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PRODUCTION FUNCTIONS WITH FACTOR ORIENTED SCALE SENSITIVITY

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PRODUCTION FUNCTIONS WITH FACTOR ORIENTED SCALE SENSITIVITY

Abstract

The analysis of economic phenomena at the wholistic (aggregative) level maintains a long tradition that assumes the neoclassical production function Q=f(K,L) (i.e., output as a function of capital and labor) satisfies the condition of constant returns to scale. The assumed absence of any (dis-)economies of scale renders the production function useless, when the scale effect is as pronounced as is typically found at the less aggregative levels of individual firm or industry analysis.

The purpose of this paper is to deduce new classes of production functions that are not limited to the constant returns to scale characteristic. More specifically, the scale effect is described by an arbitrary function of one of the factors of production, capital in this paper. This class of production functions exhibits scale sensitivity with respect to capital (SSWK).

The paper shows how different families of production functions can be derived from two basic "building blocks," a wage share function and a scale function. The Cobb-Douglas, CES and VES production functions are special cases. The Cobb-Douglas and CES functions can be expanded to incorporate non-constant returns to scale.

A sample of firms from Taiwan is used to test among various derived functional specifications. An interesting diversity of preferred specifications was found among three industries.

PRODUCTION FUNCTIONS WITH FACTOR ORIENTED SCALE SENSITIVITY*

0. Introduction and Summary

The analysis of economic phenomena at the wholistic level (e.g. general equilibrium, income distribution, international trade, and growth theories), maintains a long tradition that assumes the "Neo-Classical" production function $Q = f(K,L)$ (i.e. output as a function of capital and labor) satisfies the condition of constant returns to scale (CRTS). The assumed absence of any economy or diseconomy of scale renders the production function useless when the "scale effect" is as pronounced as is typically found at the less aggregative levels of individual firm or individual industry analysis.[1] The purpose of this paper is to deduce new classes of production functions with non-CRTS.

Intuitively scale effects can be traced either to the size of labor (L) or capital (K). In the celebrated needle factory of Adam Smith, the efficiency of large scale production is brought about mainly by the "division of labor", i.e. functional (or task) specialization rendered possible by a larger labor force (L) using simple tools (K). Rural industries in contemporary less developed countries share this feature of SSWL (scale sensitivity with respect to labor) with industries of pre-industrial days.

A modern factory in an industrially advanced country is rather different. According to Prof. Kuznets, the "modern epoch" is a scientific epoch characterized by the extended application of science to problems of economic production.[2] It is not atypical that a large number of

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engineering principles, drawn from various sciences, is embodied in an industry's capital stock K (e.g. in the refinement of petroleum from crude oil). A firm with a less sophisticated K stock will not be as efficient, nor even as feasible, from the engineering standpoint. This characteristic of production may be referred to as SSWK (scale sensitivity with respect to capital). This paper derives a new family of production functions -- i.e. the SSWK family -- incorporating this characteristic.

When the production function is given, the construction procedure begins with a scale index, or scale function, $s = s(K)$, defined for every point in the input space, such that $s > 1$ ($s=1$, or $s<1$) implies IRTS (CRTS or DRTS) in the ordinary sense. The definition of a SSWK function is that the scale function is a function of the capital stock alone (i.e., $s = s(K)$). The basic theorem, which will be proved in section 2, is that the necessary and sufficient condition for SSWK is that the wage share $\phi_L = f_L L/Q$ is a function of capital per head, $k^* = K/L$. Notice that ϕ_L is the "labor elasticity of output," which becomes the wage share when real wage w is equated with the marginal productivity of labor (f_L) (i.e., $w = f_L$). Under this assumption, both k^* and ϕ_L are statistically observable, and hence, whether or not a production function is SSWK, can be verified empirically.

Let W be the set of all SSWK functions. Many production functions familiar to economists (e.g., the Cobb-Douglas, the CES and VES functions) are homogeneous functions, which, in turn, are members (i.e., special cases) of W. Using group theoretic concepts, we shall show that W can be partitioned, or classified, into subsets Z^i, such that each subset

contains members that are a constant multiple of each other, and they all have the same share function $h(k^*)$ and scale function $s(K)$. Moreover, when the pair $[h(k^*),s(K)]$ is given, we can construct the SSWK function in any subset Z^i (see corollary 3). Using this method, we can construct not only the familiar functions mentioned above, but also <u>new</u> classes of production functions in parametric form.[3]

In addition to scale economies or diseconomies, the interests of the economists in the neoclassical production function naturally center on factor substitutability (e.g., as measured by ε, the elasticity of substitution) or the severity of the law of diminishing returns to labor (e.g., as measured by e_{LL}, the elasticity of f_L with respect to L). Almost all wholistic economic theories (e.g., population pressure, income distribution theory, etc.) emphasize these properties to reach meaningful conclusions. If any preconceived knowledge is postulated for ε and/or e_{LL}, restrictions are imposed on W, i.e., subfamilies of W can be identified. We shall show how to construct the pair $[h(k^*),s(K)]$, and hence, as indicated above, the associated SSWK function, when suitable ε and/or e_{LL} are postulated.

One suitable way to postulate the elasticity of substitution is to assume that it is a function of k^* (i.e., $\varepsilon = \varepsilon(k^*)$), rather than a function of K and L separately. For example, the existing literature has deduced the CES function (i.e., when $\varepsilon'(k^*) = 0$) and the VES function (i.e., when $\varepsilon'(k^*) \neq 0$ with a specific form). (See section 3.1). These specifications are special cases of homogeneous functions. In our paper, we shall deduce stronger results by showing that <u>any</u> function $\varepsilon = \varepsilon(k^*)$ will lead, generally, to a homogeneous function. The only exception is a constant ε (i.e., $\varepsilon'(k^*) = 0$), which

may lead to a particular type of nonhomogeneous functions.[4] In other words, when ε is constant, the production function need not be Cobb-Douglas or CES, but will take on a particular parametric form given in the text. Thus, statistical tests of scale economies under the assumption of constancy of ε are incomplete when Cobb-Douglas or CES are postulated.[5]

In section 3.2, we shall first show, that, for every SSWK function, e_{LL}, measuring the severity of the law of diminishing returns to labor, is a function of k*, i.e., $e_{LL} = \theta(k^*)$. Furthermore, when a suitable specification of $e_{LL} = \theta(k^*)$ is postulated, a share function $h(k^*)$ can be deduced, and hence a subset of W carrying arbitrarily specified scale functions is found. We should note here, that the specification of $\varepsilon = \varepsilon(k^*)$ or $e_{LL} = \theta(k^*)$ provide different entry points in the construction of a production function, but these entry points are not completely independent of each other.

In section 4, we outline the empirical investigation on basis of a particular specification of a SSWK production function. It is an application of the basic idea that an empirically observed functional relation between ϕ_L and k* constitutes inductive evidence that the production function is SSWK. But information on the scale function is necessary as well to determine the exact shape of the production function. While scale s is unobservable, estimating the production function yields estimates of the parameters of the scale function. The methodology of section 4 uses the assumption that random variation of quantity produced Q around a deterministic amount as dictated by the SSWK function is correlated with the random variation of the observed labor share ϕ_L around a deterministic value $h(k^*)$ derived from that SSWK function.

In other words, the production function and the share function must be estimated simultaneously.

This estimation procedure is implemented using data obtained from firms in Taiwan in three different industries: agricultural machinery, electronic equipment, and cotton textile. While the sample size in each industry is not large, and the quality of the data set could be better, the empirical results reported in section 5 are encouraging in regard to the validity of the SSWK methodology explored in this paper. It appears that the production processes of these three industries are characterized by different types of production functions, one of them being non-SSWK and another being a new non-homogeneous specification derived in section 3.2. Nevertheless, these results are illustrative only, and form no solid basis for rigorous conclusions.

We now turn to a technical discussion of the issues above.

1. The SSWK Production Function

Let a production function

(1.1) $Q = f(K,L)$

be given. The partial elasticities of Q with respect to K and L will be denoted by:

(1.2) $\phi_L = \frac{f_L L}{Q} > 0$, and $\phi_K = \frac{f_K K}{Q} > 0$

where $f_L = \frac{\partial Q}{\partial L} > 0$, and $f_K = \frac{\partial Q}{\partial K} > 0$. Let s, defined as:

(1.3) $s = \phi_L + \phi_K > 0$,

be referred to as the scale index, which can be defined for every point in the input space. The basic definition of a SSWK function is that the scale index s is a function of K alone, i.e.,

(1.4) $s = s(K) > 0$,

which will be referred to as the scale function.

To see the meaning of s, let $R_{x,K}$ denote the percentage increase of a variable x, when both K and L increase by one percent, i.e.,

(1.5) $R_{x,K} = \frac{dx}{dK}\frac{K}{x}$ where $L = \lambda K$ for any $\lambda > 0$.

In other words, $R_{x,K}$ is the elasticity of x under a radial expansion of the input space. In this notation, it follows immediately that:

(1.6) a) $R_{Q,K} = s$

b) $R_{p,K} = s - 1$

where $p = \frac{Q}{L}$. Thus, under a radial expansion, output Q and labor productivity p increase by s and s-1 percent respectively, when K and L increase by one percent.

If a production function is SSWK, then:

(1.7) a) $R_{f_L,K} = s - 1$

b) $R_{\phi_L,K} = 0$

(Proof: Partially differentiating $s(K)\, Q = f_K\, K + f_L\, L$ with respect to L yields:

$$s\, f_L = f_{KL}\, K + f_{LL}\, L + f_L\ ,$$

which implies:

$$(1.8) \qquad s - 1 = \frac{f_{LK}\, K}{f_L} + \frac{f_{LL}\, L}{f_L}\ .$$

$$\text{But } R_{f_L,K} = \frac{df_L}{dK}\frac{K}{f_L} = (f_{LK} + f_{LL}\frac{dL}{dK})\frac{K}{f_L} = \frac{f_{LK}\, K}{f_L} + \frac{f_{LL}\, L}{f_L}$$
$$= s - 1.$$

Here we used the fact that $L = \lambda\, K$.

$$R_{\phi_L,K} = R_{\frac{f_L}{p},K} = R_{f_L,K} - R_{p,K} = 0. \qquad \text{Q.E.D)}$$

Thus the impact of a radial expansion on f_L is the same as the impact on labor productivity p. This implies (1.7b), which states that the value of ϕ_L is uniquely determined by capital per head, $k^* = K/L$. In this paper we shall refer to any function X(K,L) as SI (scale insensitive) if it

is homogeneous of degree 0 (i.e., $X(K,L) = X(\lambda K,\lambda L) = V(k^*)$ for any $\lambda > 0$), so that $X(K,L)$ is a function of k^*. Thus:

(1.9) For any SSWK production function, the share function is SI, i.e. $\phi_L = h(k^*)$.

A homogeneous production function of the n-th degree is defined as:

(1.10) $f(\lambda K, \lambda L) = \lambda^n f(K,L)$ for all $\lambda \geq 0$

We have the following lemma:

Lemma 1: A production function is a homogeneous function, if and only if its scale function is a constant n, i.e. if and only if

(1.11) $n = s(K) = \phi_K + \phi_L$

(Proof: "only if" part:

Differentiate (1.10) with respect to λ, and apply definitions (1.2) and (1.3).

"if" part:

For every K and L: $s(K) = n = \phi_K + \phi_L$

or:

$$f_K(K,L)K/f(K,L) + f_L(K,L)L/f(K,L) = n$$

Then for every K, L, λ, the following holds:

$$f_K(\lambda K,\lambda L)\lambda K/f(\lambda K,\lambda L) + f_L(\lambda K,\lambda L)\lambda L/f(\lambda K,\lambda L) = n$$

which can be written as:

$$\lambda \frac{d}{d\lambda}\{\ell n\ f(\lambda K,\lambda L)\} = n$$

Integration yields:

$$\ell n\, f(\lambda K,\lambda L) = n\ell n\, \lambda + \ell n\, A$$

or

$$f(\lambda K,\lambda L) = A\lambda^{n}$$

To find A, set $\lambda = 1$

$$A = f(K,L)$$

Thus $f(\lambda K,\lambda L) = \lambda^{n} f(K,L)$ Q.E.D.)

Lemma 1 shows that every homogeneous fucntion has a constant scale function, $s(K) = n$, a special case of $s = s(K)$. Let W be the set of all SSWK functions and $\mathcal{H}$ be the set of all homogeneous functions. Then Lemma 1 implies Corollary 1:

Corollary 1: All homogeneous functions are SSWK, i.e., $\mathcal{H} \subset W$.

$\mathcal{H}$ contains the well-known (in economic analysis) Cobb-Douglas, CES and VES functions, which are special cases of SSWK functions. The share function of each of these is a function of k*, and their scale functions are constant. Furthermore, any nonhomogeneous function $f \in W \cap \bar{\mathcal{H}}$ has a scale function s(K) which is not a constant (i.e., $s'(K) \neq 0$).[6] In section 3.1, we shall derive nonhomogeneous versions of the Cobb-Douglas and CES functions in $W \cap \bar{\mathcal{H}}$.

2. A Classification of SSWK Functions

Let W be the set of all SSWK functions. Let $\mathcal{F}=\{Z\}$ be a family of non-empty subsets Z of W. $\mathcal{F}$ is a classification of W if

(2.1) a) $W = \bigcup_{Z\in\mathcal{F}} Z$

b) $Z_i \cap Z_j = \emptyset$ for any $Z_i, Z_j \in \mathcal{F}$

In this section we shall show that W can be classified such that all f belonging to the same subset Z not only have the same share function but take on a particular product form:

(2.2) $Q = f(K,L) = C(K)H(K,L)$

This result leads to a method for the construction of SSWK functions in parametric forms (see section 3.1 and 3.2).

In order to classify W, we shall make use of group theoretic concepts. Notice that f(K,L) = 0 and f(K,L) = 1 and f(K,L) = C(K) are all special cases of SSWK functions. This is summarized as:

Lemma 2 : 0, 1 and C(K) are SSWK functions

(Proof: The scale functions of f(K,L) = 0 and f(K,L) = 1 are s(K) = 0, and the scale function of f(K,L) = C(K) is s(K) = C'(K)K/C, which is a function of K. Q.E.D.)

We shall exclude f(K,L) = 0 from W.

W can be considered as a multiplicative Abalien group. Conditions for this to be true are[7]: (i) W must be closed under multiplicative

operation; (ii) W must contain a group identity element, which is $f(K,L) = 1$: (iii) if $f \in W$, its inverse must be an element of W, which is $f^{-1} = \frac{1}{f}$: (iv) for an Abalien (or commutative) group, $f \cdot g = g \cdot f$. Since (ii) and (iv) are satisfied, we only need to prove (i) and (iii). If $f \notin W$, we shall use the notation $(\phi_L^f , \phi_K^f , s^f)$, with a superscript f, to denote its share functions ϕ_L, its ϕ_K and its scale function s. The following lemmas prove (i) and (iii) and thus establish the fact that W is an Abalien multiplicative group.

Lemma 3: W is closed under multiplicative operation, i.e., if $f,g \in W$ then $f \cdot g \in W$. Moreever:

(2.3) a) $\phi_L^{fg} = \phi_L^f + \phi_L^g$

b) $\phi_K^{fg} = \phi_K^f + \phi_K^g$

c) $s^{fg} = s^f + s^g$

(Proof: $\frac{\partial fg}{\partial L} = g \frac{\partial f}{\partial L} + f \frac{\partial g}{\partial L}$

So: $\phi_L^{fg} = (g \frac{\partial f}{\partial L} + f \frac{\partial g}{\partial L}) \frac{L}{fg} = \phi_L^f + \phi_L^g$

This proves (2.3a). The proof for (2.3b) is similar. (2.3c) follows from (2.3a) and (2.3b) by definition (1.3). Since s^f and s^g are functions of K, their sum is a function of K, and the function fg is SSWK and a member of W. Q.E.D.)

Lemma 4: For any $f \in W$, the inverse $f^{-1} \in W$. Moreover:

(2.4) a) $\phi_L^{f^{-1}} = - \phi_L^f$

b) $\phi_K^{f^{-1}} = - \phi_K^f$

c) $s^{f^{-1}} = - s^f$

(Proof: $\frac{\partial f^{-1}}{\partial L} = -f^{-2} \frac{\partial f}{\partial L}$, from which (2.4a) follows. Similar for (2.4b). (2.4c) follows from these by definition (1.3). Q.E.D.)

Lemmas 2, 3 and 4 imply that W is a multiplicative group.[8]

Next, we seek a suitable classification $\mathcal{F}_B$ of W. If B is any subgroup of W (i.e. $B \subset W$ and B is a group), then for any $f \in W$, the coset determined by f relative to B is defined as:

$$(2.5) \quad Z_B(f) = \{f \cdot g \mid g \in B\} \subset W$$

Note that $Z_B(f) \subset W$. Let $\mathcal{F}_B$ be the set of all cosets determined by B. It is well-known,[9] that W is partitioned (or classified) by $\mathcal{F}_B$, i.e.

$$(2.6) \quad W = \bigcup_{Z \in \mathcal{F}_B} Z$$

Thus every subgroup B induces a classification of W, where a coset represents a particular class for this classification.

As an application, let $B = \{C(K)\}$ be the set of all functions of K. A member C(K) of B can be considered a production function. The economic interpretation of $Q = C(K) \in B$ is that of a production function in a "labor surplus" economy, where output depends only on the capital stock. For example, the well-known Harrod-Domar production function $Q = \frac{1}{k} K$ is a member of B, where k is a constant capital-output ratio. Lemma 5 establishes the fact that $B = \{C(K)\}$ is a subgroup of W:

Lemma 5: B is a subgroup of W. Moreover, if $C(K) \in B$:

$$(2.7) \text{ a)} \quad \phi_L^C = 0$$

$$\text{b)} \quad \phi_K^C = \frac{dC}{dK} \frac{K}{C}$$

$$\text{c)} \quad s = \frac{dC}{dK} \frac{K}{C}$$

(Proof: The proof of (2.7a) to (2.7c) is trivial. That B is a subgroup of W follows readily from: (i) $f(K,L) = 1 \in B$;

(ii) for any $C(K) \in B$, $C(K)^{-1} \in B$; (iii) for $C(K), D(K) \in B$, $C(K) \cdot D(K) \in B$. Q.E.D.)

Thus B induces a classification $\mathcal{F}_B$ of W in the sense of (2.6). If $f = H(K,L)$ is any member in a coset $Z_B(H)$, all the production functions in the coset $Z_B(H)$ take on the form:

(2.8) $\quad Q = C(K)\, H(K,L)$

where $H(K,L) \in Z_B(H)$, and $C(K) \in B$.

(2.7a) and (2.3a) imply Lemma 6:

Lemma 6: All SSWK production functions f in the same coset $Z_B(H)$ have the same share function $\phi_L^f = \phi_L^H$.

Now we can formulate the basic theorem of this section.

Theorem 1: If $Q = f(K,L)$ has a share function which is SI (i.e., $\phi_L^f = h(k^*)$), then:

(2.9) a) $f(K,L) \in W$, i.e., f is a SSWK function,

and if, in addition, the scale function of $f(K,L)$ is $S(K)$, then $f(K,L)$ takes on the form:

b) $Q = f(K,L) = C(K) \cdot H(K,L)$

where $H(K,L) \in W$ and can be calculated from $h(k^*)$ by:

c) $H(K,L) = e^R$ with $R = \int \frac{h(k^*)}{L}\, dL$

and $C(K)$ is a solution of the following differential equation:

d) $\frac{dC}{dK}\frac{K}{C} = s(K) - h(k^*) - K\frac{\partial R}{\partial K}$

(Proof: Rewrite the expression $\phi_L = h(k^*)$ as:

$$\frac{dQ}{Q} = \frac{h(k^*)}{L}\, dL$$

Integrate both sides while treating K as a constant:

$$\ell n\ Q = \ell n\ C(K) + R$$

or: $Q = f(K,L) = C(K)\cdot H(K,L)$,

where R and H(K,L) are as defined in (2.9c). Since $C(K) \in W$, $f \in W$ if $H \in W$, by Lemma 3. That $H \in W$, follows from:

$$s^H = \phi_L^H + \phi_K^H = h(k^*) + K\frac{\partial R}{\partial K}$$

so: $$\frac{\partial s^H}{\partial L} = \frac{\partial h(k^*)}{\partial L} + K\frac{\partial(\partial R/\partial L)}{\partial K}$$

$$= -\frac{h' K}{L^2} + K\frac{\partial(h(k^*)/L)}{\partial K} = 0.$$

So $f \in W$. (2.9d) follows, using (2.3c), (1,3) and (2.7c), from:

$$s(K) = s^C + s^H = \frac{dC}{dK}\frac{K}{C} + h(k^*) + K\frac{\partial R}{\partial K} \qquad \text{Q.E.D.)}$$

The following corollaries follow:

Corollary 2: $Q = f(K,L)$ is SSWK, if and only if its share function is SI.

(Proof: by (1.9) and Theorem 1)

Notice that, ideally, k^*, capital per head, and ϕ_L, wage share, are observable.[10] If there exists a high correlation between observed values of k^* (rather than K and L separately) and ϕ_L (e.g., across firms, industries, or regions), then there is a strong presumption that a production function is a member of the SSWK family.[11]

Corollary 3: Two SSWK functions f, g belong to the same coset Z_B, if and only if they have the same share function (i.e., $\phi_L^f = \phi_L^g$).

(Proof: Lemma 6 implies the "only if" statement. The "if" statement is implied by (2.9b), for f and g can be written as $f = C(K)H$ and $g = D(K)H$, where $C(K), D(K) \notin B$. Thus f and g belong to the same coset. Q.E.D.)

Let $\mathcal{T} = \{h(k^*)\}$ be the set of all functions of k^*. Then corollary 3 implies that there is a one to one correspondence between $\mathcal{F}_B$ and $\mathcal{T}$.[12] Thus, every coset is characterized by a distinct share function.

Theorem 1 suggests a computational procedure of SSWK functions of a particular parametric form from the share function and the scale function, i.e. from the pair $[h(k^*), s(K)]$, as summarized by the following corollary. In this corollary, $B = \{s(K)\}$ is now interpreted as the set of all scale functions while $\mathcal{T} = \{h(k^*)\}$ is the set of all share functions.

Corollary 4: From the pair $[h(k^*), s(K)]$ (i.e. $s(K) \notin B$, $h(k^*) \notin \mathcal{T}$) a particular SSWK function $Q = C(K)H(K,L)$ can be constructed from equations (2.9c) and (2.9d), which is unique up to a multiplicative constant.

$\mathcal{F}_B$ classifies W according to the set B of functions of K and thus leads to cosets that all have unique share functions. A still finer classification is possible. Consider the set Γ of constants, excluding 0. Γ is a subgroup of W. Similar to our discussion related to B above, Γ induces a classification of W into cosets Z_i, where functions in each cosets Z_i have the same pair $[h(k^*), s(K)]$ and differ only by a multiplicative constant. These Z_i cosets are subsets of the cosets under the $\mathcal{F}_B$ classification. Corollary 4 indicates how the production functions in each coset Z_i are computed.

This computational procedure is applied in section 3.1 and 3.2, where we consider production functions that have particular characteristics in relation to the elasticity of substitution between K and L, and the strength (severity) of the law of diminishing returns. As we shall see, these characteristics pose certain restrictions on the form of the functions $h(k^*)$ and $s(K)$ which, by corollary 4, determine the parametric form of the production function. Using the same methodology one can derive still other subfamilies of SSWK functions.

3. Two Applications

In the previous section it was shown that a particular SSWK function can be constructed from the pair $[h(k^*), s(K)]$. In this section we consider production functions that have particular characteristics in relation to the elasticity of substitution between K and L, and the strength (severity) of the law of diminishing returns to labor. These characteristics pose certain restrictions on the form of the functions $h(k^*)$ and $s(K)$, which, by corollary 3, determine the parametric form of the production functions. Using the same methodology one can derive still other subfamilies of SSWK functions.

Given a neoclassical production function $f(K,L)$, economists are interested in its behavior for three types of variations in the input space (see figure 1). Starting from point E, a radial movement (arrow 1)

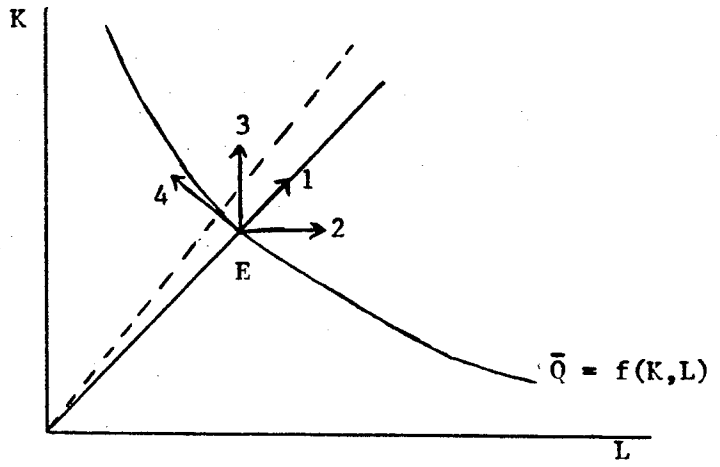


Figure 1

focuses on the scale (dis-)economy as measured by the scale index s. For a horizontal (vertical) movement of arrow 2 (3), the interest is on the law of diminishing returns to labor (capital), the strength of which is measured by $e_{LL}(e_{KK})$, which is defined in section 3.2. A pivotal variation, i.e.

a variation of k* (arrow 4), focuses on factor substitutability as measured by the elasticity of substitution ε. In short, scale economy, the laws of diminishing returns, and factor substitutability are the major engineering characteristics that have a bearing on all the social-economic problems envisioned by the neoclassicists when a production function f(K,L) is postulated. Our purpose in this section is to construct SSWK functions, of particular parametric forms, when certain "desired" properties are postulated for ε or e_{LL}. Needless to say, what is "desired," can only be a matter of econometric usefulness and/or analytical convenience, a full justification of which is beyond the scope of this paper.

The following schedule, containing certain definitions, is a classification device with 14 cells (indexed by (A), (B),..., (N) for convenience

	(1)				(2) Desired SSWK function	
	Index	Characteristic of Index			Homogeneous	Non-homogeneous
(1)	ε	SI	SI-C	ε = 1	CD_H (A)	CD_{NH} (C)
				ε ≠ 1	CES_H (B)	CES_{NH} (D)
			SI-V		(E) Ex: VES_1 VES_2 $CEDR_H$ $VEDR_H$	(F)
		non-SI			(G)	(H) Ex: ACD_{NH} $CEDR_{NH}$ $VEDR_{NH}$
(2)	e_{LL}	SI	SI-C		$CEDR_H$ (I)	$CEDR_{NH}$ (J)
			SI-V		Ex: $VEDR_H$ (K)	Ex: $VEDR_{NH}$ (L)
		non-SI			(M)	(N)

Schedule 1

of reference), that provides a guideline of the analysis in this section. Column 1 describes the index, ϵ and e_{LL}, while column 2 distinguishes homogeneous from nonhomogeneous specifications.

Let us start with ϵ, the elasticity of substitution, in row 1. In the first column, ϵ may be specified to be SI (scale insensitive) or non-SI (scale sensitive), as defined in section 1. Intuitively, the former implies the simplifying assumption that factor substitutability is the same for large and small production units, and hence the scale of production is irrelevant for factor substitutability. Economists routinely take advantage of this simplifying assumption when they work with Cobb-Douglas, CES or VES functions. These familiar functions will naturally enter into our analysis (cells (A) through (F)), but we shall also construct production functions, for which ϵ is non-SI. There is a whole family of production functions; we can only give some examples in cell (H), one of which is given the name Augmented Cobb-Douglas (ACD_{NH}), and we prove that the entire family consists of nonhomogeneous functions, i.e., cell (G) is empty (see Lemma 7 below). Moreover, we derive a condition which an arbitrary specification $\epsilon = \epsilon(K,L)$ must satisfy in order to yield a SSWK function in cell (H).

In this section, we shall refer to f(K,L) as SSI (substitution scale insensitive) when ϵ is SI. The family of SSI functions includes two classes: a SSI-C class for which ϵ is constant, and a SSI-V class for which ϵ is variable. The first (SSI-C) class contains both homogeneous and nonhomogeneous Cobb-Douglas and CES functions (cells (A) to (D)).[13] The second (SSI-V) class is more complex, because the simplifying assumption of the constancy of ϵ is dropped. We prove that this class contains only homogeneous functions (cell (F) is empty), and we give some examples,

one of which is the familiar VES function, in cell (E).

The same construction procedure and classification device is applied when "desired" properties are specified for e_{LL}, which measures the strength of the law of diminishing returns to labor (row 2 of Schedule 1). To our knowledge, this is a new exercise in economics, which is worthwhile in view of the fact that probably more social-economic problems (e.g., population pressure and determination of rent) are directly traceable to this "law," which, in any case, has a much longer history than factor substitutability. We shall prove that e_{LL} is SI for all SSWK functions (Lemma 11), so that cells (M) and (N) are empty. Two examples are presented, one where the law of diminishing returns operates at a constant strength (e_{LL} is constant, CEDR in the schedule), and one where it varies (VEDR), in cells (I) to (L). Notice that these functions also enter in cells (E) and (H), testifying to an interdependence between ε and e_{LL}, which, however, we do not explore further.

3.1 Elasticity of Substitution

In this section, the relationship between the concept of elasticity of substitution (ε) and the pair $[h(k^*), s(K)]$ will be investigated. ε measures the substitutability between K and L. Let $m = f_L/f_K$. Then, in this paper, ε is defined as:[14]

$$(3.1) \qquad \varepsilon = \frac{dm}{dk^*}\frac{k^*}{m} \qquad \text{for } Q = \bar{Q} \text{ or } dK/dL = -m$$

We have the following lemma:

Lemma 7: ε is SI for all homogeneous functions.

(Proof: Write $m = \phi_L k^*/\phi_K$. Since homogeneity implies SSWK by Corollary 1, $\phi_L = h(k^*)$. Then $\phi_K = s - \phi_L = n - h(k^*)$ implies that ϕ_K is also a function of k^*. Thus, m is a function of k^* and hence ε is SI by (3.1). Q.E.D.)

This proves that cell (G) in Schedule 1 is empty.

Within the class of SSWK production functions, the relation between ε and the pair $[h(k^*), s(K)]$ can be expressed in terms of the share elasticity (i.e., $\varepsilon_h = \frac{dh}{dk^*}\frac{k^*}{h}$), and the scale elasticity (i.e., $\varepsilon_s = \frac{ds}{dK}\frac{K}{s}$):

$$\varepsilon_h = \frac{\phi_L}{s}\varepsilon_s + \frac{\phi_K}{s}(\varepsilon - 1) \tag{3.2}$$

The proof of (3.2) is somewhat lengthy and is given in Appendix A. For a homogeneous function, $s(K) = n$ and $\varepsilon_s = 0$, so that $\varepsilon_h = \phi_K(\varepsilon - 1)/n$. For this special case, in a capital deepening process, where k^* increases, the labor (capital) share ϕ_L (ϕ_K) increases (decreases) when $\varepsilon > 1$.[15] However, for nonhomogeneous SSWK functions, the value of ε cannot unambiguously determine the direction of the change of these shares, as (3.2) contains an additional term, involving ε_s. Whether ε_s is positive (i.e., rising scale economies) or negative (i.e., falling scale economies), apparently makes a difference in income distribution theory that needs to be explored. (3.2) points out that the effect of capital deepening during a growth process contains a scale economies effect and a substitution effect on the functional income distribution.

Equation (3.2) is used for the following basic theorem of this section on the important class of SSI functions, which is proved in Appendix B.

Theorem 2: Within the class of SSWK functions, if $f \in W$ is SSI (i.e., substitution scale insensitive, so that $\varepsilon = \varepsilon(k^*)$), then:

(i) if f is SSI-V, then f is a homogeneous function (i.e., $f \in \mathcal{H}$), with a general form (2.9b), where C(K) and H(K,L) are constructed from the pair

(3.3)a) $$h(k^*) = n\ A \exp(J_0)/(1 + A \exp(J_0))$$

b) $$s(K) = n$$

where A is an integration constant, and where

c) $$J_0 = \int \{(\varepsilon(k^*) - 1)/k^*\}\ dk^*$$

using Corollary 4,

(ii) if f is SSI-C, then four specification are possible:

	Homogeneous production function	Non-homogeneous production function
$\varepsilon = 1$	(A) $\phi_L = n - \alpha = \frac{1}{A_1}$ $s = n$ $Q = AK^{\alpha}L^{n-\alpha}$	(C) $\phi_L = \frac{1}{A_1 - b\ell n\ k^*}$ $s = \frac{1}{A_4 - b\ \ell n\ K}$ $Q = A\left\{\frac{A_1 - b\ell n\ k^*}{A_4 - b\ell n\ K}\right\}^{1/b}$
$\varepsilon \neq 1$	(B) $\phi_L = \frac{(1-\varepsilon)/b}{1 + (k^*)^{1-\varepsilon}/A_2}$ $s = (1-\varepsilon)/b = n$ $Q = A\{L^{1-\varepsilon} + \frac{1}{A_2}K^{1-\varepsilon}\}^{\frac{n}{1-\varepsilon}}$	(D) $\phi_L = \frac{(1-\varepsilon)/b}{1 - A_1^*(k^*)^{1-\varepsilon}}$ $s = \frac{(1-\varepsilon)/b}{1 - A_2^* K^{1-\varepsilon}}$ $Q = A\left\{\frac{k^{*\varepsilon-1} - A_1^*}{K^{\varepsilon-1} - A_2^*}\right\}^{1/b}$

Notice that theorem 2i implies, that cell (F) in Schedule 1 is empty. Furthermore, the functions formulated in the above schedule are the homogeneous and nonhomogeneous Cobb-Douglas and CES functions indicated in cells (A) to (D) in Schedule 1.

The construction procedure of Theorem 2i allows us to construct a large number of homogeneous SSWK functions when the elasticity of substitution is suitably specified (but not constant), so that the integration

problem in (3.3c) can be solved. The following are merely examples:

$$(3.4)a) \quad \varepsilon = 1 + \frac{A_1}{1 + A_2 k^{*A_1}}$$

$$b) \quad \phi_L = \frac{\delta}{1+ \beta k^{*-A_1}}$$

$$c) \quad Q = AK^{n-\delta}L^{\delta}(1+ \beta k^{*-A_1})^{-\frac{\delta}{A_1}}$$

where, for ease of notation, $A_2 = (n-\delta)/n\beta$.

$$(3.5)a) \quad \varepsilon = \frac{\gamma}{\beta k^{*}+\gamma}$$

$$b) \quad \phi_L = \frac{\delta}{1+\beta k^{*}}$$

$$c) \quad Q = AK^{n-\delta}L^{\delta}(1+\beta k^{*})^{\delta}$$

where $\gamma = (n-\delta)/n$

The specification in (3.4c), labeled VES_1 in this paper, is the familiar VES function (Lu and Fletcher, 1968). (3.5c) is called VES_2. On closer examination, VES_2 is a special case of VES_1, by setting $A_1 = -1$ in (3.4), but we shall see why VES_2 is of special interest.

Theorem 2 is "constructive" in the sense that an SSWK production function can be constructed when the elasticity of substitution is arbitrarily specified as a function of k*, i.e., $\varepsilon = \varepsilon(k^*)$. The question still arises whether a SSWK function can be derived from a non-SI specification of ε, i.e., $\varepsilon = \varepsilon(K,L)$. Lemma 7 already shows that the SSWK function must be nonhomogeneous. The following lemma states the restriction on the function $\varepsilon(K,L)$ necessary to derive a SSWK function from the pair $[\varepsilon(K,L), s(K)]$.

Lemma 8: If $f(K,L)$ is a given SSWK function and its elasticity of substitution $\varepsilon = \varepsilon(K,L)$ is non-SI, the function $\varepsilon(K,L)$ must be such that ϕ_L is SI, where ϕ_L equals:

(3.6)a) $$\phi_L = \left[\frac{1}{s} + \left(\frac{\epsilon_s}{L}(\epsilon_s - \epsilon + 1) - \frac{k^*}{s}\frac{\partial \epsilon_s}{\partial K}\right) \Big/ \left(\frac{\partial \epsilon}{\partial L} + k^* \frac{\partial \epsilon}{\partial K}\right)\right]^{-1}$$

or in elasticity form:

b) $$\phi_L = s\left[1 + \frac{\epsilon_s}{\epsilon(\eta_K + \eta_L)}\left(s(\epsilon_s - \epsilon + 1) - \eta_{\epsilon_s}\right)\right]^{-1}$$

where:

c) $$\eta_K = \frac{\partial \epsilon}{\partial K}\frac{K}{\epsilon}, \quad \eta_L = \frac{\partial \epsilon}{\partial L}\frac{L}{\epsilon}, \quad \eta_{\epsilon_s} = \frac{\partial \epsilon_s}{\partial K}\frac{K}{\epsilon_s}$$

(Proof: If f is SSWK, then $\phi_L = h(k^*)$ is SI, so ϵ_h is SI. Then differentiate (3.2) with respect to K and L, realizing that $\partial \epsilon_h / \partial L = -k^* \partial \epsilon_h / \partial K$. Solving for ϕ_L gives (3.6a). Q.E.D.)

We see that when $\epsilon(K,L)$ satisfies the restriction, (3.6) immediately yields a solution for $h(k^*)$, from which, combined with $s(K)$, a production function can be constructed, by Corollary 4. Thus, Lemma 8 characterizes the family of SSWK functions for which ϵ is non-SI, cell (H) in Schedule 1.

We close this section with two econometric notes. Theorem 2ii implies that a constant elasticity of substitution and homogeneity are two distinct, separately testable assumptions. While not a new conclusion, it is significant, nonetheless, in the light of much empirical research (e.g., some studies cited in footnote 1). When ϵ is constant, and perfectly competitive input markets are assumed, one can integrate (3.1) to obtain:

(3.7) $$\ell n\, k^* = \frac{1}{\epsilon}\ell n\left(\frac{w}{r}\right) - \frac{1}{\epsilon}\ell n\, A_2$$

If (3.7) is estimated and the value of $\frac{1}{\epsilon}$ is found significantly different from unity, that is not evidence that CES_H is obtained, but rather, at least within SSWK, that some member of SSWK is found with $\epsilon \neq 1$, which

may or may not be homogeneous. For example, testing for IRTS under the assumption that CES_H is obtained (e.g. Griliches and Ringstad, 1970) may lead to false conclusions.

As a final note, VES_1 reduces to CES_H when $n = \delta$. VES_2 reduces to CD_H if $\beta = 0$. A straightforward generalization of VES_2 is obtained by specifying a variable scale function (3.8a) and retaining the share function (3.5b):

(3.8) a) $$s(K) = \frac{n}{1+ \alpha K}$$

b) $$\phi_L = \frac{\delta}{1+ \beta k^*}$$

c) $$Q = AK^{n-\delta}L^{\delta}(1+\beta k^*)^{\delta}(1+\alpha K)^{-n}$$

This specification, which is non-SSI, is an alternative in testing the relative importance of the homogeneity and $\epsilon = 1$ assumptions for the often used Cobb-Douglas form (and hence is called ACD_{NH}): if α is more significantly different from 0 than β, the homogeneity assumption is shown to be more restrictive than the $\epsilon = 1$ assumption in the CD_H specification, and vice versa.

3.2 Diminishing Returns to Labor

A second application of the method to construct SSWK functions relate to the "law of diminishing returns" to labor. The potential severity of this law is essential to most production-related social issues, such as population pressure, and pressure on wages and interest rates. A measure of the severity of the law can be developed as follows.

For a general production function $Q = f(K,L)$, a number of reasonable properties are usually postulated for the elasticities defined in (3.9a):

$$(3.9)\ a)\quad \begin{pmatrix} e_{LL} & e_{LK} \\ e_{KL} & e_{KK} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_L}{\partial L}\frac{L}{f_L} & \frac{\partial f_L}{\partial K}\frac{K}{f_L} \\ \frac{\partial f_K}{\partial L}\frac{L}{f_K} & -\frac{\partial f_K}{\partial K}\frac{K}{f_K} \end{pmatrix} = \begin{pmatrix} -\frac{f_{LL}\,L}{f_L} & \frac{f_{LK}\,K}{f_L} \\ \frac{f_{KL}\,L}{f_K} & -\frac{f_{KK}\,K}{f_K} \end{pmatrix}$$

b) $e_{LL} > 0$ if $f_{LL} < 0$

c) $e_{KL} = e_{LK}\,\phi_L/\phi_K > 0$ if $f_{KL} = f_{LK} > 0$

d) $e_{KK} > 0$ if $f_{KK} < 0$

$e_{LL} > 0$ is the labor elasticity of f_L, depicting the severity of the law of diminishing returns to labor. In (3.9c), e_{KL} and e_{LK} are positive, depicting the laws of factor complementarity.

Using (3.9), we obtain the following lemma.

Lemma 9: For any SSWK function f, i.e., $f \in W$:

$$(3.10)\qquad s - 1 = e_{LK} - e_{LL}$$

(Proof: see (1.8) and the definitions of e_{LK} and e_{LL}. Q.E.D.)

Thus, under SSWK, the case of IRTS ($s > 1$) is assured by the fact that the law of complementarity overwhelms the law of diminishing returns to labor ($e_{LK} > e_{LL}$).

To investigate the behavior of $h(k^*)$, let ε_h denote the elasticity of the share function, as before in equation (3.2). Then:

Lemma 10 For any SSWK function:

$$(3.11)\qquad \varepsilon_h = \phi_L + e_{LL} - 1 \qquad \text{where} \quad \varepsilon_h = (dh/dk^*)k^*/h$$

(Proof: Differentiating $\ln \phi_L = \ln f_L + \ln L - \ln Q$

yields: $d\phi_L/\phi_L = df_L/f_L + dL/L - dQ/Q$

But: $df_L/f_L = -e_{LL}\,dL/L + e_{LK}\,dK/K$

$dQ/Q = \phi_L\,dL/L + \phi_K\,dK/K$

$$dk^* = d(K/L) = (LdK - KdL)/L^2$$

or $$dL/L - dK/K = - dk^*/k^*$$

Thus

$$d\phi_L/\phi_L = (1 - e_{LL} - \phi_L)(dL/L) + (e_{LK} - \phi_K)(dK/K)$$
$$= (1 - e_{LL} - \phi_L)(dL/L - dK/K)$$

by (3.10) and (1.3). So:

$$d\phi_L/\phi_L = - (1 - e_{LL} - \phi_L)dk^*/k^* \qquad \text{Q.E.D.)}$$

Lemma 10 implies that the share function increases with k^* if $\phi_L > 1 - e_{LL}$, and decreases if $\phi_L < 1 - e_{LL}$. In the former (latter) case, the wage share increases (decreases in a capital deepening process. In view of Lemmas 9 and 10, e_{LL}, as a measure of the severity of the law of diminishing returns to labor, is a crucial characteristic of th SSWK family.

Notice from (3.11), that e_{LL} is SI, because both ε_h and ϕ_L are SI. This is summarized in Lemma 11:

Lemma 11: For any SSWK function, the labor elasticity of the marginal product of labor (f_L) is SI, i.e., $e_{LL} = \theta(k^*)$.

Therefore, in the SSWK family, e_{LL} and ϕ_L determine each other. Moreover, Lemma 11 implies that cells (M) and (N) of Schedule 1 are empty.

Now for a given $e_{LL} = \theta(k^*)$, $\phi_L = h(k^*)$ is the solution to the differential equation (3.11). Then Corollary 3 and 2 imply that $e_{LL} = \theta(k^*)$ determines a coset carrying an arbitrary scale function associated with $\theta(k^*)$. The solution to (3.11) is:[16]

$$(3.12) \qquad \phi_L = -N \{ \int (\frac{N}{k^*})\, dk^* \}^{-1}$$

where $$N = \exp \{ \int \frac{\theta(k^*)-1}{k^*}\, dk^* \}$$

Many SSWK functions in particular parametric form can be generated. Here we shall give two examples. The first is the SSWK function with constant $e_{LL} = \theta$, called the Constant Elasticity of Diminishing Returns (CEDR) function. It is given in (3.13):

(3.13)a) $e_{LL} = \theta$

b) $$\phi_L = \frac{1 - \theta}{1 + B\, k^{*1-\theta}}$$

c) $$Q = A\, e^{J}\, (k^{*\theta-1} + B)$$

where $J = \int \frac{s(K)}{K}\, dK$. There are restrictions on the parameter domain: a if $0 < \theta < 1$ and $B > 0$, then $0 \leq \phi_L \leq 1-\theta$; b if $0 < \theta < 1$ and $B < 0$, then $1-\theta \leq \phi_L \leq 1$ and $k^* \leq (-\frac{\theta}{B})^{(1/1-\theta)}$; and c if $\theta > 1$, then $B < 0$ and $k^* \leq (-\frac{\theta}{B})^{(1/1-\theta)}$. The scale function can be chosen freely; if $s(K) = n$ and $B = 0$, CEDR reduces to CD_H. One can easily check that ε is not a function of k^* unless $s(K) = n$.

The second example is intentionally chosen to link up with CES_{NH}. Now, e_{LL} varies with k^*, and hence this function is called VEDR:

(3.14)a) $$e_{LL} = \frac{1 - \varepsilon - A_2}{1 + A_1\, k^{*1-\varepsilon}} + \varepsilon$$

b) $$\phi_L = \frac{A_2}{1 + A_1\, k^{*1-\varepsilon}}$$

c) $$Q = A\, e^{J}\, (k^{*\varepsilon-1} + A_1)^{A_2/1-\varepsilon}$$

A more general form than (3.14a) seems to present difficulties when solving the integrals in (3.12). Again, one may generate homogeneous as well as nonhomogeneous functions from (3.14c) by specifying appropriate scale

functions. If one chooses $s(K) = A_5(1 + A_4 K^{1-\delta})^{-1}$, the production function, called $VEDR_{NH}$, becomes:

$$(3.14)c') \quad Q = A\,(k^{*\varepsilon-1} + A_1)^{A_2/1-\varepsilon}\,(K^{\delta-1} + A_4)^{-A_5/1-\delta}$$

If one restricts $A_2 = A_5$ and $\varepsilon = \delta$, CES_{NH} returns. Thus, one can test whether the particular scale function necessary in obtaining a constant ε (see box in Theorem 2ii) is restrictive in a statistical sense.

4. Empirical Specification

So far, we have derived analytical expressions for the production function under alternative assumptions. The next step is to subject these specifications to an empirical investigation. When ε is constant, four specifications can be tested against each other, namely the homogeneous and non-homogeneous CD and CES functions. As seen in the previous section, the difference between the most elaborate function, CES_{NH}, and the simplest function, CD_H, is two parameters.

ε need not be constant, however. When ε varies with k* (i.e., is SI-V), the scale function s(K) is constant, according to Theorem 2i, and examples of resulting production functions are VES_1 and VES_2. Looking at characteristics of e_{LL}, we found the CEDR and VEDR specifications. All these specifications are linked together by simple parameter restrictions. This section puts forth a framework that allows testing the statistical significance of these parameters. The maintained assumption is that the production function is one within the class of SSWK functions, which can be challenged, of course, at the cost of more elaborate specifications.

At the start of empirical analysis, we are faced with the question why observations do not follow one of the specifications perfectly. Since we consider the production function as a product of building blocks, the share and the scale function, a logical approach would be to assume random variation around the value of these functions. But scale (i.e., s) is an unobservable variable. Quantity produced on the other hand is observable, and its random variation around a deterministic value f

will depend in part on random variation in the share of labor and the scale of production. In this way, we are led to the behavioral model of Zellner, Kmenta and Drèze (1966) with the assumptions that the entrepreneur maximizes expected profits at a time that random variation in production is still unknown, and that the realized demand for labor contains managerial errors due to inertia, ignorance, etc.

To be more precise, let u_1 and u_2' enter exponentially into the production relation and the marginal-product-of-labor relation respectively:

$$(4.1) \quad Q = f\, e^{u_1}$$

$$(4.2) \quad w = f_L e^{u_2'}$$

Equation (4.2) can be written as a share equation:

$$(4.2)' \quad \phi_L = wL/Q = f_L L e^{u_2'}/Q = (f_L L/f) e^{u_2' - u_1} = h e^{u_2}$$

Let us take the logarithms of (4.1) and (4.2)':

$$(4.3) \quad \ell n Q = \ell n f + u_1$$

$$(4.4) \quad \ell n \phi_L = \ell n h + u_2$$

The wage share is a variable in the interval $[0,1]$. Thus $-\infty \leq \ell n \phi_L \leq 0$. This imposes a restriction on u_2:

$$(4.5) \quad u_2 \leq - \ell n h$$

As suggested above, the errors u_1 and u_2 will be correlated. We assume that (u_1, u_2) are jointly normally distributed with mean $(0,0)$ and covariance matrix Σ where:

$$(4.6) \quad \Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}$$

Using these assumptions, it is straightforward to write the log-likelihood function $\mathcal{L}$ as:

$$(4.7)\quad \mathcal{L} = \sum_{i=1}^{n} \ell n\, g(u_{1i}, u_{2i}; \Sigma) - \sum_{i=1}^{n} \ell n\, \Phi(-\sigma_{22}^{-\frac{1}{2}}\, \ell n h_i)$$

where $u_{1i} = \ell n Q_i - \ell n f_i$; $u_{2i} = \ell n \phi_{Li} - \ell n h_i$; g is a bivariate normal density function with mean (0,0) and covariance matrix Σ; and Φ is the standard normal cumulative distribution function. The second sum in (4.7) represents the truncation on u_{2i} as given in (4.5).

Maximum likelihood estimation appears the most suitable estimation technique, in the face of the nonlinearity in the parameters, the cross-equation parameter restrictions evident in every specification, and the distributional assumption (4.6). This allows straightforward testing of restrictions on parameters: if hypothesis H_1 leads to a log-likelihood value $\mathcal{L}(H_1)$, and if hypothesis H_2 restricts one parameter compared to H_1 and gives a log-likelihood value $\mathcal{L}(H_2)$, then $\lambda = -2(\mathcal{L}(H_2) - \mathcal{L}(H_1))$ is distributed (asymptotically) as $\chi^2(1)$, and significance of λ can thus be tested.

5. Empirical Results

5.1 Data Description

We now turn to a discussion of empirical results obtained from estimating the SSWK production functions constructed in the previous sections. For this estimation, we have used a set of data collected by the Census Bureau of the government in Taiwan during 1981. From this Census material, the largest three industries were chosen: agricultural machinery (industrial code DN=00), electronic equipment (DN=11) and cotton textile (DN=22). Variables used for this study are: value added (=Q), total wage cost (=wL), value of fixed assets (=K), and number of employees (=L).

The main disadvantage of these data is a problem in the measurement of wage cost and value added. For a number of firms, total wage cost exceeded value added, so that the wage share exceeded unity. For others, value added was negative.[17] Our interpretation is that presumably wage cost is overstated and/or value added is understated, as firms would have an incentive to do so in view of the highly competitive environment in Taiwan (for fear of taxation agencies and information sharing with competitors). Observations with a measured wage share outside the interval from 0 to 1 are therefore excluded from the sample, but nevertheless one may doubt the quality of the remaining data. While the results reported here are mainly meant as an illustration, they must be interpreted with caution.

In table 1, some summary statistics are given. In terms of mean value added, the electronic equipment industry operates on the largest scale, followed by cotton textile, and agricultural machinery. The same ranking would appear, when we look at the average number of employees

per firm. The last row shows the capital-labor ratio for the "average firm" in each industry. It proves that cotton textile is the most capital-intensive industry, followed by agricultural machinery. Electronic equipment is the most labor-intensive. Finally, one may note that wage shares are roughly comparable across industries.

5.2 Estimation Results

The first step in the empirical investigation of SSWK production functions is to see whether the share function in each industry is SI, i.e., requirement (1.9). Table 2 reports on the estimated relationships between ϕ_L and k*. Linear and log-linear specifications lead to the same results, so only linear regressions are discussed here. The regression gives poor results for DN=00, and somewhat better for DN=11 and DN=22. To check for scale insensitivity, one may add K, L and Q to the regression. F-statistics show the joint significance of these variables: adding K and L yields a better fit for DN=00, but not for DN=11 and DN=22, and adding Q is significantly better in each case.

The latter result is not surprising in view of the definition of ϕ_L as wL/Q. Moreover, when random errors of ϕ_L and Q (i.e., u_2 and u_1) are strongly correlated, as we shall see to be the case, the significance of Q in these regressions may be caused merely by simultaneity of the variables. The lack of significance of K and L in the regressions is thus a better indicator that the production function is SSWK. Therefore we conclude that the production processes in the electronic equipment and cotton textile industries are SSWK, while that in agricultural machinery is not.

The specifications given in section 3 were estimated for the SSWK industries DN=11 and DN=22. As emphasized before, most of the specifications are nested within each other. Rather than listing results of all variants, we compare in Table 3 the values of the log-likelihood functions. In addition to the values, this table lists the number of parameters, including the covariance matrix Σ, for each specification, and shows arrows pointing toward the preferred specification according to the likelihood ratio test between pairs of specifications.

A few notes are in order. First, the $CEDR_{NH}$ function uses a scale function that is linear in K. Nevertheless, the homogeneity assumption of CD_H appears not as objectionable as the $\epsilon=1$ assumption, in view of the acceptance of $CEDR_{NH}$ and VES_2 over CD_H and the rejection of ACD_{NH} in favor of VES_2. Second, CD_{NH} is somewhat better than CD_H for DN=11, but for DN=22 nonlinearity presents problems for estimation, due to the fact that b decreases steadily toward 0 during iterative steps of the maximization routine and thus $\frac{1}{b}$, which appears in the exponent of Q (see Theorem 2ii), increases to ∞. We have assumed, that b=0 yields the maximum, i.e., CD_{NH} reduces to CD_H.

Third, the CES assumption, embodied in CES_H and CES_{NH}, is accepted for DN=22. For this industry, there is no evidence in favor of a varying elasticity of substitution, nor of scale being a function of K. CES_H is most preferred. Fourth, the elasticity of substitution is definitely variable of DN=11. VES_1 and $VEDR_{NH}$ are both preferred over CES_H and CES_{NH}, suggesting, moreover, that scale s(K) may be constant ($s'(K) = 0$) rather than variable ($s'(K) \neq 0$).

Table 4 reports the parameter estimates of these preferred specifications. For the meaning of each parameter, one is referred back to

section 3; similar parameters in different columns have different meaning. The first column shows results of the VES_1 specification for DN=11. Significant economies of scale exist: n equals 1.0374 and is significantly different from 1. The elasticity of substitution $\varepsilon(k^*)$ increases with k^*. It rises from .548 for $k^* = 38$ (=.5 $\bar{k}^*$ for this industry) to .604 for $k^* = 152$ (= 2 $\bar{k}^*$). Note that since $\varepsilon(k^*)$ is the inverse of the common definition of the elasticity of substitution, capital and labor are relatively good substitutes for each other. As can be expected in such case, the wage share ϕ_L falls with the capital-labor ratio; its elasticity ε_h at the mean of $\bar{k}^* = 76$ equals -.205.

The second column of Table 4 shows $VEDR_{NH}$ results for DN=11. Scale is now a function of K, and is found to decrease with K. For the smallest firms (K=1000), s(K) equals 1.050, while for the average firm ($\bar{K}$ = 129361) s(K) equals 1.036, which is remarkably close to the estimate of n in the VES_1 specification. CRTS is reached at K = 903156. Only a few firms in the electronic equipment industry operate at DRTS; for the largest, s(K) equals .888. This shape of the scale function implies a U-shaped long run average cost function. The elasticity of substitution is now a complicated function of K and L, but it can easily be calculated from equation (3.2): from $\varepsilon_h = -.213$, $\phi_L = .529$, and $\varepsilon_s = -.009$ follows $\varepsilon = .574$. Notice that these numbers correlate well with the VES_1 results, so that the two sets of estimates are quite comparable.

The third column considers estimates for DN=22. The scale function is constant, at a value of $(1-\varepsilon)/b = 1.058$. A test of CRTS involves setting b equal to $1-\varepsilon$. This restriction is rejected at the .5 percent significance level by these data for this industry. The elasticity of substitution ε is quite small, indicating a "regular" elasticity of

$1/\varepsilon = 4.097$, which implies good capital-labor substitutability. Accordingly, the wage share falls during a capital-deepening process.

Finally, the estimates of the covariance matrix Σ in all three columns reveal a high degree of correlation between the errors in the two equations, u_1 and u_2. The correlation coefficients vary between -.850 and -.900. This substantiates our claim that one should not test for SSWK by including Q in the regression (see Table 2).[18]

6. Concluding Remarks

What motivated our research for the SSWK-function in general and the specified functions in particular are certain basic issues in disaggregate (i.e. individual firm and individual industry) production analysis. On the one hand, the familiar U-shaped long and/or short run average cost curves testify to a long tradition of non-CRTS study. On the other hand, the "law of diminishing returns" is a basic "law", the severity of which underlines the gravity of most production related social issues (e.g. population pressure, and pressures on interest rates or wage rates).

A systematic analysis into the characteristics lead to an identification of basic elements of the specification of any production function, namely the scale function and the share function. Several commonly used specifications fitted in this categorization, but those specifications all exhibited CRTS. Two of them, the Cobb-Douglas and the Constant-Elasticity-of-Substitution (CES) specifications, can be expanded within the SSWK framework. This study is not limited to these specifications, however. Suitable functional forms for the scale and share functions have yielded other specifications that are both parsimonious in their parameters and rich in the variety of characteristics.

An empirical investigation with firm level data from Taiwan showed interesting diversity in the characterization of the production processes among the three industries considered. One should view these results as not much more than an illustration of the richness of the SSWK class of production functions. We look forward to using more suitable data in order to continue this promising avenue of research.

Footnotes

(1) CRTS was rejected by Berndt and Khaled (1979) using time series data for the U.S. manufacturing as a whole from 1947 to 1971. Lopez (1980) rejected the implications of CRTS for systems of input demand equations for Canadian agriculture, with time series data from 1946 to 1977. Early cross-sectional studies surveyed by Walters (1963) often indicated more or less constant returns, with the exception of Klein (1974), who found large significant IRTS for the 1936 U.S. railway industry. CRTS was rejected by Griliches and Ringstad (1970) for manufacturing as a whole, as well as many individual industries, from steel to diary, in Norway in the mid 1960's. Lovell (1973) and Christensen and Greene (1976) reject the hypothesis of homotheticity, which includes CRTS, for the U.S. transportation equipment and the U.S. power industry, both leading to U-shaped average cost curves. Lau and Tamura (1972) also reject homotheticity for the Japanese petreochemical industry and find that increasing returns to scale are concentrated in the labor input.

(2) Kuznets (1966, p. 9).

(3) Arrow et al. (1961) define the class of CAP (capital-augmenting production) functions, and search for the CES function. The VES function (Lu and Fletcher, 1968) may be a generalization of the CES function, but is not a member of the CAP family. In our paper, we search for special members in the family W of SSWK functions.

(4) These non-homogeneous functions are the only members within the SSWK class with the characteristic that ε is constant. They are special cases of the most general class of CES functions examined by Sato (1975, 1977).

(5) Early tests of scale economies employed Cobb-Douglas specifications. Later on, CES specifications were used. Recently, several flexible functional forms, such as translog, were estimated and tested for scale economies and homotheticity. Such flexible forms generally do not have particular properties regarding ε and/or e_{LL}. Examples of these three methods are given in footnote 1.

(6) $\bar{\mathcal{H}}$ is the complement of $\mathcal{H}$ and consists of all nonhomogeneous functions. Then $W \cap \bar{\mathcal{H}}$ is the set of all nonhomogeneous SSWK functions.

(7) See Birkhoff and MacLane (1950, p. 130).

(8) It becomes clear at this point, that some members of W do not have economic significance, e.g., those with negative share functions. When conducting an economic analysis, such members should be excluded, as they do not satisfy basic conditions for production functions.

(9) See Birkhoff and MacLane (1950, p. 146). The cosets Z_B are disjoint: they do not have elements in common.

(10) Since we do not assume constant returns to scale, the equality of wage rates with value of marginal product prevails only in the short run.

(11) As mentioned in section 0, the same method was employed by Arrow et al. (1961) to establish the empirical validity of the capital-augmenting production functions, of which CES is a special case, when wage rates show a high correlation with labor productivity ($p = Q/L$).

(12) One can readily show that the multiplicative quotient group W/B is isomorphic to the additive group $\mathcal{T}$. See Birkhoff and MacLane (1950, p. 158).

(13) Thus Sato (1975, 1977) studies the family of SSI-C functions.

(14) The common definition of the elasticity of substitution is the inverse of ε as defined in (3.1).

(15) We emphasize again that ε is the inverse of the regularly defined elasticity of substitution. So if $d\varepsilon/dk^* > 0$ (<0), then as capital intensity ($k^*=K/L$) increases, substitutability between capital and labor decreases (increases). The fixed proportions production function would be characterized by $\varepsilon = \infty$, the perfect substitutability production function by $\varepsilon = 0$.

(16) Equation (3.11) is the so-called Bernouilli's equation. See Boyce and Dippine (1967).

(17) Such problems were most severe in the cotton textile industry. Frequency distributions in the three industries were as follows:

	DN=00	DN=11	DN=22	Total %
$\phi_L \leq 0$	16	4	43	10.2
$0 < \phi_L \leq 1$	83	144	145	60.2
$\phi_L > 1$	22	15	146	29.6

(18) For the interested reader, we note that CD_{NH} was the preferred specification for DN=00, the industry we have excluded from further discussion, after we found no evidence in favor of the SSWK characteristic. The estimates were: A=264.5218 (2.61), A_1=1.3143 (15.98), A_4=27.9913 (12.96), b=.0486 (8.42), σ_{11}=.3898 (5.17), σ_{12}=-.3973 (-5.61), and σ_{22}=.5028 (6.44), with a log-likelihood value of -45.919.

Table 1

Definitions and Descriptive Statistics

Industry code:[a]		DN=00	DN=11	DN=22
Q = value added (NT$)	mean	10143	281105	131774
	st.dev.	30090	1396508	311076
ϕ_L = wage share in value added	mean	.5910	.5194	.5690
	st.dev.	.2178	.2045	.2336
K = value of fixed assets (NT$)	mean	9914	129361	203414
	st.dev.	35226	594482	664804
L = number of employees	mean	38.9	851	551
	st.dev.	82.4	2524	1434
k^* = capital-labor ratio	mean	174.9	76.0	192.8
	st.dev.	144.9	54.2	163.5
Number of observations		83	144	145

Note: [a]The industrial codes stand for the following industries:
DN=00 : agricultural machinery
DN=11 : electronic equipment
DN=22 : cotton textile

Table 2

Analysis of SSWK relationship[a]

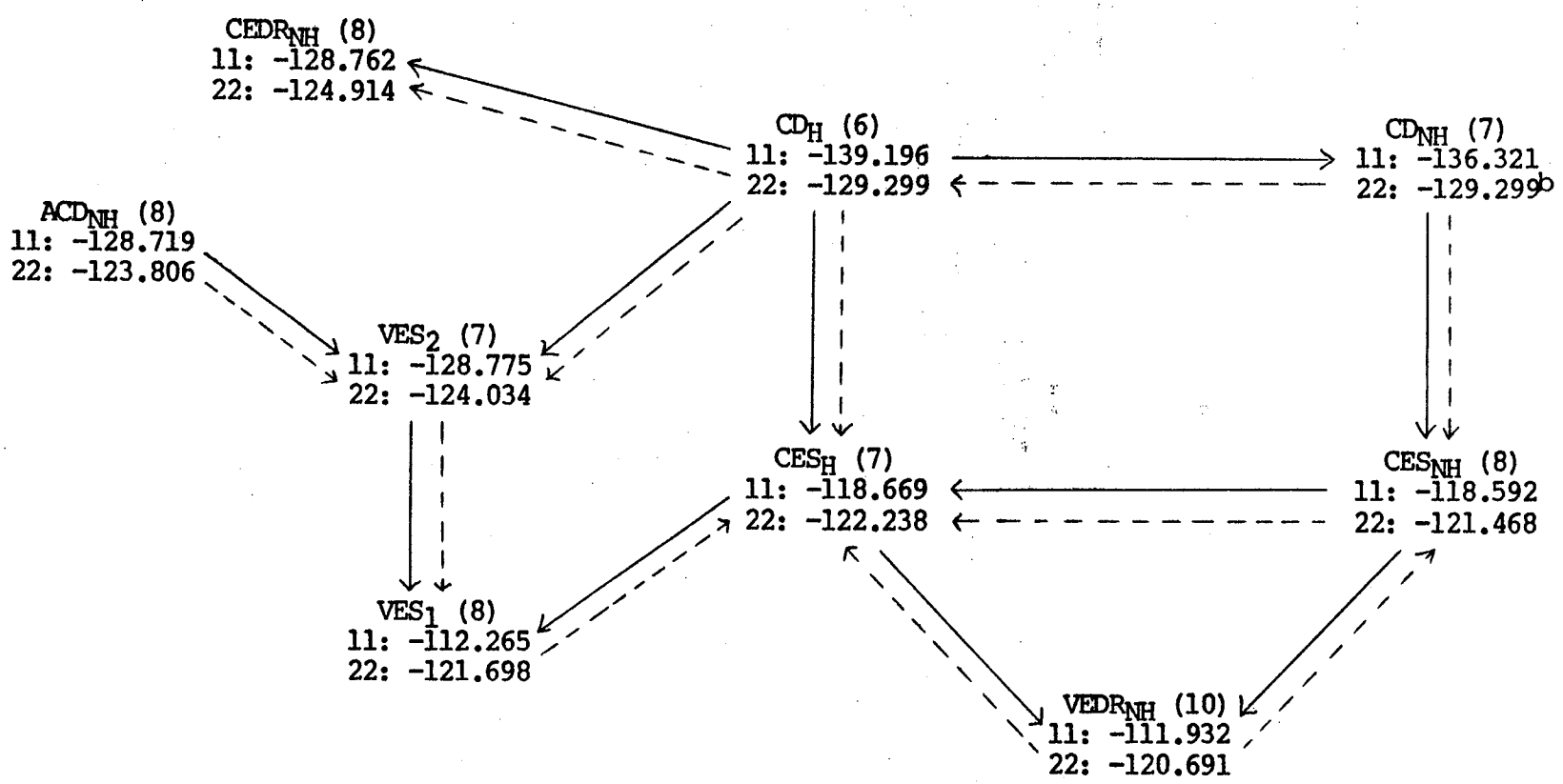
Linear Regression	DN=00	DN=11	DN=22
Intercept	.614	.611	.643
	(9.08)	(15.19)	(14.80)
K/L (*10^3)	-.061	-1.507	-.342
	(-0.09)	(-1.68)	(-0.89)
$(K/L)^2$ (10^6)	-.292	2.646	-.135
	(-.022)	(0.65)	(-0.22)
R^2	.012	.068	.088
F - equation	0.47	5.14 *	6.87 *
F - add K, L	3.60 *	.51	1.58
F - add K, L, Q	6.34 *	4.80 *	9.05 *

Log-Linear Regression	DN=00	DN=11	DN=22
Intercept	-.635	-.460	-.380
	(-1.50)	(-3.75)	(-2.43)
ln (K/L) (*10)	.987	.093	.408
	(0.49)	(0.21)	(0.92)
$(\ln (K/L))^2$ (*10)	-.183	-.191	-.195
	(-0.74)	(-2.27)	(-2.82)
R^2	.017	.065	.069
F - equation	0.68	4.93 *	5.18 *
F - add ln L	17.34 *	1.23	1.03
F - add ln L, ln Q	59.12 *	111.06 *	159.46

Note: [a]t-statistics in parentheses
F-values marked with * are significant at 5 percent level

Table 3

Comparison of Log-Likelihood Values[a]



Notes: [a]The first value under each specification acronym is the log-likelihood value for DN=11, the second refers to DN=22. The number next to the acronym is the number of parameters estimated under this specification. The solid arrow points to the preferred specification for DN=11, the dashed arrow does so for DN=22.

[b]The value for CD$_{NH}$ appears to be the same as for CD$_{H}$ for DN=22. See discussion in the text.

Table 4

Preferred Parameter Estimates for DN=11 and DN=22[a]

DN=11				DN=22	
VES_1		$VEDR_{NH}$[b]		CES_H	
A	8.2550	A	8.8187	A	52.2074
	(0.76)		(3.53)		(9.10)
n	1.0374	A_1	.3942	A_2	.0097
	(59.93)		(6.21)		(1.63)
δ	1.4364	A_2^*	4.2326	b	.7147
	(6.98)		(8.26)		(6.85)
β	.4197	ε	.6617	ε	.2441
	(1.87)		(19.88)		(2.07)
A_1	-.3251	A_4	$.47*10^{-5}$		
	(-3.93)		(0.78)		
		A_5^*	1.5521		
			(43.65)		
		δ	.3232		
			(0.99)		
σ_{11}	.3466	σ_{11}	.3490	σ_{11}	.5716
	(6.43)		(6.62)		(6.02)
σ_{12}	-.2827	σ_{12}	-.2848	σ_{12}	-.4689
	(-5.50)		(-5.80)		(-5.62)
σ_{22}	.3193	σ_{22}	.3206	σ_{22}	.4749
	(5.83)		(6.18)		(5.89)
$\mathcal{L}$	-112.265	$\mathcal{L}$	-111.932	$\mathcal{L}$	-122.238
N	144	N	144	N	145

Note: [a]Asymptotic t-statistics in parentheses

[b]$A_2^* = A_2/1-\varepsilon$; $A_5^* = A_5/1-\delta$

Appendix A: Proof of equation (3.2)

Let us define an operator E as

(A.1) $E(z) = \frac{\partial Z}{\partial k^*}\frac{k^*}{Z}$ subject to $\frac{dK}{dL} = -\frac{f_L}{f_K}$

In other words, E(z) is the elasticity of z with respect to k* subject to the condition that the total output is constant. The following relations are generally valid.

(A.2) a) $E(\phi_L) = E(s) + \mu_K(\varepsilon-1)$

b) $E(\phi_K) = E(s) - \mu_L(\varepsilon-1)$

c) where $\mu_K = \frac{\phi_K}{s}$, $\mu_L = \frac{\phi_L}{s}$ and $E(f_L/f_K) = \varepsilon = E(m)$, $s = \phi_L + \phi_K$

(Proof: $E((\phi_K/\phi_L)+1) = E(s/\phi_L) = E(s) - E(\phi_L)$

on the other hand

$E((\phi_K/\phi_L)+1) = \frac{\phi_K}{s} E(\phi_K/\phi_L) = -\mu_K E(\frac{f_L}{f_K}/k^*) = -\mu_K(\varepsilon-1)$.

Thus

$E(s) - E(\phi_L) = -\mu_K(\varepsilon-1)$

or $E(\phi_L) = E(s) + \mu_K(\varepsilon-1)$

The proof of (A.2b) is similar. Q.E.D.)

Equation (3.2) is valid for SSWK functions only. So s = s(K) and therefore:

$$E(s) = \frac{\partial s(K)}{\partial k^*}\frac{k^*}{s} = \frac{\partial s(K)}{\partial K}\frac{K}{s}\frac{\partial K}{\partial k^*}\frac{k^*}{K} = \varepsilon_s E(K)$$

Since: $dk^* = \frac{1}{L} dK - \frac{K}{L^2} dL$

and: $dK = -(f_L/f_K)\, dL = -m\, dL$

we have: $E(K) = \frac{1}{\frac{1}{L} + \frac{K}{mL^2}} \cdot \frac{K/L}{K} = \frac{\phi_L}{s} = \mu_L$

Since $E(\phi_L) = \varepsilon_h$ for SSWK functions, this completes the proof of equation (3.2).

Appendix B: Proof of Theorem 2.

Proof of (i):

By applying equation (3.2) in the text

$$\varepsilon_h = \mu_L \varepsilon_s + \mu_K(\varepsilon - 1) = \frac{\phi_L}{s}(\varepsilon_s - \varepsilon + 1) + (\varepsilon - 1)$$

we have

(B.1) $$\varepsilon_h/\phi_L + (1-\varepsilon)/\phi_L = \frac{1}{s}(\varepsilon_s - \varepsilon + 1)$$

Define $f(k^*) = \varepsilon_h/\phi_L + (1-\varepsilon)/\phi_L$

Differentiate equation (B.1) with respect to L:

$$\partial f(k^*)/\partial L = \frac{\partial f(k^*)}{\partial k^*}\frac{\partial k^*}{\partial L} = f'(k^*)(-K/L^2)$$

$$= -\frac{1}{s}\frac{\partial \varepsilon}{\partial k^*}\frac{\partial k^*}{\partial L} = \frac{1}{s}\frac{\partial \varepsilon}{\partial k^*}\left(\frac{K}{L^2}\right)$$

Which is

(B.2) $$f'(k^*) = -\frac{1}{s}\frac{\partial \varepsilon}{\partial k^*}$$

Differentiate equation (B.1) with respect to K:

(B.3) $$f'(k^*)/L = \partial(\varepsilon_s/s)/\partial K + \partial(1/s)/\partial K - \partial(\varepsilon/s)/\partial K$$

where

$$\frac{\partial(\varepsilon/s)}{\partial K} = \frac{1}{s}\frac{\partial \varepsilon}{\partial k^*}\frac{1}{L} - \varepsilon\frac{\partial s}{\partial K}\frac{1}{s^2}$$

By substituting (B.2) into (B.3) we can get

(B.4) $$\partial(\varepsilon_s/s)/\partial K + \partial(1/s)/\partial K + (\varepsilon/s^2)(\partial s/\partial K) = 0$$

Since (B.4) is a differential equation in K; ε is one of the "parameters" of the differential equation, and so will enter into the solution. The solution of s thus contains $\varepsilon = \varepsilon(k^*)$. The function is thus not SSWK. We thus conclude that: "if $\varepsilon = \varepsilon(k^*)$, then (B.4) is true if and only if $\partial s/\partial K = 0$".

So we have proved: "If $\varepsilon = \varepsilon(k^*)$ with $\partial\varepsilon(k^*)/\partial k^* \neq 0$, and the production function is SSWK, then s(K) is a constant i.e. s(K) = n".

To get a general production form we need to know $h(k^*)$, because as equation (2.9) in the text indicates, a production function can be constructed through $[h(k^*), s(K)]$. By equation (B.1), we have

$$\frac{dh}{dk^*}\frac{k^*}{h} = \frac{n-h}{h}(\varepsilon-1)$$

which is

$$\frac{n}{h(n-h)}\,dh = \frac{\varepsilon-1}{k^*}\,dk^*$$

Integrate

$$\int \left(\frac{1}{n-h} + \frac{1}{h}\right) dh = \int \frac{\varepsilon-1}{k^*}\,dk^*$$

we can get

$$h/(n-h) = AJ_1$$

where $J_1 = e^{J_0}$, and $J_0 = \int((\varepsilon(k^*) - 1)/k^*)dk^*$, $h = nAJ_1/(1+AJ_1)$

Since $\varepsilon(k^*)$ does not have explicit form, a general production form is constructed.

$$Q = C(K)H(K,L)$$

where $H(K,L) = e^R$, $R = \int (h(k^*)/L)dL$ and C(K) is found through the equation:

$$s(K) = n = h + K(\partial R/\partial K) + (dC/dK)(K/C)$$

Proof (ii):

When ε is constant (i.e. $\partial\varepsilon/\partial k^* = 0$), equation (B.2) is

(B.2)' $\quad f'(k^*) = 0$

It implies that $f(k^*) = b$ is a constant. Therefore (B.1) can be re-expressed as $sb = \varepsilon_s - \varepsilon + 1$, i.e.

(B.5) $\quad \varepsilon_s = (\partial s/\partial K)(K/s) = sb + \varepsilon - 1$

According to the property of s (i.e. constant or not), there are two possible solutions.

Case 1: $(\partial s/\partial K)(K/s) \neq 0$

Equation (B.5) can be re-expressed as

(B.6) $\quad (1/(s(sb-1+\varepsilon))ds = dK/K$

(i) $\varepsilon \neq 1$

Since $b = \varepsilon_h/h + (1 - \varepsilon)/h$, as the definition indicates in the beginning of this proof, we have

$$(\partial h/\partial k^*)(k^*/h) = bh - 1 + \varepsilon$$

which is

$$dh/((bh-1+\varepsilon)h) = dk^*/k^*$$

i.e.

$$\frac{1}{(\varepsilon-1)}\left(\frac{1}{h} - \frac{1}{bh-1+\varepsilon}\right)dh = dk^*/k^*$$

By integration we have

$$\ell n\, h - \ell n(bh+\varepsilon-1) = (\varepsilon-1)\ell n\, k^* + \ell n A_1$$

which is

$$h/(bh+\varepsilon-1) = A_1 k^{*\,\varepsilon-1}$$

The share function is thus obtained,

(B.7) $\quad h = ((1-\varepsilon)/b)/(1-A_1^* \cdot k^{*1-\varepsilon})$

where $\quad A_1^* = 1/bA_1$

In order to get the scale function, equation (B.6) can be expressed as:

$$\frac{1}{(\varepsilon-1)}\left(\frac{1}{s} - \frac{b}{\varepsilon+sb-1}\right)ds = dK/K$$

By integration

$$\ell n\ s - \ell n\ (sb+\varepsilon-1) = (\varepsilon-1)\ell n\ K + \ell n\ A_2$$

The scale function is thus obtained,

(B.8) $\quad s = ((1-\varepsilon)/b)/(1-\frac{1}{bA_2}K^{1-\varepsilon})$

Since we have $h(k^*)$ and $s(K)$, we can construct the production function by using equation (2.9) in the text.

$$R = \int (h(k^*)/L)dL = \frac{1}{b}\ell n\ (L^{1-\varepsilon} - \frac{1}{bA_1}K^{1-\varepsilon}) + \ell n\ A_3$$

and

$$H(K,L) = e^R = A_3(L^{1-\varepsilon} - \frac{1}{bA_1}K^{1-\varepsilon})$$

In order to obtain $C(K)$, equation (2.9d) has to be used. It can be readily shown that $s^H = (1-\varepsilon)/b$. Putting s^H, s^C, and (B.8) together we have

$$dC/C = -((1-\varepsilon)/b)\{1 - \frac{1}{1 - K^{1-\varepsilon}/bA_2}\}\frac{dK}{K}$$

By integration:

$$\ell n\ C = ((\varepsilon-1)/b)\ell n\ K + \frac{1}{b}\ell n\ (K^{1-\varepsilon}/(1-\frac{1}{bA_2}K^{1-\varepsilon})) + \ell n\ A_4$$

We thus get

$$C(K) = (1 - \frac{1}{bA_2}K^{1-\varepsilon})^{-1/b}A_4$$

The production function is

$$Q = C(K)H(K,L) = A_5\left(\frac{k^{*\varepsilon-1} - A_1^*}{K^{\varepsilon-1} - A_2^*}\right)$$

where $A_5 = A_3A_4$, $A_1^* = 1/bA_1$, $A_2^* = 1/bA_2$

(ii) $\varepsilon = 1$

In this case, $b = \varepsilon_h/h$. The share function can be easily calculated

(B.9) $$h = 1/(A_1 - b \ln k^*)$$

And equation (B.6) is, in this case,

$$ds/s^2 = b\, dK/K$$

The scale function is thus

$$s = 1/(A_4 - b \ln K)$$

By applying equation (2.9) in the text, a production can be constructed.

$$Q = A_6\left\{\frac{A_1 - b \ln k^*}{A_4 - b \ln K}\right\}^{1/b}$$

Case 2 $(\partial s/\partial K)(K/s) = 0$

In this case $s(K) = n$ is a constant. By lemma 1 in the text, the production must be a homogeneous one.

(i) $\varepsilon \neq 1$

By equation (B.1), we have

$$(dh/dk^*)(k^*/h) = ((n-h)/n)(\varepsilon-1)$$

The share function is calculated:

$$h = n/\left(1 + \frac{1}{A_2}(k^*)^{1-\varepsilon}\right)$$

The scale function is, by equation (B.5),

$$s = n = (1-\varepsilon)/b$$

By equation (2.9), a production function can be constructed through $[h(k^*), s(K)]$. According to (2.9c):

$$R = \int (h(k^*)/L)dL = (n/(1-\varepsilon))\ell n\ (L^{1-\varepsilon} + \frac{1}{A_2}K^{1-\varepsilon}) + \ell n\ A_3$$

and

$$H(K,L) = A_3\ (L^{1-\varepsilon} + \frac{1}{A_2}K^{1-\varepsilon})^{n/(1-\varepsilon)}$$

As to C(K), we calculate

$$K\frac{\partial R}{\partial K} = n/(1+A_2k^{*\varepsilon-1})$$

and $$s^H = h + K\frac{\partial R}{\partial K} = n$$

According to equation (2.9d), $\frac{dC}{dK}\frac{K}{C} = 0$. C(K) is thus a constant c. The production is thus

$$Q = C(K)H(K,L)$$
$$= A_4(L^{1-\varepsilon} + \frac{1}{A_2}K^{1-\varepsilon})^{\frac{n}{1-\varepsilon}}$$

where $A_4 = cA_3$

By defining $A_2 = \alpha/(1-\alpha)$, and $\varepsilon = 1+\rho$, it can be shown easily that this production function is exactly the same as the CES function.

(ii) $\varepsilon = 1$

According to (B.5), either s or b equals to zero in this case. Since s=0 is excluded in this paper (see equation (1.3) in the text),

b must be zero. By definition, $b = \varepsilon_h/h + (1-\varepsilon)/h = \varepsilon_h/h = 0$, h is thus constant.

Let $h = n-\alpha$, and $s = n$. By applying equation (2.9),

$$R = \int (h(k^*)/L)dL = (n-\alpha)\ \ell nL + \ell n\ A$$

and

$$H(K,L) = e^R = AL^{n-\alpha}$$

Calculating s^H, which is $s^H = n-\alpha$, we get $\frac{dC}{dK}\frac{K}{C} = \alpha$. C(K) is thus

$$C(K) = A_1K^{\alpha}$$

The production function is thus

$$Q = C(K)H(K,L) = A_2K^{\alpha}L^{n-\alpha}$$

where $A_2 = AA_1$

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