamic disequilibrium approach – discussed in detail in [Chiarella and Flaschel \(2000\)](#) and [Chiarella  
103 et al. \(2005\)](#) – seems like a natural complement to the behavioral WHL approach to expectation  
104 formation, see also [Chiarella et al. \(2009\)](#).

105 The remainder of the paper is organized as follows. In section 2 we lay out the macroeconomic  
106 framework. Section 3 derives the main analytical results concerning the dynamics of the economy.  
107 Section 4 illustrates the properties of the model by means of numerical simulations. Section 5 analyzes  
108 some policy measures. Section 6 concludes, and the proofs of all Propositions are in the Appendix.

## 109 2 The Model

### 110 2.1 Core Real-Financial Interactions

111 We consider a closed economy consisting of households, firms and a monetary authority. We assume  
112 that households are the sole owners of the firms’ stocks or equities  $E$  which represent claims on the  
113 firms’ physical capital stock  $K$ .

114 Unlike in [Chiarella and Flaschel \(2000\)](#) and [Chiarella et al. \(2005\)](#), we abstract from the “Met-  
115 zlerian” inventory accelerator mechanism in the modeling of goods market dynamics<sup>2</sup> in order to  
116 focus on the interaction emerging from a stock market driven by aggregate sentiment dynamics and  
117 the macroeconomy. We assume instead that aggregate production evolves according to a dynamic  
118 multiplier specification<sup>3</sup>

$$\dot{Y} = \beta_y(Y^d - Y), \quad (1)$$

119 where  $Y$  represents aggregate output,  $Y^d$  aggregate demand and  $\beta_y > 0$  the speed of adjustment of  
120 output to market disequilibrium as in the seminal paper by [Blanchard \(1981\)](#).

121 Let  $p_e$  denote the equity price, and  $p$  the price of capital goods. The [Brainard and Tobin \(1968\)](#)  
122  $q$  ratio is then given by

$$q = p_e E / p K. \quad (2)$$

123 Without loss of generality, we normalize the price of output to one,  $p = 1$ , and assume further that  
124 the horizon of our analysis is sufficiently short as to guarantee that both  $E$  and  $K$  are constant

---

<sup>2</sup>These potentially destabilizing macroeconomic channels arising from the real side of the economy could be however  
reincorporated in the present framework in a straightforward manner.

<sup>3</sup>For any dynamic variable  $z$ ,  $\dot{z}$  denotes its time derivative,  $\hat{z}$  its growth rate and  $z_o$  its steady state value.

125 magnitudes. We normalize  $K$  assuming  $K = 1$ . As a result, changes in  $q$  are determined solely by  
 126 changes in  $p_e$ . Further, we assume that financial markets dynamics affect the real economy via the  
 127 impact of Tobin's  $q$  on aggregate demand. Hence, aggregate demand is given by:

$$Y^d = a_y Y + A + a_q (p_e - p_{eo}) E, \quad (3)$$

128 where  $a_y \in (0, 1)$  is the propensity to spend,  $A$  is autonomous expenditure, and  $a_q > 0$  measures the  
 129 responsiveness of output demand to the difference between the actual value of stocks and their steady  
 130 state value  $p_{eo}$ . Inserting equation (3) into equation (1) yields

$$\dot{Y} = \beta_y [(a_y - 1)Y + a_q (p_e - p_{eo})E + A]. \quad (4)$$

131 In addition to  $E$ , we assume that there are two more financial assets, namely, as is customary,  
 132 money  $M$  and short-term fix-price bonds  $B$ .<sup>4</sup> For simplicity we assume that the monetary authorities  
 133 fix the interest rate on the bonds  $B$  at the level  $r$ , accommodating the households' excess demand  
 134 for money. This allows us to abstract from the traditional interest rate effect on aggregate output so  
 135 central in New Neoclassical Consensus models (see e.g. Woodford, 2003) and focus in isolation on the  
 136 stock price effects under aggregate sentiment dynamics, as discussed below.

137 Since in our economy profits are assumed to be entirely redistributed to firms' owners (households)  
 138 as dividends, the expected return on equity  $\rho_e^e$  is

$$\rho_e^e = \frac{bY}{p_e E} + \pi_e^e. \quad (5)$$

139 where  $b \geq 0$  is the profit share,  $bY/(p_e E)$  is the dividend rate, and  $\pi_e^e$  represents the *average*, or *market*  
 140 expectation of future capital gains  $\pi_e = \dot{p}_e/p_e$ , i.e., the growth rate of equity prices.

141 Finally, we assume that the equity market is imperfect due to information asymmetries, adjustment  
 142 costs, and/or institutional restrictions, so that the equity price  $p_e$  does not move instantaneously to  
 143 clear the market. More specifically, we assume that

$$\dot{p}_e = \beta_e (\rho_e^e - \rho_{eo}^e) = \beta_e \left( \frac{bY}{p_e E} + \pi_e^e - \rho_{eo}^e \right), \quad (6)$$

144 where  $\beta_e$  describes the adjustment speed at which the equity price reacts to discrepancies between the  
 145 expected rate of return on equity and its steady state value,  $\rho_{eo}^e$ , which is assumed to be a given and  
 146 strictly positive parameter in the model. As we will discuss below, while equation (6) seems rather  
 147 stylized at first sight, it actually describes a complex mechanism due to the intrinsic nonlinearity of  
 148 the dynamics of the capital gain expectations  $\pi_e^e$ .

---

<sup>4</sup>See Charpe et al. (2011) for an explicit analysis and also for a critique of allowing governments to issue a perfectly liquid asset  $B$ , with a given unit price.



## 149 2.2 Aggregate Sentiment Dynamics

150 Based on the empirical findings of Frankel and Froot (1987, 1990) and Allen and Taylor (1990), and  
 151 the extensive literature they sparked, we assume that traders in financial markets use various types of  
 152 heuristics when forming their expectations about future asset price developments. To be specific, we  
 153 assume that traders in the stock market use either a *fundamentalist* rule (denoted by the superscript  
 154  $f$ ) according to which they expect capital gains to converge back to their long-run-steady state value  
 155 (assumed to be zero), i.e.

$$\dot{\pi}_e^{e,f} = \beta_{\pi_e^{e,f}}(0 - \pi_e^e), \quad (7)$$

156 or a *chartist* rule (denoted by  $c$ ) given by

$$\dot{\pi}_e^{e,c} = \beta_{\pi_e^{e,c}}(\hat{p}_e - \pi_e^e), \quad (8)$$

157 where  $\beta_{\pi_e^{e,f}}$  and  $\beta_{\pi_e^{e,c}}$  are the speed of adjustment parameters of the two heuristics-based forecasting  
 158 rules, respectively.

159 Suppose that at any given time a share  $\nu_c \in [0, 1]$  of the population consists of financial market  
 160 participants using the chartist rule and a share  $\nu_f = 1 - \nu_c$  consists of traders using the fundamentalist  
 161 rule. The law of motion of aggregate capital gain expectations can then be expressed as

$$\begin{aligned} \dot{\pi}_e^e &= \nu_c(\beta_{\pi_e^{e,c}}(\hat{p}_e - \pi_e^e)) + (1 - \nu_c)(\beta_{\pi_e^{e,f}}(0 - \pi_e^e)) \\ &= \nu_c\beta_{\pi_e^{e,c}}\hat{p}_e - (\nu_c\beta_{\pi_e^{e,c}} + (1 - \nu_c)\beta_{\pi_e^{e,f}})\pi_e^e. \end{aligned} \quad (9)$$

162 According to this equation the evolution of *aggregate, market-wide* expectations of future capital gains  
 163 is given by the weighted average of the *change* of the expectations, or forecasts, resulting from the use  
 164 of the fundamentalist or chartist forecasting rule. Further, as the interplay between fundamentalists  
 165 and chartists is well understood in the literature (see e.g. Hommes, 2006), we assume in the following  
 166 that  $\beta_{\pi_e^{e,c}} = \beta_{\pi_e^{e,f}} = \beta_{\pi_e^e}$  for simplicity and in order to focus on other rather new channels which  
 167 emerge from the aggregate sentiments dynamics.<sup>5</sup> Then, the above equation becomes

$$\dot{\pi}_e^e = \beta_{\pi_e^e}(\nu_c\hat{p}_e - \pi_e^e). \quad (10)$$

168 Observe that in equations (7) and (8), both fundamentalists and chartists are assumed to use  
 169 aggregate expectations  $\pi_e^e$  as the reference value for the updating of their own expectations. This  
 170 specification is meant to reflect Keynes' (1936, p.156) famous view of the stock market as a process of  
 171 choosing the most beautiful model in a beauty contest, where the winner is the one who has selected

---

<sup>5</sup>Further, by assuming that the two heuristics are updated with the same speed or frequency we are able to focus on the implications of the use of the different heuristics *per se*. We think that the latter are more relevant behaviorally and capture the most relevant part of heterogeneity in the stock market.

172 the model who is chosen as the most beautiful by the (relative) majority of players. Winning requires  
 173 guessing the views of the other players.

174 We endogenize the variable  $\nu_c$  by adopting the aggregate sentiment dynamics approach by Weidlich  
 175 and Haag (1983) and Lux (1995) as recently reformulated in Franke (2012, 2014), which provides  
 176 behavioral microfoundations to agents' attitudes in financial markets. Accordingly, agents decide  
 177 whether to take either a chartist, or a fundamentalist stance depending on the current status of the  
 178 economy (captured by the key variables  $Y$ ,  $p_e$ ), on expectations on the evolution of financial gains  
 179 ( $\pi_e^e$ ), and – crucially – on the current composition of the market (captured by the variable  $x$ , defined  
 180 below).

181 Formally, suppose that there are  $2N$  agents in the economy. Of these,  $N_c$  use the chartist forecasting  
 182 rule and  $N_f$  use the fundamentalist rule, so that  $N_c + N_f = 2N$ . Following Franke (2012) we describe  
 183 the distribution of chartists and fundamentalists in the market by focusing on the *difference* in the  
 184 size of the two groups (normalized by  $2N$ ). To be precise, we define

$$x \equiv \frac{N_c - N_f}{2N}. \quad (11)$$

185 Therefore  $x \in [-1, +1]$ ,  $\nu_c = N_c/N = \frac{1+x}{2}$  and  $\nu_f = N_f/N = \frac{1-x}{2}$ , and  $x > 0$  indicates a dominance of  
 186 chartists, while  $x < 0$  implies a majority of fundamentalists at any given point in time.

187 Let  $p^{f \rightarrow c}$  be the transition probability that a fundamentalist becomes a chartist, and likewise for  
 188  $p^{c \rightarrow f}$ . The change in  $x$  depends on the relative size of each population multiplied by the relevant  
 189 transition probability. Given the continuous time setting of the present framework, we take the limit  
 190 of  $\dot{x}$  as the population  $N$  becomes very large as in Franke (2012), so that the intrinsic noise from  
 191 different realizations at the individual level can be neglected. Then:

$$\dot{x} = (1 - x)p^{f \rightarrow c} - (1 + x)p^{c \rightarrow f}. \quad (12)$$

192 The key behavioral assumption concerns the determinants of transition probabilities: we suppose  
 193 that they are determined by a *switching index*,  $s$ , which captures the expectations of traders on  
 194 market performance. An increase in  $s$  raises the probability of a fundamentalist becoming a chartist,  
 195 and decreases the probability of a fundamentalist becoming a chartist. More precisely, assuming that  
 196 the relative changes of  $p^{c \rightarrow f}$  and  $p^{f \rightarrow c}$  in response to changes in  $s$  are linear and symmetric:

$$p^{f \rightarrow c} = \beta_x \exp(a_x s), \quad (13)$$

197

$$p^{c \rightarrow f} = \beta_x \exp(-a_x s). \quad (14)$$

198 The switching index depends positively on market composition (capturing the herding component  
199 of agents' behavior) and on economic activity; and negatively on deviation of the market value of the  
200 capital stock and of the average capital gain expectations from their respective steady state values.  
201 As in Franke and Westerhoff (2014), this can be written as:<sup>6</sup>

$$s = s_x x + s_y (Y - Y_o) - s_{p_e} (p_e - p_{e_o})^2 - s_{\pi_e} (\pi_e^e)^2. \quad (15)$$

202 Deviations of share prices and capital gain expectations from their steady state values tend to  
203 favor fundamentalist behavior as doubts concerning the macroeconomic situation become widespread.  
204 This can be interpreted as a change in the state of confidence, whereby agents believe that increasing  
205 deviations from the steady state eventually become unsustainable.

206 The economy is described by the 4D dynamical system consisting of equations (4), (6), (10), and  
207 (12), where  $\nu_c$  results from equation (11) and  $p^{f \rightarrow c}$  and  $p^{c \rightarrow f}$  are given by equations (13) and (14),  
208 i.e.

$$\dot{Y} = \beta_y [(a_y - 1)Y + a_q (p_e - p_{e_o})E + A], \quad (16)$$

$$\dot{p}_e = \beta_e \left( \frac{bY}{p_e E} + \pi_e^e - \rho_{e_o}^e \right) p_e, \quad (17)$$

$$\dot{\pi}_e^e = \beta_{\pi_e} \left( \frac{1+x}{2} \beta_e \left( \frac{bY}{p_e E} + \pi_e^e - \rho_{e_o}^e \right) - \pi_e^e \right), \quad (18)$$

$$\dot{x} = (1-x)\beta_x \exp(a_x s) - (1+x)\beta_x \exp(-a_x s). \quad (19)$$

209 and  $s$  is given by equation (15).

210 The model provides a simple but general framework to capture some key real-financial interactions,  
211 and the feedback between economic variables and agents' attitudes and expectations.

### 212 3 Local Stability Analysis

213 Let  $\mathbf{z} = (z_1, z_2, \dots, z_n)$ . For any dynamical system  $\dot{\mathbf{z}} = g(\mathbf{z})$ , a steady state is defined as the state in  
214 which  $\dot{\mathbf{z}} = \mathbf{0}$ . Then, it is straightforward to prove the following Lemma:<sup>7</sup>

<sup>6</sup>We adopt a quadratic specification only for the sake of simplicity and expositional clarity. All of our results can be extended to more general switching index functions  $s = s(x, Y, p_e, \pi_e^e)$ , with  $s'_x > 0$ ,  $s'_y > 0$ ,  $s'_{p_e} < 0$ , and  $s'_{\pi_e^e} < 0$ , where  $s'_i$  is the derivative of the function  $s(\cdot)$  with respect to  $i$ .

<sup>7</sup>Recall that the steady state value of the expected return on equity,  $\rho_{e_o}^e$ , is assumed to be a parameter of the model. Therefore Lemma 1 can be interpreted as identifying a one-parameter *family* of steady states.

215 **Lemma 1** *The dynamical system formed by of equations (16), (17), (18), and (19) always has the*  
 216 *following steady state solution:*

$$Y_o = \frac{A}{1 - a_y}, \quad (20)$$

$$p_{eo} = \frac{bA}{(1 - a_y)\rho_{eo}^e E}, \quad (21)$$

$$\pi_{eo}^e = 0, \quad (22)$$

$$x_o = 0. \quad (23)$$

217 While Lemma 1 defines the unique steady state values of the variables  $Y$ ,  $p_e$  and  $\pi_e^e$ , which will  
 218 always exist independently of the steady state values of  $x$ , it does not rule out the existence of further  
 219 steady states which however may arise solely due to the nonlinearity of the population dynamics.

220 In the following, we shall analyze the local stability of various subparts of the model separately.  
 221 This exercise allows us to understand the sources of instability (and the stabilizing forces) in the  
 222 economy before exploring the complete model by means of numerical simulations.

### 223 3.1 Core Real-Financial Interactions

224 We begin by analyzing the interaction between the macroeconomy and the stock market under the  
 225 assumption of constant capital gains expectations  $\pi_e^e = \bar{\pi}_e^e = 0$ . This assumption reduces our macroe-  
 226 conomic model to a 2D core system formed by equations (16) and (17).<sup>8</sup>

227 **Proposition 1** *The dynamical system formed by equations (16) and (17) has a unique steady state:*

228  $Y_o = \frac{A}{1 - a_y}$  and  $p_{eo} = \frac{bA}{(1 - a_y)\rho_{eo}^e E}$  *with the following stability conditions:*<sup>9</sup>

229 (i) *if  $\frac{a_q b}{1 - a_y} < \rho_{eo}^e$ , then the steady state is (asymptotically) stable;*

230 (ii) *if  $\frac{a_q b}{1 - a_y} > \rho_{eo}^e$ , then the steady state is an (unstable) saddle point.*

231 In this model, Tobin's  $q$  plays a key role in breaking down the dichotomy between the real and  
 232 financial components of the economy. An increase in  $p_e$  has a positive effect on the rate of change of  
 233 output, but a negative effect on the expected return on equity. Similarly, real markets influence asset  
 234 markets via the role of output as the main determinant of the rate of profit of firms, and thus of the

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<sup>8</sup>The proofs of all Propositions can be found in Appendix A.

<sup>9</sup>Given the fact that this dynamical subsystem is linear, local stability implies also global stability.

235 rate of return on real capital. A higher output level has a positive effect on  $\hat{p}_e$ , but a negative effect  
 236 on the rate of change of output.<sup>10</sup>

237 Proposition 1 concerns the interaction of real and financial adjustment processes and does not  
 238 depend on the presence of capital gain expectations, which are introduced next.

### 239 3.2 Real-Financial Interactions with Constant Heterogeneous Beliefs

240 As a next step, we introduce heterogeneous expectations in the basic 2D macroeconomic model while  
 241 assuming agents' attitudes, and thus  $\nu_c$ , to be exogenously given. This allows us to analyze the  
 242 effect of expectations on the dynamics of real financial interactions. Not surprisingly, introducing  
 243 heterogeneity in agents' expectations, may play a destabilizing role in the economy.

244 The next Proposition characterizes the dynamics of the 3D model when  $\beta_e < 1$ .

245 **Proposition 2** Consider the dynamical system formed by equations (16), (17) and (18) and let  $\beta_e <$   
 246 1. For any  $\nu_c \in [0, 1]$ , at the steady state given by equations (20)-(22):

247 (i) if  $a_q b / (1 - a_y) < \rho_{eo}^e$  then the system is locally (asymptotically) stable,

248 (ii) if  $a_q b / (1 - a_y) > \rho_{eo}^e$  then the system is unstable.

249 Observe that Proposition 2 holds for any  $\nu_c \in [0, 1]$ , and so it provides some important insights  
 250 on the dynamics of the system formed by equations (16), (17) and (18). Interestingly, as in the 2D  
 251 system, the stability of the steady state depends on the relation between  $a_q$ ,  $b / (1 - a_y)$  and  $\rho_{eo}^e$ . In  
 252 the case where  $\beta_e < 1$  the introduction of heterogeneous expectations (chartist and fundamentalist)  
 253 changes neither the number of steady states, nor their stability properties.

254 The validity of Proposition 2 (the irrelevance of the *exogenous* share of chartists and fundamen-  
 255 talists in the markets for the stability of the system) depends of course on  $\beta_e < 1$ . The following  
 256 Proposition applies for the case where  $\beta_e > 1$ :

257 **Proposition 3** Consider the dynamical system formed by equations (16), (17) and (18). Further, let

$$\nu_c^* = \frac{\beta_y(1 - a_y) + \beta_e \rho_{eo}^e + \beta_{\pi_e^e}}{\beta_{\pi_e^e} \beta_e} = \frac{\beta_y(1 - a_y)}{\beta_{\pi_e^e} \beta_e} + \frac{\rho_{eo}^e}{\beta_{\pi_e^e}} + \frac{1}{\beta_e}.$$

<sup>10</sup>It is also interesting to consider briefly the dynamics of the model under perfect foresight i.e.  $\pi_e^e = \hat{p}_e$ , see e.g. Turnovsky (1995). In this case, the population dynamics and a separate law of motion for share price expectations are redundant, and the law of motion of share prices is:

$$\hat{p}_e = \beta_e \left( \frac{bY}{p_e E} + \hat{p}_e - \rho_{eo}^e \right) \iff \hat{p}_e = \frac{\beta_e}{1 - \beta_e} \left( \frac{bY}{p_e E} - \rho_{eo}^e \right).$$

It is straightforward to confirm by a standard local stability analysis that if  $\beta_e < 1$ , the conditions for local stability of the steady state are the same as those postulated in Proposition 1.

258 Under the assumption that  $\beta_e > 1$ , if  $\nu_c^* \in [0, 1]$  and  $\nu_c > \nu_c^*$ , then the steady state given by equations  
 259 (20)-(22) is unstable.

260 According to Proposition 3, if  $\beta_e > 1$  and the share of chartists in the market  $\nu_c$  is beyond the  
 261 endogenously determined threshold value  $\nu_c^*$ , the destabilizing influence of the chartists will lead to  
 262 macroeconomic instability, as higher capital gains expectations will lead to higher share prices and  
 263 higher output which will in turn translate into higher capital gain expectations. Accordingly,  $\nu_c^*$   
 264 represents an endogenous upper bound on  $\nu_c$  above which the system loses stability to exogenous  
 265 shocks. Higher values for  $\beta_{\pi_e}$  and/or  $\beta_e$  lower  $\nu_c^*$ , making the whole system more prone to overall  
 266 instability.

267 The previous analysis has only described the dynamics of the economy in a neighborhood of the  
 268 steady state characterized by equations (20), (21) and (22). The introduction of aggregate sentiments,  
 269 and by extension of a varying influence of chartist expectations, is likely to lead to explosive dynam-  
 270 ics, for instance if either the speed of adjustment in financial markets  $\beta_e$  or the coefficient  $\beta_{\pi_e}$  are  
 271 sufficiently high. This explosiveness may be tamed far off the steady state through the activation of  
 272 nonlinear policy measures or, as we will discuss below, by intrinsic nonlinear changes in behavior, thus  
 273 ensuring that all trajectories remain within an economically meaningful bounded domain.

274 We will explore the global dynamics of the system with aggregate sentiment dynamics by numerical  
 275 simulations in section 4 below. In the next section, we explore the possibility that endogenous changes  
 276 in the agents' populations,  $\nu_c$ , reduce the influence of chartists far off the steady state and thereby  
 277 create turning points in the evolution of capital gain expectations.

### 278 3.3 Real-Financial Interactions with Endogenous Aggregate Sentiments

279 As previously mentioned, while Lemma 1 characterizes a particular steady state solution that al-  
 280 ways exists, other steady states may also exist for particular parameter constellations. The following  
 281 proposition focuses on the role of the parameters  $s_x$  and  $a_x$  for the emergence of multiple steady  
 282 states.

283 **Proposition 4** Consider the dynamical system formed by equations (16)-(19). If  $s_x \leq 1/a_x$  then the  
 284 steady state given by equations (20)-(23) is unique. If  $s_x > 1/a_x$ , then there are two additional steady  
 285 state values for  $x_o$ : one characterized by a dominance of fundamentalists,  $e_f$ , and one where chartists  
 286 dominate,  $e_c$ .

287 The intuition behind Proposition 4 is captured in Figure 1, which illustrates the number of steady  
 288 states of  $x$  for different values of  $a_x$  and  $s_x$ . While the steady state is unique if  $s_x \leq 1/a_x$ , there are  
 289 multiple steady states if  $s_x > 1/a_x$ . For example, for  $s_x = 2/a_x$ , there are three steady states: one

290 with a large prevalence of fundamentalists ( $x \approx -1$ ), one with populations of equal size ( $x = 0$ ), and  
 291 one with a large prevalence of chartists ( $x \approx 1$ ).

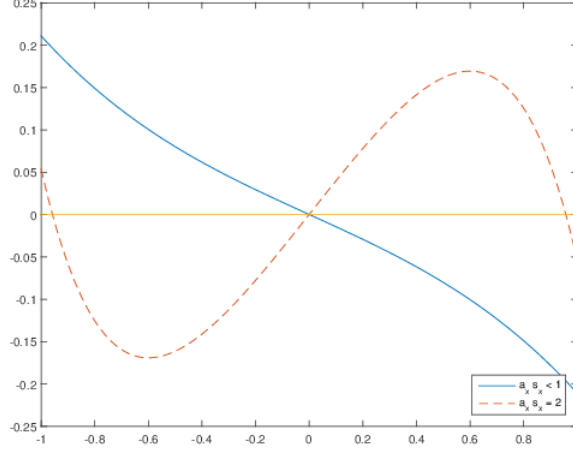


Figure 1: Steady states of population dynamics for different values of  $a_x$  and  $s_x$

292 Before analyzing the dynamics of the complete system numerically in the next section, it is inter-  
 293 esting to consider the properties of the opinion dynamics and the expectations part of the model in  
 294 isolation. We thus assume that output and dividend payments are fixed at their steady state values  
 295  $Y_o$  and  $p_{eo}$  in the rest of this section. By inserting equations (20) and (21) into (18) we get

$$\dot{\pi}_e^e = \beta_{\pi_e^e} \left[ \beta_e \frac{1+x}{2} - 1 \right] \pi_e^e, \quad (24)$$

296 and from equation (15),

$$s = s_x x - s_{\pi_e^e} (\pi_e^e)^2. \quad (25)$$

297 Inserting this expression in equation (19) yields

$$\dot{x} = \beta_x \left[ (1-x) \exp(a_x (s_x x - s_{\pi_e^e} (\pi_e^e)^2)) - (1+x) \exp(-a_x (s_x x - s_{\pi_e^e} (\pi_e^e)^2)) \right]. \quad (26)$$

298 A quick glance at equation (24) makes clear that the condition  $\dot{\pi}_e^e = 0$  can be fulfilled either when  
 299  $\pi_e^e = 0$ , or when  $\pi_e^e \neq 0$ . This means that the multiplicity of steady states arises here not only through  
 300 the nonlinear equation (26), as discussed in Proposition 4, but also through equation (24). The next  
 301 two Propositions deal with the case with  $\pi_{eo}^e = 0$ .

302 **Proposition 5** Consider the dynamical system formed by equations (24) and (26). Then:

303 (i) if  $s_x \in (0, 1/a_x)$ ,  $e_o = (\pi_{eo}^e, x_o) = (0, 0)$  is the only steady state with  $\pi_{eo}^e = 0$ ;

304 (ii) if  $s_x > 1/a_x$ , then two additional steady states exist,  $e_f = (0, x_o^f)$  and  $e_c = (0, x_o^c)$  with  $x_o^f < 0$   
 305 and  $x_o^c > 0$ , respectively.

306 In other words, if the aggregate sentiment dynamics display a strong self-reinforcing behavior,  
 307 multiple equilibria emerge in which either fundamentalists or chartists dominate. The next Proposition  
 308 describes some stability properties of the steady states identified in Proposition 5.

309 **Proposition 6** Consider the dynamical system formed by equations (24) and (26). Then:

310 (i) Let  $s_x \in (0, 1/a_x)$ . If  $\beta_e > 2$ , then  $e_o = (\pi_{e_o}^e, x_o) = (0, 0)$  is an unstable saddle point. If  $\beta_e < 2$ ,  
 311 then  $e_o$  is locally asymptotically stable.

312 (ii) Let  $s_x > 1/a_x$ . The steady state  $e_o = (0, 0)$  is unstable. The steady states  $e_c = (0, x_o^c)$  and  
 313  $e_f = (0, x_o^f)$  are locally asymptotically stable if and only if  $(1 + x_o^c)\beta_e < 2$  and  $(1 + x_o^f)\beta_e < 2$ ,  
 314 respectively.

315 By Proposition 6, it follows that sentiment dynamics may lead to local instability. This raises  
 316 the issue of the global viability of the dynamical system formed by equations (24) and (26). It is  
 317 difficult to draw any definite analytical conclusions on this issue and we shall analyze it in detail  
 318 by means of numerical methods in the next section. To be sure, opinion dynamics do incorporate  
 319 a stabilizing mechanism far off the steady state(s), as  $x$  always points inwards at the border of the  
 320  $x$ -domain  $[-1, 1]$ . Yet the global viability of the system will ultimately depend on the properties of  
 321 the *interaction* between market expectations and opinion dynamics.

322 Consider, for example, case (i) of Proposition 6 and suppose that  $\beta_e > 2$ , so that  $e_o = (0, 0)$   
 323 is unstable. It can be shown that there must be an upper and a lower turning point for  $\pi_e^e$  in the  
 324 economically relevant phase space  $[-1, 1] \times [-\infty, +\infty]$ . For suppose, by way of contradiction, that  $\pi_e^e$   
 325 tends to infinity. By equation (26) it follows that  $\dot{x}$  becomes negative and approaches  $-\infty$ . But then as  
 326  $x$  approaches  $-1$ , by equation (24) it follows that  $\dot{\pi}_e^e$  becomes negative, which contradicts the starting  
 327 assumption. A similar argument rules out the possibility that  $\pi_e^e$  becomes infinitely negative and  
 328 therefore there must always be an upper or lower turning point for capital gain inflation or deflation.  
 329 This implies that all trajectories stay within a compact subset of the phase space and the interaction  
 330 between expectation dynamics and herding mechanism would thus be bounded, if taken by itself.<sup>11</sup>

It is also worth noting that the dynamical system formed by equations (24) and (26) features two  
 additional steady states for the case where  $\pi_{e_o}^e \neq 0$ ,  $e_+ = (\pi_{e_o}^+, x_o^+)$  and  $e_- = (\pi_{e_o}^-, x_o^-)$ , with

$$x_o = \frac{2}{\beta_e} - 1, \quad \text{and} \quad \pi_{e_o}^e = \pm \sqrt{\frac{s_x \left( \frac{2}{\beta_e} - 1 \right) - \ln \left( \frac{1}{\beta_e - 1} \right) / 2a_x}{s_{\pi_e^e}}}.$$

<sup>11</sup>Given the instability of the steady state, this suggests the existence of a limit cycle.



331 These steady states<sup>12</sup> are locally asymptotically stable if

$$a_x s_x < \frac{1}{1 - x_o^2}.$$

## 332 4 Numerical Simulations

333 This section examines the properties of the model using numerical simulations.<sup>13</sup> We first illustrate  
 334 the effects of capital gain expectations on the dynamics of Tobin's  $q$  using the 3D model comprising  
 335 the output equation (16), the share price equation (17) and the capital gains equation (18) and then,  
 336 in a second step, investigate the complete 4D dynamical system including the endogenous dynamics  
 337 of aggregate sentiments.

Table 1: Baseline Parameter Calibration of the 2D model

Autonomous spending	$A$	0.128
Profit share	$b$	0.35
Elasticity of aggregate demand to income	$a_y$	0.8
Elasticity of aggregate demand to Tobin's $q$	$a_q$	0.05
Adjustment speed of Tobin's $q$	$\beta_e$	2
Adjustment speed of output	$\beta_y$	2
Parameter in population dynamics	$a_x$	0.8
Steady state capital stock	$K_o$	1
Steady state equity stock	$E_o$	1
Steady state population	$x_o$	0
Steady state expectations	$\pi_{eo}^e$	0
Steady state expected capital return	$\rho_{eo}^e$	0.14
Steady state output capital ratio	$\frac{Y_o}{K_o}$	0.64
Steady state share price	$p_{eo}$	1.6

338 The calibration of the 2D model is shown in Table 1. The profit share  $b$  is set at 0.35, in line with  
 339 the long term average in Karabarounis and Neiman (2014). Based on Bloomberg data from 2000 to  
 340 2013, the return on equity (adjusted for R&D spending) is on average 14 percent in the United States,  
 341 so we set  $\rho_{eo}^e = 0.14$ . Brooks and Ueda (2011) argue that Tobin's  $q$  has been fluctuating between  
 342 1.4 and 1.7 over the period 1990 to 2013. We set its steady state value within this range at 1.6. It  
 343 follows that the steady state output capital ratio is  $\frac{Y_o}{K_o}$  is 0.64. Mukherjee and Bhattacharya (2010)  
 344 estimate that, in 18 OECD countries, the propensity to spend out of income fluctuates between 0.6  
 345 and 1.2. We set  $a_y$  equal to 0.8. Therefore by equation (20) the autonomous spending component  
 346  $A = Y_o(1 - a_y)$  equals 0.128.

<sup>12</sup>For these steady states to be economically meaningful the following conditions must hold:  $x_o = \left[\frac{2}{\beta_e} - 1\right] \in [-1, 1]$   
 and  $2a_x s_x \left(\frac{2}{\beta_e} - 1\right) \geq \ln\left(\frac{1}{\beta_e - 1}\right)$ .

<sup>13</sup>The numerical simulation are performed using the SND package (Chiarella et al., 2002).

347 The elasticity of aggregate demand to Tobin's  $q$ ,  $a_q$ , is set equal to 0.05. The dynamic output  
 348 multiplier,  $\beta_y$ , and the speed of adjustment of Tobin's  $q$ ,  $\beta_e$ , are both set equal to 2. Unless otherwise  
 349 stated, the experiment considered in this section is a 1 percent shock on output with no auto-regressive  
 350 component. All diagrams reporting simulation results display the deviation of variables from their  
 351 steady state value in percent, unless otherwise stated.

352 Figure 2 illustrates the dynamic adjustments of the 3D model consisting of the output equation  
 353 (16), the share price equation (17) and the capital gains expectations equation (18) for  $\beta_{\pi_e} = 0$ ,  
 354  $\beta_{\pi_e} = 0.2$  and  $\beta_{\pi_e} = 4$ .<sup>14</sup> In all cases, the parameter  $a_q$  is small enough (0.05) to ensure that the  
 355 determinant is positive, and  $\nu_c = 0.5$ , which corresponds to  $\nu_c = \frac{1+x}{2}$  with  $x_o = 0$  in line with the 4D  
 356 model calibration presented below.

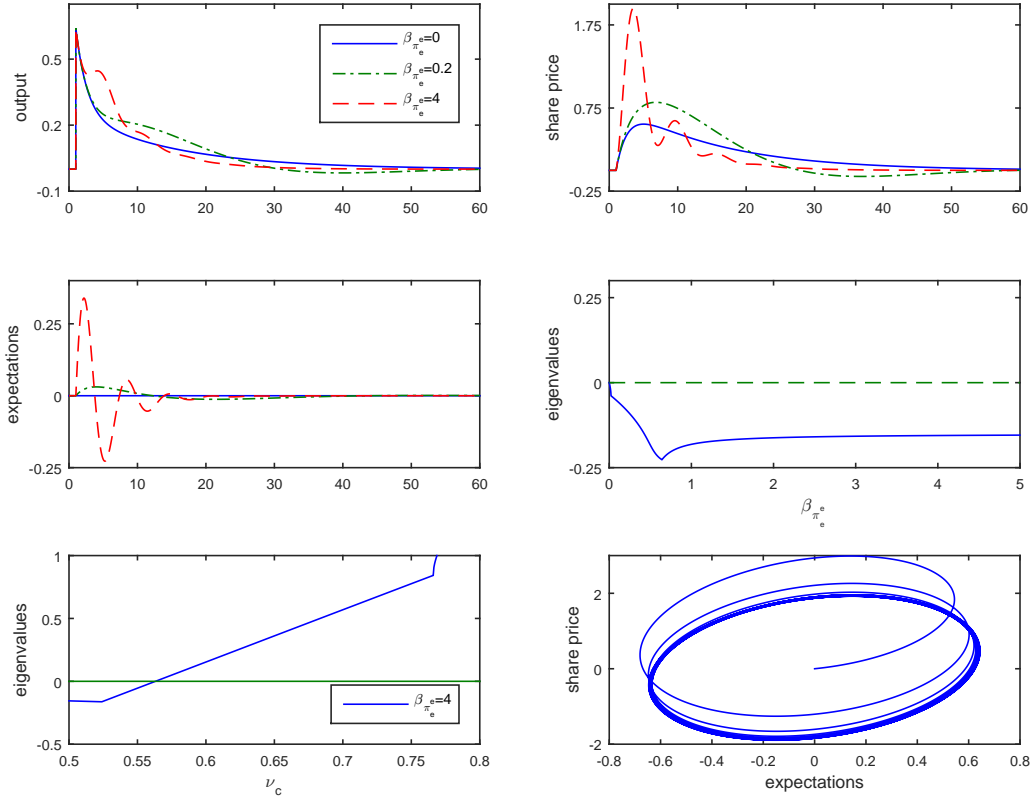


Figure 2: Dynamic responses following a positive one-percent output shock and maximum eigenvalues for the 3D model  $(Y, p_e, \pi_e^e)$ .

357 If  $\beta_{\pi_e} = 0$  the dynamics of the system is rather simple: the positive shock on output is followed  
 358 by an increase in share price  $p_e$  as the expected return on the capital stock  $\rho_e^e$  rises. The dynamics  
 359 of  $p_e$  is hump-shaped as the increase in the share price is modest at the beginning and does not

<sup>14</sup>It is worth noting that the simulations based on  $\beta_{\pi_e} = 0$  represent the dynamics of the 2D model and are thus related to the analytical stability conditions described in Proposition 1.

360 immediately reduce the return on capital. When the equity price rises enough to lower the return on  
361 equity, the economy converges back to its steady state. If  $\beta_{\pi_e} = 0.2$  the model displays an oscillatory  
362 behavior after the aggregate demand shock due to the activated feedback channel between  $\pi_e^e$  and  
363  $p_e$ , as capital gains expectations amplify *both* the increase in the price of equity initiated by a higher  
364 return on capital *and* the decline in the price of equity when the rate of return diminishes due to a fall  
365 in the price of equities. As the share price  $p_e$  undershoots its steady state value it generates further  
366 oscillations in aggregate output. These fluctuations are not, however, self-sustaining and the economy  
367 returns to the steady state.

368 The dashed red line in Figure 2 corresponds to the case where the speed of adjustment in capital  
369 gains expectations  $\beta_{\pi_e}$  is increased from 0.2 to 4 with  $a_q = 0.05$ , which implies that the stability  
370 conditions in Proposition 2 continue to hold. As the (negative) trace of the corresponding Jacobian  
371 matrix declines with  $\beta_{\pi_e}$ , the model is stable but displays oscillations around the trajectory converging  
372 back to the steady state. As shown by the solid blue line in the second row, second column graph,  
373 the maximum real part of the eigenvalues is always negative for all values of the speed of adjustment  
374 of expectations,  $\beta_{\pi_e}$ . Raising  $\beta_{\pi_e}$  increases the amplitude of the fluctuations of the expectations but  
375  $\beta_{\pi_e}$  has a stabilizing effect on output. Adaptive expectations are inherently stable given the influence  
376 of the equity price on the real return on equity. In contrast, the graphs in the third row of Figure 2  
377 highlight the importance of the parameter  $\nu_c$  for the stability of the 3D model  $(Y, p_e, \pi_e^e)$  as discussed  
378 in Proposition 3. In the left panel of the third row, the maximum real part of the eigenvalues turns  
379 positive for values of  $\nu_c$  strictly larger than 0.56. Increasing the value of  $\nu_c$  at 0.56 while keeping  
380  $\beta_{\pi_e} = 4$  produces self-sustaining oscillations of the model, as shown in the right panel of this figure.<sup>15</sup>

381 Figure 3 illustrates the case of multiple steady states described at the end of section 3 for the  
382 subsystem  $(\pi_e^e, x)$  where the steady state for expectations and population are different from zero. In  
383 the upper two panels we set  $\beta_e = 1.15$ ,  $s_x = 1.5$  and  $a_x = 1$  (so that  $s_x > 1/a_x$ ), which implies  
384  $x_o = \frac{2}{\beta_e} - 1 = 0.74$  and  $\pi_{eo}^e = 0.57$ . Following a positive shock on the population variable  $x$ , the  
385 population dynamics fluctuates around its steady state value following dampening oscillations. In this  
386 case, the prevalence of chartist expectations (as  $x_o = 0.74 > 0$ ) does not lead to explosive dynamics  
387 due to the relatively slow adjustment in the price of shares. On the contrary, as illustrated in the  
388 two lower panels in Figure 3, increasing the speed at which the price of shares adjusts,  $\beta_e = 1.5$ ,  
389 makes the steady state  $e_+ = (\pi_{eo}^+, x_o^+)$  locally unstable. Following the shock, the population features  
390 an explosive oscillatory dynamic response until the excess volatility in the financial markets leads  
391 agents to switch towards fundamentalist expectations. The economy then converges towards a stable  
392 equilibrium dominated by fundamentalists where capital gains expectations are zero.

393 The next simulation in Figure 4 considers the influence of the aggregate sentiment dynamics on  
394 the price of capital and the financial multiplier by setting  $\beta_x = 0.75$ . The choice of  $a_x = 0.8$  and

<sup>15</sup>Given the parametrization of the model, while the value of  $\nu_c^*$  is 0.585, the cut-off value for instability is 0.5635. These values corroborate Proposition 3 as identifying a *sufficient* condition for local instability.

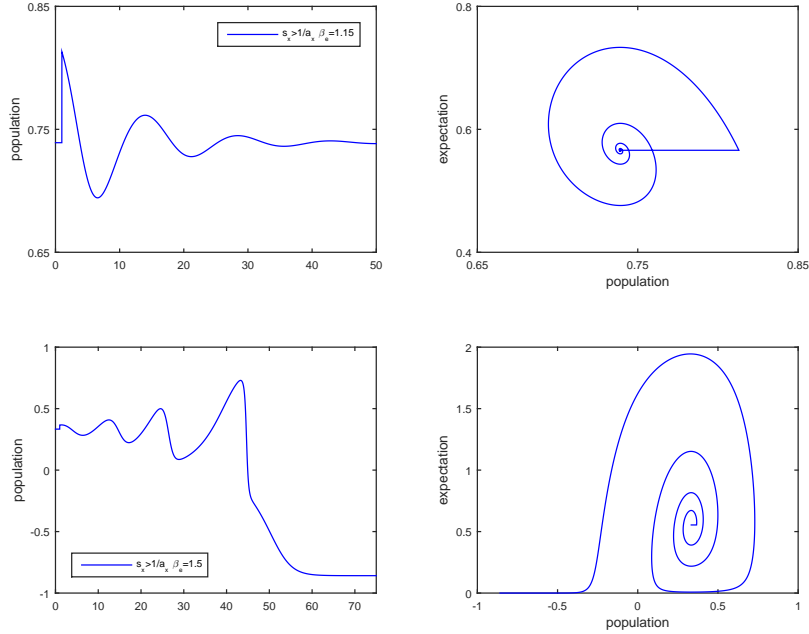


Figure 3: Dynamic response for the 2D model  $(\pi_e^e, x)$  following a positive shock on the population dynamics in the multiple (non-zero) steady state case.

395  $s_x = 0.8$  corresponds to the case of a unique steady state with  $x_o = 0$  for the relative population of  
 396 fundamentalists and chartists. We now set  $s_y = 20$  in order to incorporate the impact of real economic  
 397 activity on the aggregate sentiments of the agents. As a first step, we focus on a linear version of the  
 398 opinion switching index abstracting from the influence of price and capital gains volatility by setting  
 399  $s_{p_e} = s_{\pi_e^e} = 0$  (we analyze the general case with  $s_{p_e} \neq 0$  and  $s_{\pi_e^e} \neq 0$  in Figure 7 below). The rest of  
 400 the parameters are similar to those of the dashed green line in Figure 2 ( $\beta_{\pi_e^e} = 4$ ). Figure 4 compares  
 401 the 3D model just discussed (solid blue line) with the 4D model (green line).

402 As Figure 4 clearly shows, the addition of the population dynamics generates larger fluctuations  
 403 in output and equity prices. Following a positive output shock, the increase in chartist population  
 404 further raises capital gain expectations, which further increases the expected returns on equity and  
 405 the demand for equity. The dashed-dotted red line corresponds to the 4D model where the self-  
 406 reference parameter  $s_x$  in the aggregate sentiment index is increased from 0.8 to 1. This value of  $s_x$   
 407 still generates a unique steady state ( $x_o = 0$ ) of the population variable. But the population dynamics  
 408 now exhibits larger fluctuations between -0.2 and 0.3. These larger fluctuations translate into wider  
 409 oscillations in capital gains expectations, share prices, and economic activity, with the reversal of  
 410 expectations towards fundamentalism generating a decline in output by 6 percent.

411 Given that the stability conditions cannot be derived analytically for the 4D model, the interpreta-  
 412 tion of the numerical simulations is indicative only. In order to interpret them recall that Proposition 6

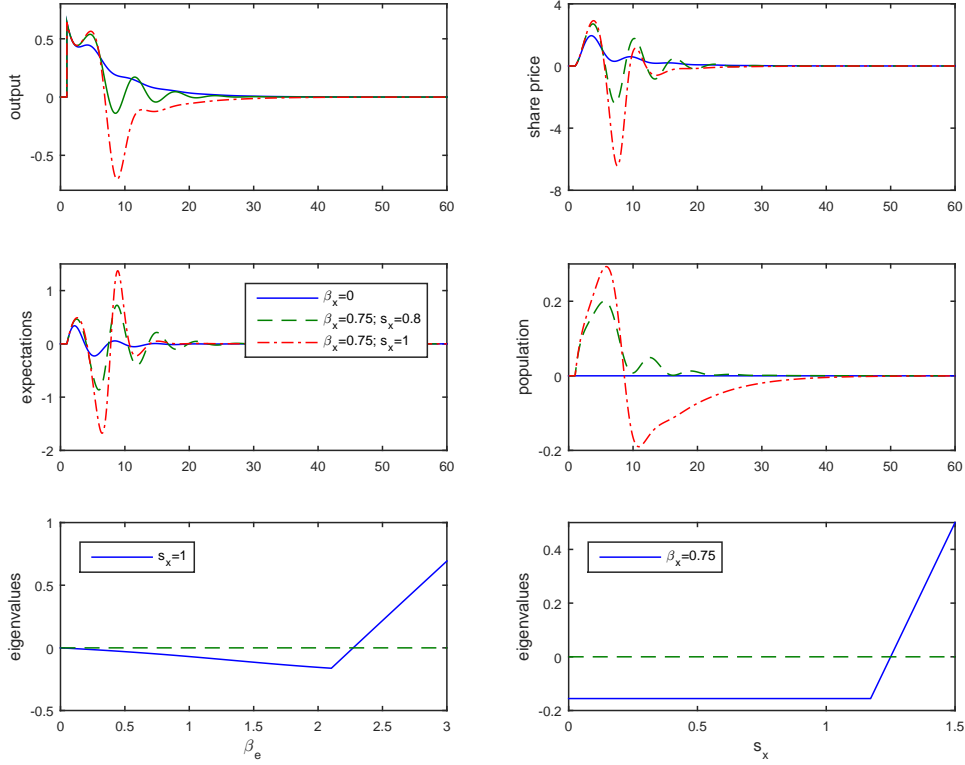


Figure 4: Dynamic adjustments to a one percent output shock in the 3D model ( $Y, p_e, \pi_e^e$ ) and the 4D model ( $Y, p_e, \pi_e^e, x$ ) (first two rows) and maximum eigenvalue diagrams (last row)

413 stated that the 2D model formed by equations (24) and (26) has a unique steady state if  $s_x \in (0, 1/a_x)$   
 414 and is stable if  $\beta_e < 2$ . Similarly, as shown in section 3.2 above, the value of  $\beta_e$  affects the stability  
 415 of the 3D dynamical system formed by equations (16)-(18). This suggests that the parameter  $\beta_e$  may  
 416 play a key role in determining the stability properties of the whole system. The left figure of the third  
 417 panel in Figure 4 confirms this intuition: it plots the maximum real part of the eigenvalues of the  
 418 system around the steady state with  $x_o = 0$  with respect to different values of  $\beta_e$ . The maximum  
 419 real part of the eigenvalues turns positive for  $\beta_e$  larger than 2.3, indicating that the 4D model loses  
 420 stability for large values of  $\beta_e$ . Comparably, the right panel of the third row displays the maximum  
 421 real part of the eigenvalues of the system around the steady state with  $x_o = 0$  for  $s_x$  varying between  
 422 0 and 1.5. In line with the previous simulation, the system is stable when  $s_x$  is smaller than 1.25.  
 423 The system of equations has a unique steady state towards which the economy converges.

424 Next we analyze the dynamics of the 4D model assuming  $s_{p_e} = s_{\pi_e^e} = 0$  with  $s_x = 1.5$ . Given  
 425  $a_x = 0.8$ , these parameter values lead to the existence of three steady states, as discussed in Proposition  
 426 4. In this case, a negative shock on output steers the population dynamics towards a steady state  
 427 dominated by fundamentalists at  $x_o = -0.65$  as illustrated in Figure 5. Given the parametrization

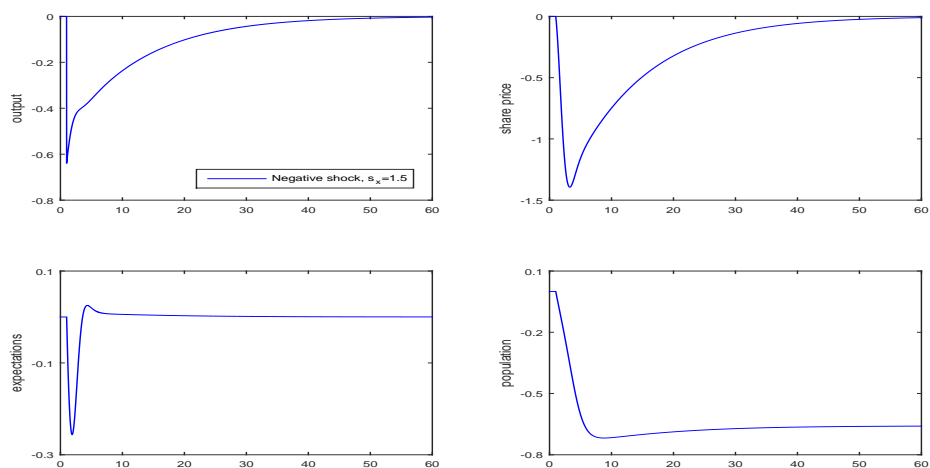


Figure 5: Dynamic adjustments to a negative one percent output shock in the 4D model.

428 of this simulation, output and share prices converge back to their corresponding steady states in a  
 429 monotonic manner.

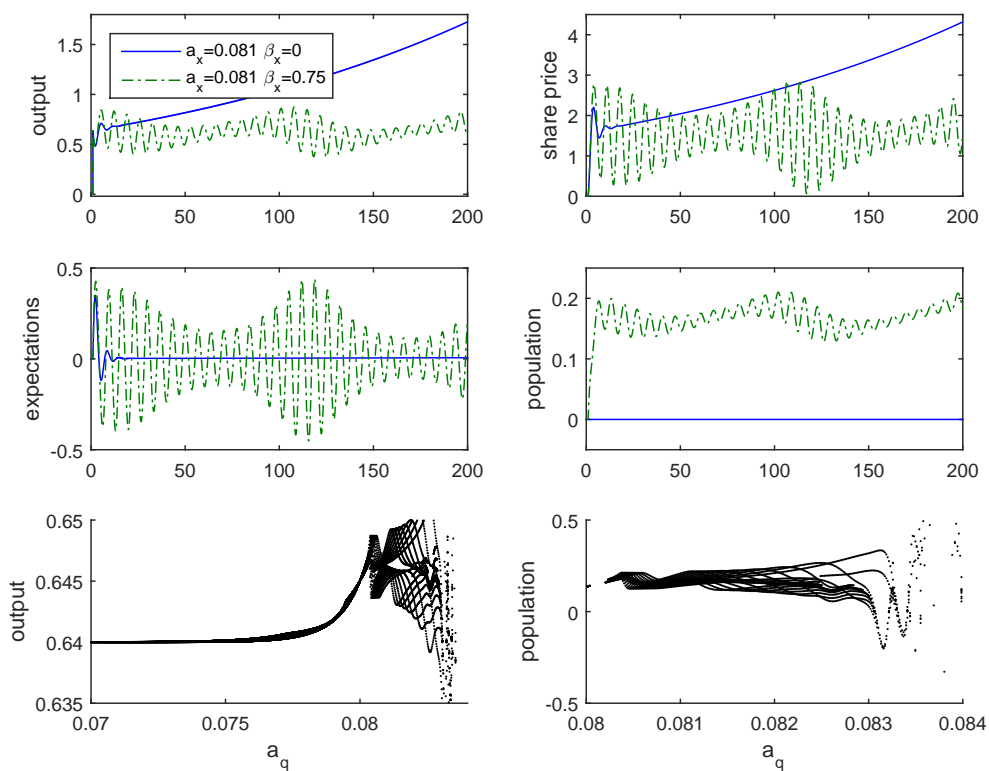


Figure 6: Explosive dynamics in the 3D model  $(Y, p_e, \pi_e^e)$  versus bounded dynamics in the 4D model  $(Y, p_e, \pi_e^e, x)$ .

430 While the aggregate sentiment dynamics tends to amplify financial instability in the proximity of  
431 the steady state, the non-linearity embedded in the population dynamics generates forces that keep the  
432 aggregate fluctuations within viable boundaries. Figure 6 illustrates how global stability is generated  
433 by the sentiment dynamics. The solid blue line corresponds to the 3D model presented in Figure 2  
434 with the parameter  $a_q$  (which represents the sensitivity of output to Tobin's  $q$ ) increased from 0.05 to  
435 0.081. For a value of  $a_q = 0.081$ , the 3D model is unstable as illustrated by the monotonically explosive  
436 trajectory of output and of the price of equities in the top row, and of the capital gain expectations in  
437 the left panel in the second row.<sup>16</sup> The instability is located in the financial sector and arises because  
438 of a positive feedback between the rate of return on equity, the price of equity, and its accelerator effect  
439 on the real economy. The dashed line corresponds to the case where the 3D model is augmented by  
440 aggregate sentiment dynamics with  $\beta_x = 0.75$ ,  $s_x = 0.8$ ,  $s_y = 12.5$  and  $s_{p_e} = s_{\pi_e} = 0$ . The economy  
441 does not display an explosive behavior now, being characterized instead by bounded cycles with high  
442 frequency oscillations taking place around lower frequency fluctuations. The non-linearity embedded  
443 in the sentiment dynamics sets an upper and a lower bound to the amplitude of the cycles. The lower  
444 two panels plot the bifurcation diagrams for output and the relative size of the two populations for  
445  $a_q \in [0.07; 0.084]$ . The diagram shows the Hopf bifurcation for  $a_q = 0.08$ , beyond which the model  
446 displays oscillations.

447 As already mentioned, the simulations of the 4D model shown in Figures 4 through 6 have all  
448 considered a linear version of the sentiment switching index with  $s_{p_e}$  and  $s_{\pi_e}$  equal to zero in equation  
449 (15). In Figure 7, we consider the case where the opinion switching index depends negatively on  
450 the volatility of capital gain expectations and of the share price. As the graphs in Figure 7 show,  
451 the activation of these nonlinear terms does modify the dynamics of the model. When the sentiment  
452 switching index also depends on these two volatility terms, there is a coordination in the expectations of  
453 financial market agents towards fundamentalism. We illustrate this emergent feature by the following  
454 two examples.

455 The first example corresponds to the case where  $\beta_e = 0.75$  and  $s_x = 1$  and is illustrated in the  
456 upper panels of Figure 7. Therein the blue line corresponds to the 4D model of Figure 4 with a linear  
457 switching index specification ( $s_{p_e} = s_{\pi_e} = 0$ ), while the green line corresponds to the case where the  
458 switching index contains also nonlinear terms ( $s_{p_e} = s_{\pi_e} = 20$ ), both with  $\beta_e = 0.75$  and  $s_x = 1$ . As  
459 it can be clearly observed, the extent of the dynamic reaction of the full nonlinear 4D model following  
460 a positive output shock is smaller than the reaction of the 4D model with a linear switching index, as  
461 the volatility in share price and capital gain expectations reduces the fluctuations in the population  
462 dynamics.

463 The second example corresponds to the dynamically explosive case discussed for the 3D model  
464 in Figure 6 and is illustrated in the lower panels of Figure 7. Therein, the blue line corresponds

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<sup>16</sup>The scale of the graph gives the impression that  $\pi_e$  returns to its initial steady state value, but in fact it diverges, too, albeit very slowly.

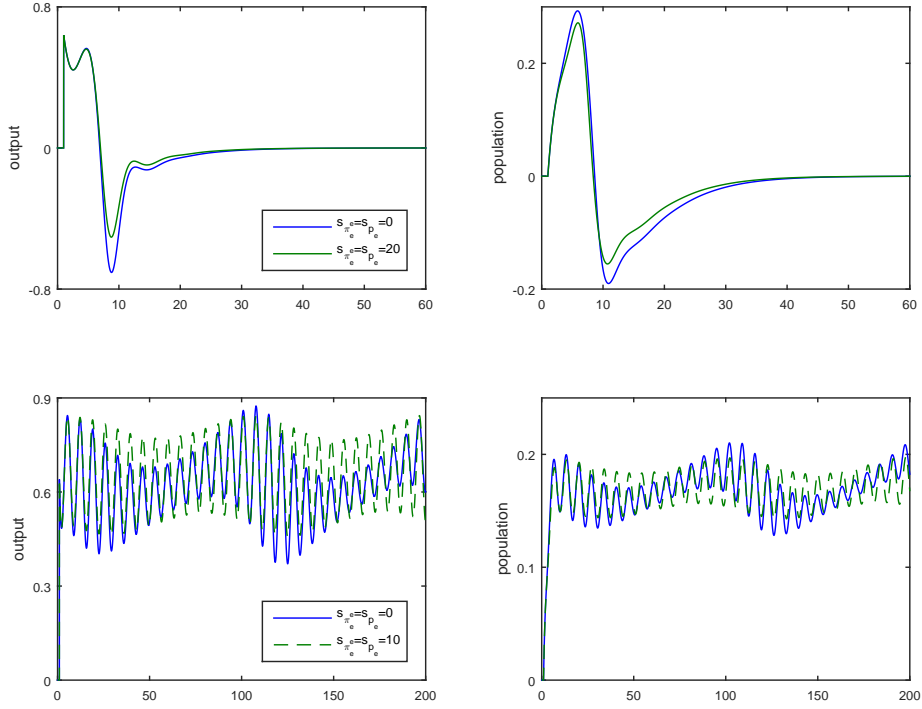


Figure 7: Dynamic adjustments of the full 4D model ( $Y, p_e, \pi_e^e, x$ ) for different values of  $s_{p_e}$  and  $s_{\pi_e}$  for the dynamically stable case (upper panels) and the explosive case (lower panels).

465 to Figure 6 where the nonlinearity in the population dynamic stabilizes an otherwise explosive 3D  
 466 model. More precisely, what characterized the dynamics of the 4D model shown in Figure 6 was that  
 467 fluctuations took place along both high and low frequencies. Adding a second type of nonlinearity in  
 468 the 4D model via the volatility terms in the sentiment switching index seems to reduce in particular  
 469 the amplitude of the low frequency population fluctuations.<sup>17</sup>

## 470 5 Dynamics under Unconventional Monetary Policies

471 The previous numerical analysis showed the ambivalent effects of the interaction between capital gains  
 472 expectations and the composition of the population of financial agents on the stability of our model  
 473 economy. In this section, we briefly outline some policies that could stabilize both real *and* financial  
 474 markets. Two policy proposals immediately come to mind, in the light of the current financial crisis  
 475 and the measures adopted to tackle it.

476 Given the economic debate of the last years about a renewed regulation of international financial  
 477 markets, it is natural to consider the impact of a tax on capital gains. Taxing finance either via a

<sup>17</sup>Appendix B contains additional simulations illustrating the properties of the full model highlighting in particular the possibility of complex dynamics and performing various robustness checks by means of bifurcation diagrams.



478 “Tobin Tax” or by increasing the marginal tax rate on capital is often suggested by policy makers as  
479 a way of curbing financial market instability, see e.g. [Admati and Hellwig \(2013\)](#). A second policy  
480 focuses on the ability of the Central Bank to reduce the pro-cyclicality of the sentiment switching  
481 index by convincing agents that it will act vigorously to prevent bubbles in financial markets. Indeed,  
482 as central banks greatly influence financial markets sentiments beyond the conventional interest rate  
483 policy via their communication policies, the ability of a central banker to coordinate financial traders’  
484 expectations on a stable equilibrium may be crucial in times of financial distress, see e.g. [Siklos and  
485 Sturm \(2013\)](#).

486 In Figure 8, the first two policies are assessed with respect to the dashed-dotted red line which  
487 corresponds to the green line in the top row of Figure 7 generated with  $\beta_x = 0.75$  and  $s_x = 1$ . Further,  
488 we assume  $s_{p_e} = s_{\pi_e} = 20$  as in Figure 7 of the previous section. In the following we thus simulate  
489 the impact of various policies in the full 4D model. Taxing capital gains is taken into account by  
490 introducing the tax rate  $\tau_{p_e}$  in the equation for capital gain expectations (equation (18)).

$$\dot{\pi}_e^e = \beta_{\pi_e^e} \left[ (1 - \tau_{p_e}) \left( \frac{1+x}{2} \right) \hat{p}_e - \pi_e^e \right]. \quad (27)$$

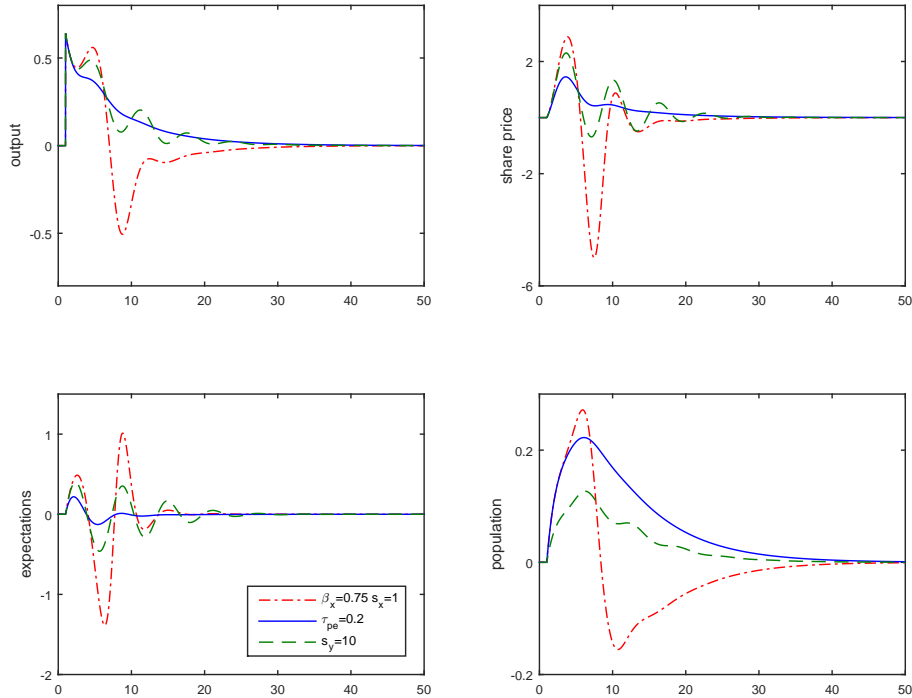


Figure 8: Dynamics under capital gains taxation and central bank communication policy in the full 4D model  $(Y, p_e, \pi_e^e, x)$ .

491 The dynamics illustrated by the continuous blue line was generated assuming a tax rate of 20%.  
492 As it can be clearly observed, taxing capital gains has a strong impact on the output dynamics as it  
493 almost entirely smooths out output fluctuations, and it also reduces the amplitude of the fluctuations  
494 in expectations. A side effect is that the sentiment dynamics now follows a humped-shaped trajectory,  
495 rather than an oscillating pattern. As a result, the fluctuations in share prices are much more limited  
496 than in the case illustrated in the top row of Figure 7.<sup>18</sup>

497 The dashed green lines describe the dynamics of the 4D model under a successful central bank  
498 communication policy which modifies the perceptions of financial market participants. We specify  
499 this scenario in our stylized framework by a reduction of the sentiment index parameter  $s_y$  from 20 to  
500 10. This type of policy has a direct impact on the volatility of financial markets and the real sector,  
501 and the reduction in  $s_y$  translates into a sharp reduction in output fluctuations.

## 502 6 Conclusions

503 We have studied in this paper a stylized dynamic macroeconomic model of real-financial market  
504 interactions with endogenous aggregate sentiment dynamics and heterogenous expectations in the  
505 tradition of the Weidlich-Haag-Lux approach as recently reformulated by Franke (2012). Following  
506 Blanchard (1981), we focused on the impact of equity prices on macroeconomic activity through the  
507 Brainard-Tobin  $q$ , leaving the nominal interest rate fixed for the sake of simplicity, and also because  
508 goods prices were assumed to be constant.

509 Using this extremely stylized but – due to the intrinsic nonlinear nature of the Weidlich-Haag-Lux  
510 approach – complex theoretical framework, we showed that the interaction between real and financial  
511 markets need not be necessarily stable, and might well be characterized by multiple equilibria (and even  
512 complex dynamics, see Appendix B below). The crucial theoretical, empirical, and policy question,  
513 then, is whether unregulated market economies contain some mechanisms ensuring the stability or  
514 global boundedness of the economy, or whether centrifugal forces may prevail, making some equilibria  
515 locally unstable and, potentially, the whole system globally unstable.

516 Our numerical simulations show that global stability can obtain if, far off the steady state, aggregate  
517 sentiment dynamics favor fundamentalist behavior during booms and busts which ensures that there  
518 are upper and lower turning points. Yet, both the local analysis and the simulations suggest that  
519 market economies can be plagued by severe business fluctuations and recurrent crises. We showed  
520 that two policy measures often advocated in the Keynesian literature, namely Tobin-type taxes (here  
521 on capital gains), and Central Bank intervention, can mitigate these problems.

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<sup>18</sup>Actually, the tax  $\tau_{pe}$  is not restricted to apply to actual transactions and is imposed on *both* actual *and* notional capital gains. Therefore, rather than a Tobin tax, it may be more appropriately interpreted as a wealth tax of the kind advocated by Piketty (2014). It is therefore quite interesting to note that, in addition to any redistributive effects, such a wealth tax may also help to mitigate business cycles and financial turbulence. We are grateful to Bruce Greenwald for pointing this out to us.

522 Our theoretical framework can be extended in a variety of directions. First, through the incorpo-  
523 ration of a varying goods price level and an active conventional interest rate policy, the interaction  
524 between macroprudential and conventional policies could be investigated. Also, given the central role  
525 of aggregate sentiments and bounded rationality, we may use the model to investigate the efficiency of  
526 these policies near or at the zero-lower bound of interest rates. Finally, we could analyze the dynamics  
527 of the model under alternative heuristics than the traditional chartist and fundamentalist rules. We  
528 intend to pursue some of these alternatives in future research.

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651 **Appendix A**

652 For any matrix  $J$ , let  $\text{tr}(J)$  be the trace of  $J$  and let  $|J|$  be its determinant.

653 **Proof of Proposition 1**

At a steady state, the Jacobian matrix  $J$  of equations (16) and (17) is:

$$J = \begin{pmatrix} -\beta_y(1 - a_y) & \beta_y a_q E \\ \frac{\beta_e b}{E} & -\beta_e \rho_{eo}^e \end{pmatrix}.$$

It is easy to see that  $\text{tr}(J) < 0$ . Furthermore, the determinant of  $J$  is

$$|J| = \beta_y(1 - a_y)\beta_e \rho_{eo}^e - \frac{\beta_y a_q E \beta_e b}{E}.$$

Therefore  $|J| > 0$  if and only if

$$(1 - a_y)\rho_{eo}^e > a_q b.$$

654 Thus,  $|J| > 0$  if and only if

$$\rho_{eo}^e > \frac{a_q b}{1 - a_y}. \quad (\text{Q.E.D.})$$

655 **Proof of Proposition 2**

656 For any  $\nu_c \in [0, 1]$ , at the steady state given by equations (20)-(22), the Jacobian of the 3D system  
657 formed of equations (16), (17) and (18) is

$$J = \begin{pmatrix} -\beta_y(1 - a_y) & \beta_y a_q E & 0 \\ \frac{\beta_e b}{E} & -\beta_e \rho_{eo}^e & \beta_e p_{eo} \\ \frac{\beta_{\pi_e} \beta_e \nu_c b}{p_{eo} E} & -\frac{\beta_{\pi_e} \beta_e \nu_c \rho_{eo}^e}{p_{eo}} & \beta_{\pi_e} (\nu_c \beta_e - 1) \end{pmatrix}. \quad (28)$$

658 According to the Routh-Hurwitz theorem, the necessary and sufficient conditions for stability of  
659 the system are:

660 (C1)  $\text{tr}(J) < 0$ ;

661 (C2)  $J_1 + J_2 + J_3 > 0$ , where  $J_i$  represents the principal minor of order  $i$  of the matrix  $J$ ;

662 (C3)  $|J| < 0$ ; and

663 (C4)  $B = -\text{tr}(J)(J_1 + J_2 + J_3) + |J| > 0$ .



664 Condition (C1) clearly holds. If  $a_q < (1 - a_y)\rho_{eo}^e$ , then (C2) and, since it can be proved that  
 665  $|J| = -\beta_{\pi_e} J_3$ , (C3) also hold. As for (C4):

$$\begin{aligned} -\text{tr}(J) (J_1 + J_2 + J_3) &= (\beta_y(1 - a_y) + \beta_e\rho_{eo}^e + \beta_{\pi_e}(\nu_c\beta_e - 1)) \\ &\cdot (\beta_e\rho_{eo}^e\beta_{\pi_e} - \beta_y(1 - a_y)\beta_{\pi_e}(\nu_c\beta_e - 1) + \beta_y(1 - a_y)\beta_e\rho_{eo}^e - \beta_y a_q \beta_e b), \end{aligned}$$

and

$$|J| = -\beta_{\pi_e} \left( \beta_y(1 - a_y)\beta_e\rho_{eo}^e - \frac{\beta_y a_q E_o \beta_e b}{E_o} \right).$$

666 Therefore, simplifying terms,  $B > 0$  if and only if

$$\begin{aligned} &[\beta_y(1 - a_y) + \beta_e\rho_{eo}^e - \beta_{\pi_e}(\nu_c\beta_e - 1)] \{ \beta_e\beta_{\pi_e}\rho_{eo}^e - \beta_y(1 - a_y)(\nu_c\beta_e - 1) + \beta_y\beta_e[(1 - a_y)\rho_{eo}^e - a_q b] \} \\ &+ \beta_e\beta_{\pi_e}\beta_y[a_q b - (1 - a_y)\rho_{eo}^e] > 0 \end{aligned}$$

667 or, equivalently, after some straightforward algebra,

$$\begin{aligned} &[\beta_y(1 - a_y) + \beta_e\rho_{eo}^e] \{ \beta_e\beta_{\pi_e}\rho_{eo}^e + \beta_y(1 - a_y)(1 - \nu_c\beta_e) + \beta_y\beta_e[(1 - a_y)\rho_{eo}^e - a_q b] \} + \beta_{\pi_e}(1 - \nu_c\beta_e) \\ &\cdot [\beta_e\beta_{\pi_e}\rho_{eo}^e + \beta_y(1 - a_y)(1 - \nu_c\beta_e)] + \nu_c\beta_e\beta_e\beta_{\pi_e}\beta_y a_q b - \nu_c\beta_e\beta_e\beta_{\pi_e}\beta_y(1 - a_y)\rho_{eo}^e > 0 \end{aligned}$$

668

669 Note that if  $1 > \beta_e$  and  $(1 - a_y)\rho_{eo}^e > a_q b$  then all terms in the previous expression except for the  
 670 last one are strictly positive. Then in order to prove that the desired inequality holds it suffices to  
 671 note that

$$\beta_y(1 - a_y)\beta_e\beta_{\pi_e}\rho_{eo}^e - \nu_c\beta_e\beta_e\beta_{\pi_e}\beta_y(1 - a_y)\rho_{eo}^e = \beta_y(1 - a_y)\beta_e\beta_{\pi_e}\rho_{eo}^e(1 - \nu_c\beta_e) > 0. \quad (\text{Q.E.D.})$$

### 672 **Proof of Proposition 3**

673 Since condition (C1) does not hold for  $\nu_c > \frac{\beta_y(1 - a_y) + \beta_e\rho_{eo}^e + \beta_{\pi_e}}{\beta_{\pi_e}\beta_e}$ , the steady state of the 3D system is  
 674 locally unstable. (Q.E.D.)

### 675 **Proof of Proposition 4**

676 Note that the steady state value of  $Y$ ,  $p_e$  and  $\pi_e$  are uniquely determined independently of  $x$  by  
 677 conditions (20)-(22) in Lemma 1. Given this, we focus on equation (19) where the probabilities and  
 678 switching index are given by equations (13), (14) and (15), respectively. Let  $Y$ ,  $p_e$  and  $\pi_e$  be equal to  
 679 their steady state values so that  $s = s_x x$ .

680 Define then the following real valued function  $g : (-1, +1) \rightarrow \Re$

$$g(x) := s_x x - \frac{1}{2a_x} [\ln(1+x) - \ln(1-x)] \quad (29)$$

681 This function has the property that  $g(x) = 0$  if and only if  $\dot{x} = 0$  as can be seen from (19) setting  
 682  $\dot{x} = 0$  and taking the logs. The equation  $g(x) = 0$  always has a solution for  $x = 0$  and thus there is  
 683 always a steady state with  $x_o = 0$ .

684 (i) Observe that

$$\lim_{x \rightarrow 1} g(x) = -\infty, \quad (30)$$

$$\lim_{x \rightarrow -1} g(x) = +\infty, \quad (31)$$

686 and the derivative of  $g(x)$  is

$$g'(x) = s_x - \frac{1}{a_x(1-x^2)}. \quad (32)$$

687 Then if  $s_x \leq \frac{1}{a_x}$ ,  $g'(x) < 0$  and  $g(x)$  is strictly decreasing for all  $x \in (-1, 1)$ . So, if  $s_x \in (0, 1/a_x]$ ,  
 688  $x_o = 0$  is the only value of  $x$  such that  $g(x) = 0$  and so  $\dot{x} = 0$ .

689 (ii) By equation (32),  $g(x)$  is increasing if and only if

$$g'(x) = s_x - \frac{1}{a_x(1-x^2)} \geq 0 \Leftrightarrow x^2 \leq \frac{s_x a_x - 1}{s_x a_x}.$$

690 Because  $s_x a_x > 1$ , it follows that  $g(x)$  is strictly increasing for  $x \in \left(-\sqrt{\frac{s_x a_x - 1}{s_x a_x}}, \sqrt{\frac{s_x a_x - 1}{s_x a_x}}\right)$  and  
 691 strictly decreasing for  $x \in \left(-1, -\sqrt{\frac{s_x a_x - 1}{s_x a_x}}\right) \cup \left(\sqrt{\frac{s_x a_x - 1}{s_x a_x}}, 1\right)$ . Then, noting that  $g(0) = 0$  and  
 692  $g'(0) > 0$ , by equations (30) and (31), and the continuity of  $g(x)$ , there exist three steady states:  
 693 one with equal populations ( $x_o = 0$ ), one where fundamentalists dominate ( $x_o < 0$ ) and one  
 694 where chartists dominate ( $x_o > 0$ ). (Q.E.D.)

#### 695 **Proof of Proposition 4**

696 The proof of Proposition 4 is a trivial modification of the proof of Proposition 3. (Q.E.D.)

#### 697 **Proof of Proposition 5**

698 At any steady state  $(x_o, \pi_{e_o}^e)$  with  $\pi_{e_o}^e = 0$ , the Jacobian of the system formed by equations (24)-(26)  
 699 is:

$$J = \begin{pmatrix} \beta \pi_e^e \left[ \frac{1+x_o}{2} \beta_e - 1 \right] & 0 \\ 0 & 2\beta_x \exp(a_x s_x x_o) \left[ (1-x_o) a_x s_x - \frac{1}{1+x_o} \right] \end{pmatrix}. \quad (33)$$

700 (i) At the steady state with  $x_o = 0$  and  $\pi_{e_o}^e = 0$ , the Jacobian becomes

$$J = \begin{pmatrix} \beta_{\pi_e^e} \left( \frac{\beta_e}{2} - 1 \right) & 0 \\ 0 & 2\beta_x (a_x s_x - 1) \end{pmatrix}. \quad (34)$$

701 Because  $s_x \in (0, 1/a_x)$ , if  $\beta_e > 2$  then  $|J| < 0$ , and the steady state is an unstable saddle point.  
 702 Conversely, if  $\beta_e < 2$  then  $\text{tr}J < 0$  and  $|J| > 0$ , and the steady state is stable.

703 (ii) The stability properties of the steady state with  $x_o = 0$  and  $\pi_{e_o}^e = 0$  can be derived with a  
 704 straightforward modification of the argument in part (i) noting that  $s_x > 1/a_x$ .

705 In order to derive the stability properties of  $e_f = (0, x_o^f)$  and  $e_c = (0, x_o^c)$ , note that  $J_{22} \lesseqgtr 0$  if  
 706 and only if  $(1 - x_o)a_x s_x \lesseqgtr \frac{1}{1+x_o}$  or equivalently

$$x_o^2 \gtrless \frac{a_x s_x - 1}{a_x s_x}. \quad (35)$$

707 By the argument in part (ii) of Proposition 3, it follows that both at  $e_c$  and at  $e_f$ ,  $x_o^2 > \frac{a_x s_x - 1}{a_x s_x}$   
 708 and therefore  $J_{22} < 0$ . (Q.E.D.)

## 709 Appendix B

710 In this appendix we present some additional simulations of the full model as well as bifurcation  
 711 diagrams. Figure 9 illustrates the case where the relative population variable displays irregular yet  
 712 persistent fluctuations. In this simulation, the adjustment speed of share price  $\beta_e$  is increased from  
 713 2 to 2.5, while the sensitivity of the sentiment switching index to the output gap,  $s_y$ , is reduced to  
 714 0.1. The fast adjustment of share price is a source of instability, which is counter-balanced by the  
 715 nonlinearity in the opinion switching index ( $s_{p_e} = 0.06$  and  $s_{\pi_e^e} = 0.5$ ). The self-reflection parameter  
 716 in the opinion switching index,  $s_x$ , is kept at 1.

717 The fluctuations in the population of traders are translated to capital gains expectations and the  
 718 real economy. The relative size of the two groups (fundamentalists and chartists) fluctuates between  
 719 -0.25 and 0 with oscillations differing in both amplitude and frequency. The stability in the fluctuation  
 720 of the sentiment dynamics is related to the two volatility parameters in the switching equation –  $s_{p_e}$   
 721 and  $s_{\pi_e^e}$  – which capture the idea that higher volatility leads agents to become fundamentalists.

722 We now turn to bifurcation diagrams based on the same calibration as in the lower panels of Figure  
 723 9 in order to further illustrate the properties of the full model. The top panel of Figure 10 show the  
 724 bifurcation diagrams of population dynamics and output with respect to the sensitivity of the opinion  
 725 switching index to the self-reference element, with  $s_x$  varying between 0.4 and 1.5. For values of  $s_x$   
 726 between 0 and 0.5 there are four local minima and maxima for  $x$ . This number doubles between 0.5  
 727 and 0.9. The number of local minima and maxima then goes back to four between 0.9 and 1 and

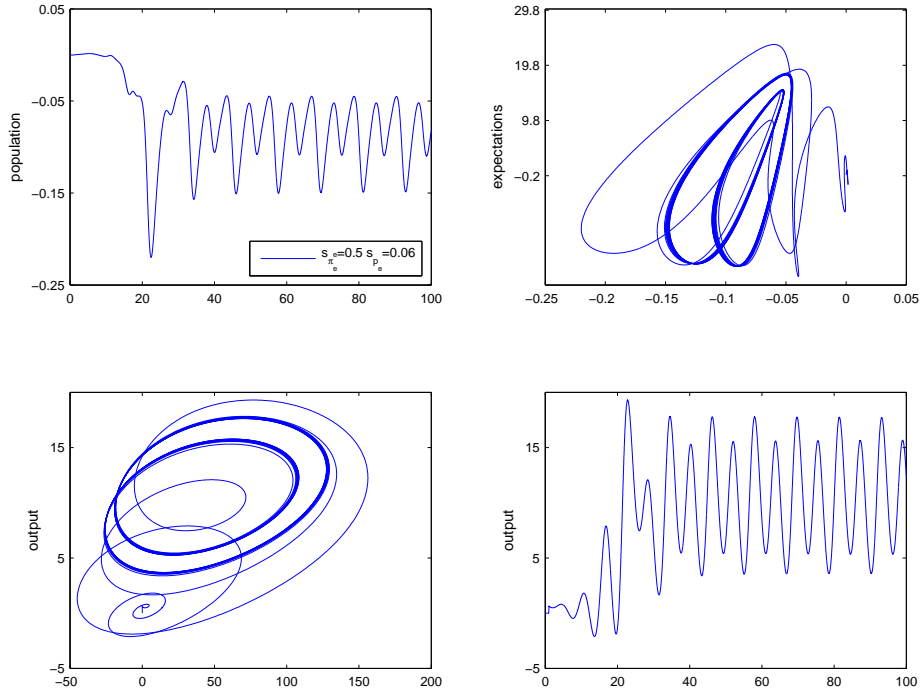


Figure 9: Complex dynamics in the 4D model  $(Y, p_e, \pi_e^e, x)$ .

728 further reduces to two between 1 and 1.25. Beyond 1.25 there is a unique steady state. A similar  
 729 pattern describes the oscillation of output.

730 As shown in the next two panels, the number of local minima and maxima decreases with  $a_x$  from  
 731 four over the range 0.7-0.8 to two over the range 0.8-1 and one when  $a_x > 1$ . This result is also  
 732 consistent with the analysis in section 3.3.

733 The third row of Figure 10 shows bifurcation diagrams of the population dynamics with respect to  
 734 the sensitivity of the opinion switching index to the output gap,  $s_y$ , and to capital gains expectations  
 735  $s_{\pi_e^e}$ . Values of  $s_y$  in the range  $[0.15; 0.2]$  and  $[0.27; 0.32]$  produce large fluctuations in the opinion  
 736 dynamic. The population variable  $x$  goes either to -1 or to positive values when  $s_y > 0.34$ . For values  
 737 of  $s_{\pi_e^e} < 0.3$ , the opinion dynamics displays large fluctuations over the range  $[-0.6; 0]$  in line with the  
 738 result that excess volatility favors fundamentalist expectations.

739 The fourth and fifth rows of Figure 10 summarize additional sensitivity analysis. The population  
 740 dynamics is stable for either low or high values of the speed of adjustment of expectations,  $\beta_{\pi_e^e}$ , and  
 741 the speed of adjustment of the price of capital,  $\beta_e$ . Interestingly, only a high speed of adjustment of  
 742 population dynamics ( $\beta_x > 0.8$ ) produces stability. Finally, the system produces oscillations when the  
 743 sensitivity of aggregate demand to Tobin's  $q$ ,  $a_q$ , is either small or larger than 0.8.

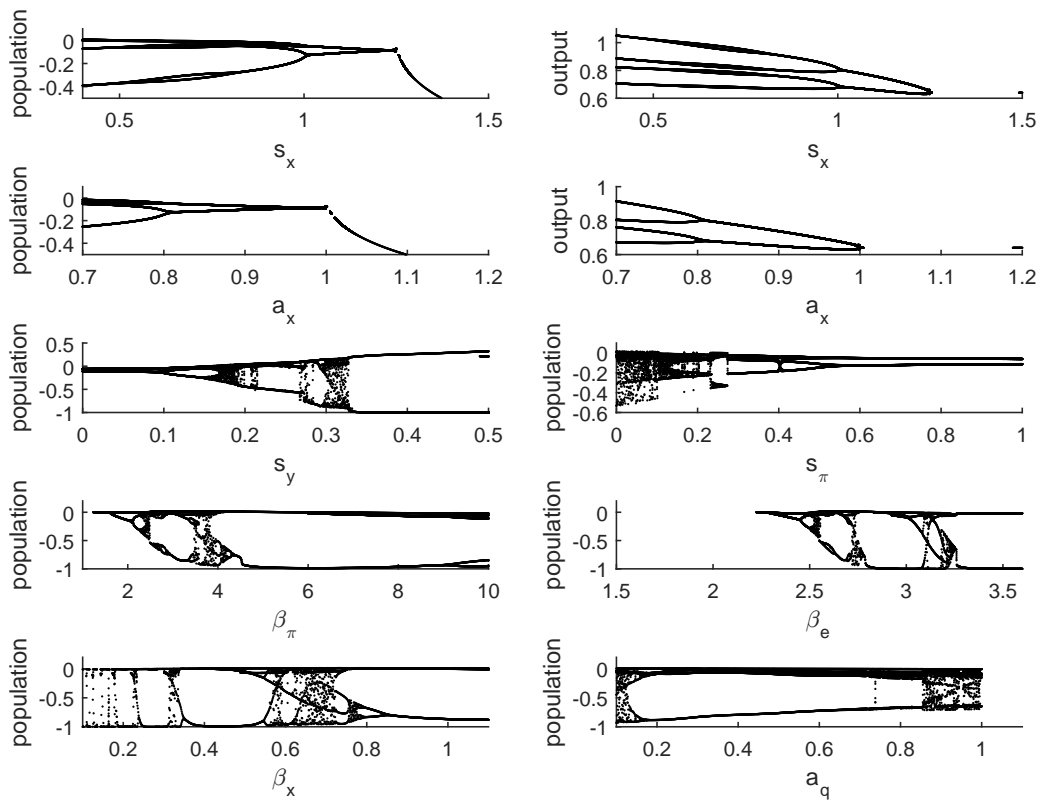


Figure 10: Bifurcation diagrams

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