

Sah, Raaj K.; Stiglitz, Joseph E.

**Working Paper**

## Qualitative Properties of Profit-Maximizing K-out-of-N Systems Subject to Two Kinds of Failures

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ECONOMIC GROWTH CENTER

YALE UNIVERSITY

Box 1987, Yale Station  
New Haven, Connecticut 06520

CENTER DISCUSSION PAPER NO. 559

QUALITATIVE PROPERTIES OF PROFIT-MAXIMIZING K-OUT-OF-N SYSTEMS

SUBJECT TO TWO KINDS OF FAILURES

Raaj Kumar Sah

Yale University

and

Joseph E. Stiglitz

Stanford University

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Abstract:

This paper derives several properties of the optimal  $k$ -out-of- $n$ :G systems where: (i) the i.i.d. components can be, with a pre-specified frequency, in one of two possible modes, (ii) components are subject to failures in each of the two modes, and (iii) the costs of two kinds of system's failures are not necessarily the same. A characterization of the optimal  $k$  which maximizes the system's expected profit is obtained (a special case of this optimization criterion is the maximization of the system's reliability). We show how one can predict, based directly on the parameters of the system, whether the optimal  $k$  is smaller or larger than one-half of  $n$ . Also, the directions of change in the optimal  $k$  resulting from changes in the system's parameters are ascertained.

## 1. INTRODUCTION

A central concern of reliability studies is to help understand how, using unreliable components, better systems can be designed. This paper studies the following problem. The system consists of  $n$  identical and statistically independent components that can be, with a pre-specified frequency, in one of the two possible modes: mode 1 (closed) and mode 2 (open). The components are subject to failures in each mode. Thus, the two types of a component's failures are: failure in mode 1 (failure to close), and failure in mode 2 (failure to open). The system is closed if  $k$  out of  $n$  components are closed. The two types of system's failures, therefore, are: (i) failure to close, which occurs if fewer than  $k$  components close when commanded to close, and (ii) failure to open, which occurs if  $k$  or more components do not open when commanded to open. In general, these two kinds of system failures can have quite different costs. The objective of this paper is to analyze the optimal  $k$  which maximizes the system's expected profit.

Two categories of results are obtained below. First, a characterization of the optimal  $k$  is derived. This characterization is then employed to identify the circumstances under which one can predict, based directly on the parameters of the system, whether the optimal  $k$  is smaller than or larger one-half of  $n$ . The same characterization also predicts the effect on the optimal  $k$  of a change in the costs of the two kinds of system failure.

The second category of results consists of: (i) several propositions concerning the nature of change in the optimal  $k$  from a change in  $n$ ,

and (ii) the effect on optimal  $k$  of changes in the probabilities of a component's failures. The latter set of analytical results are based on an approximation in which the binomial cumulative density is approximated by a normal density. These results are supported with a range of numerical simulations.

A special case of the optimization criterion employed in this paper is one where: (i) the costs of the two kinds of system failure are identical, and (ii) the system is in the two modes with equal frequency. In this case, our optimization criterion becomes the same as maximizing the system's reliability. This special case has been studied by many authors (e.g., Ansell and Bendell [1], Ben-Dov [2], Phillips [3]). By contrast, our analysis deals with cases which may not satisfy the above restrictive conditions.

Among the common examples of the type of systems studied in this paper are relay circuits (subject to failures to energize and failures to de-energize) and monitoring safety systems (subject to failures in detecting a break-in and failures leading to a false alarm). Similar systems are also relevant in the context of economic organizations where managers' judgments (concerning, for instance, the acceptance or rejection of projects or ideas) are subject to fallibility. Sah and Stiglitz ([4], [5]) have studied the economic consequences of such fallibility for the structures of different kinds of organizations. For example, a committee with  $n$  members which approves a project only if  $k$  or more members approve it is analogous to a  $k$ -out-of- $n$ :G system. Further, if  $k$  equals one-half of  $n$ , then the decisions of such a committee are based on commonly observed simple majority rule.

## 2. NOTATIONS

- $p_1$  Probability of a component's success in mode 1 (succeeding to close). Thus, the probability of a component's failure in mode 1 is  $1 - p_1$ .  $1 > p_1 > 0$ .
- $p_2$  Probability of a component's failure in mode 2 (failing to open). Thus, the probability of a component's success in mode 2 is  $1 - p_2$ .  $1 > p_2 > 0$ .  $p_1 > p_2$ .
- $n$  Number of components in the system.  $n > 1$ .
- $h_i(k)$   $\sum_{j=k}^n \binom{n}{j} p_i^j (1 - p_i)^{n-j}$ , where  $i = 1$  and  $2$ .
- $h_1(k)$  Probability of the system's success in mode 1 (succeeding to close). Thus, the probability of the system's failure in mode 1 is  $1 - h_1(k)$ .
- $h_2(k)$  Probability of the system's failure in mode 2 (failing to open). Thus, the probability of the system's success in mode 2 is  $1 - h_2(k)$ .
- $\alpha$  Probability (or frequency) with which the system will be in mode 1. Thus,  $1 - \alpha$  is the probability with which the system will be in mode 2.  $1 > \alpha > 0$ .
- $B^1$  Gain from the system's success in mode 1.
- $B^2$  Gain from the system's failure in mode 1.  $B^1 > B^2$ .
- $B^3$  Gain from the system's success in mode 2.
- $B^4$  Gain from the system's failure in mode 2.  $B^3 > B^4$ .
- $\beta$   $(1 - \alpha)(B^3 - B^4)/\alpha(B^1 - B^2)$ . From above,  $\beta > 0$ .
- $k^*$  Optimal  $k$ . That is,  $k$  which maximizes the system's expected profit.

### 3. ASSUMPTIONS

1. The system consists of  $n$  i.i.d. components.
2. The components are commanded, with a pre-specified frequency, to be in one of the two possible modes: closed and open.
3. In each of the two modes, the components are subject to failures (the failure to close when commanded to close, and the failure to open when commanded to open).
4. The system is closed if  $k$  or more components are closed. Thus, the system is subject to two kinds of failures (the failure to close when commanded to close, and the failure to open when commanded to open). The two kinds of system failures can have different costs.

### 4. GLOBAL PROPERTIES OF OPTIMAL $k$

The criterion for judging the performance of the system is the expected profit. Using the notations defined above, the expected profit is

$$\alpha[B^1 h_1(k) + B^2(1 - h_1(k))] + (1 - \alpha)[B^3(1 - h_2(k)) + B^4 h_2(k)] . \quad (1)$$

This expression is to be maximized with respect to  $k$ . The maximization of (1) is same as the maximization of

$$Y(k) = h_1(k) - \beta h_2(k) . \quad (2)$$

The summary parameter  $\beta$  can thus be viewed as the cost of the system's failure in mode 2, expressed relative to the gain from the system's success in mode 1. A special case of the above formulation is one where  $\beta = 1$ . In this case,  $Y(k)$  is the same as the reliability of the

system,  $h_1 - h_2$ . The optimization of this special case has been studied by Ansell and Bendell [1], Ben-Dov [2] and Phillips [3]. In our more general analysis below, it is straightforward to identify the results corresponding to this special case.

It is shown in the Appendix that  $Y(k)$  is a single-peaked function of  $k$ . Therefore an interior  $k$  (that is, where  $n - 1 \geq k \geq 1$ ) is optimal if and only if it satisfies

$$Y(k) - Y(k - 1) \geq 0, \text{ and} \quad (3)$$

$$Y(k) - Y(k + 1) \geq 0, \text{ with at least one strict inequality.} \quad (4)$$

By the same logic,  $k = 0$  is optimal if and only if

$$Y(0) \geq Y(1); \quad (5)$$

and  $k = n$  is optimal if and only if

$$Y(n) \geq Y(n - 1). \quad (6)$$

Substitution of the definition of  $h_i(k)$  into (3) to (6), and a rearrangement of the resulting expressions yields the following result.

#### Theorem 1

Recalling that  $k^*$  denotes the optimal  $k$ , the necessary and sufficient conditions for the determination of  $k^*$  are as follows:

$$(i) \quad \left( \frac{1 - p_1}{1 - p_2} \right)^{n-k^*} \left( \frac{p_1}{p_2} \right)^{k^*} \geq \beta \geq \left( \frac{1 - p_1}{1 - p_2} \right)^{n-k^*+1} \left( \frac{p_1}{p_2} \right)^{k^*-1},$$

for  $n - 1 \geq k^* \geq 1$ . (7)



$$(ii) \quad \left( \frac{1 - p_1}{1 - p_2} \right)^n \geq \beta, \quad \text{for } k^* = 0, \quad (8)$$

$$(iii) \quad \beta \geq \left( \frac{1 - p_1}{1 - p_2} \right) \left( \frac{p_1}{p_2} \right)^{n-1}, \quad \text{for } k^* = n. \quad (9)$$

One of the uses of the above theorem is that it can help predict certain properties of the magnitude of  $k^*$ , based directly on the values of the parameters  $\beta$ ,  $p_1$  and  $p_2$ . In particular, the following result delineates sufficient conditions under which  $k^*$  is smaller than, larger than, or equal to one-half of  $n$ .

#### Theorem 2

$$(i) \quad k^* < \frac{n}{2} + 1; \quad \text{if } \beta < 1, \quad \text{and } p_2 \leq 1 - p_1. \quad (10)$$

$$(ii) \quad k^* > \frac{n}{2}; \quad \text{if } \beta > 1, \quad \text{and } p_2 \geq 1 - p_1. \quad (11)$$

$$(iii) \quad k^* = \frac{n+1}{2} \quad \text{for odd } n, \quad k^* = \frac{n}{2} \quad \text{or} \quad \frac{n}{2} + 1 \quad \text{for even } n, \\ \text{if } \beta = 1, \quad \text{and } p_2 = 1 - p_1. \quad (12)$$

(See the Appendix for a proof.)

Theorem 1 also yields the following result concerning the effect of  $\beta$  on  $k^*$ .

### Theorem 3

An interior  $k^*$  is non-decreasing in  $\beta$ . If the increase (respectively, decrease) in  $\beta$  is sufficiently large, then an interior  $k^*$  must increase (respectively, decrease).

(See the Appendix for a proof.)

## 5. LOCAL EFFECTS OF THE PARAMETERS ON THE OPTIMAL $k$

This section presents some results concerning how an interior  $k^*$  is altered due to a change in the number of components,  $n$ , or due to a change in the parameters  $p_1$  and  $p_2$ . Here the emphasis is on identifying qualitative aspects of these effects; for example, under what circumstances  $k^*$  increases or decreases if a particular parameter changes. The analysis is based on two analytical simplifications: (i)  $k$  is treated as a continuous variable, and (ii) the binomial cumulative density in the definition of  $h_i(k)$  is approximated by the following normal distribution:

$$h_i(k) = 1 - \phi(z_i) , \quad (13)$$

where  $\phi$  is the unit normal distribution function, and

$$z_i = (k - np_i) / [np_i(1 - p_i)]^{1/2} .$$

As is well known, there are other approximations of a binomial cumulative density which can be more accurate than (13) for particular ranges of parameters. Also, even when (13) provides a satisfactory approximation of  $h_i$ , it might not be the case that the derivatives of  $h_i$  (to be used in the analysis below) are satisfactorily approximated when (13) is used. To support our results, therefore, a range of numerical

simulations is presented at the end of the paper. Moreover, our main objective here is to derive qualitative results concerning the direction of change in  $k^*$  due to a change in parameters. These results are less likely to be sensitive to the approximation than, for instance, the results concerning the magnitude of change in  $k^*$ .

The derivative of  $Y(k)$  with respect to  $k$  is denoted by

$$Y_k(k) = h_{1k}(k) - \beta h_{2k}(k) \quad (14)$$

where  $Y_k(k) = \partial Y(k)/\partial k$ , and  $h_{ik}(k) = \partial h_i(k)/\partial k$  for  $i = 1$  and  $2$ .

Thus, the interior extreme points of  $Y(k)$  are those which satisfy

$$Y_k(k) = 0. \quad (15)$$

Next, it is easily shown that  $Y(k)$  is strictly concave in  $k$  (that is,  $\partial Y_k(k)/\partial k < 0$ ) at any  $k$  which satisfies (15) (for a confirmation, see (A.16) in the Appendix). It follows therefore that, for an interior  $k^*$ , expression (15) represents the necessary and sufficient condition for optimality.

Now if  $\theta$  represents a parameter (that is,  $\theta$  is  $n$ ,  $p_1$ , or  $p_2$ ), then a perturbation in (15) yields

$$\frac{dk^*}{d\theta} = - \frac{\partial Y_k}{\partial \theta} / \frac{\partial Y_k}{\partial k}, \quad (16)$$

where the right hand side of (16) is evaluated at  $k^*$ ; that is, at the value of  $k$  which satisfies (15). Using the above method, we obtain the following expression for  $dk^*/dn$  (see the Appendix for a derivation).

$$\frac{dk^*}{dn} = \frac{1}{2n} \frac{(k^*)^2(1-p_1)(1-p_2) + (n^2 - (k^*)^2)p_1p_2}{k^*(1-p_1)(1-p_2) + (n - k^*)p_1p_2} . \quad (17)$$

The above expression yields several qualitative conclusions concerning the local change in an interior  $k^*$  as  $n$  changes. These are summarized below.

Theorem 4

$$(i) \quad 1 > \frac{dk^*}{dn} > 0 . \quad (18)$$

$$(ii) \quad \frac{dk^*}{dn} > \frac{1}{2} , \quad \text{if } p_2 > 1 - p_1 . \quad (19)$$

(See the Appendix for proofs.)

Part (i) of the above theorem has a clear meaning.  $k^*$  increases if  $n$  increases, but the increase in  $k^*$  is smaller than that in  $n$ . Part (ii) shows that  $dk^*/dn$  is larger or smaller than one-half, depending on whether the probability of a component's failure in mode 2 is larger or smaller than the probability of a component's failure in mode 1.

The final result, stated below, ascertains the direction of local change in  $k^*$  from a change in the probabilities of a component's failure, when these probabilities are the same in the two modes.

Theorem 5

$$\frac{dk^*}{dp_1} > 0 , \quad \text{if } \beta < 1 , \quad \text{and } p_2 = 1 - p_1 . \quad (20)$$

(See the Appendix for a proof.)

This result also has the following implication. Note that (10), (11) and (12) imply, in the present case where  $k$  is treated as a continuous variable, that

$$\frac{k^*}{n} \begin{matrix} < \\ > \end{matrix} \frac{1}{2} ; \text{ if } \beta \begin{matrix} < \\ > \end{matrix} 1 , \text{ and } p_2 = 1 - p_1 . \quad (21)$$

Thus, (20) in combination with (21) implies that  $k^*/n$  becomes closer to  $1/2$  as the probabilities of a component's failure become smaller.

**Numerical Simulations.** The results (18), (19) and (20) are based on the approximation (13). To test the suitability of this approximation for these results, we have undertaken numerical simulations, for a range of parameters, of exact changes in  $k^*$  due to changes in parameters. The exact changes in  $k^*$  are computed as follows. For each combination of parameters, the integer value of  $k^*$  is first calculated directly from (7), (8) and (9). One of the parameters is then altered, and the new integer value of  $k^*$  is similarly calculated.

Our simulations support, within the ranges of parameters we have considered, the results (18), (19) and (20). It should be emphasized, however, that the approximation (13) may not always be suitable for the results under consideration, if the parameters are sufficiently different from those we have considered.

The first set of simulations was for each of the following 540 combinations of parameters:  $n = (25, 45, 65, 85, 105)$  ,  
 $\beta = (0.05, 0.1, 0.75, 1.5)$  ,  $p_1 = (0.3, 0.35, 0.4, 0.45, 0.5, 0.55, 0.6, 0.65, 0.7)$  , and  $p_2 = (p_1 - 0.05, p_1 - 0.1, p_1 - 0.2)$  . In each case, the value of  $n$  was increased by 2 and the resulting increase in the

integer value of  $k^*$  was computed. For all cases for which the pre- and post-change  $k^*$  had an interior value (which was true for 529 out of 540 cases), the change in  $k^*$  was (i) non-negative, (ii) not larger than 2, (iii) not smaller than 1 if  $p_2 \geq 1 - p_1$ , and (iv) not larger than 1 if  $p_2 \leq 1 - p_1$ . These results are consistent with (18) and (19) in which, it will be recalled,  $n$  and  $k^*$  were treated as continuous variables.

The second set of simulations was for each of the following 64 combinations of parameters:  $n = (25, 50, 75, 100)$ ,  $\beta = (0.05, 0.1, 0.75, 1.5)$ ,  $p_1 = (0.55, 0.6, 0.65, 0.7)$  and  $p_2 = 1 - p_1$ . For all cases, the change in  $k^*$  was calculated by increasing  $p_1$  by 0.05. As (20) suggests, this change in  $k^*$  was non-negative for  $\beta \leq 1$ , and non-positive for  $\beta \geq 1$ .

## 6. SUMMARY AND CONCLUSIONS

This paper has examined  $k$ -out-of- $n$  systems constructed from i.i.d. components which experience two kinds of failures. Our aim has been to identify some of the qualitative properties of the optimal  $k$  which maximizes the system's expected profit, subject to the consideration that the costs of the two kinds of system failures are not necessarily the same.

We have obtained a characterization of the optimal  $k$ . This has yielded results which allow a prediction (based directly on the parameters of the system) of certain bounds on the magnitude of the optimal  $k$ . We have also obtained results concerning the directions of change in the optimal  $k$  if the system's parameters change. A sub-class of our formulation and results corresponds to the case examined in the literature in which the optimal  $k$  is chosen to maximize the system's reliability.

## APPENDIX

Proof that  $Y(k)$  is a single-peaked function of  $k$

From the definition of  $h_i(k)$ , we obtain

$$h_i(k) - h_i(k-1) = -\binom{n}{k-1} p_i^{k-1} (1-p_i)^{n-k+1} . \quad (A.1)$$

Expressions (2) and (A.1) yield

$$Y(k) - Y(k-1) = \binom{n}{k-1} \{-p_1^{k-1} (1-p_1)^{n-k+1} + \beta p_2^{k-1} (1-p_2)^{n-k+1}\} . \quad (A.2)$$

$$Y(k+1) - Y(k) = \binom{n}{k} \{-p_1^k (1-p_1)^{n-k} + \beta p_2^k (1-p_2)^{n-k}\} . \quad (A.3)$$

Using (A.3), we can rewrite (A.2) as

$$Y(k) - Y(k-1) = [Y(k+1) - Y(k)] a_1(k) + a_2(k) , \quad \text{where} \quad (A.4)$$

$$a_1(k) = (1-p_1) \binom{n}{k-1} / p_1 \binom{n}{k} > 0 , \quad \text{and}$$

$$a_2(k) = \binom{n}{k-1} \beta p_2^k (1-p_2)^{n-k} \left\{ \frac{1-p_1}{p_1} \left[ \frac{p_1(1-p_2)}{p_2(1-p_1)} - 1 \right] \right\} > 0 .$$

The sign of  $a_1(k)$  is obvious. The sign of  $a_2(k)$  is established by noting that  $p_1(1-p_2)/p_2(1-p_1) > 1$  because  $p_1 > p_2$ .

Expression (A.4) implies that  $Y(k-1) - Y(k-2)$   
 $= [Y(k) - Y(k-1)] a_1(k-1) + a_2(k-1)$ . Since  $a_1(k-1)$  and  $a_2(k-1)$  are both positive, it follows that:

$$\text{If } Y(k) \geq Y(k-1) , \text{ then } Y(k-1) > Y(k-2) . \quad (\text{A.5})$$

Another implication of (A.4) is that  $Y(k+1) - Y(k) = [Y(k+2) - Y(k+1)]a_1(k+1) + a_2(k+1)$  . Since  $a_1(k+1)$  and  $a_2(k+1)$  are both positive, it follows that:

$$\text{If } Y(k) \geq Y(k+1) , \text{ then } Y(k+1) > Y(k+2) . \quad (\text{A.6})$$

Expressions (A.5) and (A.6) show that  $Y(k)$  is single-peaked in  $k$  .

### Proof of Theorem 2

For brevity, we use the symbols:  $q = p_1/p_2$  , and  $r = (1 - p_1)/(1 - p_2)$  . Then, since  $p_1 > p_2$  , it is easily established that

$$rq \begin{matrix} > \\ < \end{matrix} 1 , \text{ if } p_2 \begin{matrix} < \\ > \end{matrix} 1 - p_1 . \quad (\text{A.7})$$

Next, using natural logarithms, expression (7) can be rewritten as

$$\begin{aligned} (n - k^*)\ln(rq) + (2k^* - n)\ln(q) &\geq \ln(\beta) \\ &\geq (n - k^* + 1)\ln(rq) + (2k^* - n - 2)\ln(q) . \end{aligned} \quad (\text{A.8})$$

Now suppose (10) is not true. That is,  $k \geq \frac{n}{2} + 1$  , when  $\beta < 1$  , and  $p_2 \leq 1 - p_1$  . Then, the right hand side of (A.8) is nonnegative because  $rq \geq 1$  from (A.7),  $n - k^* + 1 > 0$  ,  $q > 1$  , and  $(2k^* - n - 2) \geq 0$  . On the other hand,  $\ln(\beta) < 0$  . Expression (A.8) is thus contradicted. An analogous argument shows that (A.8) is contradicted if (11) is not true.

Finally consider the case where  $\beta = 1$  , and  $p_2 = 1 - p_1$  . Then  $rq = 1$  from (A.7). Thus (A.8) becomes



$$(2k^* - n)\ln(q) \geq 0 \geq (2k^* - n - 2)\ln(q) . \quad (\text{A.9})$$

Now since  $\ln(q) > 0$  , it follows from (A.9) that  $k^* = \frac{n+1}{2}$  if  $n$  is odd. If  $n$  is even, then (A.9) is satisfied for either  $k^* = \frac{n}{2}$  , or  $k^* = \frac{n}{2} + 1$  .

### Proof of Theorem 3

Using symbols  $r$  and  $q$  defined in the proof of Theorem 2, expression (7) can be rewritten as

$$(q/r)^{k^*} \geq \beta/r^n \geq (q/r)^{k^*-1} , \quad (\text{A.10})$$

where  $q/r = p_1(1 - p_2)/p_2(1 - p_1) > 1$  because  $p_1 > p_2$  . Now suppose  $\beta$  is changed to  $\bar{\beta}$  such that  $\bar{\beta} > \beta$  . If the corresponding optimal  $k$  is denoted by  $k^{**}$  , then

$$(q/r)^{k^{**}} \geq \bar{\beta}/r^n \geq (q/r)^{k^{**}-1} . \quad (\text{A.11})$$

Since  $\bar{\beta} > \beta$  , the second part of the inequality (A.10) yields:  $\bar{\beta}/r^n > (q/r)^{k^*-1}$  . The preceding expression implies that the first part of the inequality (A.11) will be contradicted if  $k^{**} \leq k^* - 1$  . Therefore  $k^{**} \geq k^*$  . Next, if  $\bar{\beta}$  is sufficiently larger than  $\beta$  (specifically, if  $\bar{\beta}/r^n > (q/r)^{k^*} \geq \beta/r^n$  ) then it is obvious that (A.11) will be satisfied only if  $k^{**} > k^*$  .

Expression for  $dk^*/dn$

For brevity in the derivations below, we suppress the arguments of functions  $Y_k$ ,  $h_{ik}$  and  $z_i$ . Differentiation of (13) with respect to  $k$  yields

$$h_{ik} = -\phi_z(z_i)[np_i(1 - p_i)]^{-1/2} < 0, \quad (\text{A.12})$$

where  $\phi_z(z_i)$  is the unit normal probability density at  $z_i$ , and it is always positive.

Next,  $Y_k$  in (14) can be rewritten as  $Y_k = h_{2k}[(h_{1k}/h_{2k}) - \beta]$ . Thus, the derivative  $\partial Y_k/\partial k$ , evaluated at  $Y_k = 0$ , is

$$\partial Y_k/\partial k = h_{2k} \frac{\partial}{\partial k} (h_{1k}/h_{2k}). \quad (\text{A.13})$$

Using the above expression and (A.12), one obtains

$$\partial Y_k/\partial k = b h_{1k} (p_1 - p_2) / np_1 p_2 (1 - p_1)(1 - p_2), \quad (\text{A.14})$$

where for brevity, we have used the symbol

$$b = k^*(1 - p_1)(1 - p_2) + (n - k^*)p_1 p_2. \quad (\text{A.15})$$

It follows from (A.14) that

$$\partial Y_k/\partial k < 0, \quad (\text{A.16})$$

because  $b > 0$ ,  $h_{1k} < 0$ , and  $p_1 > p_2$ .

Analogous to (A.13), the expression for  $\partial Y_k/\partial \theta$ , evaluated at  $k^*$ , is

$$\partial Y_k / \partial \theta = h_{2k} \frac{\partial}{\partial \theta} (h_{1k} / h_{2k}) . \quad (A.17)$$

If  $\theta$  is  $n$ , then the above expression and (A.12) yield

$$\begin{aligned} \partial Y_k / \partial n = & -h_{1k}(p_1 - p_2)[(k^*)^2(1 - p_1 - p_2) \\ & + n^2 p_1 p_2] / 2n^2 p_1 p_2 (1 - p_1)(1 - p_2) . \end{aligned} \quad (A.18)$$

Substituting (A.14) and (A.18) into (16), and rearranging the resulting expression, one obtains

$$\frac{dk^*}{dn} = [(k^*)^2(1 - p_1)(1 - p_2) + \{n^2 - (k^*)^2\}p_1 p_2] / 2nb . \quad (A.19)$$

Recalling the definition of  $b$  in (A.15), the above expression is the same as (17).

#### Proof of Theorem 4

Expression (A.19) is obviously positive. It also yields

$$1 - \frac{dk^*}{dn} = [k^*(2n - k^*)(1 - p_1)(1 - p_2) + (n - k^*)^2 p_1 p_2] / 2nb > 0 . \quad (A.20)$$

This establishes part (i).

The following expression is obtained from (A.19)

$$\frac{dk^*}{dn} - \frac{1}{2} = k^*(k^* - n)(1 - p_1 - p_2) / 2nb \quad (A.21)$$

Parts (ii) follows immediately from (A.21).

Proof of Theorem 5

Under the assumption that  $p_2 = 1 - p_1$ , (A.12) and (A.17) yield the following derivative for  $\theta = p_1$

$$\partial Y_k / \partial p_1 = -h_{1k}(n - 2k^*)[(1 - p_1)^2 + p_1^2] / 2p_1^2(1 - p_1)^2. \quad (\text{A.22})$$

Using (16), (A.16) and (A.22), it follows that

$$\frac{dk^*}{dp_1} \begin{matrix} > \\ < \end{matrix} 0, \quad \text{if } \frac{k^*}{n} \begin{matrix} < \\ > \end{matrix} \frac{1}{2}. \quad (\text{A.23})$$

Next, since  $k$  is being treated as a continuous variable, (10), (11) and (12) imply

$$\frac{k^*}{n} \begin{matrix} < \\ > \end{matrix} \frac{1}{2}, \quad \text{if } \beta \begin{matrix} < \\ > \end{matrix} 1 \quad \text{and} \quad p_2 = 1 - p_1. \quad (\text{A.24})$$

Combination of the above with (A.23) yields (20).

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