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RECENT DEVELOPMENTS IN MODELLING PREFERENCES: UNCERTAINTY AND AMBIGUITY

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ABSTRACT

In subjective expected utility (SEU) the decision weights people attach to events are their beliefs about the likelihood of events. Much empirical evidence, inspired by Ellsberg (1961) and others, shows that people prefer to bet on events they know more about, even when their beliefs are held constant. (They are averse to "ambiguity", or uncertainty about probability.) We review evidence, recent theoretical explanations, and applications of research on ambiguity and SEU.

Key Words: Ambiguity, uncertainty, Ellsberg paradox, non-expected utility

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In the last 40 years the leading theories of choice in economics and psychology have been the expected utility theory (EU) of von Neumann and Morgenstern (1947) and the subjective expected utility theory (SEU) of Savage (1954). Empirical violations have led to reexaminations of both kinds of theory. In Weber and Camerer (1987) we reviewed the evidence, axioms, and application of alternatives to EU. Here we do the same for SEU.

EU assumes the probabilities of outcomes are known. If preferences follows a set of simple axioms they can be represented by a real-valued utility function-- preferred choices have higher utility numbers-- and the utility of a choice is the expected utility of its possible outcomes, weighted by their probabilities.

In SEU, probabilities are not necessarily objectively known, so SEU applies more widely than EU. (Indeed, it is hard to think of an important natural decision for which probabilities are objectively known.) In SEU decision makers choose acts, which have consequences that depend on which of several uncertain "states" occurs. People are assumed to have subjective, or "personal", probabilities of the states (which may legitimately differ across people). The SEU axioms show the conditions under which preferences can be represented by a numerical expected utility which uses subjective probabilities of states to weight consequence utilities. The theory combines the von Neumann and Morgenstern (1947) EU approach with de Finetti's (1937) calculus of subjective probabilities.

Much of the empirical evidence against SEU (as a description of choices) concerns precisely the distinction between whether probability is known or unknown. This basic distinction goes by many names: risk vs. uncertainty (Knight, 1921); unambiguous vs. ambiguous probability (Ellsberg, 1961); precise or sharp vs. vague probability (Savage, 1954, p. 59), epistemic reliability (Gardenfors & Sahlin, 1982), and so forth. We generally use the term "ambiguity", purely from tradition.

In SEU the distinction between known and unknown probability is pointless because subjective probabilities are never unknown--they are always known to decision makers (or inferrable from their choices). But empirical evidence suggests that how much people know about a state's probability does influence their willingness to bet on the state.

For example, suppose you must choose between bets on two coins. After flipping the

first coin thousands of times you conclude it is fair. You throw the second coin twice; the result is one head and one tail. Many people believe both coins are probably fair (p(head)=p(tail)=.5) but prefer to bet on the first coin, because they are more confident or certain that the first coin is fair. Ambiguity about probability creates a kind of risk in betting on the second coin-- the risk of having the wrong belief.[1] SEU effectively requires that decision makers be indifferent toward such a risk.

Most of the research we review tests whether SEU is a good descriptive theory, or suggests alternative descriptions. There is relatively little discussion about whether SEU is normatively adequate.[2] We suspect most alternatives to SEU are meant to be normative improvements too, but unclear standards for what makes a theory normative inhibit such claims. A clearer standard and more debate would be useful.

Our goal in this paper is to review recent literature on ambiguity in decision making. We will cover both empirical and theoretical work, and we will try to point out the relevance of ambiguity for a wide range of professions and disciplines. There are many important related areas we ignore. We will not review generalizations of EU. We will also ignore the literatures on probability elicitation (e.g. Spetzler and von Holstein, 1975), psychology of probability judgments (e.g., Kahneman, Slovic, Tversky, 1982), organizational choice under ambiguity (e.g., March and Olsen, 1976), and ambiguity intolerance as a personality trait (e.g., Budner, 1962). More technical reviews include Fishburn (1988b, pp. 190-193, 1989), Karni and Schmeidler (1990), and Kischka and Puppe (1990). Smithson (1989) offers an eclectic, broad review.

The paper proceeds as follows. Section 1 is a brief formal overview of SEU. Section 2 reviews empirical work demonstrating ambiguity effects in individual decisions. Some

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[1] Savage (1954) recognized such a risk but did not know how to model it: "...there seem to be some probability relations about which we feel relatively "sure" as compared with others...The notion of "sure" and "unsure" introduced here is vague, and my complaint is precisely that neither the theory of personal probability, as it is developed in this book, nor any other device known to me renders the notion less vague." (pp. 57-8)

[2] Exceptions include Ellsberg (1961), Fellner (1961), Hazen (1987) and Gilboa (1989a, p. 1; 1989b, p. 412), and Neehring (1990, p. 4).

conceptions and sources of ambiguity are mentioned in Section 3. Recent generalization of SEU are described in Section 4. Applications of these recent developments to several areas, mostly in economics and business, are discussed in Section 5. Some conclusions and suggestions for future research are drawn in Section 6.

1. SUBJECTIVE EXPECTED UTILITY AND THE ELLSBERG PARADOX

We first describe SEU very briefly, to motivate our review of experimental studies and the discussion that follows. Section 4 gives more details of the SEU axioms.

1.1 Subjective Expected Utility Theory

SEU was first developed by Savage (1954) (inspired by Ramsey, 1931, and de Finetti, 1937), then derived by Anscombe and Aumann (1963) in an approach that combined EU and SEU.

In SEU a decision maker must choose between "acts" denoted by uppercase letters (e.g., X). The consequences of an act X depend on which state s occurs, from the set S of possible states. (The consequence of X if s occurs is denoted x(s).) For simplicity we assume the sets of acts and states are finite.[3] If we include subjective probabilities of the states, denoted p(s), then an act X will be described by a vector $(x(s_1),p(s_1);...;x(s_n),p(s_n))$ (where states are indexed $s_1, s_2,...s_n$). Preferences between a pair of acts X and Y will be denoted by $X \sim Y$ (X is indifferent to Y) and $X \succeq Y$ (X is weakly preferred-- preferred to or indifferent-- to Y).

The mathematical goal of SEU is to represent preferences over acts by a numerical utility index u and a probability measure on the states, p, such that act X is preferred to act Y if and only if the subjective expected utility (SEU) of X is larger than the SEU of Y. The SEU of X is defined as

$$(1) \quad SEU(X) = \sum_{s \in S} p(s)\, u(x(s))$$

[3] This is technically wrong for Savage's SEU formulation, since Savage's P6 axiom implies an uncountable set of states.

If preferences satisfy certain axioms then there are numerical utilities and probabilities which represent acts by their SEU.

1.2 The Challenge to SEU: The Ellsberg Paradox

As innocuous as the SEU form (1) looks, there is a long, rich tradition of questioning whether it describes behavior adequately. Keynes (1921) drew the distinction between the implications of evidence-- the likelihood judgment evidence implies-- and the weight of evidence, or the confidence in assessed likelihood. Keynes wondered whether a single probability number could express both dimensions of evidence.

Knight (1921) distinguished "risk", or known probability, and "uncertainty". He suggested that economic returns were earned for bearing uncertainty, but not for bearing risk.

The modern attack on SEU as a descriptive theory was made most directly by the "Ellsberg paradox" (Ellsberg, 1961). Two similar problems were posed in that remarkable paper. (One was mentioned much earlier by Knight, 1921, pp. 218-219.)

In the first problem, a decision maker has to choose from an urn which contains 30 red balls and 60 balls in some combination of black and yellow. We call this the three-color problem. There are two pairs of acts, X and Y, and X' and Y'. Acts have consequences W (for "win") or 0, as shown in Table 1.

Many people choose $X \succ Y$ and $Y' \succ X'$. The number of black balls which yield a win if act Y is chosen is unknown (or ambiguous); people prefer the less ambiguous act X. The same principle, applied to the second choice, favors Y' because exactly 60 balls yield W. (The same preference pattern is common for losses, $W < 0$.)

In the three-color problem, people prefer acts with a known probability of winning. That is, they take confidence in estimates of subjective probability into account when making choices. Such a pattern is inconsistent with the sure-thing principle of SEU. Both pairs of acts only differ in consequences when the yellow state occurs. That consequence is the same for X and Y (you win 0) and for X' and Y' (you win W). The sure-thing principle assumes a state with a consequence common to both acts is irrelevant in determining preference between the acts. According to SEU, $X \succ Y$ if and only if $X' \succ Y'$. The common

pattern $X \succ Y$ and $Y' \succ X'$ violates the sure-thing principle because ambiguity affects choices and the ambiguity inherent in one state-- red, for example-- may disappear when the state is combined with an equally ambiguous state, like yellow.

More formally, suppose p(r), p(b), and p(y) are the subjective probabilities of drawing a red, black or yellow ball. Under SEU, $X \succ Y$ if and only if $p(r)u(W) > p(b)u(W)$, or $p(r) > p(b)$.[4] Similarly, $Y' \succ X'$ implies $p(b \cup y) > p(r \cup y)$. If we assume probabilities are additive then $p(b \cup y) = p(b) + p(y)$ (since $p(b \cap y) = 0$). Then $Y' \succ X'$ implies $p(b) > p(r)$, which conflicts with the inequality $p(r) > p(b)$ implied by $X \succ Y$.

Ellsberg also posed a two-color problem using two urns, one containing 50 red and black balls and one containing 100 balls in an unknown combination of red and black (see Table 2). Many people prefer to bet on red from urn 1 (rather than betting on red from urn 2) and prefer to bet on black from urn 1 too, but they are indifferent between the two colors when betting on only one of the two urns (i.e., "bet red i"~"bet black i" for i=1, 2). That pattern violates SEU.

2. CONCEPTIONS AND SOURCES OF AMBIGUITY

A working definition of ambiguity is useful to guide theorizing and empirical studies. Researchers have followed three strategies in developing definitions.

2.1 Banishing Ambiguity

The first strategy is to banish ambiguity by simply denying that "ambiguous" and "unambiguous" are distinctive categories of events. To a staunch subjectivist, there is no such thing as unknown probability-- all probabilities are equally well-known, to ourselves-- so ambiguity is meaningless (de Finetti, 1977). This may be a reasonable normative position, but it does not help explain descriptive evidence of ambiguity-aversion.

2.2 Expressing Ambiguity as Second-order Probability

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[4] We have the freedom to set $u(0) = 0$ for simplicity, since utility is only unique up to positive linear transformations.

The second strategy is reductionist: Express ambiguity about a probability $p(s_i)$ as a "second-order" probability (SOP) distribution of its possible values, denoted $\Phi(p(s_i))$ (e.g., Marschak, 1975). For example, in the two-color Ellsberg problem p(black) might be uniformly distributed between 0 and 1 rather than setting p(black)=.50. Since EU and SEU are linear functions of probabilities, only the expected value of an SOP should matter for choice, so ambiguity should not matter.

The SOP view is routinely used in many kinds of reasoning. Recall the coin example given in the introduction: One coin is flipped twice (the result is one head and one tail); another is flipped many times (half heads, half tails). If one takes true, or objective, probability to be the long-run limiting relative frequency of heads, then every subjective probability is an SOP of objective probabilities. The many-flip coin simply has a tighter SOP around p=.5 than the two-flip coin does.

In some of the research reviewed below, the SOP view goes further, by presuming subjective second-order probabilities of (first-order) probabilities which might also be subjective. When three oddsmakers give different odds that a horse will win the Preakness Stakes-- a one-time event for which a limiting-frequency interpretation of probability is unnatural-- a person could have an SOP over the oddsmakers' subjective beliefs. (Or more peculiarly, she could have an SOP over three of her own possible beliefs.)

The SOP view is popular (e.g., Howard, 1988) but has some drawbacks. Certain kinds of ambiguity, like the urn in Ellsberg's two-color problem, do not appear to completely captured by SOP because subjects prefer bets on known SOPs to bets on ambiguous urns (stylized fact 4 in Table 3). Furthermore, known probability and SOP will only lead to the same choices if compound lotteries are reduced to equivalent single-stage bets. But the reduction principle is often violated in experiments (Camerer & Ho, 1991).

Other objections to SOP are philosophical and practical. Since SOP does not describe observed departures from SEU well, its best use might be as a normative theory, but the normative case for replacing single subjective probabilities with SOPs has not been made. As a practical matter, if a person cannot express a precise probability she may not be able to confidently express a second-order distribution either, or a third-order distribution over second-order distributions, ad infinitum (see Savage, 1954, p. 59).

2.3 Defining Ambiguity: Missing Information

The third strategy is to construct a pragmatic definition of ambiguity which captures its psychological essence. Ellsberg's (1961) definition is typical: Ambiguity is the "quality depending on the amount, type, reliability, and 'unanimity' of information". We favor a more general definition mentioned by Fellner (1961) and many others, and elaborated by Frisch & Baron (1988):

> Ambiguity is uncertainty about probability created by missing information that is relevant and could be known.

Not knowing important information is upsetting and scary; it makes people shy away from taking either side of a bet (see Heath & Tversky, 1991). Indeed, one explanation of ambiguity-aversion is that people transfer a heuristic which is helpful in many natural situations-- "avoid betting when you lack information others might have"-- to other situations in which their fears are unfounded (Frisch and Baron, 1988, p. 153).

3.4 Other Definitions and Types of Ambiguity

Many popular definitions and types of ambiguity can be traced to missing information. We mention a few.

Ambiguity about probability. In Ellsberg problems the composition of the ambiguous urn is missing information which is relevant and could be known, but isn't.

Source credibility and expert disagreement. Credibility of sources creates an important kind of ambiguity (e.g., Einhorn and Hogarth, 1985). In legal proceedings, for example, observations by witnesses, attorneys, and judges must be weighed by a jury to reach a verdict.[5] Disagreements among experts, often stemming from controversy about the causal mechanisms generating physical or social activity, cause ambiguity too. In these settings, ambiguity is caused by missing information about whose belief should be believed.

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[5] One could model credibility concerns in a Bayesian framework by attaching probabilities to likelihood evidence from different sources which reflect their truthfulness, performing a "cascaded inference" (e.g., Schum, 1989). But ambiguity about which sources are most truthful then requires truthfulness probabilities of truthfulness probabilities, ad infinitum. It is not clear such a procedure captures the kind of ambiguity generated by credibility concerns, or does so in a practical way.

Weight of evidence. Evidence has both implications, and weight or amount (Keynes, 1921; Shafer, 1976; Cohen, 1977). Standard probabilities should express only the implications of evidence, not its weight, but it seems reasonable for choices to sometimes depend on both.[6] The weight of evidence can be defined as the amount of available information relative to the amount of conceivable information (see Keynes, 1921). The gap is the amount of missing information.

In all the experiments reviewed in the next section, and most of the theories and applications described in sections 4 and 5, ambiguity is simply uncertainty about a probability. We refer back to the more general idea of ambiguity as missing information when appropriate.

3.5 Degrees of ambiguity

Before we proceed it is useful to distinguish precisely between various degrees of uncertainty before we proceed. Suppose the the utilities of states $u(s_i)$ are known, so we can focus only on the probabilities $p(s_i)$.

When a person knows one state will occur with certainty ($p(s_i)=1$ for some i) her distribution of $p(s_i)$'s is the spike shown in Figure 1a. We call this "certainty".

When a person is not sure which state will occur, but knows the probabilities of each state precisely, her distribution is like the one shown in Figure 1b. We call this "risk", or unambiguous probability.[7]

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[6] For instance, in Scottish law there are three verdicts: guilty, innocent, and unproven. All the evidence might imply guilt, but if there is too little of it the verdict will be unproven. Choices should depend on the weight of evidence when more evidence might be gathered; then a reluctance to bet allows time to get more information and bet more wisely. Ambiguity-aversion observed in experiments, and in some everyday choices, might reflect an overapplication of this principle to situations in which there is no time or possibility of getting missing information (Frisch & Baron, 1988).

[7] In many definitions, risk and uncertainty differ because risk involves objectively-known probabilities (or "roulette lotteries") and uncertainty involves subjectively-known probabilities over states (or "horse lotteries"). Since we are exclusively interested in subjective

When a person is not sure which distribution of probabilities over states is correct, we call the state probabilities "ambiguous". The definitions mentioned above distinguish two kinds of ambiguity: When the probability distributions in the set of conceivable distributions can themselves be assigned probabilities, ambiguity can be expressed as second-order probability, as in Figure 1c. When the distributions cannot be assigned probability, as in Figure 1d, ambiguity is expressed by a set of probability distributions.

Recall the three views mentioned earlier in this section. Banishing ambiguity means assuming that choices are made as if ambiguous sets of distributions (Figure 1c and 1d) are collapsed into a single distribution (Figure 1b). The SOP view implies that knowledge about probabilities can always be expressed as in Figure 1c. If ambiguity is caused by missing information, then the number of possible distributions in Figure 1d might vary as the amount or nature of missing information varies.

Figures 1a-d also illustrate a small confusion about ambiguity over probability versus ambiguity over outcomes. Ambiguity about which outcome will occur is too coarse a category, because risk (Figure 1b) and ambiguous probability (Figures 1c-d) both exhibit ambiguity about outcomes. And it is misleading to suppose ambiguity about outcomes and ambiguity about probabilities are parallel conditions or treatment variables. If people are averse to ambiguity about which outcome will occur, but outcome probabilities are known (Figure 1b), they are risk-averse and consistent with EU. But if people are averse to ambiguity about the probability of an outcome, they are ambiguity-averse and inconsistent with SEU. The two kinds of "ambiguity" are fundamentally different.

3. EMPIRICAL STUDIES OF AMBIGUITY

Ellsberg did not run careful experiments.[8] But the intuitive appeal of his thought

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probabilities in this review, we blur the distinction by calling known subjective probabilities, as in Figure 1b, "risk" instead of uncertainty.

[8] However, he alludes to "a large number of responses, under absolutely nonexperimental conditions" which suggest ambiguity-aversion is the majority pattern of choice.

experiments, and varying reactions from famous theorists of the time (mentioned in his paper) were enough to initiate a lively debate. Since then, many others have studied ambiguity empirically.

There are roughly three kinds of empirical work on ambiguity. The first kind is Ellsberg's original thought experiment and replications of it which vary parameters. The second kind tries to determine the psychological causes of ambiguity. The third kind tests specific definitions or models of ambiguity, or studies ambiguity in an applied setting (these are mostly discussed in section 5).

Several stylized facts about ambiguity-aversion have been established by studies with chance devices, typically bingo cages or decks of cards. Table 3 summarizes some of the stylized facts. (The reader who is busy or bored by experimental details can glance at Table 3 and skip ahead to the summary section 3.4.) The table only reports findings replicated by more than one study, using chance devices (or stated probabilities). Many other interesting empirical results are mentioned below but left out of Table 3.

2.1 Ellsberg experiments and extensions

Chipman (1960) was the first to study ambiguity empirically[9], in a setting slightly different from Ellsberg's. His ten subjects chose bets on boxes with known proportions of 100 matchstick heads and stems (say, 60-40, inducing $p=.6$) and ambiguous boxes with unknown proportions from which small samples were drawn. Subjects acted roughly like Bayesians who thought the unknown proportions were centered around 50-50 and updated their beliefs using a 10-stick sample. For example, 67% preferred betting on an ambiguous box with a 4-6 sample to a 40-60 box. They exhibited some inherent ambiguity-aversion too, sinc 70% preferred a bet on the 50-50 box to a bet on the ambiguous box from which a 5-5 sample was drawn.

Gigliotti & Sopher (1990) replicated most of Chipman's results with a wider variety of urn and samples sizes. Their subjects obeyed some statistical principles in judging samples from unknown urns.

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[9] According to Ellsberg (1961), footnote 8.

Becker and Brownson (1964) did the first study of ambiguity effects in Ellsberg-type settings. Ambiguity was operationalized as the range of the number of red balls in an urn. Subjects were given a list of ten pairs of urns differing in ambiguity One of the ten choices was picked randomly and played for $1.

Before the experiment began, subjects were screened for ambiguity-aversion using the two-color Ellsberg problem. About half the subjects were ambiguity-averse; they then chose between pairs of urns. They always picked the less ambiguous urn and paid substantial amounts to avoid ambiguity. For example, to avoid an ambiguous urn and choose from an urn with exactly 50 red balls, they paid an average of 72% of expected value when the ambiguous urn had 0 to 100 red balls, and 28% when the ambiguous urn had 40 to 60 red balls. Becker & Brownson estimated that the amount paid to avoid ambiguity, or "ambiguity premium", was about 60% of the difference in the ranges of two urns.[10]

MacCrimmon (1968) gave 35 business executives a series of three Ellsberg problems, involving choices between bets on chance devices or on natural events (a stock price change, or the level of GNP). Only 10% exhibited the Ellsberg pattern. However, almost half were ambiguity-averse when choosing whether to make investments in countries with historical frequencies ("risky") or no historical frequencies ("uncertainty"). Exposure to written arguments for and against the Ellsberg pattern did not change choices.

Sherman (1974) found modest correlations between ambiguity-aversion (in the Ellsberg two-color problem) and a psychometric scale measuring "intolerance of ambiguity". (He also noted that the intolerance scale correlates with some intelligence measures.)

Yates and Zukowski (1976) studied whether the range of possible probabilities is a reasonable measure of ambiguity. They compared a "known urn" with 5 red and 5 blue poker chips, a "uniform urn" with the number of red chips uniformly distributed from 0 to 10, and an "ambiguous urn" with chips in unknown proportion. The range explanation of ambiguity predicts that subjects would like the uniform urn least (since it has the largest

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[10] These ambiguity premia are much higher than those observed in other studies, but recall that Becker & Brownson only allowed subjects who were ambiguity-averse in the initial two-color Ellsberg problem to participate in the rest of their experiment.

possible range). Subjects chose between pairs of urns, and stated minimum selling prices for bets on urns (using the Becker, DeGroot, Marschak, 1964) procedure. The unknown urn was least preferred and lowest-priced. Subjects were willing to pay an average ambiguity premium of 20% of expected value to bet on the known urn instead of the uniform urn, showing that ambiguity-aversion extends to bets with known second-order distributions of probability (SOP's).

MacCrimmon and Larsson (1979) studied the three-color Ellsberg problem. Fifteen of their 19 subjects committed the standard paradox (see their Figure 7). They lowered the known-probability of a red ball, to measure how much of a probability premium subjects were willing to pay to avoid ambiguity. Only six subjects committed the paradox when the known probability was .25 (instead of its original value of 1/3), suggesting a probabilistic ambiguity premium of .05-.10.

Larson (1980) tested aversion to differing degrees of ambiguity using decks of cards with (truncated) known normal distributions of winning probability. The decks had expected probabilities, E(p), of .2, .5, and .8. Subjects chose between two decks with the same E(p), but different distributions of probability. (They played one choice for $3.) About 2/3 of the subjects preferred the less ambiguous distribution in a pair, roughly independently of E(p).

Curley and Yates (1985) also studied the effects of probability range and E(p) ("center of range") on choices. Students chose between 30 pairs of urns, and stated their strengths of preference. Each pair had two urns with the same E(p) and different ranges, with unknown distributions of probability. They played one choice for $5.

Ambiguity-aversion increased with E(p). About 80% disliked the ambiguous distribution when E(p) = .80 but they were indifferent to ambiguity for E(p) below .4. The strongest aversion to an increase in range occurred for intermediate values of E(p). In addition, ambiguity-aversion was stronger (and most sensitive to E(p)) when comparing an urn with no ambiguity (e.g., p = .4) to an urn with a little ambiguity (e.g., (.2,.6)), than when comparing a low-ambiguity urn, like (.2,.6), to a high-ambiguity urn like (0,.8).

Boiney (1990) found a significant effect of skewness in the second-order distribution of probability (as did Viscusi & Magat, 1991). Small majorities preferred positive skewness

(53%) and disliked negative skewness (57%), for E(p)=.2, .5, and .8. (These patterns parallel skewness preference for distributions of outcomes.)

Bernasconi & Loomes (in press) used a two-stage lottery operationalization of the Ellsberg three-color problem. Drawing the color red (R) was the unambiguous event (p(R)=1/3); blue (B) and yellow (Y) were ambiguous separately, and unambiguous together (p(B ∪ Y)=2/3). About half the subjects indicated ambiguity-aversion by betting on R (for £ 10). About 60% of the subjects were unwilling to switch their £ 10 bet on an ambiguous color to a £ 12 bet on any other color. Nearly 90% of those who chose the unambiguous bet R refused to switch (implying an ambiguity premium of more than 20%).

When told they could bet on any <u>two</u> colors (the second choice in the three-color Ellsberg problem, essentially), those who had bet on ambiguous colors, either B or Y, mostly chose the unambiguous combination B∪Y. However, those who bet the unambiguous R typically bet ambiguous combinations, R∪B or R∪Y, rather than the unambiguous B∪Y. These odd choice patterns do not reflect a clear preference for or against ambiguity. Instead, subjects mostly chose one color to bet on, then coupled their choice with another color, thus switching from apparent ambiguity-aversion to ambiguity-preference or vice versa.

In several studies, parameters were varied more widely.

Cohen, Jaffray and Said (1985) elicited certainty equivalents for a 50-50 chance of winning 10 French francs and an unknown chance of winning the same amount. They conducted the same experiment for losses of 10 francs. (Subjects whose certainty equivalents only differ by half a franc were classified as indifferent.) For gambles over gains, 59% of the subjects were ambiguity-averse, 35% were indifferent and 6% ambiguity-preferring. For losses, 25% were averse, 42% indifferent, and 33% preferring. Ambiguity attitudes for gains and losses were not significantly correlated. Neither were risk attitudes and ambiguity attitudes. (However, in this study and others below both types of correlations could be low simply because risk and ambiguity attitudes are measured with error.[11])

[11] The appropriate comparison is between the risk-ambiguity correlation, for example, and the correlation between two separate measures of risk attitude or ambiguity attitude.

Kahn and Sarin (1988) (detailed in their 1987 paper) ran several experiments in which subjects made choices and stated ambiguity premiums (increments of known probability they would give up to avoid ambiguity). Kahn & Sarin replicated the Ellsberg paradox and found that ambiguity premiums were increasing in probability range, roughly linearly. They also observed modest ambiguity-seeking at low probabilities for gains (.1-.3) and high probabilities for losses (.7-.9). In another experiment subjects preferred an 80% chance of an urn with 63 winning balls (of 100) to a 50-50 chance of urns with 25 or 75 balls. Since the two choices have the same mean and variance of probability, the observed preference for the first urn (61% chose it) suggests mean and variance of second-order probability are not the only determinants of ambiguity-aversion (cf. Boiney, 1990, and Viscusi & Magat, 1991, on skewness). In decisions about natural contexts-- pregnancy, product breakdown, scholarship applications-- MBA subjects were roughly ambiguity-neutral on average, except in the scholarship context (premium=.02).

Curley and Yates (1989) had subjects rank a large variety of gambles which varied by stake (gain or loss), expected probability, and ambiguity. (Their methods are like the risk-measurement approach of Coombs & Lehner, 1981.) In an iterated choice task, subjects were willing to pay 5-10% of expected value to avoid ambiguity when probability was around .5 or .75, but they demanded a similar premium to give up ambiguity when the probability was low (.25). The subjects' rankings of gambles also ruled out a variety of simple additive and multiplicative models in which ambiguity, probability, and outcome were independent in various ways.

Hogarth and Einhorn (1990) had subjects choose among urns with single outcomes and distributions of outcomes (to measure risk-aversion), and known probabilities or ambiguous probabilities (to measure ambiguity-aversion). To create ambiguity, subjects were told they had been allowed to look into the urn and estimate its composition, "but [you] are not too sure of your estimate". Subjects made choices for two outcome levels ($1

We know of no studies that adjust risk-ambiguity correlations for measurement error in this way.

and $10,000 in one experiment, $.10 and $10 in an experiment with real payoffs), for gains and losses, and for three probability levels (.10, .50, .90). Subjects were generally ambiguity-averse. There was some ambiguity-preference for low probabilities of gain and high probabilities of loss. (Einhorn & Hogarth, 1986, found the same pattern.) Ambiguity-aversion was weaker for losses than for gains, and slightly weaker for small payoffs than for large payoffs. There was no correlation between risk-attitudes and ambiguity-attitudes.

Goldsmith & Sahlin (1983) report a study using bets on natural events, like the occurrence of a bus strike in Verona next week. Holding first-order (or mean) probability constant, about half the subjects preferred bets on less ambiguous events for gains and bets on more ambiguous events for losses. When ambiguity-preference switched across the range of probabilities, it usually switched from ambiguity-preference at low probability to ambiguity-aversion at high probability for gains, and oppositely for losses.

Many people think the Ellsberg paradox is an error in judgment, like an arithmetic or logic mistake, which people will correct when their error is made clear (Howard, in press). A study by Slovic and Tversky (1974) suggests subjects are immune to certain kinds of persuasion. They showed the three-color Ellsberg problem to 49 students. Students read two statements before making choices. One statement explained the psychological appeal of ambiguity-aversion; the other explained the sure-thing-principle. Most subjects said the statement advocating ambiguity-aversion was more compelling. Eighty percent of them committed the paradox. Curley, Yates & Abrams (1986, Table 4) replicated this finding with slightly different arguments. Axiom popularity polls of this sort are controversial. If subjects do not accept an argument for SEU, perhaps a poor argument may have been used. (The statements used in the experiments are given in the papers so readers can judge them.) More strongly-worded arguments might work better, but there is a fine line between simply presenting subjects with an argument and creating subtle experimental demands for conformity. We think axiom polls are of some help in answering a simple question: Will people abandon appealing principles, or stick by them, when the principles conflict with specific choices that are appealing and the conflict is made clear?

3.2 Psychological Determinants of Ambiguity-aversion

Three empirical studies explored the psychological roots of ambiguity-aversion in special detail.

Einhorn & Hogarth (1985, 1986) proposed a two-parameter descriptive model (discussed further in section 4.4) based on the idea that subjects anchor on an ambiguous probability and adjust upward or downward depending on their imagination of other possible probabilities. Their subjects judged the likelihood of an event based on conflicting evidence (e.g., 2 witnesses to an accident said a car was green and 1 said it was blue). Judgments fit the model well. In one experiment, parameters estimated from their likelihood judgments were used to predict subjects' choices between bets on events and bets on unambiguous chance devices. The model predicted 60% of the choices correctly.

Curley, Yates, Abrams (1986) used variants of the two-color Ellsberg problem to test several hypotheses about sources of ambiguity-aversion. Subjects who said the urn could not be biased against them were ambiguity-averse too; their ambiguity-aversion was not due to belief in "hostile" generation of outcomes. As in other studies, many subjects were ambiguity-averse even when indifference was allowed (disproving the conjecture of Roberts, 1963, and others), indicating a strict preference for avoiding ambiguity. There was no correlation between the risk-attitudes and ambiguity-attitudes of individual subjects. Subjects were no more ambiguity-averse when the contents of the urn were revealed afterward (contrary to some regret-based arguments). However, subjects were significantly more ambiguity-averse (using a 7-point strength-of-preference scale) when the gamble they chose would be played, and the urn's contents revealed, in front of other subjects.

Heath & Tversky (1991) suggest that competence-- knowledge, skill, comprehension-- is what causes the gap between belief and decision weight. Subjects gave probability assessments for natural events (like the temperature in Tokyo). Then they chose between betting on the event and betting on a chance device constructed to have the same subjective probability as the event. If people are ambiguity-averse they should prefer the matched-probability chance bets to bets on events (which are inherently ambiguous). In one experiment subjects were generally ambiguity-averse: The sum of certainty-equivalents for a bet on an event and a bet against the same event was less than the sum for bets on chance devices. Keppe & Weber (1991) replicated this result using probability-equivalents.

But subjects were not uniformly ambiguity-averse. They preferred betting on events they knew a lot about, holding beliefs constant: In one experiment, those who knew a lot about football preferred bets on football-related events to matched-probability chance bets (at all levels of probability), and those who knew little about football preferred the chance bets (cf. Fellner, 1961, p. 687).[12]

The competence hypothesis broadens the study of choice anomalies in SEU, by suggesting that ambiguity about probability is just one of many forces that undermines competence and makes people reluctant to bet. For example, people would rather bet on future events than on past events, because not knowing what happened undermines competence (Rothbart & Snyder, 1970; Brun & Teigen, 1990). They would also rather bet on skill (which creates ambiguous probability of winning) than on chance (Cohen & Hansel, 1959; Howell, 1971; cf. Langer, 1975, on the "illusion of control").

Competence also provides an interesting, long-awaited bridge between the psychology of choice and the psychology of group and organizational decisions. Heath & Tversky conjecture that competence influences betting because social (and personal) assignments of credit and blame are asymmetric-- competent people can take credit for winning but incompetent people can only take blame for losing. (Alternatively, competent people might get more blame for losing than incompetent people do.) The fact that subjects were more ambiguity-averse when bets were resolved in front of others, in the study by Curley, Yates & Abrams (1986), is consistent with the credit-blame hypothesis. The influence of competence and justification in group decision-making under ambiguity should be an important new area of research.

3.3 Experimental Markets

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[12] In another experiment, subjects chose the most likely answer to a question with four possible answers and gave the probability their answer was right. Then they chose between betting on their answer or betting on matched-probability chance devices. Since a high subjective probability is an indication of knowledge about the right answer, if knowledge increases decision weight then preference for betting on answers should rise with probability. It did. Taylor (1991) replicated this result with a different probability-elicitation method, but Goldsmith & Sahlin (1983) found the opposite result (p. 464).

Ambiguity has been studied in two market experiments. Camerer and Kunreuther (1989) created a simple market for insurance, in which some traders were endowed with potential losses which they could transfer to other traders by paying a negotiated insurance premium. In some periods the probability of loss was .1 (unambiguous), and in other periods it was equally likely to be 0, .1, or .2 (ambiguous). Ambiguity had no systematic effect on prices, but it did create concentration in the insurance-seller's market (increasing the number of losses insured by each active insurer).

Sarin and Weber (1989) tested whether ambiguity affected prices in an experimental asset market, with German business students and bankers as subjects. The assets were draws from urns, with a known .5 chance of winning or an ambiguous chance of winning. The market price of the known .5 bet was considerably larger than the market price of the ambiguous bet, in both sealed-bid and double oral auctions, whether the two assets were traded sequentially or simultaneously. Prices of known and ambiguous bets were about the same when the probability was .05. Over several periods the ambiguity effects got slightly smaller, but did not disappear.

The difference in these two studies could be due to several factors. In the Camerer-Kunreuther insurance market study the subjects were American undergraduates, ambiguity was clearly operationalized as second-order probability, and prices were close to expected value. In the Sarin-Weber study the subjects were German business students, ambiguity was operationalized a la Ellsberg, and prices were further from expected value (often above it). There were some differences in exchange institutions too.

3.4 The difficulty of establishing equivalence of ambiguous and unambiguous probability

Preferring bets on unambiguous events is only a violation of SEU if equivalence between the likelihoods of the ambiguous and unambiguous events has been established.[13] In many experiments subjects might guess the ambiguous probability has a skewed

[13] For example, in the Ellsberg two-color problem Frisch (1988) found that many subjects said the probabilities of red or black draws from the ambiguous urn were not .5. Those who said the ambiguous probabilities were .5 were generally not ambiguity-averse.

distribution which biases its mean. An event with ambiguous probability around .1, for instance, might have a positive skew and a mean above .1 (even if the median and mode are .1). Betting on such an event, instead of a chance device with p=.1, is consistent with SEU.

To check for equivalence, Heath & Tversky (1991) induced ambiguity by telling subjects that probabilities were around .01 (as in many experiments), then asked subjects whether an ambiguous probability around .01 was above, below, or exactly .01. A majority (75%) said it was above .01. Nearly 60% said an ambiguous .9 was most likely to be below .9. Other data suggest that perceptions of skewness, and hence non-equivalence of known probabilities and mean ambiguous probabilities, are widespread (e.g., Larson, 1980, p. 301; Goldsmith & Sahlins, 1983, p. 459; Curley, Eraker & Yates, 1984, p. 507; Frisch, 1988). These data warn researchers to include simple manipulation checks-- ask subjects whether they think the mean of the ambiguous event probability is the same as the known probability.[14] We should not be surprised if the two are different.

3.5 Synthesis: Stylized Facts From Empirical Work

Several stylized facts emerge from the empirical work (see Table 3). We start with the simplest findings and proceed to the most subtle (and controversial) ones.

Ambiguity-aversion is found consistently in variants of the Ellsberg problems (many of them using small actual payoffs). Ambiguity-averters have not been swayed in experiments which offered written arguments against their paradoxical choices. Indeed, subjects pay substantial premiums to avoid ambiguity-- around 10-20% of expected value or expected probability.

Subjects typically prefer to bet on known probabilities, instead of known distributions of probability (SOPs) with the same expected probability. Increasing the range of possible

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[14] It is conceivable that the probability estimates given by subjects in manipulation checks are actually decision weights, which already reflect forces that create ambiguity-aversion (like psychological adjustment of stated probabilities, nonlinear weighting of SOPs, or non-additivity of probability). Then the onus is on researchers to better disentangle true subjective beliefs and reported beliefs.

probabilities increases ambiguity-aversion.

There is some evidence of ambiguity-preference for betting on gains with low ambiguous probability, or betting on losses with high probability. (This may be due to perceived skewness, which distorts the mean of the ambiguous distributions of high and low probabilities.)

Broader studies show that ambiguity about probability is simply one determinant of competence, and hence of decision weight. Other determinants include the presence of knowable missing information and overall knowledge about event domains.

Several phenomena should be studied further. Betting on gains (rather than losses) and making choices in the presence of others both seem to increase ambiguity-aversion, but the effects are weak and should be replicated. There is also weak evidence that ambiguity-aversion increases with outcome size,[15] which should certainly be explored further because outcome-dependency distinguishes sharply between theories. The correlation between risk-attitudes and ambiguity-attitudes appears to be low, but studies have not carefully corrected for measurement error. And in many studies, the failure to induce or establish a sharp equivalence between unambiguous and ambiguous probabilities makes it difficult to know whether results violate SEU or not. Simple, reliable techniques for inducing equivalence would be very useful.

4. FORMAL MODELS OF AMBIGUITY

Since many of the models proposed to describe ambiguity effects generalize the axioms underlying SEU, it is useful to give some details of the axioms. (See Fishburn, 1970, 1982, 1988b, for more detail.)

The lottery-act formulation of Anscombe & Aumann (1962), which encompasses both EU and SEU as special cases. In their formulation each consequence is a lottery over

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[15] Hogarth & Einhorn (1990) report a weak stakes effect (hypothetical stakes range from $1 to $10,000). Goldsmith & Sahlin (1983) report no difference from multiplying stakes by twenty. Unpublished data collected by the first author and Murray Low, using 54 MBA's, show no stakes effect between bets of $5 (70% ambiguity-averse) and $1000 (71%).

outcomes with objective probabilities. A consequence, an outcome lottery $\alpha(s)$, will be written as a vector $(x_1,p_1;...;x_m,p_m)$, where the p_i's denote objective probabilities (perhaps generated by physical devices like coins or roulette wheels). In the lottery-act framework, the final outcome of act X depends on which state s_i occurs, then on which outcome the lottery $\alpha(s_i)$ yields. They call the first stage of consequences $(\alpha(s_1),p(s_1);...;\alpha(s_n),p(s_n))$ a "horse lottery" and the second stage $(x_1,p_1;...;x_m,p_m)$ a "roulette lottery".[16]

Anscombe and Aumann use standard EU axioms to establish existence of state-dependent utility functions which represent preferences: Order (completeness and transitivity); continuity; and independence. We define independence (because some of the theories reviewed below relax it):

Independence

If $X \succeq Y$, then for any number $r \in [0,1]$ and any Z, $rX+(1-r)Z \succeq rY+(1-r)Z$.

The independence axiom states that preferences between two lottery-acts composed of roulette lotteries between X or Y and a common act should be independent of the common (or "irrelevant") act.

These axioms yield a state-dependent SEU representation in which the utility of consequences depends on the state in which the consequence occurs. Two other axioms restrict the utility function to be the same for all states. The SEU representation theorem states that preferences over lottery-acts satisfy the five axioms if and only if there exists a unique additive probability measure (or distribution) for all states: $p: S \rightarrow [0,1]$ and utility function on the lotteries $\alpha(s)$, $\beta(s)$, etc., unique up to a positive linear transformation, so that:

$$(2) \quad X \succeq Y \leftrightarrow \sum_{s \in S} p(s)\, u(\alpha(s)) \geq \sum_{s \in S} p(s) u(\beta(s))$$

[16] Sarin & Wakker (1990) use an analogous representation with only one stage. States can be either unambiguous, like roulette lotteries, or ambiguous, like horse lotteries.

The Anscombe-Aumann representation has EU and Savage-style SEU as special cases.[17] Note also that the probability measure must be additive[18]: $P(A \cup B) = P(A) + P(B)$ if $A \cap B = \varnothing$ (e.g., the probabilities of heads and tails in a coin flip must add to one.)

Savage (1954) used six axioms (for acts with finitely-many outcomes). His second, the "sure-thing principle" is analogous to independence in EU (see Fishburn, 1987a):

Sure-Thing Principle

> Let X, Y, X' and Y' be acts and let S' be a subset of the set of states S. If $x(s) = x'(s)$ and $y(s) = y'(s)$ for $s \in S'$ and $x(s) = y(s)$ and $x'(s) = y'(s)$ for $s \in S / S'$, then $X \succeq Y$ if and only if $X' \succeq Y'$.

The sure-thing principle requires one to ignore states in which acts yield the same consequence when choosing between the acts.[19]

Several formal models have been proposed to accomodate ambiguity effects. (Note that these models attempt to describe attitudes toward ambiguity, revealed by choices or judgments, rather than just define ambiguity.) Some models invoke psychological principles or propose ad hoc decision rules. Others generalize the axioms of SEU. Of course, there is no reason that weakening SEU axioms necessarily leads to a better descriptive theory, but having an axiomatic underpinning for a theory provides a simple way to test it, and might provide a transparent way to judge its plausibility. (Unfortunately, some of the axioms given below are not transparent!)

The models can be roughly grouped into four classes:

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[17] Savage's formulation applies if all lotteries $\alpha(s)$ have sure outcomes; then there are only horse lotteries. EU applies if there is only one state; then there are only roulette lotteries.

[18] In SEU, additivity of probability is implied by the conjunction of the sure-thing principle and Savage's fourth axiom, "comparative probability". Machina & Schmeidler (1990) show that abandoning the sure-thing principle, but strengthening the comparative probability axiom by adding a dash of the sure-thing principle to it, yields axioms which guarantee that preferences satisfy additivity without necessarily satisfying SEU. Nonadditive probability explains some SEU violations (as we see below). The Machina-Schmeidler axioms show the conditions under which SEU violations can be explained without invoking nonadditivity.

[19] In the axiom, ignore states S/S'. Since $X = X'$ and $Y = Y'$ in the other states, if $X \succeq Y$ then $X' \succeq Y'$ too.

(1) Some theories account for ambiguity in consequence utilities (Smith, 1969; Sarin & Winkler, 1990). The other three classes assume varying degrees of knowledge about second-order probability (SOP).

(2) Some theories assume a single SOP distribution (with mean E(p)), but relax the axiom of compound lottery reduction and weight SOPs nonlinearly to explain ambiguity-aversion or ambiguity-preference (Segal, 1987a; Kahn & Sarin, 1988; Becker & Sarin, 1990). These theories treat possible probabilities the way possible outcomes are treated in EU and SEU.

(3) Other theories accept the idea of sets of probabilities, but do not assume a unique distribution of probability over elements of the set (as the SOP approach does). They assume preferences are generated by considering some or all of the possible probability distributions in the set (Hodges & Lehmann, 1952; Ellsberg, 1961; Gilboa & Schmeidler, 1989; Gardenfors & Sahlin, 1982; Weber, 1987; Nau, 1988; Neehring, 1990).

(4) Still other theories avoid unique SOPS or sets of probabilities entirely. In some theories, the expected probability E(p) is assumed to be known (or measurable) and is transformed to express ambiguity-aversion (Fellner, 1961; Einhorn & Hogarth, 1985; cf. Viscusi, 1989). When E(p) is not known, nonadditive probabilities of events can be used to express ambiguity-aversion (Schmeidler, 1982, 1986, 1989; Gilboa, 1987; Hazen, 1987; Fishburn, 1988a; Wakker, 1989a; cf. Luce and Narens, 1985; Tversky and Kahneman, 1990; Luce & Fishburn, 1991). A similar approach, designed to be easily tested with market data, separates decision weight into risky probability (based on a sample of observations) and uncertainty which depends on the sample sizes (Phillipson, 1991).

We now review each of the four classes of models in turn.

4.1 Utility-based Models

A simple way to express ambiguity aversion is to allow the utilities from winning bets on ambiguous and unambiguous events to be different (cf. Karni, 1985, on state-dependent utility). If the utility from winning an ambiguous bet is lower, aversion to ambiguity is consistent with utility maximization (Smith, 1969; Franke, 1978).

Recall the three-color Ellsberg problem from section 1.2. Suppose utilities of the

ambiguous events b and r ∪ y are $u_A(W)$ and utilities of the unambiguous events r and b ∪ y are $u_U(W)$. Then assuming additivity, the paradoxical pattern of preferences implies $p(r)u_U(W) > p(b)u_A(W)$ and $p(b \cup y)u_U(W) > p(r \cup y)u_A(W)$. These inequalities need not be inconsistent if $u_U(W) > u_A(W)$. While it expresses ambiguity-aversion in a direct way, the state-dependent utility approach is not parsimonious and borders on tautology. It cannot explain many of the empirical facts, especially variation in ambiguity-aversion across probability levels, unless those observations are built directly into utilities.

Sarin & Winkler (1990) advocate the utility-based approach and offer axioms which imply more restrictive models. In their models, the utility of a prize depends on the other possible prizes. In the Ellsberg problem, for example, a person dislikes betting on the ambiguous urn because the disutility of losing is increased by the amount one could have won (reflecting disappointment). (To model aversion when probabilities are ambiguous, they assume there is no disappointment when probabilities are known.)

To some extent, it is a matter of modelling taste whether ambiguity-aversion is located in modified utilities, reflecting how people feel about consequences, or in modified decision weights, reflecting feelings about likelihoods. The choice between the two approaches turns crucially on whether one believes likelihood estimates and decision weights must be equal. Those who advocate modifying utilities are reluctant to let likelihood and decision weight differ. Those who are willing to allow a difference may find utility modification cumbersome or tautological.

4.2 Models Based On Unique SOPs

Several models assume unique SOPs, but relax the reduction of compound lottery assumption[20] and weight the possible probabilities nonlinearly. These models exhibit risk-aversion toward (second-order) distribution of probability, akin to risk-aversion toward distributions of outcomes in EU (Chew, Karni & Safra, 1987). Segal (1987a) uses the "rank-

[20] Reduction is violated in systematic ways (Camerer & Ho, 1991). One class of violations, called "probabilistic risk-aversion", corresponds to nonlinear weighting of possible probabilities in an SOP distribution.

dependent" generalization of EU originally developed by Quiggin (1982) and expanded by Yaari (1987), Jullien & Green (1988), Chew (1989), Segal (1989), Luce (1988, 1991), and Luce and Fishburn (1991). Segal's application represents an interesting bridge between the EU and SEU approaches. Like all SOP approaches, his assumes that subjective uncertainty about a state's occurrence can be reduced to a distribution of possible probabilities, and probability can be assigned to the elements of that distribution, effectively transforming uncertainty and risk. Whether the SOP assumption represents a useful unification of EU and SEU, or deliberate ignorance of an essential distinction, is for the reader to judge.

Segal supposes that the subjective distribution of balls in the Ellsberg two-color problem is discretely uniform between 0 and 100. Since a discrete uniform distribution has 101 possible outcomes, the decumulative second-order distribution (one minus the cumulative distribution) assigns a (101-i)/101 chance of having i or more winning balls in the urn.[21] His approach differs from others based on SOP because he does not assume that the compound lottery generated by the SOP is reduced to its single-stage equivalent. Therefore, the utility of betting on either color from the ambiguous urn is

$$(3)\quad u(W)\{\sum_{i=0}^{100} f(i/100)[f((101-i)/101)-f((101-i-1)/101)]\}$$

The unambiguous urn has utility

$$(4)\quad u(W)f(50/100)$$

The function f(p) allows nonlinear probability weights, between zero and one, to express ambiguity-aversion or -preference. Tedious algebra shows that if f(p)=p then (3) and (4)

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[21] The incremental probabilities in the decumulative distribution run in the opposite direction than you might think, beginning at 0/101 for 100 winning balls, because the decumulative distribution arranges the outcomes from best (100 winning balls) to worst (0 winning balls) and takes increments in weighted decumulative probabilities from best to worst.

are the same; people should not be ambiguity-averse.

The terms in (3) represent weighted probabilities of drawing i balls and winning. For example, the i=63 summation term is u(W)f(63/100)[f(38/101)-f(37/101)], the weighted utility of winning if the urn has 63 winning balls, u(W)f(63/100), times the incremental weighted probability of getting 63 or more winning balls (f(38/101)) instead of 64 or more winning balls (f(37/101)). When f(p) is convex, the increments in (3) are largest when i is small, overweighting the possibility of an unfavorable urn with few winning balls. However, a slightly stronger condition than convexity of f(p) is needed to guarantee ambiguity-aversion.[22] Luce and Narens (1985) and Luce (1988) axiomatized related forms of rank-dependent utilities, though the connection to ambiguity-aversion is not made explicit.

Kahn and Sarin (1988) also posit a nonlinear weighting function. The weight for an event with second-order probability distribution $\Phi(p)$ (with mean E(p) and variance $\sigma^2(\Phi(p))$ is

$$(5) \qquad w(\Phi(p)) = E(p) + \int_0^1 \Phi(p)(p-E(p))e^{-\lambda(p-E(p))/\sigma(\Phi(p))}\,dp$$

If λ is zero, $w(\Phi(p))=E(p)$; the model reduces to SEU. If λ is positive, the function (5) adjusts the probability weight by underweighting the chance of higher-than-average values of p and overweighting lower-than-average values, producing $w(\Phi(p))<E(p)$ and expressing ambiguity-aversion. A negative λ does the opposite, expressing ambiguity-preference.[23]

The Kahn-Sarin model resembles theories of disappointment and elation in choice

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[22] A convex f(p) underweights the probability f(i/100) if i is small (the urn has few winning balls), and overweights f(i/100) when i is large, so convexity is not sufficient for ambiguity-aversion. Instead, f(p) must be convex and have nondecreasing elasticity $pf'(p)/f(p)$ (or equivalently, $f(p)f(q)\leq f(pq)$) and 1-f(1-p) must have nonincreasing elasticity (Segal, 1987a, Theorem 4.2). The "common ratio effect" observed in studies of EU can be explained by similar restrictions on elasticity of f(p) (Segal, 1987b).

[23] Note that ambiguity-aversion for both gains and losses requires λ to be positive for gains and negative for losses.

(Bell, 1985; Loomes and Sugden, 1986; Gul, 1991), except that disappointment results from bad probability outcomes (from the second-order distribution $\Phi(p)$) instead of bad consequences. Put differently, the adjustments in (5) reflect aversion to the probability risk inherent in the second-order distribution. The same risk function has been applied elsewhere (e.g., to asset pricing by Weber, 1990).

Becker & Sarin (1990) take a more general approach. Leaning heavily on the assumption of a well-specified SOP, their paper builds up analytical tools reflecting the natural analogy between aversion toward spreads in outcomes (risk-aversion) and aversion toward spread in probabilities (ambiguity-aversion). They first derive the existence of a decision weight function on events, w(e) (much like the non-additive probabilities described in section 4.4 below). They assume the event e has an SOP, $f_e(p)$. Then they posit a "probability equivalence" function $\phi(.)$, which gives decision-weight equivalents, $\phi(w(e))$ equal to the expectation of the weighted SOP probabilities, $E(\phi(f_e(p)))$. The function $\phi(p)$ is simply a utility function on the second-order probabilities in $f_e(p)$. Ambiguity-aversion corresponds to concavity of $\phi(p)$; ambiguity-preference corresponds to convexity. They also derive ambiguity-premia from properties of $\phi(p)$, much as risk-premia are related to u(x) in EU. The value of their approach depends on whether the formal analogy between risk-aversion and ambiguity-aversion proves theoretically useful and empirically tenable.

4.3 Models Based On Sets of Probabilities

Hodges and Lehmann (1952) and Ellsberg (1961) suggested people choose using a weighted average of a gamble's expected utility (averaged over possible distributions) and its minimum expected utility over those distributions.[24] In Ellsberg's three-color example, suppose uncertainty about the ambiguous urn is characterized by a set of equally-likely possible probabilities for W, from 0 to 2/3. Then a bet on the ambiguous urn has an expected utility of (1/3)u(W) and a minimum expected utility of 0 (which occurs if

[24] Gilboa (1986) derives a related generalization of EU (but not SEU) in which preferences are represented by a function of both a lottery's expected utility and the utility of its worst consequence.

p(W)=0), whereas the unambiguous urn has an expected utility of (1/3)u(W) and a minimum expected utility of (1/3)u(W). Any average which puts some weight on minimum expected utility favors the unambiguous urn.

Others have proposed models based on similar decision criteria, typically combining expectation with some other moment or parameter, or taking an expectation on a limited set of distributions. Gardenfors and Sahlin (1982) propose choosing according to the minimum expected utility over all probability distributions which satisfy some threshold level of "epistemic reliability". (The threshold reflects the "epistemic risk", or risk of error in probability judgment, one is willing to take.) Their criterion resembles Hurwicz's (1951) "generalized Bayes-minimax principle". Gigliotti & Sopher (1990) suggest a rule, of limited applicability, based on hypothesis testing: The null hypothesis that a known probability and ambiguous probability are the same is tested, using a sample from the ambiguous urn or process, against the alternative hypothesis that the ambiguous probability is different. One bets on the known-probability event unless the hypothesis is rejected.

Minimax decision rules also emerge from axiomatic analyses in which people are assumed to have probabilities which are additive but not unique. In Gilboa and Schmeidler (1989), preferences are represented by the minimum of all the expected utilities of a lottery over its possible probability distributions. (Their representation thus justifies formally the "maximin" criterion first suggested by Wald (1950) for choice under uncertainty.)

The crucial axioms for the Gilboa-Schmeidler maximin representation are "uncertainty-aversion" and "certainty-independence". Uncertainty-aversion requires that $f \sim g$ imply $pf+(1-p)g \succeq f$ (that is, mixing f and g using objective probabilities can only be an improvement). The maximin rule is consistent with uncertainty-aversion because the minimum EU for pf+(1-p)g can be no worse than the minimums for f and g taken separately.

Certainty-independence is a independence restricted to mixtures of acts with sure outcomes (denoted h): $f \succeq g$ iff $pf+(1-p)h \succeq pg+(1-p)h$. Intuitively: The sure outcome h has the same expected utility for <u>any</u> distribution of probabilities, so mixing it with f and g does not affect determination of the distributions that minimize EU for f and g. Therefore, the minimum EU's for f and g will be ranked the same way as the minimum EU's for

pf+(1-p)h and pg+(1-p)h.

A closely related model is proposed by Bewley (1986). He suggests that the distinction between risk and uncertainty is best captured by allowing lotteries to be incomparable when their consequences are uncertain. Incomparability implies preferences are incomplete. Aumann (1962) showed that incomplete preferences can be represented by expected utilities over sets of probabilities. Then $X \succeq Y$ iff $E(U(X)|\pi) > E(U(Y)|\pi)$ for all distributions π; otherwise X and Y are incomparable.

When lotteries are incomparable, Bewley assumes choices are made by inertia: The current choice, or "status quo", is only abandoned if a new choice appears which is certainly better (i.e., which has higher expected utility for all possible probability distributions). Bewley admits it is probably hard to distinguish his inertial theory from the maximin SEU approach of Gilboa and Schmeidler (1989), but experimental evidence of "status quo bias" (Samuelson & Zeckhauser, 1988; Knetsch, 1989) supports Bewley's inertia assumption.

Neehring (1990) offers a related approach in which sets of beliefs are assumed as a primitive construct (rather than simply implied, as they are by the maximin SEU and Bewley approaches). Imagine that each of the most extreme beliefs in the set are the beliefs of a different alter ego within a person's mind. Neehring's "simultaneous EU" rule then chooses an act as if resolving bargaining among the alter egos holding the most extreme beliefs by using the Kalai-Smorodinsky bargaining solution. In contrast, maximin SEU uses a Rawlsian bargaining solution, by picking the act which makes the alter ego made worst-off by that act as well-off as possible.

There is also a large statistical and philosophical literature concerning sets of probabilities, or "upper" and "lower" probabilities (Koopman, 1940; Good, 1950; Smith, 1961). The main objection to this approach (as with SOP) is that it replaces unrealistic precision in probability estimates with precision in estimates of probability bounds. But people might be comfortable giving precise bounds. And even if bounds are imprecise, decisions might be more robust to errors in upper and lower estimates than to errors in unique probability estimates.

Nau (1986) derived a choice representation using upper and lower probabilities which are "confidence-weighted" by the amount of money a decision maker will bet at the odds

implied by the upper or lower probabilities. The confidence weights are like a membership function (Zadeh, 1965) for specific upper and lower probabilities. They also measure the Keynesian "weight of evidence", in a manner consistent with de Finetti and Savage's principle of inferring judgments from choice. Nau's model also avoids objections to precise assessment of upper and lower probabilities because the confidence weight measures their imprecision. His model can also allow imprecision of utilities.

4.4 Models Without SOPs

4.4.1 Nonlinear weighting of expected probability E(p)

Einhorn and Hogarth (1985) propose an anchoring and adjustment model underlying nonlinear weights. They begin by positing weights equal to an anchor p_A, adjusted up by k_+ and down by k_- to express the imagination of possible probabilities greater and smaller than the anchor. (The Kahn-Sarin model can be interpreted as making such adjustments, with the anchor p_A equal to the expected probability.) Upward adjustment k_+ is assumed to be proportional to the range between one and the anchor $(1-p_A)$; downward adjustment is proportional to the anchor, raised to a power β to reflect ambiguity attitude. The resulting probability weight is

(6) $S(p) = (1-\theta)p_A + \theta(1-p_A^{\beta}) = p_A + \theta(1-p_A-p_A^{\beta})$

The Kahn-Sarin and Einhorn-Hogarth models allow probability weights to depend on outcomes through the parameters λ and β. Outcome-dependence is important because people are ambiguity-averse for both gain and loss gambles. Models like Fellner's (1961), in which ambiguous events simply have a lower probability weight, fail descriptively because they predict preference for ambiguous bets on losses.

Viscusi (1989) proposes a probability adjustment model in which subjective probabilities s, like the imagined probabilities in the Einhorn-Hogarth approach or elements of the set of possible probabilities, are weighted and combined with a weighted objective probability q (perhaps a stated probability or subjective best-guess). The combination rule corresponds to Bayesian updating of a beta distribution, which generates a simple posterior probability $p^* = (\gamma s + \xi p)/(\gamma + \xi)$. The parameters γ and ξ are weights that represent

confidence, or the amount of information backing each probability (formally operationalized as a number of observations based on the probability). For example, in the two-color Ellsberg problem s for the ambiguous urn would be less than .5, and q=.5, giving $p^* < .5$ (depending on the weights γ and ξ).

Hazen (1987) axiomatized a "subjectively weighted linear utility" (SWLU) model which permits direct outcome-dependence of ambiguity. As in weighted utility under risk (Chew and MacCrimmon, 1979; Chew, 1983), in SWLU subjective probabilities are weighted by a function of their consequences. SWLU assumes independence and substitution for roulette lotteries, and two unique axioms which dictate how uncertain probability-equivalents of risky lotteries are combined. Then the SWLU of an act f is

$$(7) \quad \frac{\sum_{i=1}^{n} p(s_i)\psi[u(f(s_i))]u(f(s_i))}{\sum_{i=1}^{n} p(s_i)\psi[u(f(s_i))]}$$

The function $\psi[u(f(s_i))]$ weights the subjective probability $p(s_i)$ in a way that depends on the utility of the outcome $f(s_i)$. The denominator normalizes the weighted probabilities so they add to one. Note that if $\psi(\bullet)$ is constant, all subjective probabilities are weighted equally, so SWLU reduces to SEU.

SWLU expresses both features of evidence that Keynes wrote about: Subjective probabilities $p(s_i)$ have the classical interpretation as degrees of belief (implications of evidence), but the function $\psi(\bullet)$ allows one to be hesitant or eager to bet according to those beliefs (reflecting the weight of evidence). Furthermore, one's hesitance to bet can depend on the size of the outcomes (though recall that outcome-dependence is weak in experiments). Hazen and Lee (1989) spell out some other implications of the SWLU model.

4.4.2. Models with non-additive probabilities

Schmeidler (1982, 1989) axiomatized SEU with non-additive probability distributions in an extension of the Anscombe-Aumann framework (using both objective and subjective

probabilities). In his model, probabilities vary between 0 and 1 and are "monotonic" (i.e., $P(E) \leq P(F)$ if E and F are sets of events with $E \subseteq F$) but not necessarily additive. That is, $P(E \cup F) \neq P(E) + P(F) - P(E \cap F)$.

Non-additive probability allows the probabilities of two equally-likely events to be equal ($P(A)=P(B)$), but does not force them have the same probabilities as two events C and D which are also thought to be equally likely, based on a richer set of information. Non-additivity allows P(A) and P(B) to measure likelihood of events (implications of evidence), while 1-P(A)-P(B) measures faith in those likelihoods (weight of evidence).

Schmeidler showed SEU can be generalized to allow non-additive probabilities when independence is weakened to apply only to "comonotonic" events. Acts f and g are comonotonic if $f(s) \succ f(t)$ implies $g(s) \succeq g(t)$ (for states s and t). If f and g are comonotonic they (weakly) rank states, according to their consequences, in the same way. Violations of independence seem to occur when $g \succ f$ but the mixture $pf+(1-p)h$ is preferred to the mixture $pg+(1-p)h$ because h "hedges" f more than it hedges g. (An act h hedges f if it has a good outcome when f has a poor outcome, and vice versa.) By definition, comonotonic acts cannot hedge each other because a state which pays off well for one act pays well for every act that is comotonic.

Comonotonicity is crucial in deriving nonadditive probability. To see the link, reconsider the three-color Ellsberg example, rewritten slightly in Table 4. No pair of the three acts X, Y, and Z are comonotonic because there are always two states in which the acts' consequences are ranked oppositely. (For example, $X(\text{red}) \succ Y(\text{red})$ but $X(\text{black}) \prec Y(\text{black})$.) Now note that the act X' is a mixture which yields a .5 chance of W (a probability mixture between X and Z) if red or yellow occurs. Y' is a similar mixture of Y and Z.

Ambiguity about the black and yellow states is what makes Y and Z unappealing. Mixing Y with Z hedges that ambiguity (or probability risk), producing the appealing act Y', precisely because Y and Z are <u>not</u> comonotonic. The black and yellow states hedge each other because each state "cancels out" the ambiguity in the other when the states are unionized. A way to express the value of hedging mathematically is to make the decision weight on the union (black $\cup$ yellow) greater than the sum of the weights on black and on

yellow. The mathematical expression of the value of hedging probability risk is precisely what nonadditive probability allows.

When probabilities are nonadditive, expected utilities must be calculated in an unorthodox way, introduced by Choquet (1953-4) and first applied to utility theory in Schmeidler (1982). First rank the states s_i from 1 to n based on their utility $u(f(s_i))$ (for a particular act f). Then for lotteries with finitely-many outcomes, non-additive subjective expected utility is

$$(8) \qquad u(f(s_1))p(s_1) + \sum_{i=2}^{n} u(f(s_i)) \left[p\left(\bigcup_{j=1}^{i} s_j\right) - p\left(\bigcup_{j=1}^{i-1} s_j\right)\right]$$

Note that if the probability measure $p(\bullet)$ is additive, the bracketed difference is simply $p(s_i)$ and the expression reduces to SEU.

The rank-dependent extension of prospect theory to many-outcome lotteries, called cumulative prospect theory (Tversky & Kahneman, 1990), uses Choquet integrals to compute weighted values of consequences. The twist is that the Choquet weighting reflects around the origin (or reference point). The decumulative distribution function of positive outcomes above the reference point (or its event-based equivalent) is weighted, the cumulative distribution of negative outcomes is weighted separately, and the two are added together to determine the weighted value of a gamble.

Gilboa (1987) axiomatized SEU with non-additive probability in a Savage framework (using only subjective probabilities, with an infinite state space). His proof uses a variant of the sure-thing principle restricted to comonotonic acts, which is shown in Table 5. In Gilboa's axiom, indifference between the acts in the first row of the table, combined with preference for the left act in the third row (and the fact that $y_1 > x_1$) implies that A is a better state to bet on than B. His axiom then requires that indifference between acts in the third row implies the preference pattern in the fourth row. Note that the axiom only applies if the acts in the left and right halves of the table are comonotonic with other acts in the same half. Since acts in each half are comonotonic, the probabilities of A and not-A could be nonadditive without disturbing the axiom's implication.

Wakker (1984) found an alternative axiomatic route to the proof of EU, using a complicated axiom called "cardinal coordinate independence". In Wakker (1989a,b) he uses the same axiom, restricted to comonotonic acts, to derive SEU with non-additive probabilities.

Table 5 shows Wakker's axiom. The crucial idea is whether tradeoffs across different state outcomes are stable. In the first two rows, preferences for acts are reversed by substituting γ for α in state A, and δ for β in state B. The bottom two rows imply that the same substitutions should not reverse preference in the opposite direction when the consequences in the complementary states not-A and not-B are changed. Prohibiting such contradictory substitution effects is enough to prove an SEU representation. Limiting the prohibition only to comonotonic acts allows probabilities to be nonadditive.[25]

Fishburn (1988a) derived a generalization of SEU allowing nonadditive probabilities with nontransitive preferences.

Sarin & Wakker (1990) derive nonadditive SEU in yet another way. Their paper introduces a new axiom which generates nonadditive probability without mentioning comonotonicity. First, they introduce the useful idea of "cumulative consequence sets", sets of consequences E such that if x is in E then all better consequences, $y \succeq x$ are in E too. Denote the set of states which give consequences in E, for a particular act f, by $f^{-1}(E)$. Then induce a preference relation on states A and B from preferences over acts: $A \succ B$ if an act which pays off an amount on A is preferred to an act which pays off the same amount on B. Sarin & Wakker show that the usual Savage axioms (with the sure-thing principle restricted to a set of unambiguous events), along with a "cumulative dominance" axiom, imply a nonadditive SEU representation.

Cumulative dominance: If $f^{-1}(E) \succeq g^{-1}(E)$ for all E then $f \succeq g$

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[25] In Wakker's world the state space can be either finite or infinite, and the set of consequences must be a connected separable topological space. Nakamura (1990) proved the same result for finite state spaces, with an infinite set of consequences that is not necessarily a connected separable topological space.

Cumulative dominance requires that an act which gives good consequences (in E) in more likely states should be preferred. (It is a reasonable SEU analogue to stochastic dominance in EU.) In conjunction with the other axioms, cumulative dominance is strong enough to force probabilities to be monotonic-- it would be violated if $P(A) \geq P(B)$ when $A \subset B$-- but not strong enough to require additivity.

We note that it is difficult to judge the normative appeal or empirical descriptiveness of cumulative dominance or the two axioms shown by Table 5. But all three axioms are easy to test experimentally (and all three are implied by the standard SEU axioms, but they do not imply all the standard axioms).

Luce and Narens (1985) studied even more abstract models in which outcome utilities were weighted by (rank-dependent) event weights. But their study was limited only to binary gambles. Luce (1988) extended the rank-dependent model to gambles with many events and provided axioms. Luce (1991) and Luce and Fishburn (1991) extended it further, to gambles with both gains and losses.

Non-additive probabilities have also been prominently discussed in the belief theories of Dempster (1967), Shafer (1976) and others (e.g., Levi (1984)). In the Dempster-Shafer theory, a portion of belief can be committed precisely to event A (the amount of belief is denoted m(A)), or to subsets of events which include A. Belief committed to sets of events need not satisfy additivity. In the Ellsberg two-color example, for example, we might have $m(\text{red}) = m(\text{black}) = 0$-- we refuse to commit any belief specifically to red or black-- but $m(\text{red} \cup \text{black}) = 1$. Dempster-Shafer beliefs (and related approaches) are widely used in applications such as artificial intelligence, where elicitation of conditional probabilities and Bayesian updating is tedious and unappealing to users (see Buchanan and Shortliffe, 1984).

Nonadditive probabilities create some curious problems (Gilboa, 1989b). If probabilities are nonadditive, then maximizing u(x) is not necessarily the same as minimizing -u(x)[26], preference orders can differ when two different kinds of Choquet expectations are

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[26] The two are not the same because the Choquet integral in (10) runs in the opposite direction when minimizing -u(x) (since the negative utilities are ranked oppositely of positive ones). Tversky & Kahneman's (1990) formulation can prevent the problem by weighting gains and losses differently.

taken, and the standard conditional probability $p(A|B) \equiv p(A \cap B)/P(B)$ can violate intuitively appealing properties (such as $P(B|A_1) \geq P(B|A_1 \cup A_2) \geq P(B|A_2)$). Gilboa (1989b, p. 412) interprets these problems as normative reasons to prefer additive probability.

In a novel approach, Fishburn (1990) supposes the ambiguity of an event can be measured directly (acts can be ordered by their degree of ambiguity), not merely inferred from choices. He proposes several axioms on ambiguity judgments, shows what numerical properties of ambiguity the axioms imply, and connects event ambiguity with subadditivity of event probability. His axioms are easy to test.

Phillipson (1991) constructs a model designed to be testable using observable market data. In his model, risky probabilities arise from a sample of N observations of outcomes in the set $Z \equiv \{z_1, z_2, ..., z_n\}$. (For example, insurance companies observe accident frequencies of drivers in demographically-determined risk classes.) Uncertainty arising from the sample of size N is represented by putting a decision weight M(N) on the set of all possible outcomes Z (as in the Dempster-Shafer approach). The weight placed on z_i is the relative frequency of observations of that outcome (n_i/N), times the weight placed on all risks rather than uncertainty, 1-M(N). The scheme therefore expresses both uncertainty (through M(N)) and risk (through n_i/N). Then the model can be tested using observable prices and relative frequencies, if assumptions are made about M(N) and an uncertainty-aversion parameter.

4.4.3 Non-additive probabilities and maximin rules on sets of SOPs

There is an important kinship between probabilities which are unique but non-additive, and sets of additive probabilities (section 4.3 above). The "core" of a non-additive probability distribution v generates a set of distributions. For example, suppose A and B are two events with $v(A) = .2$, $v(B) = .3$, $v(A \cup B) = 1$, and $v(A \cap B) = 0$. Then the core of v is all p(A) and p(B) satisfying $p(A) \geq .2$, $p(B) \geq .3$ and $p(A) + p(B) = 1$. If v is convex (i.e., $v(A \cup B) \geq v(A) + v(B) - v(A \cap B)$, as in the example), then the nonadditive SEU determined by v is the same as the maximin SEU determined by the set of probabilities which is the core of v. Intuitively, one can think of the unique non-additive distribution v as a compact way of representing maximin preferences. Weighting outcomes by subadditive probabilities

expresses precisely the same kind of pessimism that taking the minimum SEU over possible probabilities does. But strictly speaking, neither approach is a special case of the other.[27]

4.4.4 Pseudo-Bayesian updating with non-additive and maximin SEU

Gilboa & Schmeidler (1991) study a crucial problem in generalized SEU approaches: How to update nonadditive probabilities and sets of probabilities. They define a family of pseudo-Bayesian updating rules in which decision makers update after an event A occurs by implicitly assuming that an outcome f would have happened if A hadn't. (Preferences over the acts updated this way reveal updated subjective beliefs about A.) Each f gives a different updating rule. When probabilities are additive each rule coincides with Bayes' rule.

Probabilities updated using pseudo-Bayesian rules are nonadditive if and only if the possible outcomes f are the best and worst possible outcomes (or combinations of them across states). If f is the worst possible outcome, the updating rule corresponds to the familiar formula $p(B|A)=p(B\cap A)/P(A)$. If f is the best possible outcome the updating rule corresponds to the Dempster-Shafer rule, $p(B|A)=[p((B\cap A)\cup A^c)-p(A^c)]/(1-P(A^c))$ (where A^c denotes the complement of A).[28]

When there is a set of additive probabilities (as in the section 4.3 models), a

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[27] Maximin SEU is not more general because a nonadditive v which is not convex may not have a core (e.g., $v(A)=.4$ and $v(B)=.7$ in the example above has no core), and therefore does not correspond to maximin SEU over a set of probabilities. And nonadditive SEU is not more general because some sets of probabilities (e.g., $1\geq P(A)+P(B)\geq .8$) do not correspond to the core of any convex non-additive distribution, so maximinimizing over the set is not the same as maximizing over a nonadditive distribution.

[28] Recall that the Choquet integral in (9) weights the state with the best outcome by $p(s_1)$, then weights other states i by $p(s_1\cup\ldots\cup s_i)-p(s_1\cup\ldots\cup s_{i-1})$. Then roughly speaking, if the outcome f in A^c is the worst possible outcome, it comes last in the Choquet weighting and does not disturb the standard Bayesian updating of the probability of A. If the outcome f is the best outcome, it comes first in the Choquet weighting and creates the complicated non-Bayesian Dempster-Shafer rule.

maximum likelihood updating rule picks out those distributions which give maximum probability to an observed event A, updates them, and gives zero probability to all other distributions. When preferences can be expressed either by nonadditive SEU or by maximin SEU, then the maximum likelihood rule gives the same updated probabilities as the Dempster-Shafer rule. The equivalence to maximum-likelihood (which is widely used in classical statistics) gives a novel underpinning to the Dempster-Shafer rule.

4.5 Synthesis

The theories reviewed in this section are diverse and numerous. Unlike generalizations of EU, which weaken one of a few crucial axioms, attempts to modify SEU to allow ambiguity-aversion have gone in many different directions.

The available evidence on ambiguity (some of which was reviewed in section 2) casts doubt on features of some of the theories. For example, since the degree of observed ambiguity-aversion seems to be roughly independent of the consequences of bets, approaches which modify probability appear, so far, to be more descriptively apt than utility-based approaches. Other studies indicate that subjects are sensitive to features of the distribution of probability beyond the minimum and mean, which casts doubt on the maximin-SEU approach (and on some variants, including Ellsberg's). Most available studies find no correlation between risk attitudes and ambiguity-attitudes, which suggests that conceiving of ambiguity-aversion as simply an implication of risk-aversion toward probability-- as in the nonlinear weighting approaches-- might be wrong.

So far, we know of no evidence that directly contradicts the nonadditive SEU, or the psychological theories based on distortion or modification of "expected" probability. The two kinds of theorizing could be productive complements. Psychological notions of distortion arising from anchoring and adjustment, or pessimism, might explain why probabilities are nonadditive and suggest tests in natural settings. At the same time, giving nonadditive SEU an axiomatic underpinning supplies a sharp new way to test for nonadditivity (by testing axioms like comonotonic independence or cumulative dominance).

Finally, all the theories in this section were initially inspired by experiments on the Ellsberg paradox (or thought experiments described by Ellsberg and earlier sources). As

a result, none of the theories incorporate the newer ideas that ambiguity springs from missing information, and gaps between decision weight and belief arise because of competence (which is connected to missing information through decision maker knowledge). It is time for a second generation of richer theories which are stretched, or specifically designed, to reach beyond Ellsberg-type problems into broader domains of missing information and competence.

5. APPLICATIONS AND SPECULATIONS

5.1 Medicine and Health

Hamm & Bursztajn (1979) were the first to replicate the Ellsberg paradox in a medical setting, using a hypothetical clinical scenario. They found a substantial minority of ambiguity-averters.

Curley, Eraker & Yates (1984) gave a hypothetical case, describing a patient with stiffness and pain in the legs after walking, to patients waiting for treatment in a clinic. The patients first gave a probability of success P at which they would choose a specified treatment (if they were the patient in the case). Then they said whether they would choose treatments with ambiguous probabilities distributed around P.

About 20% avoided the ambiguous treatments. That degree of ambiguity-aversion may seem low, but only 25% of the same patients avoided ambiguity in an Ellsberg-type question with monetary outcomes. (Ambiguity attitudes in the medical and monetary settings were only weakly correlated.) There was no variation in ambiguity-aversion for three brands of ambiguity (the treatment was new, patients' reactions were variable, or doctors' estimates were variable). The best predictor of ambiguity-aversion was a lack of confidence that the ambiguous treatment would work as well as the unambiguous treatment.

Ritov & Baron (1990) studied the effect of ambiguity on decisions about vaccinating children. Vaccination reduces the risk of dying from a disease, but the vaccination itself might be harmful. When ambiguity about vaccination risk was caused by salient missing information about the risks from vaccination-- a child had a high risk of being harmed by the vaccine, or no risk at all, but it was impossible to find out which-- subjects were more

reluctant to vaccinate.[29] Viscusi, Magat & Huber (1991) studied reactions to ambiguous information about the risk of nerve disease and lymphatic cancer among 650 customers at a shopping mall. Subjects were told that two studies gave different estimates about the risk in town A (150 and 200 cases per million, for example). Then they were asked what certain risk level in town B (or risk-equivalent) would make them indifferent between living in A and living in B. A person neutral (averse) toward ambiguity should have a risk-equivalent equal to (greater than) 175, the mean of the estimates in the two studies. There was a small amount of ambiguity-aversion: The median risk-equivalent was usually 175, but the mean was 178.1 for estimates of (150,200) and 180.7 for when estimates were more widely-dispersed (110,240). Regression estimates of individual risk-equivalents (Viscusi & Magat, 1991) suggested that the risk estimate given first received more weight than the second estimate (a "primacy effect"), unless subjects were explicitly told that the second study was done after the first. The effect of dispersion in risk estimates also appeared to be concave (the dispersion of 130 between (110,240) had only a little more effect than the dispersion of 50 in (150,200).)

Curley, Young & Yates (1989) tried to measure doctors' ambiguity about the chance of coronary obstruction in several hypothetical cases, in three different ways: expressions of confidence, interquartile ranges around estimated probability, and ranges between minimum and maximum "plausible" probability values. Their goal was to study validity of the three measures of ambiguity, rather than to investigate ambiguity-aversion in choice. Validity was unimpressive. While precise probability estimates did reflect the contents of the case, and changed when additional case information was given, the range measures changed in unpredicted ways when additional information was given. The interquartile and plausible ranges were correlated across doctors ($r = .54$) but only weakly

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[29] Their experiment also circumvented the problem of subjects transforming ambiguous probabilities they are told (see section 3.4). Their subjects gave overall risk thresholds (probabilities) at which they would vaccinate. Then the comparison between control-group thresholds and missing information-group thresholds does not assume subjects believe or understand any probability information given by the experimenters, since the only probability information is supplied by the subjects themselves.

correlated with confidence. The ambiguity measures were also correlated with precise probability estimates, suggesting that disentangling measures of ambiguity from measures of probability is difficult.

These medical and health studies are a little discouraging, because they show less ambiguity-aversion, and less reliable measurement of ambiguity, than is observed or assumed in laboratory experiments (and in theory). But applied work of this sort is crucial to bridging the gap between the world of theory and the natural world.

5.2 Insurance, Liability, and Taxes

In the 1980's American insurance firms began raising premiums dramatically (or refusing to sell insurance at all) for several classes of highly ambiguous risks, like environmental hazards or manufacturing defects (e.g., Priest, 1987). Intrigued by the insurance companies' behavior, Hogarth and Kunreuther (1985, 1989) conducted surveys and discovered ambiguity-aversion in hypothetical decisions by both professional actuaries and experienced insurance underwriters. Indeed, many actuaries set rates by multiplying expected value by a multiple which reflects both administrative costs and unanticipated risks (an ambiguity premium; see Lemaire, 1986). This pricing rule is predicted by Phillipson's (1991) model, in which rates reflect expected losses and a premium depending on the firms' ambiguity-aversion and on their sample size. (His model is easy to test because it links the degree of ambiguity explicitly to an observable variable, the sample size of observed accidents.)

Hogarth & Kunreuther (1990) also found that actuaries responded to perfect correlation of risks by adding a large ambiguity premium to rates, contrary to the SEU prediction that ambiguity should not matter when risks are perfectly correlated.

Ambiguity may be important in litigation. Hogarth (1989) used experimental scenarios to study willingness to settle litigation before trial when ambiguity about winning was high or low. (Ambiguity was defined as an attorney's uncertainty about the probability of winning.) Plaintiffs filing suits were always eager to settle rather than litigate. Defendants' decisions were more subtle. When the (expected) $p = .5$, ambiguity made defendants more willing to settle; at $p = .8$ ambiguity made defendants less willing to settle (consistent with

ambiguity-seeking at high probabilities of loss observed in other studies).

Willham & Christensen-Szalanski (in press) gave subjects actual medical liability cases and manipulated ambiguity about the probability of winning. As in the Hogarth (1989) study, ambiguity was high when lawyers had "little confidence" in the assessed probabilities, and low when the lawyers had "extreme confidence". When ambiguity was high, subjects were willing to settle for less as plaintiffs, and offered more as defendants. As a result, pairs of subjects were three times as likely to settle out of court when ambiguity was high. In the legal framework ambiguity appears to increase the zone of settlements subjects prefer to a court battle. Compliance with tax reporting requirements is another legal choice where ambiguity about the probability of getting caught may matter. Casey & Scholz (in press) conducted an experiment with tax compliance scenarios in which the risks and penalties from getting caught were clear or ambiguous. Their data replicate patterns observed in more abstract settings (e.g., Einhorn & Hogarth, 1986): Subjects were typically ambiguity-averse, but were ambiguity-preferring when expected probabilities of getting caught were high (.9) and the expected penalty was near its stated maximum.

5.3 Marketing

Kahn & Meyer (in press) studied choices of consumer products when the usefulness of a new feature was ambiguous[30]. For example, subjects could buy a VCR with or without a stereo feature; the fraction of stereo videos available for rent was 25% (no ambiguity) or 0 to 50% (high ambiguity). When features enhanced the value of a product, corresponding to a gain in utility, subjects were ambiguity-averse (they were less likely to buy the enhanced product when ambiguity was high than when ambiguity was low). But when features were necessary to preserve the product's value, they preferred ambiguity.

Ross (1989) gave subjects a choice of whether to make a new product or not, given five expert estimates of the probabilities that other firms would make a competing product. Ambiguity was created by dispersion of the experts' estimates. A computer system recorded

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[30] Ambiguity about the incremental utility from a product feature corresponds to ambiguity about the feature's weight in a multiattribute utility evaluation.

the information subjects looked at while they made their decisions. When ambiguity was higher, subjects took longer to make a decision, looked longer at "bad information" (the experts' assessed probabilities that two other firms would compete), but made roughly the same choice as when ambiguity was low. These data are the first direct evidence that increasing ambiguity shifts measured attention from good outcomes to bad outcomes, providing a psychological underpinning to theories in which ambiguous probabilities of bad outcomes are overweighted, or a maximin SEU is calculated over pessimistic probabilities.

5.4 Financial markets

Dow and Werlang (1988) applied SEU with nonadditive probability to asset markets[31] (see also Simonsen & Werlang, 1990). Consider an ambiguity-averse trader who begins owning no shares of a stock. They showed that if an asset's value is ambiguous, there will be a price B at which a trader will buy shares and a price S at which a trader will sell shares short (with B < S, a positive "bid-ask" spread). In the range from B to S, the trader will not take a position. In SEU with risk-neutrality B must equal S.

The Dow and Werlang model implies that trading volume on organized exchanges will be affected by changes in ambiguity. For instance, when the US-led forces attacked Iraq in January 1991, ambiguity presumably rose because the invasion created missing relevant information (viz.: Who would win the war?). Because of the ambiguity, many stockbrokers told investors that it was a bad time to either buy or sell. Careful studies are needed to test whether ambiguity shocks like these actually do raise bid-ask spreads and reduce trading volume. The crucial empirical step is finding a sensible observable measure of ambiguity which predicts bid-ask spreads.

On the other hand, when Iraq invaded Kuwait in August, 1990-- raising ambiguity about future oil prices-- there was enormous trading volume in oil markets. This curious contrast in trading volume-- high in oil, low in stocks-- suggests willingness to trade in ambiguous situations may be moderated by knowledge or confidence (as in Heath &

[31] In their setting, nonadditive probability gives the same result as maximin EU.

Tversky, 1991): professional oil speculators traded more while individual stockholders traded less.

While the Dow-Werlang theorem suggests a kind of portfolio inertia, or reluctance to trade, other applications of ambiguity-aversion suggest an irrational eagerness to trade. Milgrom & Stokey (1982) showed that two traders whose only motive for trade was speculation based on private information should not trade with each other. Their "Groucho Marx theorem"[32] implies no speculative trade, roughly speaking, because the willingness of A to trade with B indicates the superiority of A's information, which should make B unwilling to trade. Dow, Madrigal & Werlang (1989) showed that, surprisingly, if one trader has nonadditive beliefs then she will be willing to trade (violating the Groucho Marx theorem). Taken together, the Dow-Werlang and Dow-Madrigal-Werlang results show that compared to SEU, ambiguity-aversion creates irrational reluctance to trade against "nature" (when asset values are exogeneous) and irrational eagerness to trade against others who may be better-informed.

Ambiguity might be especially important for certain kinds of assets about which little is known, like shares of smaller firms or "initial public offerings" (IPOs) of small privately-held companies. Indeed, stock prices of smaller firms do rise more than price of large firms, adjusting for market risk (Keim, 1983), and IPOs tend to jump in price about 10% when the market for their shares first opens. The apparent excess returns to small firms and IPOs might be premiums paid to investors who dislike ambiguity (see Yoo, 1990), but there are many competing explanations (e.g., Koh & Walter, 1989 and others they cite).

Another indication of financial ambiguity-aversion comes from personal holdings of domestic and foreign securities. French & Poterba (1991) estimate that people in several countries sacrifice about 3% in annual expected returns-- a substantial amount, since stocks rise about 8% per year-- by holding too many shares of domestic firms, and not enough foreign shares. One explanation is that people feel less ambiguity, or have more knowledge, about their own country's economy (cf. MacCrimmon, 1968, pp. 13-14). The 3% loss they

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[32] The theorem is named after a joke told by comedian Groucho Marx, who would 'never belong to a club that would have him as a member.'

accept is the premium they pay for avoiding ambiguity about foreign investments.

5.5 Economic Applications

5.5.1 Entrepreneurship

Knight (1921) distinguished between risk and uncertainty (or ambiguity) because he thought entrepreneurs earned economic rents from bearing uncertainty rather than risk. Bewley (1986) made a similar conjecture. (In technical terms, entrepreneurs are those with "fatter cones".) Some studies have found that entrepreneurs have a higher "tolerance for ambiguity" than non-entrepreneurs (e.g., Begley & Boyd, 1987), as measured by psychometric scales, but no studies have looked specifically for differences in ambiguity-aversion.

5.5.2 Contracting

There is a large literature discussing "incomplete contracting" in economic situations. The presumption is that there are many contingencies which could affect the terms of an economic relationship, but the probabilities of all contingencies cannot be sharply specified. Ambiguity about state probabilities might force agents to use simplified contracts with flexible, heuristic methods for resolving disputes. Williamson (1985), Grossman and Hart (1986) and others note that the inability to make complete contracts will favor ex-ante specification of authority as a mechanism for resolving disputes-- transactions will be "internalized" within firms-- when disputes are costly to resolve. When disputes are less costly, vague long-term contracts, enforced by reputational incentives, will be common.[33]

Bewley (1986) applied his choice framework to labor contracting.[34] Contracts are often simple (keeping wages rigid, for instance) but extend for long terms. Bewley suggests

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[33] As Hellwig (1989) wrote: "It may well be that the 'house bank' relation emphasized by Mayer owes some of its stability to the firm's reluctance to exchange a known partner for one whose behaviour in the course of the overall relation it can less well predict." (p 284)

[34] Indeed, he began considering uncertainty-aversion models after "exasperation and defeat" in trying to explain how simple long-term contracts with rigid wages could be optimal under asymmetric information or risk-aversion.

that ambiguity creates a special kind of bargaining cost, which can be reduced if contracts are simple.

5.6 Demand for Ambiguity-Reducing Information

If people are averse to ambiguity because they dislike not having missing information, it follows that people will value provision of any information which reduces their ambiguity (or increases perceived competence), even if it will not change their decisions. This result is in sharp contrast to the economic model of demand for information, which assumes that demand for information is derived from its value in making decisions. (One could model demand for ambiguity-reducing information in the economic model simply by positing direct preferences over the amount of information known, or distaste for unknown information. Asch, Patton & Hershey, 1990, give such a model for medical choices.)

For example, medical tests are sometimes conducted even when they are not accurate enough to change a doctor's diagnosis.[35] And surveys show that patients prefer to know more than doctors typically tell them (Strull, Lo & Charles, 1984), but the patients do not want a greater role in making medical decisions. Perhaps the patients simply dislike ambiguity.

5.7 Decision Analysis

Decision analysis is the practice of engineering better decisions. In decisions which involve risk or uncertainty, analysts usually begin by assuming, or trying to persuade people, that SEU is an appealing rule for making decisions.[36]

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[35] Many doctors realize the tests have no decision-theoretic value, but conduct them to avoid legal liability. Then we wonder: Is the legal system which induces doctor to overtest guided by a desire for irrelevant information, induced by ambiguity-aversion among patients, jurors, judges, attorneys, or others?

[36] The typical presumption among decision analysts is that people make paradoxical choices because they don't see the conflict between specific behavior and general axioms (or the "desiderata" the axioms imply). When properly educated, they'll switch to conform to the axioms (e.g., Howard, in press). But there is no scientific agreement on what proper education is (except that some small doses do not work; MacCrimmon, 1968; Slovic and

Many decision analysts have tried to incorporate concern for ambiguity, or at least second-order probability, in their analyses. In risk estimation it is common to include three (or more) levels of estimated risk, rather than collapsing them into a single estimate (e.g., Pate-Cornell, 1987). Brown (1990) describes an "assessment uncertainty technology" for computing the chance that probability estimates are accurate. Strategic planners in businesses and government often work with an expected, "best case", and "worst case" scenario in forecasting variables that are probabilities. Others have developed decision-analytic methods which allow users to specify ranges on probabilities, rather than point estimates (Weber, 1987). Users are then told whether the expressed range is informative enough to pick one alternative, and how narrowing the expressed range would affect the optimal choice.

5.8 Dutch Books

de Finetti (1937) showed that a person who made incoherent probability estimates could be offered a series of bets, each of which she would take, which would certainly make her worse off. Such "Dutch books" (or "money pumps") are often thought to provide an external discipline which might enforce principles of rationality when people mistakenly violate the principles or consciously reject them.

A Dutch book can be constructed to exploit ambiguity-aversion too. Consider the Ellsberg two-color problem. In step 1, give an ambiguity-averter a bet on a red draw from the ambiguous urn. She will pay to exchange it for a red bet on the unambiguous urn; collect her payment. In step 2, do the same with a bet on black. Now if she had kept both of the bets she got initially (ambiguous red and ambiguous black), she would have certainly won W. With the unambiguous bets she bought, she certainly wins W too, but she paid money in the process. (To complete the cycle, persuade her to give the two unambiguous bets back in exchange for the two ambiguous ones.) Heath & Tversky (1991) and Keppe & Weber (1991) report empirical observations of this sort.

Tversky, 1974; Curley, Yates, & Abrams, 1986). The intuitive appeal of the Ellsberg paradoxes and SEU generalizations designed to explain it raise at least some doubt about whether people should be helped to conform to SEU.

The construction of Dutch books like this one is delicate. If our ambiguity-averter had known in step 1 that she was getting a matching ambiguous black bet in step 2, she would not have exchanged the ambiguous red bet. So the success of the Dutch book relies on isolating each choice in a sequence (cf. Fishburn, 1988b, pp. 43-44): If the victim had asked at step 1 whether a second step was coming up, and what it was, the Dutch bookie would have to close up shop (or lie). Furthermore, a victim would have to be myopic to be led through the entire cycle repeatedly.

A less slippery Dutch book arises from an observation first made by Raiffa (1961): A person who dislikes ambiguity in the two-color Ellsberg problem can always flip a coin to decide whether to bet red or black, transforming the choice between ambiguous bets into a gamble similar to a bet on the unambiguous urn. In some theories proposed to account for ambiguity-aversion, the coin flip is distinctly preferred to an ambiguous bet. (If it is not, the Dutch book will not work.) For instance, Gilboa & Schmeidler's (1989) maximin SEU theory explicitly assumes such an axiom (called uncertainty-aversion): $f \sim g$ imply $pf+(1-p)g \succeq f$, where $pf+(1-p)g$ is a probabilistic mixture of f and g. For people that exhibit uncertainty-aversion, a Dutch book works as follows: Give them f. Then let them pay to exchange it for the gamble $pf+(1-p)g$. Play the gamble: If it yields g, switch g for f (since $f \sim g$); if it yields f do nothing. The Dutch book always leaves them with f, where they began[37], and leaves them a little poorer.

A Dutch book is an extreme example of violating stochastic dominance (since, by definition, one is certainly left worse off by a Dutch book). Violations of dominance which might leave a person worse off, because of ambiguity-aversion, can also be constructed.

Tversky & Kahneman (undated) give an elegant example: Two boxes each contain red and green marbles. A marble is drawn from each box; if their colors match you win $60. In Game A both boxes have 50% red and 50% green marbles. In Game B one box

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[37] A possible objection is that getting f from the resolution of the gamble $pf+(1-p)g$ is worse than getting f at the start, violating "consequentialism". But since $f \sim g$, it is hard to imagine that the taste of getting f as an outcome of the gamble instead of g is soured by disappointment, as might occur in most other counterexamples to consequentialism (e.g., Machina, 1989).

has 50% red and 50% green and the other box's composition is unknown. In Game C both boxes have the same composition of red and green marbles, but their composition is unknown. The games can be easily ranked by their ambiguity, based on the amount of missing information about the boxes-- A is least ambiguous, and C most ambiguous. Asked to choose which game they would like to play, about 2/3 of their subjects ranked A first and 2/3 ranked C last. But C is the best bet: While A and B have the same chance of winning (5.), C has the greatest chance of winning (because the chance of matching colors rises if there are many green balls, or many red balls, in both boxes).[38] Ambiguity-aversion leads most people to reject the bet with the highest chance of winning, violating stochastic dominance.

6. CONCLUSION

In expected utility (EU), a numerical utility function over outcomes represents preferences by establishing a correspondence between numbers (utilities) and bets in which higher-numbered bets are preferred. But EU assumes events have known objective probabilities, which is rarely true. The elegance of the Ramsey-de Finetti-Savage approach to subjective expected utility (SEU) is that preferences among bets simultaneously reveal beliefs about the probability of events, and utilities of the consequences of events. (Conversely, the approach implies that combining utilities and subjective probabilities using the SEU rule-- as decision analysis helps people to do-- guarantees satisfaction of certain appealing axioms.)

The crucial presumption in SEU is that people bet on events only because they think the events are likely. Betting weights must be the same as beliefs, which can reflect the implications of evidence but not its weight. But as Ellsberg (1961) showed (following the intuitions of Keynes and Knight), people demonstrate a persistent preference for betting on events whose likelihoods they know more about, when perceived likelihoods are held constant. "Ambiguity-aversion" has been observed in a dozen or so experimental studies,

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[38] Define p as the proportion of green marbles in the C boxes. Since both boxes have the same proportion p, the chance of winning is $p^2+(1-p)^2$, which is at least .5.

using various methods and parameters. (In about half the studies, subjects actually played one gamble for money.)

In most studies, subjects bet on chance devices with varying amounts of information about the probability of winning. This brand of ambiguity is narrow. More generally, ambiguity means that information, which could be known, is missing and salient. Uncertainty about the composition of an urn of balls is just one kind of missing information. Feeling ignorant about football or politics, having doubts about which of several experts is right, wondering whether your child has a predisposition to vaccine side-effects, or being unsure about another country's economy, are all manifestations of missing information. Some newer studies suggest these other kinds of missing information about events make people reluctant to bet on the events.

Theories

Many theories have been proposed to explain ambiguity-aversion. Some theories are utility-based: Ambiguity about events lowers the utility of the consequences those events have (keeping beliefs and betting weights the same, but making utilities event-dependent). Other theories assume a second-order distribution of belief about an event's probability and allow the second-order beliefs to be weighted nonlinearly. (Those theories explain ambiguity-aversion in much the same way that risk-aversion is explained in EU, weighting possible probabilities nonlinearly instead of outcomes.) A third class of theories assumes there is a set of possible event probabilities and bases choice on the minimum SEU taken over all probabilities in the set. And a fourth class of theories assume no second-order beliefs about probability. These theories either take an expected probability, then adjust it or weight it nonlinearly to reflect ambiguity, or allow probabilities which are unique but nonadditive ($P(E \cup F) \neq P(E) + P(F) - P(E \cap F)$).

Purposes and directions

The research reviewed in this paper does not always cohere because the researchers' purposes are different. Psychologists are mostly interested in how people think and behave: Their goal is to build parsimonious models which are psychologically plausible and fit

individual-level data (typically from experiments). Decision theorists are curious about the mathematical connections between axioms and numerical representations of preferences. Economists want predictions that are testable using observable market-level data. Decision analysts want to help people make better decisions.

The differences in researchers' purposes sometimes limit communication and cross-fertilization. For example, psychologists are sometimes annoyed that decision theorists rely on unrealistic axioms. Theorists see more realistic axioms as inelegant and difficult to work with. A review like this is meant to promote cross-fertilization by telling people with different purposes about other kinds of research, so they can draw inspiration and ideas from others. Since psychologists and decision theorists are not as curious about market implications as economists, economists who find the psychology described here inspiring must figure out its market implications, and test them using market data, themselves.

Research directions

We see several fruitful directions for research. Experimental studies which do not directly test a specific theory should contribute to a broader understanding of betting on natural events in a wider variety of conditions where information is missing information. There are diminishing returns to studying urns!

There are many open empirical questions. Valuable contributions could be made by measuring the dependence of ambiguity-aversion on the size of outcomes, the correlation between risk-attitudes and ambiguity-attitudes (adjusted for measurement error), the influence of credit and blame (by oneself or by others) on ambiguity-aversion, the information processing people do when making decisions under ambiguity (e.g., Ross, 1989), or by testing the axioms underlying new theories. Measurement of the parameters or functions posited by different theories is important too because theories can be approximately true (and useful) even if the axioms they are based on are false (see Kahn & Sarin, 1987, 1988; Einhorn & Hogarth, 1985).

Only measurement can tell how good the approximation is. (Such measurement also gives decision analysts an idea of how robust reliance on SEU is.)

Some applications have already been made to a wide variety of topics, including

medicine, law, consumer behavior, finance, and economics. Each of these fields wrestles with questions of individual and collective choice in the face of uncertainty. Many anomalies in these fields might be explained by ambiguity theories, and the anomalies can refine theories in return.

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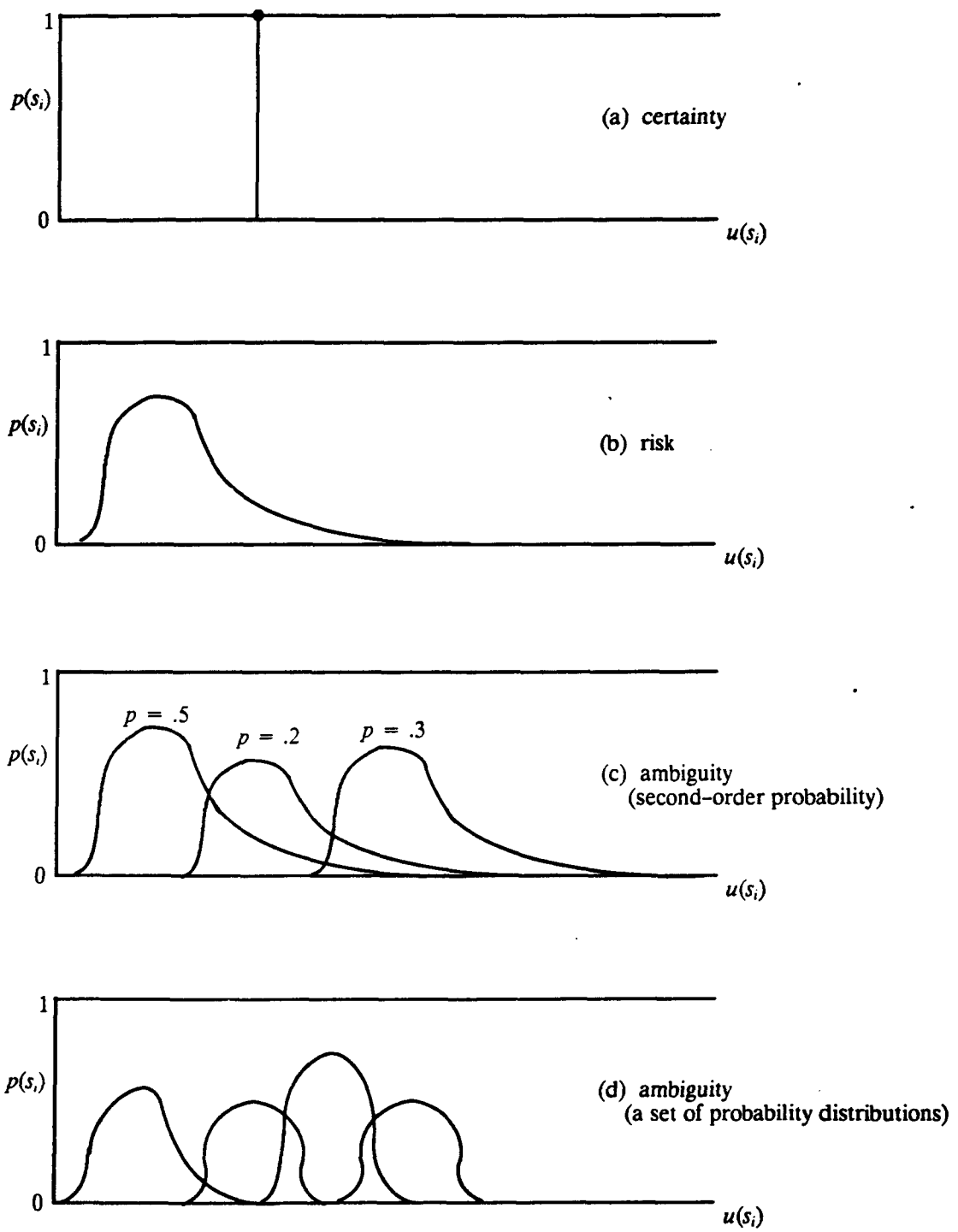

Figure 1 : Distributions under certainty, risk, and ambiguity

Table 1: The three-color Ellsberg problem.

	number of balls		
	30	60	
act	red	black	yellow
X	W	0	0
Y	0	W	0
X'	W	0	W
Y'	0	W	W

Table 2: The two-color Ellsberg problem.

urn 1	number of balls	
	50	50
act	red	black
bet red 1	W	0
bet black 1	0	W

urn 2	number of balls	
	100	
act	red	black
bet red 2	W	0
bet black 2	0	W

Table 4: The three-color Ellsberg problem, revisited.

	number of balls		
	30	60	
act	red	black	yellow
X	W	0	0
Y	0	W	0
Z	0	0	W
X'	.5W	0	.5W
Y'	0	.5W	.5W

Table 3: Stylized Empirical Facts About Ambiguity-Aversion

Stylized fact	Formal definition*	Studies	Comments
1. Replication of Ellsberg	$.5 > (0,1)$	Becker & Brownson (1964), Table 2 Slovic & Tversky (1974), MacCrimmon & Larsson (1979) Einhorn & Hogarth (1986), Table 1 Curley & Yates(1989) Kahn & Sarin(1988)	Ambiguity premium = 70% of EV = 20% of p = 5-10% of EV
2. Immunity to persuasion	$p > (p_L, p_U)$ after written arguments in favor of $p \sim (p_L, p_U)$	Slovic & Tversky (1974) Curley, Yates & Abrams (1986), Table 4	
3. Strict aversion to ambiguity	$p > (p_L, p_U)$ not $p \geq (p_L, p_U)$	Einhorn & Hogarth(1986), Table 1 Curley, Yates & Abrams(1986), Table 2 Cohen, Jaffray & Said (1985) Curley & Yates (1989)	Test by allowing indifference
4. Aversion to SOP	$p > \phi(p)$ $\phi(p)$ uniform $E[\phi(p)] = p$	Yates & Zukowski (1976) Bernasconi & Loomes (in press)	Ambiguity premium = 20% of EV $\geq$ 20% of EV.
5. Aversion to increasing range of probability	$(p_L, p_U) > (q_L, q_U)$ with $p_U - p_L < q_U - q_L$	Becker & Brownson (1964) Yates & Zukowski (1976) Larson (1980) Curley & Yates (1985)	SOP with $\phi(p)$ normal
6. Ambiguity-preference at low probabilities(gains) high probabilities(losses)	$(p_L, p_U) > p$ for $E(p) \leq P_{low}$(gains) $E(p) \geq P_{high}$(losses)	Curley & Yates (1985) Einhorn & Hogarth (1986) Hogarth & Einhorn (1990) Casey & Scholz (in press) Kahn & Sarin (1988) Curley & Yates (1989)	$P_{low} \leq .4$ $P_{low} = .001$ $P_{low} = .10$, $P_{high} = .90$ $P_{high} = .90$ $P_{low} = .1\text{-}.3$, $P_{high} = .7\text{-}.9$ $P_{low} = .25$
7. Independence of risk-attitude and ambiguity attitude	$x < E(x)$ uncorrelated with $p > (p_L, p_U)$	Cohen, Jaffray & Said (1985) Curley, Yates & Abrams (1986), Table 1 Hogarth & Einhorn (1990), p. 797	
8. Less ambiguity-aversion for losses than for gains	Less $p > \phi(p)$ for $x < 0$ than for $x > 0$	Cohen, Jaffray & Said (1985) Einhorn & Hogarth (1986), Table 1 Hogarth & Einhorn (1990), Table 4 Kahn & Sarin (1988) (no gain-loss difference)	

* p denotes a lottery with known probability p. $\phi(p)$ denotes a second-order probability distribution (SOP) of p. (p_L, p_U) denotes a lottery with probabilities bounded by lower and upper values p_L and p_U.

Table 5: Generalized Sure-Thing Principles

Gilboa (1987)

	A	not - A		B	not - B
	x_1	f_1	~	x_1	f_2
	x_2	g_1	~	x_2	g_2
	y_1	f_1	>	y_1	f_2
implies	y_2	g_2	>	y_2	g_2

(for $y_i > x_i$, all acts in each column are pairwise co-monotonic)

Wakker (1989)

A	not - A		A	not - A
α	f	$\leq$	β	g
γ	f	$\geq$	δ	g

B	not - B		B	not - B
α	s	$\geq$	β	t
γ	s	$\geq$	δ	t

implies

(all acts in the top two rows are pairwise co-monotonic, and all acts in the bottom two rows are pairwise co-monotonic)