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Finding all pure-strategy equilibria in games with continuous strategies

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Static and dynamic games are important tools for the analysis of strategic interactions among economic agents and have found many applications in economics. In many games, equilibria can be described as solutions of polynomial equations. In this paper, we describe state-of-the-art techniques for finding all solutions of polynomial systems of equations, and illustrate these techniques by computing all equilibria of both static and dynamic games with continuous strategies. We compute the equilibrium manifold for a Bertrand pricing game in which the number of equilibria changes with the market size. Moreover, we apply these techniques to two stochastic dynamic games of industry competition and check for equilibrium uniqueness.

Keywords. Polynomial equations, multiple equilibria, Bertrand game, dynamic games, Markov-perfect equilibria.

JEL classification. C63, C73, L13.

1. Introduction

Static and dynamic games are important tools for the analysis of strategic interactions among economic agents and have found many applications in economics. Such models are used both for policy experiments and for structural estimation studies. It is well

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known that equilibrium multiplicity poses a serious threat to the validity of such analyses. This threat is particularly acute if not all equilibria of the examined model are known. Often equilibria can be described as solutions of polynomial equations (which perhaps also must satisfy some additional inequalities). In this paper, we describe state-of-the-art techniques developed in algebraic geometry for finding all solutions of polynomial systems of equations, and illustrate these techniques by computing all equilibria of both static and dynamic games with continuous strategies. We compute the equilibrium manifold for a Bertrand pricing game in which the number of pure-strategy equilibria changes with the market size. Moreover, we apply these techniques to two stochastic dynamic games of industry competition and check for equilibrium uniqueness. Our examples show that the all-solution methods can be applied to a variety of applied static and dynamic models.

Multiplicity of equilibria is a prevalent problem in equilibrium models with strategic interactions. This problem has long been acknowledged in the theoretical literature but has in the past been largely ignored in applied work, even though simple examples of multiple equilibria have been known for decades; see, for example, the model of strategic investment in Fudenberg and Tirole (1983a). Until recently, this criticism was also true for one of the most prolific literatures of applied game-theoretic models, namely the literature based on the framework for the study of industry evolution introduced by Ericson and Pakes (1995). This framework builds the foundation for very active research areas in industrial organization, marketing, and other fields; see the survey by Doraszelski and Pakes (2007). Some recent work in this literature is a great example of the growing interest in equilibrium multiplicity in active areas of modern applied economic analysis. Besanko, Doraszelski, Kryukov, and Satterthwaite (2010) stated that to their knowledge "all applications of Ericson and Pakes' (1995) framework have found a single equilibrium." They then showed that multiple Markov-perfect equilibria can easily arise in a prototypical model in this framework. Borkovsky, Doraszelski, and Kryukov (2008) and Doraszelski and Satterthwaite (2010) presented similar examples with multiple Markov-perfect equilibria. But findings of multiple equilibria are not confined to stochastic dynamic models. Bajari, Hong, Krainer, and Nekipelov (2010) showed that multiple equilibria may arise in static games with incomplete information and they discussed a possible approach to estimating such games. Clearly, the difficulty of equilibrium multiplicity is not restricted to the cited papers. In fact, in many other economic applications, we may often suspect that there could be multiple equilibria.

In many economic models, equilibria can be described as solutions of polynomial equations (which perhaps also must satisfy some additional inequalities). Recent advances in computational algebraic geometry have led to several powerful methods and their easy-to-use computer implementations that find all solutions to polynomial systems. Two different solution approaches stand out, all-solution homotopy methods and Gröbner basis methods, both of which have advantages and disadvantages. The methods using Gröbner bases (Cox, Little, and O'Shea (2007), Sturmfels (2002)) can solve only rather small systems of polynomial equations, but can analyze parameterized systems. For an application of these methods to economics, see the analysis of parameterized

general equilibrium models in Kubler and Schmedders (2010). The all-solution homotopy methods (Sommese and Wampler (2005)) are purely numerical methods that cannot handle parameters, but can solve much larger systems of polynomial equations. It is these homotopy methods that are the focus of the present paper.

All-solution homotopy methods for solving polynomial systems derived from economic models have been discussed previously in both the economics and the mathematics literature on finite games. McKelvey and McLennan (1996) mentioned the initial work on the development of all-solution homotopy methods such as Drexler (1977, 1978) and Garcia and Zangwill (1977). Herings and Peeters (2005) outlined how to use all-solution homotopies for finding all Nash equilibria of generic finite n-person games in normal form, but neither implemented an algorithm nor solved any examples. Sturmfels (2002) surveyed methods for solving polynomial systems of equations and applied them to finding Nash equilibria of finite games. Datta (2010) showed how to find all Nash equilibria of finite games by polyhedral homotopy continuation. Turocy (2008) described progress on a new implementation of a polyhedral continuation method via the software package PHCpack (Verschelde (1999)) in the software package Gambit (McKelvey, McLennan, and Turocy (2007)). The literature on computing one, some, or all Nash equilibria in finite games remains very active; see the introduction to a recent symposium by von Stengel (2010) and the many citations therein. In the present paper, we do not consider finite games, but instead analyze static and dynamic games with continuous strategies. Such games have many important economic applications. To our knowledge, the present paper is the first application of state-of-the-art all-solution homotopy methods to such games. In addition, this paper presents the first application of advanced techniques such as the parameter-continuation method and the system splitting approach to economic models.[1]

The application of homotopy methods has a long history in economics; see Eaves and Schmedders (1999). Kalaba and Tesfatsion (1991) proposed an adaptive homotopy method to allow the continuation parameters to take on complex values to deal with

[1]In this paper, we neither prove any new theorems nor present the most recent examples of frontier applications. Instead, we follow the traditional approach in computational papers and describe a numerical method and apply it to examples that are familiar to most readers. This paper, as many previous computational papers have done, aims to educate the reader about the key ideas underlying a useful numerical method and illustrates these techniques in the context of familiar models. It does so in a way that makes it easy for readers to see how to apply these methods to their own particular problems, and points them to the appropriate software. To clarify what we mean by "traditional method," we give a few examples. First, the paper by Kloek and van Dijk (1978) introduced Monte Carlo methods to basic econometrics using examples from the existing empirical literature and also focused on the methods as opposed to examining breakthrough applications. Second, Fair and Taylor (1983) showed how to use Gauss–Jacobi methods to solve rational expectation models. Again, their paper neither presented new theorems nor used frontier applications as examples. Instead it focused on very simple examples that clarified the mathematical structure of the algorithm and related it to the standard structure of rational expectations models. Third, Pakes and McGuire (2001) showed how to use stochastic approximation to accelerate the Gauss–Jacobi algorithm that they had previously introduced (Pakes and McGuire (1994)) for the solution of stochastic dynamic games. Again, their paper did not analyze new applications and proved only one (convergence) theorem. Instead, their paper educates the reader about stochastic ideas and illustrates their value in a well known example. In this paper, we follow the tradition of this literature.

singular points along the homotopy path. Berry and Pakes (2007) used a homotopy approach for the estimation of demand systems. The homotopy approach was first applied to stochastic dynamic games by Besanko et al. (2010) and Borkovsky, Doraszelski, and Kryukov (2008, 2009). These three papers report results from the application of a classical homotopy approach to the computation of Markov-perfect equilibria in stochastic dynamic games. They show how homotopy paths can be used to find multiple equilibria. When the homotopy parameter is itself a parameter of the economic model, then all points along the path represent economic equilibria (if the equilibrium equations are necessary and sufficient). Whenever the path bends back on itself, there exist multiple equilibria. While this approach can detect equilibrium multiplicity, it is not guaranteed to find all equilibria. Only the all-solution homotopy techniques presented in this paper allow for the computation of all equilibria. However, the classical homotopy approach has the advantage that it can find (at least) one equilibrium of much larger economic models with thousands of equations that do not have to be polynomial. Currently available computer power does not allow us to solve systems with more than a few dozen equations, depending on the degree of the polynomials. As we explain below, however, the all-solution homotopy methods are ideally suited for parallel computations. Our initial experience with an implementation on a computer cluster is very encouraging.

The remainder of this paper is organized as follows. Section 2 depicts a motivating economic example. We both provide some intuition and describe the theoretical foundation for the all-solution homotopy methods in Section 3. Section 4 briefly comments on an implementation of such methods. In Section 5, we provide more details on the computations for the motivating example. Section 6 provides a description of the general setup of dynamic stochastic games. In Section 7, we present an application of the all-solution methods to a stochastic dynamic learning-by-doing model. Similarly, Section 8 examines a stochastic dynamic model of cost-reducing investment with the all-solution homotopy. Finally, Section 9 concludes the paper and provides an outlook on future developments. The Appendices provide more mathematical details on four advanced features of all-solution homotopy methods.

2. Motivating example: Duopoly game with two equilibria

Before we describe details of all-solution homotopy methods, we motivate the application of such methods in economics by reporting results from applying such a method to a static duopoly game. Depending on the value of a parameter, this game may have no, one, or two pure-strategy equilibria. This example illustrates the various steps that are needed to find all pure-strategy Nash equilibria in a simple game with continuous strategies.

2.1 *Bertrand price game*

We consider a Bertrand price game between two firms. There are two products, x and y, two firms with firm x (y) producing good x (y), and three types of customers. Let p_x (p_y) be the price of good x (y). $Dx1$, $Dx2$, and $Dx3$ are the demands for product x by

customer types 1, 2, and 3, respectively. Demands $Dy1, \ldots$ are similarly defined. Type 1 customers only want good x and have a linear demand curve

$$Dx1 = A - p_x; \qquad Dy1 = 0.$$

Type 3 customers only want good y and have a linear demand curve

$$Dx3 = 0; \qquad Dy3 = A - p_y.$$

Type 2 customers want some of both. Let n be the number of type 2 customers. We assume that the two goods are imperfect substitutes for type 2 customers with a constant elasticity of substitution between the two goods and a constant elasticity of demand for a composite good. These assumption imply the demand functions

$$Dx2 = np_x^{-\sigma}(p_x^{1-\sigma} + p_y^{1-\sigma})^{(\gamma-\sigma)/(-1+\sigma)};$$
$$Dy2 = np_y^{-\sigma}(p_x^{1-\sigma} + p_y^{1-\sigma})^{(\gamma-\sigma)/(-1+\sigma)},$$

where σ is the elasticity of substitution between x and y, and γ is the elasticity of demand for the composite good $(q_1^{(\sigma-1)/\sigma} + q_2^{(\sigma-1)/\sigma})^{\sigma/(\sigma-1)}$. Total demand for good x (y) is given by $Dx = Dx1 + Dx2 + Dx3$ ($Dy = Dy1 + Dy2 + Dy3$). Let m be the unit cost of production for each firm. Profit for good x is $R_x = (p_x - m)Dx$; R_y is similarly defined. Let MR_x be marginal profits for good x; similarly for MR_y. Equilibrium prices satisfy the necessary conditions $MR_x = MR_y = 0$.

Firm x (y) is a monopolist for type 1 (3) customers. The two firms only compete in the large market for type 2 customers, and so we may envision two different pricing strategies for the firms. The mass-market strategy chooses a low price so that the firm can sell a large quantity to the large number of type 2 customers that would like to buy both goods but are price sensitive. Such a low price leads to small profits from the customers dedicated to the firm's product. The niche strategy is to just sell at a high price to the few customers who want only its good. Such a high price leads to small demand for its product among the price-sensitive type 2 customers.

We want to demonstrate how we can find all solutions even when there are multiple equilibria. The idea of our example is to find values for the parameters where each firm has two possible strategies. We examine a case where one firm goes for the high-price, small-sales (niche) strategy and the other firm goes after type 2 customers with a mass-market strategy. Let

$$\sigma = 3, \qquad \gamma = 2, \qquad n = 2700, \qquad m = 1, \qquad A = 50.$$

The marginal profit functions are

$$MR_x = 50 - p_x + (p_x - 1)\left(-1 + \frac{2700}{p_x^6(p_x^{-2} + p_y^{-2})^{3/2}} - \frac{8100}{p_x^4\sqrt{p_x^{-2} + p_y^{-2}}}\right)$$
$$+ \frac{2700}{p_x^3\sqrt{p_x^{-2} + p_y^{-2}}},$$

$$MR_y = 50 - p_y + (p_y - 1)\left(-1 + \frac{2700}{p_y^6(p_x^{-2} + p_y^{-2})^{3/2}} - \frac{8100}{p_y^4\sqrt{p_x^{-2} + p_y^{-2}}}\right) + \frac{2700}{p_y^3\sqrt{p_x^{-2} + p_y^{-2}}}.$$

2.2 *Polynomial equilibrium equations*

We first construct a polynomial system. The system we construct must contain all the equilibria, but it may have extraneous solutions. The extraneous solutions present no problem because we can easily identify and discard them.

We need to eliminate the radical terms. Let Z be the square root term

$$Z = \sqrt{p_x^{-2} + p_y^{-2}},$$

which implies

$$0 = Z^2 - (p_x^{-2} + p_y^{-2}).$$

This is not a polynomial. We gather all terms into one fraction and extract the numerator, which is the polynomial we include in our polynomial system to represent the variable Z,

$$0 = -p_x^2 - p_y^2 + Z^2 p_x^2 p_y^2. \tag{1}$$

We next use the Z definition to eliminate radicals in MR_x and MR_y. Again we gather terms into one fraction and extract the numerator. The second and third equations of our polynomial are

$$0 = -2700 + 2700 p_x + 8100 Z^2 p_x^2 - 5400 Z^2 p_x^3 + 51 Z^3 p_x^6 - 2 Z^3 p_x^7, \tag{2}$$

$$0 = -2700 + 2700 p_y + 8100 Z^2 p_y^2 - 5400 Z^2 p_y^3 + 51 Z^3 p_y^6 - 2 Z^3 p_y^7. \tag{3}$$

Any pure-strategy Nash equilibrium is a solution of the polynomial system (1)–(3).

2.3 *Solution*

Solving the above system of polynomial equations (see Section 5.1 for details), we find 18 real and 44 complex solutions. Nine of the 18 real solutions contain at least one variable with a negative value and are thus economically meaningless. Table 1 shows the remaining 9 solutions. We next check the second-order conditions of each firm. This

TABLE 1. Real, positive solutions of (1)–(3).

p_x	1.757	8.076	22.987	2.036	5.631	2.168	25.157	7.698	24.259
p_y	1.757	8.076	22.987	5.631	2.036	25.157	2.168	24.259	7.698

check eliminates five more real solutions and reduces the set of possible equilibria to four, namely

$$(p_x^1, p_y^1) = (1.757, 1.757), \qquad (p_x^2, p_y^2) = (22.987, 22.987),$$
$$(p_x^3, p_y^3) = (2.168, 25.157), \qquad (p_x^4, p_y^4) = (25.157, 2.168).$$

We next need to check global optimality for each player in each potential equilibrium. The key fact is that the global max must satisfy the first-order conditions given the other player's strategy. So all we need to do is to find all solutions to a firm's first-order condition at the candidate equilibrium and then find which one produces the highest profits. We keep the candidate equilibrium only if it is the global maximum.

First consider (p_x^1, p_y^1). We first check to see if player x's choice is globally optimal given p_y. Since we take p_y as given, the equilibrium system reduces to the Z equation and the first-order condition for player x, giving us the polynomial system

$$0 = 0.32410568484991703 p_x^2 + 1 - Z^2 p_x^2,$$
$$0 = -2700 + 2700 p_x + 8100 Z^2 p_x^2 - 5400 Z^2 p_x^3 + 51 Z^3 p_x^6 - 2 Z^3 p_x^7.$$

This system has 14 finite solutions, 8 complex solutions, and 6 real solutions. One of the solutions is $p_x = 25.2234$, where profits equal 607.315. Since this exceeds 504.625, firm x's profits at (p_x^1, p_y^1), we conclude that (p_x^1, p_y^1) is not an equilibrium. A similar approach shows that (p_x^2, p_y^2) is not an equilibrium. Given $p_y^2 = 22.987$, firm x would receive a higher profit from a low price than from p_x^2. When we examine the remaining two candidate equilibria, we find that they are two asymmetric equilibria, (p_x^3, p_y^3) and (p_x^4, p_y^4). This may not appear to be an important multiplicity since the two equilibria are mirror images of each other. However, it is clear that if we slightly perturb the demand functions to eliminate the symmetries, there will still be two equilibria that are not mirror images.

In the equilibrium $(p_x^3, p_y^3) = (2.168, 25.157)$, firm x chooses a mass-market strategy and firm y chooses a niche strategy. The low price allows firm x to capture most of the market of price-sensitive type 2 customers, while it forgoes most of the possible (monopoly) profits in its niche market of type 1 customers. Firm y instead charges a high price (just below the monopoly price for the market of type 3 customers) to capture most of its niche market. In the equilibrium $(p_x^4, p_y^4) = (25.157, 2.168)$ the strategies of the two firms are reversed.

This example demonstrates that the problem of finding all Nash equilibria reduces to solving a series of polynomial systems. The first system identifies a set of solutions for the firms' first-order conditions, which are only necessary but not sufficient. The second step is to eliminate all candidate equilibria where some firm does not satisfy the local second-order condition for optimization. The third step is to check the global optimality of each firm's reactions in each of the remaining candidate equilibria. This step reduces to finding all solutions of a set of smaller polynomial systems.

Figure 1 displays the manifold of a firm's equilibrium prices for values of the market size parameter n between 500 and 3400. For $500 \le n \le 2470$, there is a unique equilibrium. The competitive market of type 2 customers is so small that each firm chooses a

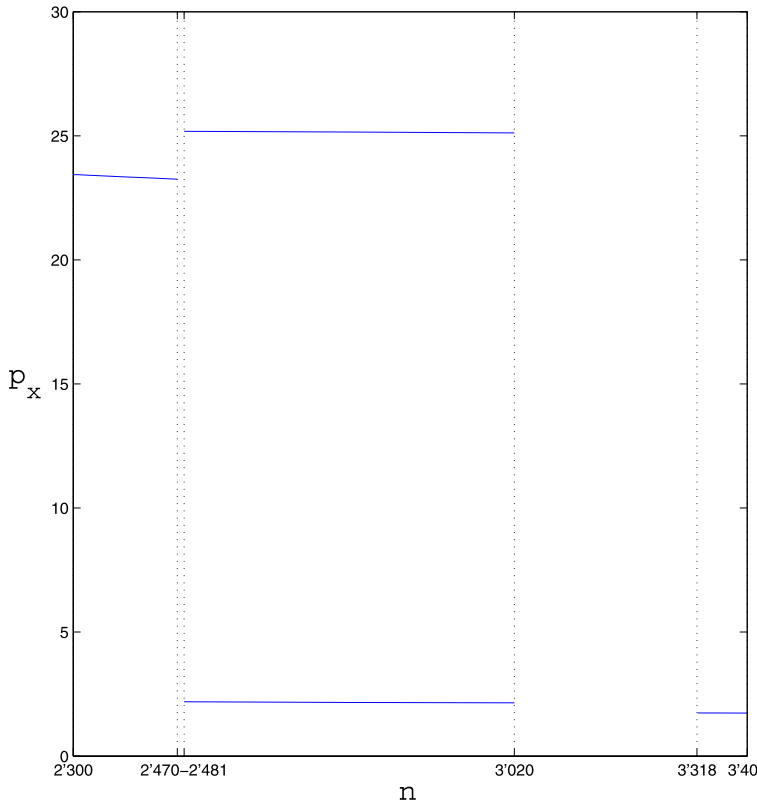

FIGURE 1. Equilibrium prices as a function of n.

niche strategy and charges a high price to focus on the few customers that only want its good. For $3318 \leq n \leq 3400$, there is again a unique equilibrium. The competitive market of type 2 customers is now sufficiently large so that each firm chooses a mass-market strategy and charges a low price to sell a high quantity into the mass market of type 2 customers. For $2481 \leq n \leq 3020$, there are two equilibria. At these intermediate values of n, the two firms prefer complementary strategies: one firm chooses a (high-price) niche strategy and the other firm chooses a (low-price) mass-market strategy. Finally, there are two regions with no pure-strategy equilibria, namely for $2471 \leq n \leq 2480$ and also for $3021 \leq n \leq 3317$.

3. All-solution homotopy methods

In this section, we introduce the mathematical background of all-solution homotopy methods for polynomial systems of equations. Polynomial solution methods rely on results from complex analysis and algebraic geometry. For this purpose, we first review some basic definitions.

3.1 *Mathematical background*

We define a polynomial in complex variables.

Definition 1. A polynomial f over the variables $z_1, \dots, z_n$ is defined as

$$f(z_1, \dots, z_n) = \sum_{j=0}^{d} \left(\sum_{d_1+\cdots+d_n=j} a_{(d_1,\dots,d_n)} \prod_{k=1}^{n} z_k^{d_k} \right) \quad \text{with } a_{(d_1,\dots,d_n)} \in \mathbb{C},\ d \in \mathbb{N}.$$

For convenience we denote $z = (z_1, \dots, z_n)$. The expression $a_{(d_1,\dots,d_n)} \prod_{k=1}^{n} z_k^{d_k}$ for $a_{(d_1,\dots,d_n)} \neq 0$ is called a term of f. The degree of f is defined as

$$\deg f = \max_{a_{(d_1,\dots,d_n)} \neq 0} \sum_{k=1}^{n} d_k.$$

The term $\sum_{d_1+\cdots+d_n=j} a_{(d_1,\dots,d_n)} \prod_{k=1}^{n} z_k^{d_k}$ is called the homogeneous part of degree j of f and is denoted by $f^{(j)}$.

Note that $f^{(j)}$ being homogeneous of degree j means $f^{(j)}(cz) = c^j f^{(j)}(z)$ for any complex scalar $c \in \mathbb{C}$. We now regard a polynomial f in the variables $z_1, \dots, z_n$ as a function $f : \mathbb{C}^n \to \mathbb{C}$. Then f belongs to the following class of functions.

Definition 2. Let $U \subset \mathbb{C}^n$ be an open subset and let $f : U \to \mathbb{C}$ be a function. Then we call f analytic at the point $b = (b_1, \dots, b_n) \in U$ if and only if (iff) there exists a neighborhood V of b such that

$$f(z) = \sum_{j=0}^{\infty} \left(\sum_{d_1+\cdots+d_n=j} a_{(d_1,\dots,d_n)} \prod_{k=1}^{n} (z_k - b_k)^{d_k} \right) \quad \forall z \in V,$$

where $a_{(d_1,\dots,d_n)} \in \mathbb{C}$, that is, the above power series converges to the function f on V. It is called the Taylor series of f at b.

Obviously every function given by polynomials is analytic with one Taylor expansion on all of $\mathbb{C}^n$. However, note that, in general, $V \subsetneqq U$ and that the power series is divergent outside of V. For functions in complex space, we can state the implicit function theorem analogously to the case of functions in real space.

Theorem 1 (Implicit Function Theorem). *Let*

$$H : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n \quad \textit{with } (t, z_1, \dots, z_n) \mapsto H(t, z_1, \dots, z_n)$$

be an analytic function. Denote by $D_z H = (\frac{\partial H_j}{\partial z_i})_{i,j=1,\dots,n}$ *the submatrix of the Jacobian of* H *containing the partial derivatives with respect to* z_i, $i = 1, \dots, n$. *Furthermore, let* $(t_0, x_0) \in \mathbb{C} \times \mathbb{C}^n$ *such that* $H(t_0, x_0) = 0$ *and* $\det D_z H(t_0, x_0) \neq 0$. *Then there exist neighborhoods* T *of* t_0 *and* A *of* x_0, *and an analytic function* $x : T \to A$ *such that* $H(t, x(t)) = 0$ *for all* $t \in T$. *Furthermore, the chain rule implies that*

$$\frac{\partial x}{\partial t}(t_0) = -D_z H(t_0, x_0)^{-1} \cdot \frac{\partial H}{\partial t}(t_0, x_0).$$

Next we define the notion of a path.

Definition 3. Let $A \subset \mathbb{C}^n$ be an open or closed subset. An analytic[2] function $x : [0, 1] \to A$ or $x : [0, 1) \to A$ is called a path in A.

Definition 4. Let $H(t, z) : \mathbb{C}^{n+1} \to \mathbb{C}^n$ and let $x : [0, 1] \to \mathbb{C}^n$ be an analytic function such that $H(t, x(t)) = 0$ for all t. Then x defines a path in $\{(t, x) \in \mathbb{C}^{n+1} \mid H(t, x) = 0\}$. We call the path *regular*, iff $\{t \in [0, 1) \mid H(t, x(t)) = 0, \det D_z H(t, x(t)) = 0\} = \emptyset$.[3]

Note that for general homotopy methods the regularity definition is less strict. One usually only wants the Jacobian to have full rank. Here we also impose which part of it has full rank. Such a definition is reasonable for polynomial homotopy methods since, as we see later, we can ensure this property for our paths.

Definition 5. Let $A \subset \mathbb{C}^n$. We call A *pathwise connected*, iff for all points $a_1, a_2 \in A$, there exists a continuous function $x : [0, 1] \to A$ such that $x(0) = a_1$ and $x(1) = a_2$.

Last, we need the following notion from topology.

Definition 6. Let $U, V \subset \mathbb{C}^n$ be open subsets and let $h_0 : U \to V$, $h_1 : U \to V$ be continuous functions. Let

$$H : [0, 1] \times U \to V,$$
$$(t, z) \mapsto H(t, z)$$

be a continuous function such that $H(0, z) = h_0(z)$ and $H(1, z) = h_1(z)$. Then we call H a homotopy from h_0 to h_1.

3.2 *Building intuition from the univariate case*

Homotopy methods have a long history in economics (see Eaves and Schmedders (1999)), for finding one solution to a system of nonlinear equations. Recent applications of such homotopy methods in game-theoretic models include Besanko et al. (2010) and Borkovsky, Doraszelski, and Kryukov (2008). Homotopy methods for finding all solutions of polynomial systems were first introduced by Garcia and Zangwill (1977) and Drexler (1977). Their papers initiated an active field of research that is still advancing today; see Sommese and Wampler (2005) for an overview. In this subsection, following Sommese and Wampler (2005) and the many cited works therein, we provide some intuition for the theoretical foundation underlying all-solution homotopy continuation methods.

The basic idea of the homotopy approach is to find an easier system of equations and continuously transform it into our target system. We first illustrate this for univariate polynomials. Consider the univariate polynomial $f(z) = \sum_{i \le d} a_i z^i$ with $a_d \neq 0$ and $\deg f = d$. The fundamental theorem of algebra states that f has precisely d complex

[2]The usual definition of a path only requires continuity, but all paths we regard are automatically given by analytic functions.

[3]We see below why we can exclude $t = 1$ from our regularity assumption.

roots, counting multiplicities.[4] A simple polynomial of degree d with d distinctive complex roots is $g(z) = z^d - 1$, whose roots are $r_k = e^{2\pi ik/d}$ for $k = 0, \dots, d-1$. (These roots are called the dth roots of unity.) Now we can define a homotopy H from g to f by setting $H = (1-t)g + tf$. Thus H is a polynomial in t, z and, therefore, is an analytic function. Under the assumption that $\frac{\partial H}{\partial z}(t, z) \neq 0$ for all (t, z) satisfying $H(t, z) = 0$ and $t \in [0, 1]$, the implicit function theorem (Theorem 1) states that each root r_k of g gives rise to a path that is described by an analytical function. The idea is now to start at each solution $z = r_k$ of $H(0, z) = 0$ and to follow the resulting path until a solution z of $H(1, z) = 0$ has been reached. The path-following can be done numerically using a predictor–corrector method (see, for example, Allgower and Georg (2003)). For example, Euler's method is a so-called first-order predictor and obtains a first step along the path by choosing an $\varepsilon > 0$ and calculating

$$\tilde{x}_k(0+\varepsilon) = x_k(0) + \varepsilon \frac{\partial x_k}{\partial t}(0),$$

where the $\frac{\partial x_k}{\partial t}(0)$ are implicitly given by Theorem 1. Then this first estimate is corrected using Newton's method with starting point $\tilde{x}_k(0+\varepsilon)$. So the method solves the equation $H(\varepsilon, z) = 0$ for z and sets $x_k(\varepsilon) = z$.

Example 1. As a first example, we look at the polynomial $f(z) = z^3 + z^2 + z + 1$. The zeros are $\{-1, -i, i\}$. As a starting polynomial, we choose $g(z) = z^3 - 1$. We define a homotopy from g to f as

$$H(t, z) = (1-t)(z^3 - 1) + t(z^3 + z^2 + z + 1).$$

This homotopy generates the three solution paths shown in Figure 2. The starting points of the three paths, $-\frac{1}{2} - \frac{\sqrt{3}}{2}i$, $-\frac{1}{2} + \frac{\sqrt{3}}{2}i$, and 1, respectively, are indicated by circles; the respective end points, $-i$, i, and -1 are indicated by squares.

This admittedly rough outline captures the fundamental idea of the all-solution homotopy methods. This method can potentially run into difficulties: first, the paths might cross; second, the paths might bend sideways and diverge. We illustrate these problems with an example and also show how to circumvent them.

Example 2. Let $f(z) = 5 - z^2$ and $g(z) = z^2 - 1$. Then a homotopy from g to f can be defined as

$$H(t, z) = t(5 - z^2) + (1-t)(z^2 - 1) = (1-2t)z^2 + 6t - 1. \tag{4}$$

Now $H(\frac{1}{6}, z) = \frac{2}{3}z^2$ has the double root $z = 0$, so $\det D_z H(\frac{1}{6}, 0) = 0$. Such points are called nonregular and the assumption of the implicit function theorem is not satisfied. Nonregular points are also problematic for the Newton corrector step in the path-following algorithm. But matters are even worse for this homotopy since $H(\frac{1}{2}, z) = 2$,

[4]Any univariate polynomial of degree d over the complex numbers can be written as $f(z) = c(z - b_1)^{r_1}(z - b_2)^{r_2} \cdots (z - b_l)^{r_l}$ with $c \in \mathbb{C} \setminus \{0\}$, $b_1, b_2, \dots, b_l \in \mathbb{C}$, and $r_1, r_2, \dots, r_l \in \mathbb{N}$. The exponent r_j denotes the multiplicity of the root b_j. For example, the polynomial z^3 has the single root $z = 0$ with multiplicity 3.

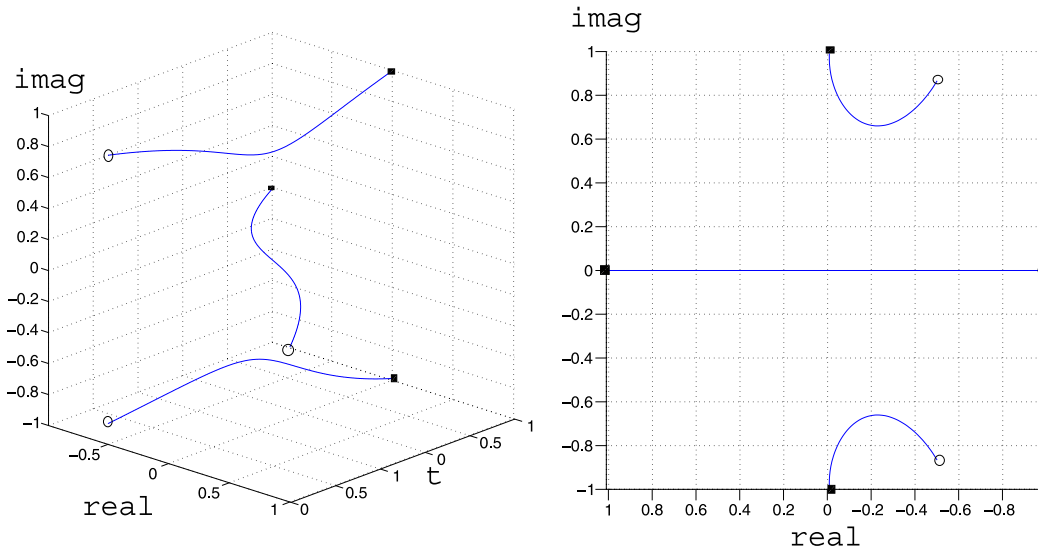

FIGURE 2. Homotopy paths in Example 1 and the projection to $\mathbb{C}$.

which has no zero at all, that is, there can be no solution path from $t = 0$ to $t = 1$. The coefficient of the leading term $(1 - 2t)z^2$ has become 0 and so the degree of the polynomial H drops at $t = \frac{1}{2}$. Figure 3 displays the set of zeros of the homotopy. The two paths starting at $\sqrt{5}$ and $-\sqrt{5}$ diverge as $t \to \frac{1}{2}$.

The general idea to resolve the technical problems illustrated in Example 2 is to "walk around" the points that cause us trouble. For a description of this idea, we need

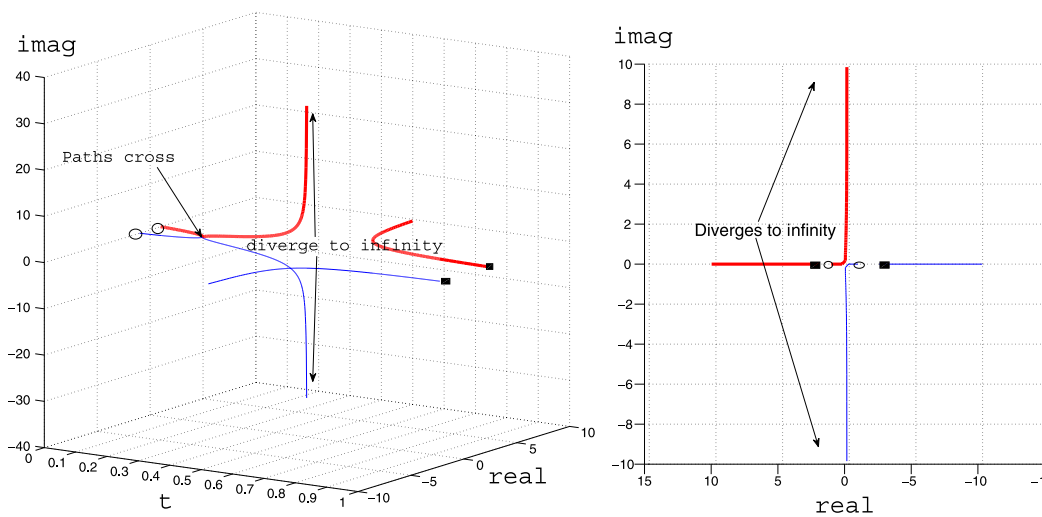

FIGURE 3. Homotopy paths in Example 2 and the projection to $\mathbb{C}$.

the following theorem, which describes one of the differences between complex and real spaces.

THEOREM 2. *Let $F = (f_1, \dots, f_k) = 0$ be a system of polynomial equations in n variables, with $f_i \neq 0$ for some i. Then $\mathbb{C}^n \setminus \{F = 0\}$ is a pathwise connected and dense subset of $\mathbb{C}^n$.*[5]

This statement does not hold true over the reals. Take for instance $n = 2$ and $k = 1$, and set $f_1(x_1, x_2) = x_1$. (Note that f_1 is not identically zero.) Now we restrict ourselves to the real numbers $(x_1, x_2) \in \mathbb{R}^2$. If we remove the zero set $\{(x_1, x_2) \in \mathbb{R}^2 : f_1(x_1, x_2) = 0\}$, which is the vertical axis, then the resulting set $\mathbb{R}^2 \setminus \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\}$ consists of two disjoint components. Thus it is not pathwise connected.

EXAMPLE 3. Returning to Example 2, we temporarily regard t also as a complex variable and thus $\{(t, z) \mid H(t, z) = 0\} \subset \mathbb{C}^2$. Due to Theorem 1, we only have a path if the determinant is locally nonzero. The points that are not regular are characterized by the equations

$$\begin{aligned} &(1-2t)z^2 + 6t - 1 = 0, \\ &\det D_z H = 2z(1-2t) = 0. \end{aligned} \tag{5}$$

Points at which our path is interrupted are given by

$$1 - 2t = 0. \tag{6}$$

In this case, we can easily determine that the only solution to (5) is $(\frac{1}{6}, 0)$ and the solution to (6) is $\{t = \frac{1}{2}\}$. The union of the solution sets to the two equations is exactly the solution set of the system of equations

$$\begin{aligned} &((1-2t)z^2 + 6t - 1)(1-2t) = 0, \\ &(2z(1-2t))(1-2t) = 0. \end{aligned} \tag{7}$$

Theorem 2 now implies that the complement of the solution set to system (7) is pathwise connected. In other words, we can find a path between any two points without running into problematic points. To walk around those problematic points, we define a new homotopy by multiplying the start polynomial $z^2 - 1$ by $e^{i\gamma}$ for a random $\gamma \in [0, 2\pi)$:

$$H(t, z) = t(5 - z^2) + e^{i\gamma}(1-t)(z^2 - 1) = (e^{i\gamma} - t - te^{i\gamma})z^2 + te^{i\gamma} - e^{i\gamma} + 5t. \tag{8}$$

Now we obtain $D_z H = 2(e^{i\gamma} - t - te^{i\gamma})z$, which has $z = 0$ as its only solution if $e^{i\gamma} \notin \mathbb{R}$ and $t \in [0, 1]$. Furthermore, if $e^{i\gamma} \notin \mathbb{R}$, then $H(t, 0) = te^{i\gamma} - e^{i\gamma} + 5t \neq 0$ for all $t \in [0, 1]$. Additionally, the coefficient of z^2 in (8) does not vanish for $t \in \mathbb{R}$ and thus $H(t, x) = 0$

[5]This is a simpler version of the theorem than is actually needed, but for simplicity, we avoid the general case.

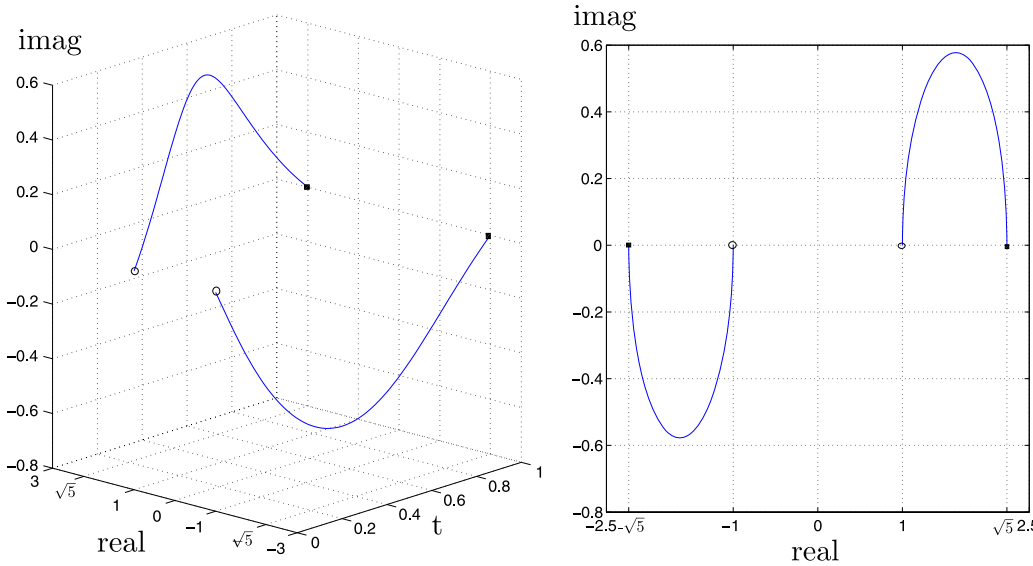

FIGURE 4. Homotopy paths in Example 3 after application of the gamma trick.

always has two solutions for $t \in [0, 1]$ due to the fundamental theorem of algebra. Therefore, this so-called gamma trick yields only paths that are not interrupted and are regular. Figure 4 displays the two paths: the left graph shows the paths in three dimensions; the right graph shows a projection of the paths on $\mathbb{C}$. It remains to check how strict the condition $e^{i\gamma} \notin \mathbb{R}$ is. We know $e^{i\gamma} \in \mathbb{R} \Leftrightarrow \gamma = k\pi$ for $k \in \mathbb{N}$. Since $\gamma \in [0, 2\pi)$, these are only two points. Thus for a random γ, the paths exist and are regular with probability 1.

This example concludes our introductory discussion of the all-solution homotopy approach. In the next subsection, we describe technical details of the general multivariate homotopy approach. A reader who is mainly interested in the quick implementation of homotopies as well as economic applications may want to skip this part and continue with Section 4.

3.3 *The multivariate case*

When we attempt to generalize the outlined approach from the univariate to the multivariate case, we encounter a significant difficulty. The fundamental theorem of algebra does not generalize to multiple equations and so we do not know a priori the number of complex solutions. However, we can determine upper bounds on the number of solutions. For the sake of our discussion in this paper, it suffices to introduce the simplest such bound.

DEFINITION 7. Let $F = (f_1, \dots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ be a polynomial function. Then the number

$$d = \prod_i \deg f_i$$

is called the *total degree* or *Bezout number* of F.

THEOREM 3 (Bezout's Theorem). *Let d be the Bezout number of F. Then the polynomial system $F = 0$ has at most d isolated solutions counting multiplicities.*

This bound is tight. In fact, García and Li (1980) showed that generic polynomial systems have exactly d distinct isolated solutions. But this result does not provide any guidance for specific systems, since systems arising in economics and other applications typically are so special that the number of solutions is much smaller.

Next we address the difficulties we observed in Example 2 for the multivariate case. Consider a square polynomial system $F = (f_1, \dots, f_n) = 0$ with $d_i = \deg f_i$. Construct a start system $G = (g_1, \dots, g_n) = 0$ such that

$$g_i(z) = z_i^{d_i} - 1. \tag{9}$$

Note that the polynomial $g_i(z)$ only depends on the variable z_i and has the same degree as $f_i(z)$. The polynomial functions F and G have the same Bezout number. Now construct a homotopy $H = (h_1, \dots, h_n) : \mathbb{C} \times \mathbb{C}^n \to \mathbb{C}^n$ from the square polynomial system $F(z) = 0$ and the start system $G(z) = 0$ that is linear in the homotopy parameter t. As a result, $h_i(z)$ is a polynomial of degree d_i in the variables $z_1, \dots, z_n$ and coefficients that are linear functions in t,

$$h_i(z) = \sum_{j=0}^{d_i} \left(\sum_{c_1+\cdots+c_n=j} a_{(i,c_1,\dots,c_n)}(t) \prod_{k=1}^{n} z_k^{c_k} \right).$$

In a slight abuse of notation we denote by $a_i(t)$ the product of the coefficients of the highest-degree monomials of $h_i(z)$. As before, we need to rule out nonregular points and values of the homotopy parameter for which the system $H(t, z) = 0$ may have no solution. Nonregular points are solutions to the system of equations

$$\begin{aligned} &h_i = 0 \quad \forall i, \\ &\det D_z H = 0. \end{aligned} \tag{10}$$

Additionally, values of the homotopy parameter for which one or more of our paths might get interrupted are all t that satisfy

$$\prod_i a_i(t) = 0. \tag{11}$$

For a t' satisfying the above equation, it follows that the polynomial $H(t', z)$ has a lower Bezout number than $F(z)$.[6] Analogously to Example 3, we can cast (10) and (11) in one system of equations:

$$\begin{aligned} &h_i \prod_j a_j(t) = 0 \quad \forall i, \\ &\det(D_z H) \prod_i a_i(t) = 0. \end{aligned} \tag{12}$$

[6]Note that after homogenization, which we introduce in Appendix A, this no longer poses any problem.

Theorem 2 states that the complement of the solution set to this system of equations is a pathwise connected set, so, as before, we can "walk around" those points that cause difficulties for the path-following algorithm. In fact, if we choose our paths randomly just as in Example 3, then we do not encounter those problematic points with probability 1.

THEOREM 4 (Gamma Trick). *Let $G(z):\mathbb{C}^n \to \mathbb{C}^n$ be our start system and let $F(z):\mathbb{C}^n \to \mathbb{C}^n$ be our target system. Then for almost all*[7] *choices of the constant $\gamma \in [0, 2\pi)$, the homotopy*

$$H(t, z) = e^{\gamma i}(1-t)G(z) + tF(z) \tag{13}$$

has regular solution paths and $|\{z \mid H(t_1, z) = 0\}| = |\{z \mid H(t_2, z) = 0\}|$ for all $t_1, t_2 \in [0, 1)$.

We say that a path *diverges to infinity* at $t = 1$ if $\|z(t)\| \to \infty$ for $z(t)$ satisfying $H(t, z(t)) = 0$ as $t \to 1$, where $\|\cdot\|$ denotes the Euclidean norm. The gamma trick leads to the following theorem.

THEOREM 5. *Consider the homotopy H as in (13) with a start system as in (9). For almost all parameters $\gamma \in [0, 2\pi)$, the following properties hold.*

(i) *The preimage $H^{-1}(0)$ consists of d regular paths, that is, no paths cross or bend backward.*

(ii) *Each path either diverges to infinity or converges to a solution of $F(z) = 0$ as $t \to 1$.*

(iii) *If $\bar{z}$ is an isolated solution with multiplicity*[8] *m, then there are m paths converging to it.*

By construction the easy system $G(z) = 0$ has exactly d isolated solutions. Each of these solutions is the starting point of a smooth path along which the parameter t increases monotonically, that is, the Jacobian has full rank and the path does not bend backward. To find all solutions of $F(z) = 0$, we need to follow all d paths and check whether they diverge or run into a solution of our system. In light of the aforementioned result by García and Li (1980) that generic polynomial systems $F(z) = 0$ have d isolated solutions, Theorem 5 implies that the homotopy H gives rise to d distinct paths that terminate at the d isolated roots of F. So, generically, the intuition of the univariate case carries over to the multivariate case.

3.4 *Advanced features*

The described method is intuitive, but has two major drawbacks that make it impractical. First, the paths diverging to infinity are of no interest in economic applications.

[7]Throughout this paper, the terminology "almost all" means an open set of measure 1. All stated results in fact hold on so-called Zariski open sets, but for simplicity we omit a proper definition of this term.

[8]Multiplicity of a root for a system of polynomial equations is similar to multiplicity in the univariate case. We forgo any proper definition for the sake of simplicity.

Second, the number of paths grows exponentially in the number of nonlinear equations. A practical homotopy method needs to spend as little time as possible on diverging paths. In addition, it always is advantageous to keep the number of paths as small as possible. Advanced all-solution homotopy methods address both these problems. In the Appendices, we describe the underlying mathematical approaches.

The diverging paths are of no interest for finding economically meaningful solutions to systems of equations derived from an economic model. The diverging paths typically require much more computational effort than converging paths, and their potential presence requires a computer program following the paths to decide whether a path is diverging or only very long but converging. The decision when to declare that a path is diverging cannot be made without the risk of actually truncating a very long converging path. A reliable and robust computational method thus needs some feature to handle diverging paths. It is possible to "compactify" the diverging path through a homogenization of the polynomials. Appendix A describes this approach.

The number of paths d grows rapidly with the degree of individual equations. It also grows exponentially in the number of equations (if the equations are not linear). For many economic models, we believe that there are only a few (if not unique) equilibria, that is, our systems have few real solutions and usually even fewer economically meaningful solutions. As a result, we may have to follow a large number of paths that do not yield useful solutions. Also, if there are only a few real and complex solutions, then many paths must converge to solutions at infinity. There may even be continua of solutions at infinity which can cause numerical difficulties; see Example 4 in Appendix A. Therefore, it would be very helpful to reduce the number of paths that must be followed as much as possible. Appendices B and C describe two methods for a reduction of the number of paths.

4. Implementation

We briefly describe the software package Bertini and the potential computational gains from a parallel version of the software code.

4.1 *Bertini*

The software package Bertini, written in the programming language C, offers solvers for a few different types of problems in numerical algebraic geometry; see Bates, Hauenstein, Sommese, and Wampler (2005). The most important feature for our purpose is Bertini's homotopy continuation routine for finding all isolated solutions of a square system of polynomial equations. In addition to an implementation of the advanced homotopy of Theorem 7 (see Appendix A), it also allows for m-homogeneous start systems as well as parameter-continuation homotopies as in Theorem 8; see Appendices B and C. Bertini has an intuitive interface that allows the user to quickly implement systems of polynomial equations; see Sections 5.1 and 5.2 for examples of code that a user must supply. Bertini can be downloaded free of charge at http://www.nd.edu/~sommese/bertini/.

All results in this paper were computed with Bertini on a laptop, namely an Intel Core 2 Duo T9550 with 2.66 GHz and 4 GB RAM.

4.2 *Alternatives*

Two other all-solution homotopy software packages are PHCpack (Verschelde (1999)) written in ADA, and POLSYS_PLP (Wise, Sommese, and Watson (2000)) written in FORTRAN90 and which is intended to be used in conjunction with HOMPACK90 (Watson, Sosonkina, Melville, and Walker (1997)), a popular homotopy path solver. Because of its versatility, stable implementation, great potential for parallelization on large computer clusters, and friendly user interface, we use Bertini for all our calculations.

4.3 *Parallelization*

The overall complexity of the all-solution homotopy method is the same as for other methods used for polynomial system solving. The major advantage of this method, however, is that it is naturally parallelizable. Following each path is a distinct task, that is, the paths can be tracked independently from each other. Moreover, the information gathered during the tracking process of a path cannot be used to help track other paths.

This advantage coincides with recent developments in processing technology. The performance of a single processor no longer grows as in the years before, since power consumption and core temperature have become big issues in the production of computer chips. The new strategy of computer manufacture's is to use multiple cores within a single machine to spread out the workload.

The software package Bertini is available in a parallel version. As of this writing, we have already successfully computed examples via parallelization on 200 processors at the CSCS cluster (Swiss Scientific Computing Center). To spread the work across many more processors, a modest revision of the Bertini code is necessary. We are optimistic that we will soon be able to solve problems on clusters with thousands of processors. Such a setup will allow us to solve problems that are orders of magnitude larger than those described below.

5. Bertrand price game continued

We return to the duopoly price game from Section 2. We now show how to solve the problem with Bertini. We also show how to use some of the advanced features from Appendices A–C.

5.1 *Solving the Bertrand price game with Bertini*

To solve the system (1)–(3) in Bertini, we write the input file

```
CONFIG
MPTYPE: 0;
END;

INPUT
variable_group px,py,z;
function f1, f2, f3;
```

```
f1 = -(px^2)-py^2+z^2*px^2*py^2;
f2 = -(2700)+2700*px+8100*z^2*px^2-5400*z^2*px^3
     +51*z^3*px^6-2*z^3*px^7;
f3 = -(2700)+2700*py+8100*z^2*py^2-5400*z^2*py^3
     +51*z^3*py^6-2*z^3*py^7;
END;
```

The option `MPTYPE: 0` indicates that we are using standard path-tracking. The polynomials `f1`, `f2`, `f3` define the system of equations. The Bezout number is $6 \times 10 \times 10 = 600$. Thus, Bertini must track 600 paths. With the above code, we obtained 18 real solutions, 44 complex solutions, 270 truncated infinite paths, and 268 failures.[9] In Appendix A, we show that if we homogenize the above equations, then we have continua of solutions at infinity as illustrated in Example 4. Such solutions are responsible for the large number of failures, since at these solutions the rank of the Jacobian drops. Of course, such paths with convergence failures represent a serious concern. Fortunately, Bertini offers the option `MPTYPE: 2` for improved convergence. This command instructs Bertini to use adaptive precision, which handles singular solutions much better but needs more computation time. We then find the same 18 real and 44 complex solutions as before, but in contrast to the previous run, we now have 538 truncated infinite paths and no failures. Bertini lists the real solution in the file `real_finite_solutions` and all finite solutions in `finite_solutions`.

Next we show how to reduce the number of paths with m-homogenization (see Appendix B). Replace `variable_group px,py,z;` by

```
variable_group px;
variable_group py;
variable_group z;
```

By separating the variables in the different groups, we indicate how to group them for the m-homogenization. As a result, we have only 182 paths to track. However, each new variable group adds another variable to the computations[10] and decreases numerical stability. Therefore, we always have to consider the problem of reducing the number of paths versus increasing the number of variables.

A key point to note is that the number of solutions is much smaller than the Bezout number. The Bezout number of the system (1)–(3) is 600, but there are only 62 finite solutions. This fact may be surprising in light of the theorem that says that systems such as (1)–(3) generically have 600 finite complex solutions; see Garcia and Li (1980). However, system (1)–(3) is not similar to the generic system, since most monomials of degree 6 are missing from (1), and most monomials of degree 10 are missing from (2) and (3).

[9] In those cases, the path tracker failed to converge on a solution at infinity. Note that Bertini uses random numbers to define the homotopy, so the number of failed paths varies.

[10] We repeatedly solve square systems of linear equations. Bertini performs this task with conventional methods with a complexity of roughly $\frac{1}{3}n^3$, where n is the number of variables. Thus increasing the number of variables by m adds $\frac{1}{3}(m^3 + 3m^2n + 3n^2m)$ to the complexity for each iteration of Newton's method.

The absence of so many monomials often implies a far smaller number of finite complex solutions. For many games this fact makes our strategy much more practical than we would initially think.

Another key point to note is that the all-solution methods can only be applied to polynomial systems, that is, when all variables have exponents with nonnegative integer values. We cannot apply such a method to equations with irrational exponents. Such systems would occur in the Bertrand game, for example, if an elasticity were an irrational number such as π. In addition, an important prerequisite for Bertini to be able to trace all paths is that the Bezout number remains relatively small. The conversion of systems with rational exponents with large denominators to proper polynomial systems, however, leads to polynomial systems with large exponents. For example, the conversion of equations with exponents such as 54,321/10,000 leads to very difficult systems that require tracing a huge number of paths. In addition, such polynomial terms with very large exponents will likely generate serious and perhaps fatal numerical difficulties for the path tracker. Therefore, we face some practical constraints on the size of the rational exponents appearing in our economic models.

5.2 *Application of parameter continuation*

To demonstrate parameter continuation, which we describe in Appendix C, we choose n as the parameter and vary it from 2700 to 1000. Note that in Bertini the homotopy parameter goes from 1 to 0, so to do this, we define a homotopy just between those two values:

$$n = 2700t + (0.22334546453233 + 0.974739352i)t(1-t) + 1000(1-t).$$

Thus for $t = 1$, we have $n = 2700$, and if $t = 0$, then $n = 1000$. The complex number in the equation is the application of the gamma trick. We also have to provide the solutions for our start system. We already solved this system. We just rename Bertini's output file `finite_solutions` to `start`, which now provides Bertini with the starting points for the homotopy paths. In addition, we must alter the input file as follows:

```
CONFIG
USERHOMOTOPY: 1;
MPTYPE: 0;
END;
INPUT
variable px,py,z;
function f1, f2, f3;
pathvariable t;
parameter n;
n = t*2700+(0.22334546453233+0.974739352*I)*t*(1-t)+(1-t)*1000;
f1 = -(px^2)-py^2+z^2*px^2*py^2;
f2 = -(n)+n*px+3*n*z^2*px^2-2*n*z^2*px^3+51*z^3*px^6-2*z^3*px^7;
f3 = -(n)+n*py+3*n*z^2*py^2-2*n*z^2*py^3+51*z^3*py^6-2*z^3*py^7;
END;
```

TABLE 2. Real, positive solutions for $n = 1000$.

p_x	3.333	2.247	3.613	2.045	24.689
p_y	2.247	3.333	3.613	2.045	24.689

If we run Bertini, we obtain 14 real and 48 complex solutions. Note that the number of real solutions has dropped by 4. Thus if we had not used the gamma trick, some of our paths would have failed. There are only five positive real solutions. The first three solutions in Table 2 fail the second-order conditions for at least one firm. The fourth solution fails the global-optimality test. Only the last solution in Table 2 is an equilibrium for the Bertrand game for $n = 1000$.

5.3 *The manifold of real positive solutions*

The parameter-continuation approach allows us to compare solutions and thus equilibria for two different (vectors of) parameter values q_0 and q_1 of our economic model. Ideally we would like to push our analysis even further and, in fact, compute the equilibrium manifold for all convex combinations $sq_1 + (1-s)q_0$ with $s \in [0, 1]$.

Observe that Theorem 8 in Appendix C requires a path between q_0 and q_1 of the form

$$\varphi(s) = e^{i\gamma} s(s-1) + sq_1 + (1-s)q_0$$

with a random $\gamma \in [0, 2\pi)$. Note that for real values q_0 and q_1, the path $\varphi(s)$ is not real and so all solutions to $F(z, \varphi(s)) = 0$ are economically meaningless for $s \in (0, 1)$. This problem would not occur if we could drop the first term of $\varphi(s)$ and instead use the convex combination

$$\tilde{\varphi}(s) = sq_1 + (1-s)q_0$$

in the definition of the parameter-continuation homotopy. Now an examination of the real solutions to $F(z, \tilde{\varphi}(s)) = 0$ would provide us with the equilibrium manifold for all $\tilde{\varphi}(s)$ with $s \in [0, 1]$. Unfortunately, such an approach does not always work. As we have seen in the previous section, while the number of isolated finite solutions remains constant with probability 1, the number of real solutions may change. A parameter-continuation homotopy with $\tilde{\varphi}(s)$ does not allow for this change.

To illustrate the described difficulty, we examine two parameter-continuation homotopies in Bertini. We vary the parameter n first from 2700 to 3400 and then from 2700 to 500. Figure 5 displays the positive real solutions as a function of n over the entire range from 500 to 3400. For a clear view of the different portions of the manifold, we separate it into two graphs.

For the first homotopy, the number of positive real, other real, and complex (within nonzero imaginary part) solutions does not change as n is increased from 2700 to 3400. Therefore, in this case, the described approach to obtain the manifold of (positive) real solutions encounters no difficulties. Things are quite different for the second homotopy when n is decreased from 2700 to 500. As n approaches 1188.6, the paths for the two

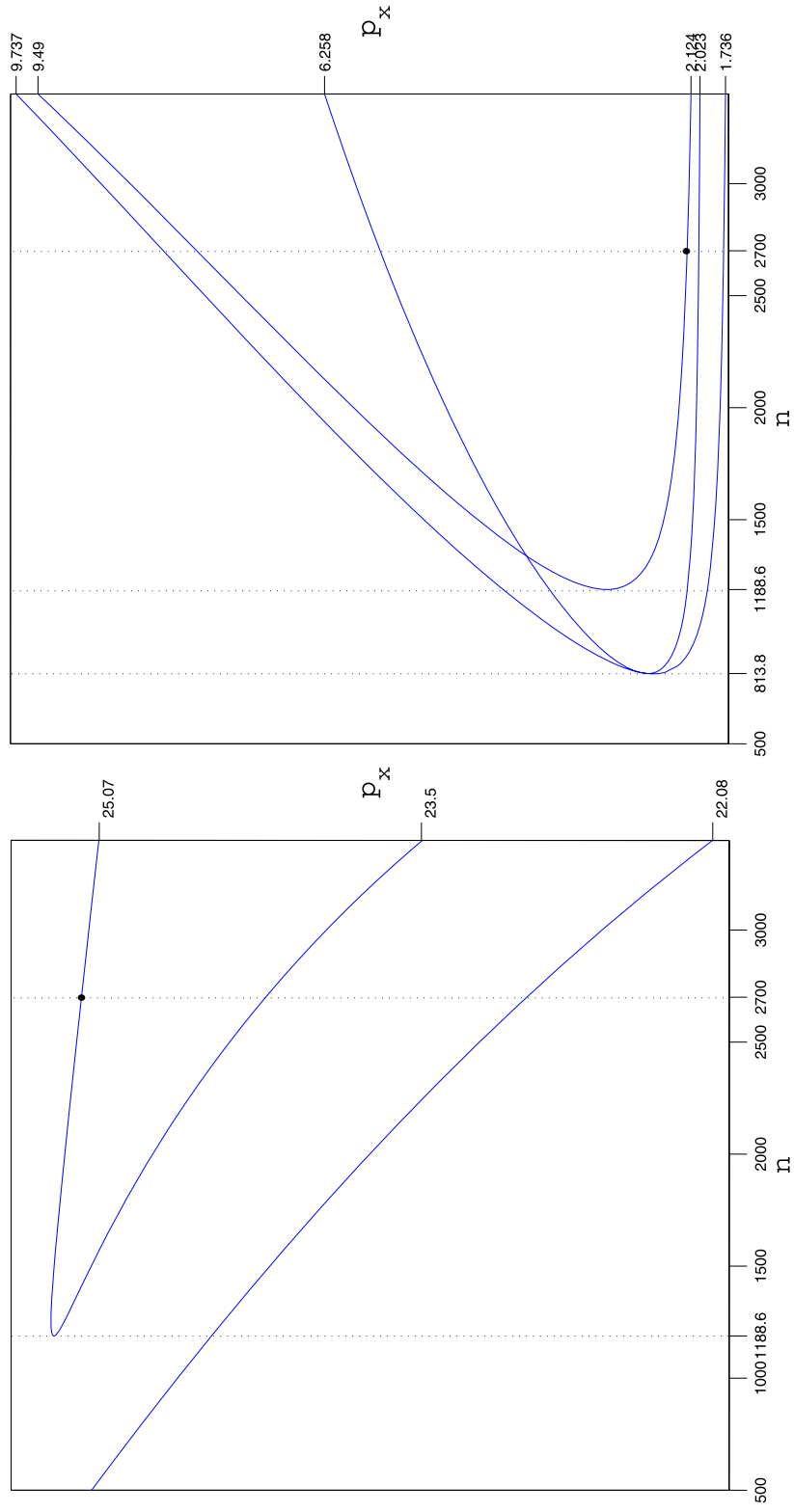

FIGURE 5. Real positive solutions as a function of n.

largest production quantities converge and then, when n is decreased further, move into complex space. The same is true for two paths in the lower graph of Figure 5. Bertini reports an error message for all four paths and stops tracking them. At $n = 1188.6$, the number of real solutions changes from 18 to 14, while the number of (truly) complex solutions with a nonzero imaginary part increases from 44 to 48. A similar change in the number of real and complex solutions occurs for $n = 813.8$.

To determine the equilibrium manifold, we need to check the second-order and global-optimality conditions for all positive real solutions. Doing so yields the equilibrium manifold in Figure 1 in Section 2.

In sum, we observe that a complete characterization of the equilibrium manifold is not a simple exercise. When we employ the parameter continuation approach with a path of parameters in real space, then we have to allow for the possibility of path-tracking failures whenever the number of real and complex solutions changes. The determination of the entire manifold of positive real solutions may, therefore, require numerous homotopy runs. Despite these difficulties, we believe that the parameter continuation approach is a very helpful tool for the examination of equilibrium manifolds.

We can continue our analysis for larger values of the market size n. Figure 6 shows the unique equilibrium price $p_x = p_y$ for $3400 \leq n \leq 10{,}000$. The market of type 2 customers is so large that both firms choose a mass-market strategy and charge a low price. While the number of equilibria remains constant for large values of n, the number of real

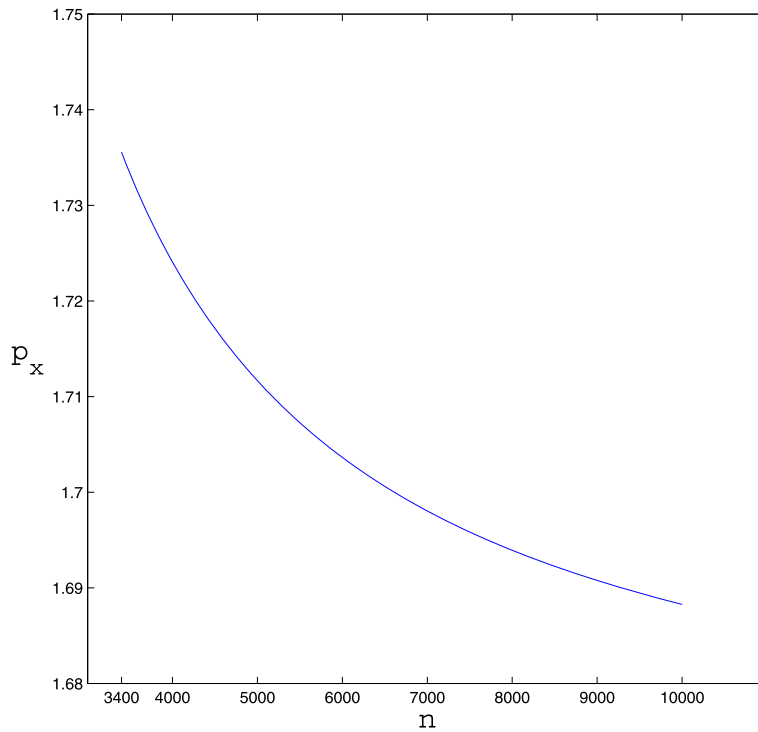

FIGURE 6. Unique equilibrium for large values of n.

solutions changes twice in the examined region. Recall that there are 18 real solutions for $n = 3400$. This number decreases to 16 at about $n = 5104.5$ and further to 14 at about $n = 5140.8$.

6. Equilibrium equations for dynamic stochastic games

In this section, we first briefly describe a general setup of dynamic stochastic games. Such games date back to Shapley (1953); for a textbook treatment, see Filar and Vrieze (1997). Subsequently, we explain how Markov-perfect equilibria (MPE) in these games can be characterized by nonlinear systems of equations.

6.1 *Dynamic stochastic games: General formulation*

We consider discrete-time infinite-horizon dynamic stochastic games of complete information with N players. In period $t = 0, 1, 2, \dots,$ player $i \in \{1, 2, \dots, N\}$ is characterized by its state $\omega_{i,t} \in \Omega_i$. The set of possible states, Ω_i, is finite and, without loss of generality, we thus define $\Omega_i = \{1, 2, \dots, \hat{\omega}_i\}$ for some number $\hat{\omega}_i \in \mathbb{N}$. The product $\Omega = \prod_{i=1}^{N} \Omega_i$ is the state space of the game; the vector $\omega_t = (\omega_{1,t}, \omega_{2,t}, \dots, \omega_{N,t}) \in \Omega$ denotes the state of the game in period t.

Players choose actions simultaneously. Player i's action in period t is $a_{i,t} \in \mathcal{A}_i(\omega_t)$, where $\mathcal{A}_i(\omega_t)$ is the set of feasible actions for player i in state ω_t. In many economic applications of dynamic stochastic games, $\mathcal{A}_i(\omega_t)$ is a convex subset of $\mathbb{R}^M$, $M \in \mathbb{N}$, and we adopt this assumption here to employ standard first-order conditions in the analysis. We denote the collection of all players' actions in period t by $a_t = (a_{1,t}, a_{2,t}, \dots, a_{N,t})$ and denote the collection of all but player i's actions by $a_{-i,t} = (a_{1,t}, \dots, a_{i-1,t}, a_{i+1,t}, \dots, a_{N,t})$.

Players' actions affect the probabilities of state-to-state transitions. If the state in period t is ω_t and the players choose actions a_t, then the probability that the state in period $t+1$ is ω^+ is $\Pr(\omega^+ | a_t; \omega_t)$. In many applications, the transition probabilities for player i's state are assumed to depend on player i's actions only and to be independent of other players' actions and transitions in their states. We follow this custom and make the same assumption. Denoting the transition probability for player i's state by $\Pr_i((\omega^+)_i | a_{i,t}; \omega_{i,t})$, the transition probability for the state of the game therefore satisfies

$$\Pr(\omega^+ | a_t; \omega_t) = \prod_{i=1}^{N} \Pr_i((\omega^+)_i | a_{i,t}; \omega_{i,t}).$$

If the state of the game is ω_t in period t and the players choose actions a_t, then player i receives a payoff $\pi_i(a_t, \omega_t)$. Players discount future payoffs using a discount factor $\beta \in (0, 1)$. The objective of player i is to maximize the expected net present value of all its future cash flows,

$$\mathrm{E}\left\{ \sum_{t=0}^{\infty} \beta^t \pi_i(a_t; \omega_t) \right\}.$$

Economic applications of dynamic stochastic games typically rely on the equilibrium notion of a pure strategy Markov-perfect equilibrium (MPE). That is, attention is

restricted to pure equilibrium strategies that depend only on the current state and are independent of the history of the game. We can thus drop the time subscript. Player i's strategy A_i maps each state $\omega \in \Omega$ into its set of feasible actions $\mathcal{A}_i(\omega)$. The actions of all other players in state ω prescribed by their respective strategies are denoted $A_{-i}(\omega) = (A_1(\omega), \dots, A_{i-1}(\omega), A_{i+1}(\omega), \dots, A_N(\omega))$. Finally, we denote by $V_i(\omega)$ the expected net present value of future cash flows to player i if the current state is ω. The mapping $V_i : \Omega \to \mathbb{R}$ is player i's value function.

For given Markovian strategies A_{-i} of all other players, player i faces a discounted infinite-horizon dynamic programming problem. As Doraszelski and Judd (2012) pointed out, Bellman's principle of optimality implies that the optimal solution for this dynamic programming problem is again a Markovian strategy A_i. That is, a Markov-perfect equilibrium remains subgame perfect even without the restriction to Markovian strategies. The Bellman equation for player i's dynamic programming problem is

$$V_i(\omega) = \max_{a \in \mathcal{A}_i(\omega)} \left\{ \pi_i(a, A_{-i}(\omega); \omega) + \beta \mathrm{E}[V_i(\omega^+) | a, A_{-i}(\omega); \omega] \right\}, \tag{14}$$

where the expectation operator $\mathrm{E}[\cdot|\cdot]$ determines the conditional expectation of the player's continuation values $V_i(\omega^+)$, which are a function of the next period's state ω^+, which in turn depends on the players current action a, the other players' actions $A_{-i}(\omega)$, and the current state ω. We denote by

$$h_i(a, A_{-i}(\omega); \omega; V_i) = \pi_i(a, A_{-i}(\omega); \omega) + \beta \mathrm{E}[V_i(\omega^+) | a, A_{-i}(\omega); \omega]$$

the maximand in the Bellman equation. Player i's optimal action $A_i(\omega) \in \mathcal{A}_i(\omega) \subset \mathbb{R}^M$ in state ω is given by

$$A_i(\omega) = \arg \max_{a \in \mathcal{A}_i(\omega)} h_i(a, A_{-i}(\omega); \omega; V_i). \tag{15}$$

For each player $i = 1, 2, \dots, N$, equations (14) and (15) yield optimality conditions on the unknowns $V_i(\omega)$ and $A_i(\omega)$ in each state $\omega \in \Omega$. A Markov-perfect equilibrium (in pure strategies) is now a simultaneous solution to equations (14) and (15) for all players and states.

6.2 *Equilibrium conditions*

Doraszelski and Satterthwaite (2010) developed sufficient conditions for the existence of a Markov-perfect equilibrium for a class of dynamic stochastic games. A slightly modified version of the existence result in their Proposition 2 holds in the described model under the assumptions that both actions and payoffs are bounded, and the maximand function $h_i(\cdot, A_{-i}(\omega); \omega; V_i)$ is strictly concave for all $\omega \in \Omega$, other players' strategies A_{-i}, and value functions V_i satisfying the Bellman equation. Under these assumptions, the maximand $h_i(\cdot, A_{-i}(\omega); \omega; V_i)$ has a unique maximizer $A_i(\omega)$. This unique maximizer could lie on the boundary of or be an interior solution of the set of feasible actions $\mathcal{A}_i(\omega)$. (As V_i changes, so does the maximizer, and there could be several consistent solutions and thus equilibria.)

For the purpose of this paper, we restrict attention to models that satisfy two further assumptions that are frequently made in economic applications. First, the function $h_i(\cdot, A_{-i}(\omega); \omega; V_i)$ is continuously differentiable. Second, we assume that the maximizer in equation (15) is always an interior solution. Under these assumptions, we can equivalently characterize players' optimality conditions (14) and (15) by a set of necessary and sufficient first-order conditions,

$$0 = \frac{\partial}{\partial a}\big\{\pi_i(a, A_{-i}(\omega); \omega) + \beta \mathrm{E}[V_i(\omega_+)|a, A_{-i}(\omega); \omega]\big\}\Big|_{a=A_i(\omega)}, \tag{16}$$

$$V_i(\omega) = \pi_i(a, A_{-i}(\omega); \omega) + \beta \mathrm{E}[V_i(\omega_+)\big|a, A_{-i}(\omega); \omega]|_{a=A_i(\omega)}. \tag{17}$$

Thus we have $M+1$ equations for each state $\omega \in \Omega$ and for each player $i = 1, 2, \ldots, N$. Any simultaneous solution of pure strategies $A_1(\omega), \ldots, A_N(\omega)$ and values $V_1(\omega), \ldots, V_N(\omega)$ for all states $\omega \in \Omega$ yields an MPE.

If the payoff functions π_i and the probability functions $\Pr(\omega^+|\cdot; \omega)$ are rational functions, then the nonlinear equilibrium equations can be transformed into a polynomial system of equations. In the next two sections we examine two economic models that satisfy these assumptions.

7. Learning curve

In many industries, the marginal cost of production decreases with the cumulative output: this effect is often called learning-by-doing. The impact of learning-by-doing on market equilibrium has been studied in the industrial organization literature for decades. Early work in this area includes Spence (1981) and Fudenberg and Tirole (1983b). Besanko et al. (2010) analyzed learning-by-doing and organizational forgetting within the framework of Ericson and Pakes (1995).

In this section, we examine a basic learning-by-doing model in the Ericson and Pakes (1995) framework. Although the functional forms for the price functions and transition probabilities are not polynomial, we can derive a system of polynomial equations such that all positive real solutions of this system are Markov-perfect equilibria.

7.1 *A learning-by-doing model*

There are $N = 2$ firms and two goods. Firm i produces good i, $i = 1, 2$. The output of firm i is denoted by q_i, which is the firm's only action. (In the language of our general formulation, $a_i = q_i$.) The state variable ω_i for firm i is a parameter in the firm's production cost function $c_i(q_i; \omega_i)$. In our numerical example, we assume $c_i(q_i; \omega_i) = \omega_i q_i$, implying that the state ω_i is firm i's unit cost of production. For simplicity, we assume, without loss of generality, that $\omega_i \in \Omega_i = \{1, 2, \ldots, \hat{\omega}_i\}$.

In each period, the two firms engage in Cournot competition. Customers' utility function over the two goods (and money M) is

$$u(q_1, q_2) = w\frac{\gamma}{\gamma - 1}\big(q_1^{(\sigma-1)/\sigma} + q_2^{(\sigma-1)/\sigma}\big)^{(\gamma-1)\sigma/(\gamma(\sigma-1))} + M,$$

where σ is the elasticity of substitution between goods 1 and 2, γ is the elasticity of demand for the composite good $(q_1^{(\sigma-1)/\sigma} + q_2^{(\sigma-1)/\sigma})^{\sigma/(\sigma-1)}$, and w is a weighting factor. The resulting market clearing prices for the two goods are then

$$P_1(q_1, q_2) = wq_1^{-1/\sigma}\big(q_1^{(\sigma-1)/\sigma} + q_2^{(\sigma-1)/\sigma}\big)^{(\gamma-\sigma)/(\gamma(\sigma-1))},$$
$$P_2(q_1, q_2) = wq_2^{-1/\sigma}\big(q_1^{(\sigma-1)/\sigma} + q_2^{(\sigma-1)/\sigma}\big)^{(\gamma-\sigma)/(\gamma(\sigma-1))},$$

where $P_i(q_1, q_2) = \frac{\partial}{\partial q_i}u(q_1, q_2)$ denotes the price of good i if sales of the two goods are (q_1, q_2). And so if the two firms produce the quantities (q_1, q_2) in state $\omega = (\omega_1, \omega_2)$, their resulting payoffs are

$$\pi_i(q_i, q_{-i}; \omega) = P_i(q_1, q_2)q_i - c_i(q_i; \omega_i). \tag{18}$$

Note that in this model, firm i's payoff does not explicitly depend on the other firm's state, but only implicitly via the other firm's production quantity.

The dynamic aspect of the model arises from changes in the unit cost ω_i. Through learning-by-doing the firms can reduce their unit cost. In our numerical example, we use the popular functional form (see Pakes and McGuire (1994), Borkovsky, Doraszelski, and Kryukov (2009), and many other papers) for the transition probabilities

$$\begin{aligned} &\Pr_i[\omega_i - 1|q_i; \omega_i] = \frac{Fq_i}{1+Fq_i}, \qquad \Pr_i[\omega_i|q_i; \omega_i] = \frac{1}{1+Fq_i}, \\ &0 \quad \text{otherwise} \end{aligned} \tag{19}$$

with some constant $F > 0$ for $\omega_i \geq 2$. The lowest-cost state $\omega_i = 1$ is an absorbing state. Note that outside the absorbing state, the higher is a firm's production quantity, the higher is its probability to move to the next lower-cost state. We assume that the transition probability functions are independent across firms.

Substituting the expressions (18) and (19) into the equilibrium equations (16) and (17) yields a system of equilibrium equations for the learning-by-doing model. This system has four equations for each state $\omega = (\omega_1, \omega_2)$ and thus a total of $4|\hat{\omega}_1||\hat{\omega}_2|$ equations and unknowns.

Solving the system of equations is greatly simplified by the observation that the nature of the transitions in this model induces a partial order on the state space Ω. The unit cost ω_i can only decrease, but never increase, during the course of the game. Instead of solving one large system of equations, we can successively solve systems of four equations state by state. For the lowest-cost state $(1, 1)$, we only need to find the static Cournot equilibrium and calculate the values $V_i(1, 1)$. Next we can successively solve the systems for the states $(\omega_1, 1)$ with $\omega_1 = 2, 3, \ldots, \hat{\omega}_1$ and for the states $(1, \omega_2)$ with $\omega_2 = 2, 3, \ldots, \hat{\omega}_2$. Next we can do the same for all $(\omega_1, 2)$ with $\omega_1 = 2, 3, \ldots, \hat{\omega}_1$, for all nodes $(2, \omega_2)$ with $\omega_2 = 3, \ldots, \hat{\omega}_1$, and so on. For symmetric games, we can further reduce the workload. We only need to solve system of equations for the states (ω_1, ω_2) with $\omega_2 \leq \omega_1$, that is, for $(1, 1)$, solve $(\omega_1, 2)$ for $\omega_1 = 2, 3, \ldots, \hat{\omega}_1$, $(\omega_1, 3)$ for $\omega_1 = 3, \ldots, \hat{\omega}_1$, and so on.

7.2 *Solving the equilibrium equations with Bertini*

We compute Markov-perfect equilibria for the learning-by-doing game for the following parameter values. We consider a utility function with $\sigma = 2$, $\gamma = 3/2$, and $w = 100/3$. The parameter for the transition probability function is $F = 1/5$. The firms use the discount factor $\beta = 0.95$. We only examine symmetric cases with $\Omega_1 = \Omega_2$.

Similar to the static game in Section 5, the equilibrium equations in this model contain fractions and radical terms. The transformation of the equations into polynomial form forces us to introduce auxiliary variables Q_1, Q_2, and Q_3 that are defined as

$$Q_1^2 = q_1, \qquad Q_2^2 = q_2, \qquad Q_3^2 = Q_1 + Q_2.$$

The introduction of these new variables enables us to eliminate the value function terms $V_i(q_1, q_2)$ of both firms. For each state (ω_1, ω_2), we obtain a system of five equations in the five unknowns q_1, q_2, Q_1, Q_2, and Q_3. There is a multiple root at 0. To remove it, we add another variable t and a normalization equation $tQ_1 - 1 = 0$, thereby obtaining a system with six variables and six equations.

We solve four different types of polynomial systems. First, we solve the system of the absorbing state $(1, 1)$. The monomials with the highest degrees of the six equations are

$$tQ_1, \qquad Q_3^3, \qquad Q_1^2, \qquad Q_2^2, \qquad -Q_1Q_3(Q_1 + Q_2), \qquad -Q_2Q_3(Q_1 + Q_2),$$

respectively, resulting in a Bezout number of $2^3 \cdot 3^3 = 216$. Using m-homogeneity, the number of paths to track reduces to 44. Bertini tracks these 44 paths in just under 4 seconds.

Next we solve the equations for the states $(1, \omega_2)$ for $\omega_2 \geq 2$. The highest degree terms of the six equations are

$$\begin{aligned} &tQ_1, \qquad Q_3^3, \qquad Q_1^2, \qquad Q_2^2, \qquad -Q_1Q_3(Q_1 + Q_2), \\ &(9F^2\omega_2)Q_1Q_2Q_3q_2^2 + (9F^2\omega_2)Q_2^2Q_3q_2^2, \end{aligned}$$

respectively, resulting in a Bezout number of $2^3 \cdot 3^2 \cdot 5 = 360$. Thanks to m-homogeneity, we need to track 140 paths and this takes us, with Bertini, about 1 minute for each ω_2.

Then we solve the equations for state $(2, 2)$, where the highest degree terms are

$$\begin{aligned} &tQ_1, \qquad Q_3^3, \qquad Q_1^2, \qquad Q_2^2, \qquad (9F^4\omega_1)Q_1^2Q_3q_1^2q_2^2 + (9F^4\omega_1)Q_1Q_2Q_3q_1^2q_2^2, \\ &(9F^4\omega_2)Q_1Q_2Q_3q_1^2q_2^2 + (9F^4\omega_2)Q_2^2Q_3q_1^2q_2^2, \end{aligned}$$

so the Bezout number is $2^3 \cdot 3 \cdot 7^2 = 1176$. Exploiting m-homogeneity, we have to track 364 paths, which takes about 5 minutes. There are 152 real and complex (finite) solutions.

For the remaining states, we can now use parameter continuation, since the degree structure of the systems is identical to that of the equations for state $(2, 2)$. The Bezout number remains the same as for state $(2, 2)$, but now we only have to track 152 paths, since that was the number of solutions to the system at $(2, 2)$. (To check whether 152 is indeed the maximal number k of isolated finite solutions as in Theorem 8, we solve a few systems with randomly chosen coefficients but the same degree structure. In all cases

TABLE 3. Production quantities q_1 and value function V_1 of firm 1.

$\omega_1 \backslash \omega_2$	5		4		3		2		1	
5	7.202	874	7.108	861	7.009	851	6.889	843	6.626	838
4	8.850	939	8.748	925	8.620	913	8.464	905	8.137	899
3	11.475	996	11.385	982	11.233	969	11.016	959	10.573	953
2	16.921	1042	16.840	1027	16.699	1014	16.401	1003	15.714	997
1	38.228	1072	38.171	1057	38.056	1043	37.773	1032	36.600	1025

there are 152 isolated finite solutions.) Tracking these 152 paths takes about 25 seconds for each state. Again we observe that tracking paths ending at finite solutions takes much less time than tracking paths that end at points at infinity. The reason is again that some of the solutions at infinity lie within continua of solutions and thus cause numerical difficulties.

We solved instances of the described learning-by-doing model with many states for each firm. We wrote a C++ script that solved the problem by backward induction by calling Bertini at each state.[11] To keep the presentation of the results manageable, we report here the results for a symmetric game with $\hat{\omega}_1 = 5$. In all our systems there was a unique real positive solution for all variables. Therefore, we found a unique Markov-perfect equilibrium for the learning-by-doing model. Table 3 reports the production quantities q_1 and the values of the value function V_1 of firm 1. For example, in state $(\omega_1, \omega_2) = (3, 4)$, firm 1 produces $q_1 = 11.385$ and the game has a value of $V_1 = 982$ for the firm. By symmetry, the corresponding values for firm 2 are $(q_2, V_2) = (8.620, 913)$.

Table 4 reports running times on a laptop (Intel Core 2 Duo T9550 with 2.66 GHz and 4 GB RAM) for the learning-by-doing model. The running times grow approximately linearly in the number of states $\hat{\omega}_1 \times \hat{\omega}_2$ and so we could easily solve games with many more states per firm.

8. COST-REDUCING INVESTMENT AND DEPRECIATION

In models of cost-reducing investment, spending on investment reduces future production cost; see, for example, Flaherty (1980) and Spence (1984). In models of irreversible investment, current investment spending increases future production capacity; see Fudenberg and Tirole (1983a). Besanko and Doraszelski (2004) presented a model with both capacity investments and depreciation within the Ericson and Pakes (1995) framework. Depreciation tends to offset investment. In this section, we describe

TABLE 4. Running times.

$\hat{\omega}_1 = \hat{\omega}_2$	3	5	7	10
Time (sec)	477	745	1359	2852

[11]The script is available at http://www.business.uzh.ch/professorships/qba/publications/Software.html.

a stochastic dynamic game model in which the marginal cost of production may decrease through investment or increase through depreciation.

8.1 *A cost-reducing investment model*

The model of Cournot competition is the same as in the learning-by-doing model with the only exception being that a firm's production quantity does not affect its unit cost. The dynamic aspect of the model arises again from changes in the unit cost ω_i. Both increases and decreases of the unit cost are possible. Firms may be hit by a depreciation shock that results in a cost increase, but they can also make a cost-reducing investment. A depreciation shock increases the unit cost from ω_i to $\omega_i + 1$ and has probability $\delta > 0$. If firm i makes a cost-reducing investment y_i at a cost $\mathrm{cr}_i(y_i)$, then it achieves a probabilistic reduction of its cost state. In our numerical examples, we assume a quadratic investment cost function, $\mathrm{cr}_i(y) = D_i y^2$. Total per-period payoff is then the difference of the Cournot profit and the investment cost,

$$\begin{aligned}\pi_i(q_i, y_i, q_{-i}, y_{-i}; \omega) &= \pi_i^C(q_i, q_{-i}; \omega) - \mathrm{cr}_i(y_i)\\ &= P_i(q_1, q_2)q_i - c_i(q_i; \omega_i) - D_i y_i^2.\end{aligned}$$

We assume a transition function of the form (19) with the investment level y_i replacing the Cournot quantity. Assuming independence of the depreciation probabilities and the investment transition function then results in the transition probabilities (see also Besanko and Doraszelski (2004))

$$\Pr_i[\omega_i - 1|y_i; \omega_i] = \frac{Fy_i}{1 + Fy_i}(1 - \delta) \quad \text{for } 2 \le \omega_i \le \hat{\omega}_i, \tag{20}$$

$$\Pr_i[\omega_i + 1|y_i; \omega_i] = \frac{1}{1 + Fy_i}\delta \quad \text{for } 1 \le \omega_i \le \hat{\omega}_i - 1, \tag{21}$$

$$\begin{aligned}\Pr_i[\omega_i|y_i; \omega_i] &= 1 - \Pr_i[\omega_i - 1|y_i; \omega_i]\\ &\quad - \Pr_i[\omega_i + 1|y_i; \omega_i] \quad \text{for } 2 \le \omega_i \le \hat{\omega}_i - 1.\end{aligned} \tag{22}$$

The remaining transition probabilities are

$$\Pr_i[1|y_i; 1] = 1 - \Pr_i[2|y_i; 1], \tag{23}$$

$$\Pr_i[\hat{\omega}_i|y_i; \hat{\omega}_i] = 1 - \Pr_i[\hat{\omega}_i - 1|y_i; \hat{\omega}_i]. \tag{24}$$

Substituting the expressions for payoffs and transition probabilities into the equilibrium equations (16) and (17) yields a system of equilibrium equations for the model. The static Cournot game played in each period does not affect the transition probabilities and so we can solve the two equations at each state that are derived from differentiating with respect to the production quantities q_1 and q_2 independently from the remaining equations. The remaining system consists of four equations for each state $\omega = (\omega_1, \omega_2)$ and thus has a total of $4|\hat{\omega}_1||\hat{\omega}_2|$ equations and unknowns. The degree of each equation is 4.

8.2 *Solving the equilibrium equations with Bertini*

Since the unit cost ω_i may increase or decrease, we cannot solve the equations state by state as in the learning-by-doing model. Instead we need to solve a single system of equations.[12]

8.2.1 *Two states for each firm* We describe the solution of the cost-reducing investment game with depreciation for the parameter values $\beta = 0.95$, $D_1 = D_2 = 1$, $F = 0.2$, and $\delta = 0.1$. The parameters for the utility functions are again $\sigma = 2$, $\gamma = 3/2$, and $w = 100/3$. Each firm can be in one of two states. We set $\Omega_1 = \Omega_2 = \{1, 5\}$ (in a slight abuse of previous notation).

We first solve the Cournot game for each state. The production quantities of firm 1 are

$$q_1(5,5) = 3.2736, \qquad q_1(5,1) = 2.4664,$$
$$q_1(1,5) = 38.224, \qquad q_1(1,1) = 36.600.$$

For this model with $2 \times 2 = 4$ states, there are 16 equations and variables. The resulting Bezout number is $4^{16} = 4{,}294{,}967{,}296$. By utilizing symmetry, we simplify our problem to 8 equations and variables with a total Bezout number of $4^8 = 65{,}536$. Utilizing m-homogeneity, we reduce the number of paths to 3328. It took us 1 hour 40 minutes to solve this problem. We found a total of 589 finite, that is, complex and real, solutions that lie in affine space, 44 of which are real. We had no path failures when using adaptive precision.[13] Only one of those real solutions is economically relevant. The investment levels of firm 1 are

$$y_1(5,5) = 3.306, \qquad y_1(5,1) = 3.223, \qquad y_1(1,5) = 0.763, \qquad y_1(1,1) = 0.736,$$

resulting in the following values of the value function:

$$V_1(5,5) = 816.313, \qquad V_1(5,1) = 794.329,$$
$$V_1(1,5) = 926.059, \qquad V_1(1,1) = 895.570.$$

8.2.2 *Three states for each firm* We choose $\Omega_1 = \Omega_2 = \{1, 5, 10\}$ and our other parameters as in the two-state case. The production quantities of firm 1 in the additional high-cost states are $q_1(10,10) = 1.1574$ and

$$q_1(10,5) = 1.0648, \qquad q_1(10,1) = 0.70015,$$
$$q_1(5,10) = 3.3975, \qquad q_1(1,10) = 37.915.$$

[12]We perform all calculations and derive the final system in Mathematica. The Mathematica file is available at http://www.business.uzh.ch/professorships/qba/publications/Software.html.

[13]If we do not use adaptive precision, we can finish computations in just under 3 minutes. However, then 396 paths fail to converge. Nevertheless we still obtain all finite solutions. Clearly, if we could prove that all equilibria are regular solutions to the polynomial system of equilibrium equations, then we could relax the precision parameters in Bertini and thus significantly reduce both the computational effort and the running time.

TABLE 5. Equilibrium investment levels y_1 and value function V_1.

$\omega_1 \backslash \omega_2$	10		5		1	
10	3.42	705.00	3.31	680.31	3.24	663.01
5	3.78	820.36	3.70	789.73	3.62	765.53
1	0.86	945.48	0.83	911.76	0.80	878.89

Solving the system of equilibrium equations for the three-state model now poses significantly more problems than the two-state case. The initial system has 36 equations and unknowns. The Bezout number is $4^{36} \approx 4.72 \cdot 10^{21}$. After exploiting symmetry and using some algebraic operations to simplify some equations, we obtain a system that has 21 equations and unknowns. Its Bezout number is 1,528,823,808. This system, however, is still unsolvable on a single laptop if we use the standard homotopy approach. For this reason, we now apply the splitting approach from Appendix D. We split the system into two subsystems that are both small enough to be solvable. In our example, the first system has $M_1 = 358$ nonsingular solutions. The second system has $M_2 = 4510$ nonsingular solutions. Therefore, if we focus only on the nonsingular solutions, we have $358 \times 4510 = 1{,}614{,}580$ paths to track when we combine the two subsystems via a parameter continuation homotopy. Note that this is an order of magnitude smaller than taking the system as a whole. We obtain a unique nonsingular equilibrium; see Table 5. The time to solve this on a single core is over a week.[14]

9. Conclusion

We summarize the paper and discuss the current limitations of all-solution methods.

9.1 *Summary*

This paper describes state-of-the-art techniques for finding all solutions of polynomial systems of equations, and illustrates these techniques by computing all equilibria of both static and dynamic games with continuous strategies. The requirement of polynomial equations may, at first, appear very restrictive. In our first application—a static Bertrand pricing game—we show how certain types of nonpolynomial equilibrium conditions can be transformed into polynomial equations. We also show how, with repeated application of the polynomial techniques, we can deal with first-order conditions that are necessary but not sufficient. Finally, this example also depicts the power of the parameter-continuation homotopy approach. This approach greatly reduces the number of homotopy paths that need to be traced and, therefore, increases the size of models that we can analyze. When handled carefully, it even allows us to trace out the equilibrium manifold.

[14]The files are available at http://www.business.uzh.ch/professorships/qba/publications/Software.html.

We also apply the all-solution techniques to two stochastic dynamic games of industry competition and check for equilibrium uniqueness. In the first application—a learning-by-doing model of industry competition—the equilibrium system separates into many small systems of equations that can be solved sequentially. As a result, we can solve specifications of this model with many states. In our second application—a model with cost-reducing investment and cost-increasing depreciation—such a separation of the equilibrium system is impossible. Solving the resulting equilibrium system requires tracing a huge number of paths. On a single laptop, we can solve specifications of the model with only a small number of states.

9.2 *Current limitations and future work*

For stochastic dynamic games, the number of equations grows exponentially in the number N of players and polynomially (with degree N) in the number of states. In turn, the Bezout number grows exponentially in the number of nonlinear equations. Additionally, the degree of the polynomials is essential, which limits the parameter choice for the exponents in the utility functions. As a result, the number of paths that an all-solution method must trace grows extremely fast in the size of the economic model. This growth clearly limits the size of problems we can hope to solve.

Modern policy-relevant models quickly generate systems of polynomial equations with thousands of equations. For example, the model in Besanko et al. (2010) has up to 900 states and 1800 equations. Finding all equilibria of models of this size is impossible with the computer power available as of the writing of this paper and it will remain out of reach for the foreseeable future. However, we will likely be able to solve smaller models such as the dynamic model of capacity accumulation of Besanko and Doraszelski (2004) with at most 100 states within a few years. Progress will come on at least three frontiers. First, computer scientists have yet to optimize the performance of software packages such as Bertini. Second, the all-solution homotopy methods are ideally suited for parallel computations. Our initial experience has been very promising. So as soon as the existing software has been adapted to large parallel computing systems, we will see great progress in the size of the models we can analyze with the methods described in this paper. Third, methodological advances such as the equation splitting approach will also help us to solve larger systems.

Appendix A: Homogenization

The all-solution homotopy method presented in Section 3.3 has the unattractive feature that it must follow diverging paths. Homogenization of the polynomials greatly reduces the computational effort to track such paths.

Definition 8. The homogenization $\hat{f}_i(z_0, z_1, \dots, z_n)$ of the polynomial $f_i(z_1, \dots, z_n)$ of degree d_i is defined by

$$\hat{f}_i(z_0, z_1, \dots, z_n) = z_0^{d_i} f_i\left(\frac{z_1}{z_0}, \dots, \frac{z_n}{z_0}\right).$$

Effectively, each term of $\hat{f}_i$ is obtained from multiplying the corresponding term of f_i by the power of z_0 that leads to a new degree of that term of d_i. So if the term originally had degree d_{ij}, then it is multiplied by $z_0^{d_i-d_{ij}}$. Performing this homogenization for each polynomial f_i in the system

$$F(z_1, \dots, z_n) = 0 \tag{25}$$

leads to the transformed system

$$\hat{F}(z_0, z_1, \dots, z_n) = 0. \tag{26}$$

For convenience, we use the notation $\hat{z} = (z_0, z_1, \dots, z_n)$ and write $\hat{F}(\hat{z}) = 0$. By construction, all polynomials $\hat{f}_i$, $i = 1, \dots, n$, are homogeneous and so for any solution $\hat{b}$ of $\hat{F}(\hat{z}) = 0$, it holds that $\hat{F}(\lambda\hat{b}) = 0$ for any complex scalar $\lambda \in \mathbb{C}$. So the solutions to system (26) are complex lines through the origin in $\mathbb{C}^{n+1}$.

Definition 9. The n-dimensional complex projective space CP^n is the set of lines in $\mathbb{C}^{n+1}$ that go through the origin. The space $\mathbb{C}^{n+1}$ is called the affine space.

A point in projective space CP^n corresponds to a line through the origin of the affine space $\mathbb{C}^{n+1}$. Let $[\hat{b}] \in \mathrm{CP}^n$ denote a point in CP^n. Then there is a point $\hat{b} = (\hat{b}_0, \hat{b}_1, \dots, \hat{b}_n) \in \mathbb{C}^{n+1} \setminus \{0\}$ that determines this line. We denote the line $[\hat{b}]$ by $(\hat{b}_0 : \hat{b}_1 : \cdots : \hat{b}_n)$ to distinguish it from a single point. The notation $(z_0 : z_1 : \cdots : z_n)$ is called the *homogeneous coordinates* of CP^n. Note, however, that this notation is not unique: we can take any $\lambda\hat{b}$ with $\lambda \in \mathbb{C} \setminus \{0\}$ as a representative. Furthermore, $(0 : 0 : \cdots : 0)$ is not a valid point in projective space. Thus for any point $(\hat{b}_0 : \cdots : \hat{b}_n)$, there exists at least one element $\hat{b}_i \neq 0$.

There is a one-to-one relationship between the solutions of system (25) in $\mathbb{C}^n$ and the solutions of system (26) in $\mathbb{C}^{n+1}$ with $\hat{b}_0 \neq 0$. If b is a solution to (25), then the line through $\hat{b} = (1, b)$, that is, $[\hat{b}] \in \mathrm{CP}^n$, is a solution to (26). For the converse, if $(\hat{b}_0 : \hat{b}_1 : \cdots : \hat{b}_n)$ with $\hat{b}_0 \neq 0$ is a solution to (26), then the point $(\frac{\hat{b}_1}{\hat{b}_0}, \dots, \frac{\hat{b}_n}{\hat{b}_0})$ is a solution of (25).

One of the advantages of the homogenized system (26) is that it can model "infinite" solutions. If we have a line $\{(\lambda b) \mid \lambda \in \mathbb{C}\} \subset \mathbb{C}^n$, $b \in \mathbb{C}^n \setminus \{0\}$, and look at the corresponding line $\{(1 : \lambda b_1 : \cdots : \lambda b_n) \mid \lambda \in \mathbb{C}\}$ in projective space, then for any λ, $(\frac{1}{\lambda} : b_1 : \cdots : b_n)$ is also a valid representation of that point on the projective line. So if $\|\lambda\| \to \infty$, then $\|\frac{1}{\lambda}\| \to 0$ and we are left with the point $(0 : b_1 : \cdots : b_n)$. Note that $\|\lambda\| \to \infty$ in the affine space means $\|\lambda b\| \to \infty$. Thus we traverse the line to "infinity." This observation leads to the following definition.

Definition 10. Consider the natural embedding of $\mathbb{C}^n$ with coordinates $(z_1, \dots, z_n)$ in the projective space CP^n with homogeneous coordinates $(z_0 : \cdots : z_n)$. Then we call points $(0 : b_1 : \cdots : b_n) \in \mathrm{CP}^n$ *points at infinity*.

The value $\hat{b}_0 = 0$ for a solution $\hat{b}$ to $\hat{F}$ implies $\hat{f}_i(\hat{b}_0 : \hat{b}_1 : \cdots : \hat{b}_n) = f_i^{(d_i)}(\hat{b}_1, \dots, \hat{b}_n) = 0$. Therefore, the solutions at infinity of $\hat{F}(\hat{z}) = 0$ correspond to the solutions to the system

$(f_1^{(d_1)}, \dots, f_n^{(d_n)}) = 0$. The fact that we now have a representation of solutions at infinity leads to a new version of Bezout's theorem for projective space.

THEOREM 6 (Bezout's Theorem in Projective Space CP^n). *If system* (26) *has only a finite number of solutions in* CP^n *and if* d *is the Bezout number of* F, *then it has exactly* d *solutions* (*counting multiplicities*) *in* CP^n.

If we view the system of equation (26) in affine space $\mathbb{C}^{n+1}$ instead of in complex projective space CP^n, then it is actually underdetermined because it consists of n equations in $n+1$ unknowns. For a computer implementation of a homotopy method, however, we need a determinate system of equations. For this purpose, we add a simple normalization. Using the described relationship between solutions of the two systems (25) and (26), we can now introduce a third system to find the solutions of system (25). Define a new linear function

$$u(z_0, z_1, \dots, z_n) = \xi_0 z_0 + \xi_1 z_1 + \cdots + \xi_n z_n$$

with random coefficients $\xi_i \in \mathbb{C}$. (The nongeneric cases are where the normalization line is parallel to a solution "line.") Now define

$$\begin{aligned} \tilde{f}_i(z_0, z_1, \dots, z_n) &:= \hat{f}_i(z_0, z_1, \dots, z_n), \quad i = 1, \dots, n, \\ \tilde{f}_0(z_0, z_1, \dots, z_n) &:= u(z_0, z_1, \dots, z_n) - 1. \end{aligned} \tag{27}$$

The resulting system of equations

$$\tilde{F} = (\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_n) = 0 \tag{28}$$

has $n+1$ equations in $n+1$ variables. Note that the system $\tilde{F}(\hat{z})$ has the same total degree d as the system $F(z)$ in the original system of equations (25). As a start system, we choose

$$\begin{aligned} G_i(z_0, z_1, \dots, z_n) &= z_i^{d_i} - z_0^{d_i}, \quad i = 1, \dots, n, \\ G_0(z_0, z_1, \dots, z_n) &= u(z_0, z_1, \dots, z_n) - 1. \end{aligned} \tag{29}$$

We write the resulting system as $G(\hat{z}) = 0$ and define the homotopy

$$H(t, \hat{z}) = t\tilde{F}(\hat{z}) + e^{\gamma i}(1-t)G(\hat{z}) \tag{30}$$

for a $\gamma \in [0, 2\pi)$. To illustrate a possible difficulty with this approach, we examine the system of equations (1)–(3) that we derived for the Bertrand price game in Section 2.2.

EXAMPLE 4. After homogenization of the equilibrium system (1)–(3) in the variables p_x, p_y, and Z with the variable x_0, we obtain the polynomial equations

$$\begin{aligned} 0 &= -p_x^2 x_0^4 - p_y^2 x_0^4 + Z^2 p_x^2 p_y^2, \\ 0 &= -2700 x_0^{10} + 2700 p_x x_0^9 + 8100 Z^2 p_x^2 x_0^6 - 5400 Z^2 p_x^3 x_0^5 + 51 Z^3 p_x^6 x_0^1 - 2 Z^3 p_x^7, \\ 0 &= -2700 x_0^{10} + 2700 p_y x_0^9 + 8100 Z^2 p_y^2 x_0^6 - 5400 Z^2 p_y^3 x_0^5 + 51 Z^3 p_y^6 x_0^1 - 2 Z^3 p_y^7. \end{aligned}$$

The solutions at infinity are those for which $x_0 = 0$. In this case, the system simplifies as

$$Z^2 p_x^2 p_y^2 = 0, \qquad -2Z^3 p_x^7 = 0, \qquad -2Z^3 p_y^7 = 0.$$

After setting $Z = 0$, all equations hold for any values of p_x and p_y. There is a continuum of solutions at infinity. Such continua can cause numerical difficulties for the path-following procedure.

The following theorem now states that in spite of the previous example, our paths converge to the relevant isolated solutions.

Theorem 7. *Let the homotopy H be as in (30) with Bezout number d. Then the following statements hold for almost all $\gamma \in [0, 2\pi)$:*

(i) *The homotopy has d continuous solution paths.*

(ii) *Each path converges either to an isolated nonsingular solution or to a singular*[15] *solution, that is, one where the rank of the Jacobian drops.*

(iii) *If b is an isolated solution with multiplicity m, then there are m paths converging to it.*

(iv) *Paths are monotonically increasing in t, that is, the paths do not bend backward.*

Now we can apply the homotopy H as defined in equation (30) and find all solutions of the system (28). There will be no diverging paths. From the solutions of (28) we easily obtain the solutions of the original system (25).

An additional advantage of the above approach lies in the possibility to scale our solutions via u. If a solution component z_i becomes too large, then this will cause numerical problems, for example, the evaluation of polynomials at such a point becomes rather difficult. Thus if something like this happens, we pick a new set of ξ_i. Furthermore, we eliminated the special case of infinite paths and we do not have to check whether the length of the path grows too large. Instead every diverging path has become a converging path, so while tracking a path, we do not need to check whether the length of the path exceeds a certain bound.

Theoretically we have eliminated the problem of solutions at infinity. Note that the problem of diverging paths still remains in practice. A solution b belongs to a diverging path if $b_0 = 0$. We still need to decide when b_0 becomes zero numerically. Thus there is no absolute certainty if a path converges to a solution at infinity or if the solution is extremely large. However, we are in the convergence zone of Newton's method and can quickly sharpen our solutions to an arbitrary precision.

Remark 1. Here we attempt to give some intuition for the problem of infinite paths. Take two lines $L_1 = \{(x_1, x_2) \mid x_1 + a_{12}x_2 + b_1 = 0\}$ and $L_2 = \{(x_1, x_2) \mid x_1 + a_{22}x_2 + b_2 = 0\}$ with $a_{12}, a_{22} \in \mathbb{R}$. Then there are three possibilities for $L_1 \cap L_2$. First $L_1 \cap L_2 = L_1$,

[15] This might be an isolated root with multiplicity higher than 1, for example, a double root of the system F, or a nonisolated solution component as in Example 4.

so $a_{12} = a_{22}$ and $b_1 = b_2$. Second, $L_1 \cap L_2 = \{p\}$ for some point $p \in \mathbb{R}^2$. Last, we have $L_1 \cap L_2 = \emptyset$, that is, the lines are parallel and so $a_{12} = a_{22}$ but $b_1 \neq b_2$. By using projective space, we eliminate the last possibility by adding infinity where the two lines can meet. So in projective space, the lines are given by the zero sets of the two polynomials $x_1 + a_{12}x_2 + b_1x_0$ and $x_1 + a_{22}x_2 + b_2x_0$. Clearly $(0 : -a_{12} : 1)$ is a common zero for these polynomials if $a_{12} = a_{22}$. So in projective space CP^n, n linear homogeneous polynomials that are not pairwise identical intersect at exactly one point.

Bezout's theorem generalizes this idea to n polynomials. However, the theorem implicitly embeds the system of polynomials in projective space. Therefore, we have to consider the possibility that solutions are at infinity and thus the paths that belong to them diverge. The case that one of those intersection points lies at infinity is equivalent to demanding that $z_0 = 0$. This is clearly a nongeneric case. But the systems that interest us are highly nongeneric, the reason being that they are sparse. This means that for a degree d polynomial in n variables there are $\binom{n+d}{d}$ monomials of degree equal to or smaller than d, but most of their coefficients are zero, which is a nongeneric condition. Thus those systems tend to have many solutions at infinity.

Appendix B: m-homogeneous Bezout number

The number of paths d grows rapidly with the degree of individual equations. For many economic models, we believe that there are only a few (if not unique) equilibria, that is, our systems have few real solutions and usually even fewer economically meaningful solutions. As a result, we may have to follow a large number of paths that do not yield useful solutions. As we have seen in Example 4, there may be continua of solutions at infinity, which can cause numerical difficulties. Therefore, it would be very helpful to reduce the number of paths that must be followed as much as possible.

Two approaches for a reduction in the number of paths exist. The first approach sets the homogenized polynomial system not into CP^n but into a product of m projective spaces $\mathrm{CP}^{n_1} \times \cdots \times \mathrm{CP}^{n_m}$. For this purpose, the set of variables is split into m groups. In the homogenization of the original polynomial F, each group of variables receives a separate additional variable; thus this process is called m-homogenization. The resulting bound on the number of solutions, called the m-homogeneous Bezout number, is often much smaller than the original bound and thus leads to the elimination of paths tending to solutions at infinity. In this paper, we do not provide details on this approach, but show only its impact in our computational examples. We refer the interested reader to Sommese and Wampler (2005) and the citations therein. The first paper to introduce m-homogeneity appears to be Morgan and Sommese (1987).

The second approach to reducing the number of paths is the use of parameter-continuation homotopies. We believe that this approach is perfectly suited for economic applications.

Appendix C: Parameter-continuation homotopy

Economic models typically make use of exogenous parameters such as risk aversion coefficients, price elasticities, cost coefficients, or many other prespecified constants. Often we do not know the exact values of those parameters and so we would like to solve

the model for a variety of different parameter values. Clearly solving the model each time "from scratch" proves impractical whenever the number of solution paths is very large. The parameter continuation homotopy approach enables us to greatly accelerate the repeated solution of an economic model for different parameter values. After solving one instance of the economic model, we can construct a homotopy that alters the parameters from their previous to their new values and allows us to track solution paths from the previous solutions to new solutions. Therefore, the number of paths we need to follow is greatly reduced.

The parameter continuation approach rests on the following theorem, which is a special case of a more general result; see Sommese and Wampler (2005, Theorem 7.1.1).

Theorem 8 (Parameter Continuation). *Let $F(z,q) = (f_1(z,q),\dots,f_n(z,q))$ be a system of polynomials in the variables $z \in \mathbb{C}^n$ with parameters $q \in \mathbb{C}^m$,*

$$F(z,q) : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n.$$

Additionally, let $q_0 \in \mathbb{C}^m$ be a point in the parameter space, where $k = \max_q |\{z \mid F(z,q) = 0; \det(\frac{\partial F}{\partial z}(z,q_0)) \neq 0\}|$ is the number of nonsingular isolated solutions. For any other set of parameters q_1 and a random $\gamma \in [0,2\pi)$, define

$$\varphi(s) = e^{i\gamma} s(s-1) + sq_1 + (1-s)q_0.$$

Then the following statements hold.

(i) $k = |\{z \mid F(z,q) = 0; \det(\frac{\partial F}{\partial z}(z,q)) \neq 0\}|$ *for almost all* $q \in \mathbb{C}^m$.

(ii) *The homotopy $F(z,\varphi(s)) = 0$ has k nonsingular solution paths for almost all $\gamma \in [0,2\pi)$.*

(iii) *All solution paths converge to all isolated nonsingular solutions of $F(z,\varphi(1)) = 0$ for almost all $\gamma \in [0,2\pi)$.*

The theorem has an immediate practical implication. Suppose we already solved the system $F(z,q_0) = 0$ for some parameter vector q_0. Under the assumption that this system has the maximal number k of locally isolated solutions across all parameter values, we can use this system as a start system for solving the system $F(z,q_1) = 0$ for another parameter vector q_1. The number of paths that need to be tracked is k instead of the Bezout number d or some m-homogeneous Bezout number. In our applications, k is much smaller (sometimes orders of magnitude smaller) than these upper bounds. As a result, the parameter continuation homotopy drastically reduces the number of paths that we must track. More importantly, no path ends at a solution at infinity for almost all $q_1 \in \mathbb{C}^n$. As we observe in our examples, exactly these solutions often create numerical problems for the path-tracking software, in particular if there are continua of solutions at infinity as in Example 4. And due to those numerical difficulties, the running times for tracking these paths is often significantly larger than for tracking paths that end at finite solutions. In sum, we believe that the parameter-continuation homotopy approach is of great importance for finding all equilibria of economic models.

A statement similar to that of Theorem 8 holds if we regard isolated solutions of some fixed multiplicity, but we then have to track paths that have the same multiplicity. Track-

ing such paths requires a lot more computational effort than nonsingular paths. The homotopy continuation software Bertini enables the user to track such paths since it allows for user-defined parameter-continuation homotopies.

Appendix D: A splitting approach for solving larger systems

In our application of the all-solutions methods to dynamic stochastic games, we quickly run into problems that are too large to be solved on a single computer. We now briefly describe an approach that enables us to increase the size of problems we can solve.

A splitting approach[16] breaks the square system

$$F(z_1, z_2, \dots, z_n) = (f_1, f_2, \dots, f_n)(z_1, z_2, \dots, z_n) = 0$$

of polynomial equations into two subsystems $F_1 = (f_1, \dots, f_p)$ and $F_2 = (f_{p+1}, \dots, f_n)$. Similarly, the variables are grouped:

$$(z_1, z_2, \dots, z_n) = (x, y) = (x_1, \dots, x_p, y_1, \dots, y_{n-p}).$$

Thus, we can write the entire system as

$$F_1(x_1, \dots, x_p, y_1, \dots, y_{n-p}) = (f_1, \dots, f_p)(x_1, \dots, x_n, y_1, \dots, y_{n-p}) = 0,$$
$$F_2(y_1, \dots, y_{n-p}, x_1, \dots, x_p) = (f_{p+1}, \dots, f_n)(y_1, \dots, y_{n-p}, x_1, \dots, x_p) = 0.$$

Clearly, F_1 and F_2 are not square systems of polynomial equations. We now solve the systems

$$F_1(x_1, \dots, x_p, y_1, \dots, y_{n-p}) = 0,$$
$$y_i = a_i, \quad i = 1, \dots, n-p,$$

and

$$F_2(y_1, \dots, y_{n-p}, x_1, \dots, x_p) = 0,$$
$$x_j = b_j, \quad j = 1, \dots, p,$$

where $a \in \mathbb{C}^{n-p}$ and $b \in \mathbb{C}^p$ are random complex numbers. Each of these two new square systems has a smaller (m-homogeneous) Bezout number than the original system.

Now suppose that we obtain finite solution sets M_1 and M_2 for each of the two systems, respectively. Any pair $(x^*, a, y^*, b) \in M_1 \times M_2$ is a solution to the following square system of polynomial equations in the unknowns $x_1, \dots, x_p$, $y_1, \dots, y_{n-p}$, $r_1, \dots, r_{n-p}$, and $s_1, \dots, s_p$:

$$F_1(x_1, \dots, x_p, r_1, \dots, r_{n-p}) = 0,$$
$$r_i - a_i = 0, \quad i = 1, \dots, n-p,$$
$$F_2(y_1, \dots, y_{n-p}, s_1, \dots, s_p) = 0,$$
$$s_j - b_j = 0, \quad j = 1, \dots, p.$$

[16]We thank Jonathan Hauenstein for suggesting this method to us.

This system is now the start system for the parameter-continuation homotopy

$$
\begin{aligned}
&F_1(x_1,\dots,x_p,r_1,\dots,r_{n-p})=0,\\
&(1-t)(r_i-a_i)+t(r_i-y_i)+(1-t)te^{i\gamma}=0, \quad i=1,\dots,n-p,\\
&F_2(y_1,\dots,y_{n-p},s_1,\dots,s_p)=0,\\
&(1-t)(s_j-b_j)+t(s_j-x_j)+(1-t)te^{i\gamma}=0, \quad j=1,\dots,p,
\end{aligned}
$$

where r and s are the parameters, and all elements in $M_1 \times M_2$ are start points. Thus there are $|M_1| \cdot |M_2|$ paths to track. Observe that for $t=1$, we obtain a system that is equivalent to the original system $F(z)=0$.

To see why this approach works, note that our parameters r and s were chosen randomly. Statement (i) of Theorem 8 states that for almost all choices of those parameters, we have the maximal number of isolated roots. Thus all the requirements of the theorem are met and our homotopy converges to all isolated solutions.

A judicious separation of the original equations produces two subsystems with respective Bezout numbers that are roughly equal to the square root of the Bezout number of the original system. This significant reduction in the number of paths to be tracked may make it feasible to solve the subsystems even if the complete system cannot be solved in reasonable time. Additionally, if the number of finite solutions of the subsystems is also not too large, then the parameter-continuation homotopy will generate all finite solutions of the original system of equations.

In Section 8.2.2, this splitting approach enables us to solve a system of polynomial equations that otherwise would have been too large to be solvable on a single laptop.

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