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## Article

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# Rationalizable partition-confirmed equilibrium

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Rationalizable partition-confirmed equilibrium (RPCE) describes the steady-state outcomes of rational learning in extensive-form games when rationality is common knowledge and players observe a partition of the terminal nodes. RPCE allows players to make inferences about unobserved play by others. We discuss the implications of this using numerous examples, and discuss the relationship of RPCE to other solution concepts in the literature.

**KEYWORDS.** Rationalizability, common knowledge, extensive-form games, self-confirming equilibrium.

**JEL CLASSIFICATION.** C70, C72.

## 1. INTRODUCTION

Most applications of game theory suppose that the observed outcomes will correspond to equilibria, so it is important to consider which sorts of equilibrium concepts are applicable to various situations. The most compelling general explanation for equilibrium is that it arises as the long-run outcome of some sort of nonequilibrium process of learning and adjustment. If the game in question is simply one round of simultaneous moves and participants observe the outcome each time the game is played, then if play converges, we expect the long-run outcomes to correspond to Nash equilibria.<sup>1</sup> However, when the game has a nontrivial extensive form, observed play need not reveal the actions that would be taken at information sets that have never been reached, so even if play converges, incorrect beliefs may persist and play may not converge to a Nash equilibrium.<sup>2</sup>

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<sup>1</sup>For example, this is true for processes that are “asymptotically myopic” and “asymptotically empirical” in the sense of Fudenberg and Kreps (1993).

<sup>2</sup>We do not explicitly study dynamics here, but one motivation for the solution concepts we propose here is the idea that a large population of agents plays the game repeatedly, with anonymous random matching,

Self-confirming equilibrium (SCE) formalizes the idea that incorrect off-path beliefs can persist for settings where players observe the terminal node of the game each time it is played, and the only restrictions placed on the players' beliefs is that they be consistent with the equilibrium distribution on terminal nodes. However, because SCE places no a priori restrictions on the players' beliefs, it does not capture the idea that players use prior information about opponents' payoff functions to predict the opponents' play. To capture such predictions, [Dekel, Fudenberg, and Levine \(1999\)](#) (hereafter DFL) define "rationalizable self-confirming equilibrium" (RSCE), which requires that players make certain inferences based on their knowledge of the other players' payoff functions and observation structure. For example, RSCE requires that player 1's conjecture about how player 2 thinks player 3 is playing be consistent with player 1's information about what player 2 observes.

Both SCE and RSCE apply to situations where all participants see the realized terminal node at the end of each play of the game. In some cases, though, players do not observe the exact terminal node that is reached. For example, in a sealed-bid uniform-price  $k$ -unit auction for a good of known value, the terminal node is the entire vector of submitted bids, but agents might only observe the winning price and the identity of the winning bidders. Alternatively, this information might only be made available to those who submitted nonzero bids, with the others only told that their bid was not high enough. Terminal node partitions are also natural when there are many agents in each player role: If we model each agent as a distinct player, then a given agent in the role of player  $i$  need not observe the play of other agents in that role.

The rationalizable partition-confirmed equilibrium (RPCE) defined in this paper generalizes RSCE by supposing that each player has a partition over terminal nodes and that players' beliefs are consistent with the observed distribution over the partition but not necessarily consistent with the true distribution on terminal nodes. We should stress that both of these equilibrium concepts implicitly suppose that equilibrium play corresponds to an objective distribution; the main difference is that in RSCE, all players observe the distribution over terminal nodes, while RPCE allows each player to have a different partition of the terminal nodes and supposes that each player sees the objective distribution over the cells of their own partition. In this case, there is no longer a publicly observed outcome path, so the implications of common knowledge of the observation structure are less immediate. Roughly speaking, RPCE describes situations where players know that the outcome of play has converged, even when they do not observe all aspects of this outcome themselves. RPCE is of interest in its own right; it also serves to provide additional support for the use of Nash and subgame perfect equilibrium in games where it coincides with one or the other. In particular, we will see that players can do a fair bit of reasoning about play they do not observe, even when we do not assume that players know one another's strategies.

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and no strategic links between repetitions. See [Fudenberg and Levine \(1993b\)](#) and [Fudenberg and Takahashi \(2011\)](#) for examples of the sorts of learning models we have in mind, and see [Fudenberg and Levine \(2009\)](#) for a survey of related work. As this literature shows, incorrect beliefs are more of an issue when players are relatively impatient and so have less incentive to "experiment" with off-path actions; very patient players will experiment enough to rule out non-Nash outcomes (although not necessarily enough to justify backward induction; see [Fudenberg and Levine 2006](#).)

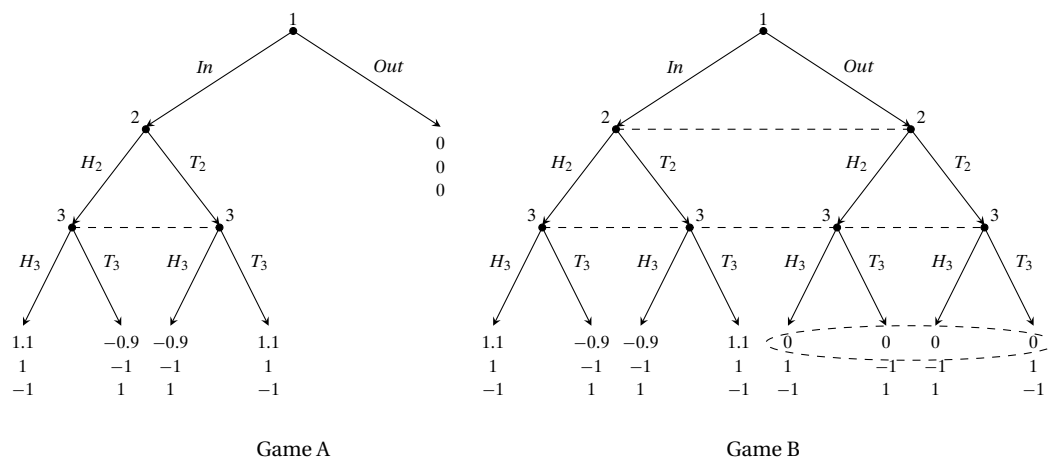


FIGURE 1. The dots connecting payoffs denote terminal node partitions.

Before proceeding to the formal part of the paper, we provide an informal illustration of RPCE in the two extensive-form games in Figure 1 (Example 1). In game A, player 1 moves first, choosing between *In* and *Out*. If he chooses *In*, players 2 and 3 play matching pennies, with player  $i$  choosing between  $H_i$  and  $T_i$ . Player 1's payoffs are the amount that player 2 gets plus an "extra" of 0.1 if player 1 plays *In*. When player 1 plays *Out*, all players obtain the payoff of 0. At the end of each play of the game, players observe the exact terminal node that is reached, as in self-confirming equilibrium.

In game B, player 1 moves first, again choosing between *In* and *Out*. Instead of players 2 and 3 only acting when player 1 plays *In*, now they play the matching pennies game regardless of player 1's action. The map from action profiles to payoffs is exactly the same as in game A. The important assumption is that if 1 plays *Out*, she observes only her own action and payoff but not the action of the other player: the corresponding cell of her terminal node partition contains four elements corresponding to the four possible choices of players 2 and 3. Players 2 and 3 observe the exact terminal nodes. Note that the observation structures for player 1 are the same in games A and B.

Note that even though player 1 receives the same information in these games, the observation structures of players 2 and 3 differ. In game A, players 2 and 3 do not observe each other's play when player 1 plays *Out*, so there is no reason for player 1 to expect their play to resemble a Nash equilibrium. Consequently, an impatient player 1 might choose to play *Out*, fearing that player 2 would lose to player 3. In game B, however, players 2 and 3 observe each other's play, whatever player 1's action is. Thus they should be playing as in the Nash equilibrium of the matching pennies game, and player 1 knows this, so she should play *In*.

In what follows, we give a formal definition of RPCE, and provide results to show that RPCE behaves as expected and to relate it to past work, but much of our contribution comes from examples that illustrate various implications of RPCE. Many (but not all) of these examples use what we call *participation games*; we explore the impact of different terminal node partitions in these games and also compare them to closely related games with sequential moves. The distinguishing feature of participation games

is that some players have the option of an action called *Out*: If a player plays *Out*, his payoff is 0 regardless of the play of the others, and he observes only his own action and payoff. Roughly speaking, the idea of RPCE is that if player 1 (say) always plays *Out*, but knows that players 2 and 3 play every period and observe the terminal node at the end of each round, and player 1 believes that play has converged, then she can use her knowledge of the payoff functions and observation structure to place restrictions on the (unobserved) play of her opponents; in particular, player 1's belief about their play must be concentrated on the set of Nash equilibria of the "subgame" between them. In contrast, if player 1's choice of *Out* ends the game and prevents players 2 and 3 from acting, then when player 1 always plays *Out*, players 2 and 3 do not have the chance to learn; here the only restriction on player 1's belief when she plays *Out* is that the play of 2 and 3 is rationalizable.

In addition to the partition over terminal nodes, this paper differs from DFL by allowing players to have correlated beliefs about unobserved play of their opponents, as advocated by Fudenberg and Kreps (1988). As we argue in Example 8, terminal node partitions make the restriction to independent beliefs less compelling, even as a simplifying assumption: When a player knows that her opponents have repeatedly played a coordination game, but she has not seen their actions, it seems odd to require that the player's beliefs about the opponents correspond to a product distribution. Put differently, with partitions on terminal nodes, play of the game on its own may provide some of the players access to a common signal that is not observed by others.

Hahn's (1977) conjectural equilibrium is a forerunner of SCE in a specific setting, as it allows firms to misperceive demand at out-of-equilibrium prices. Battigalli (1987) defines what we call self-confirming equilibrium with independent, unitary beliefs, where "unitary" means that every action in the support of a player's mixed strategy is a best response to the same belief about play of the opponents, and "independent" means that each player's subjective uncertainty about the play of the others corresponds to a product distribution. Fudenberg and Kreps (1988) give the first example where this sort of SCE has an outcome that cannot arise in Nash equilibrium. In the large-population learning models used to provide foundations for SCE, it is natural (though not necessary) to allow different agents to have different beliefs. The general definition of SCE, due to Fudenberg and Levine (1993a), allows beliefs to be heterogeneous as well as correlated.<sup>3</sup>

Allowing for heterogeneous beliefs about play when players use payoff information to make predictions is more complicated, so DFL restrict attention to unitary beliefs. This paper too restricts attention to unitary beliefs, so as to cut down on the number of new issues that need to be addressed at one time; note that unitary beliefs correspond to steady states of large-population learning systems when all agents in a given player role pool their information. Alternatively one can view our solution concept as providing predictions as a result of repeated interactions among a fixed set of players when the

<sup>3</sup>Kalai and Lehrer (1993) give a version that corresponds to independent, unitary beliefs. Lehrer's (2012) "partially specified equilibrium" is similar as it also allows players to only partially know their opponents' strategies. Ryall (2003) and Dekel, Fudenberg, and Levine (2004) develop SCE variants that in our terminology have specific sorts of partitions over terminal nodes.

discount factor is small. In the companion paper [Fudenberg and Kamada \(2015\)](#), we allow for heterogeneous beliefs.

The paper is organized as follows. [Section 2](#) defines a model of extensive-form games with terminal node partitions. [Section 3](#) revisits [Example 1](#) and analyzes other examples to show the implications of RPCE. [Section 4](#) further motivates the RPCE definition by exploring the consequences of alternative specifications. [Section 5](#) explains the connection between RPCE and other concepts from the literature, notably the rationalizable conjectural equilibrium (RCE) of [Rubinstein and Wolinsky \(1994\)](#). Appendixes A–C follow. Additional Appendixes D–F are available in a supplementary file on the journal website, <http://econtheory.org/supp/1362/supplement.pdf>.

## 2. THE MODEL

### 2.1 Extensive-form games with terminal node partitions

Let  $X$  denote the finite set of nodes, with  $Z \subseteq X$  being the set of terminal nodes. The set of players is  $I = \{1, \dots, n\}$ ;  $H_i$  is the collection of player  $i$ 's information sets,  $H = \bigcup_{i \in I} H_i$ , and  $H_{-i} = H \setminus H_i$ . Let  $A(h)$  be the set of available actions at  $h \in H$ , and let  $A_i = \times_{h \in H_i} A(h)$ ,  $A = \times_{i \in I} A_i$ , and  $A_{-i} = \times_{j \neq i} A_j$ . For each  $z \in Z$ , player  $i$ 's payoff is  $u_i(z)$ .

In the main text, we restrict attention to “one-move games,” in which for any path of play, each player moves at most once and there are no moves by Nature. In addition, we assume that for every  $h, h'$ , if there is  $x \in h$  and  $x' \in h'$  such that  $x < x'$  (where  $<$  is the precedence order on nodes), then there is no  $x'' \in h$  and  $x''' \in h'$  such that  $x''' < x''$ . We then say that  $h'$  is after  $h$  if some  $x' \in h'$  is after some  $x \in h$ , and we assume that this partial order on information sets is transitive.

To model what players observe at the end of each round of play, let  $\mathbf{P}_i = (P_i^1, \dots, P_i^{L_i})$  be a partition over  $Z$  and let  $\mathbf{P} = (\mathbf{P}_1, \dots, \mathbf{P}_n)$ . We assume that the extensive form has perfect recall in the usual sense, and we extend perfect recall to terminal node partitions by requiring that two terminal nodes must be in different cells of  $\mathbf{P}_i$  if they can be reached by different sequence of pure actions by player  $i$ . If every terminal node is in a different cell of  $\mathbf{P}_i$ , the partition  $\mathbf{P}_i$  is said to be discrete. If the cell player  $i$  observes depends only on  $i$ 's actions, the partition is called trivial. Except where otherwise noted, we will require that  $u_i(z) = u_i(z')$  if terminal nodes  $z$  and  $z'$  are in the same partition cell, so that payoffs are measurable with respect to terminal node partitions.

Because we want to model equilibrium as an objective, steady-state distribution, while maintaining the simplicity of “unitary” beliefs (defined below), we need to allow for mixed strategies as outcomes of play. Here we adopt the simplest method, namely to let the players use mixed strategies, as in [Rubinstein and Wolinsky \(1994\)](#) and DFL.<sup>4</sup> Player  $i$ 's behavioral strategy  $\pi_i$  is a map from  $H_i$  to probability distributions over actions, satisfying  $\pi_i(h) \in \Delta(A(h))$  for each  $h \in H_i$ , where and subsequently, for any set  $X$ , we let  $\Delta(X)$  denote the set of probability distributions on  $X$  with finite support. The

<sup>4</sup>In [Fudenberg and Kamada \(2015\)](#), we show that we can replace mixed strategies with a distribution of players each of whom uses a pure strategy.

set of all behavioral strategies for  $i$  is  $\Pi_i$ , and the set of behavioral strategy profiles is  $\Pi = \times_{i \in I} \Pi_i$ . Let  $\Pi_{-i} = \times_{j \neq i} \Pi_j$  and  $\Pi_{-i,k} = \times_{j \neq i,k} \Pi_j$ , with typical elements  $\pi_{-i}$  and  $\pi_{-i,j}$ , respectively. Say that an information set  $h \in H_i$  is reachable under  $\pi_{-i}$  if there exists  $\pi_i$  such that  $h$  has a positive probability under  $(\pi_i, \pi_{-i})$ .

A strategy profile  $\pi$  completely determines a probability distribution over terminal nodes; let  $d(\pi)(z)$  be the probability of reaching  $z \in Z$  given  $\pi$ , and let  $D_i(\pi)(P_i^l) = \sum_{z \in P_i^l} d(\pi)(z)$  for each cell  $P_i^l$  of player  $i$ 's partition.

## 2.2 Beliefs, consistency, and best responses

We will impose some restrictions on beliefs about off-path play, so we will need to specify assessments at off-path information sets: Player  $i$ 's assessment at  $h \in H_i$  is a probability distribution over nodes in  $h$ , so that the assessment at  $h$  is an element of  $\Delta(h)$ . For any  $h \in H_i$ ,  $i$ 's assessment at  $h$  and her opponents' behavioral strategies  $\pi_{-i}$  completely determine  $i$ 's expected payoff for playing any strategy  $\pi_i$ , conditional on  $h$ . Denote by  $\mu_i \in \Delta(\Pi_{-i}) \times [\times_{h \in H_i} \Delta(\Delta(h) \times \Pi_{-i})]$  the *belief* held by player  $i$ . That is, player  $i$ 's belief consists of two terms. The first is a finite-support probability distribution over the opponents' strategy profiles, and the second is a vector that specifies, at each information set  $h$  of player  $i$ , a probability distribution over the product space of pairs of the form (assessments at that information set, opponents' strategy profiles). We denote by  $b(\mu_i)$  the marginal of the belief  $\mu_i$  on the first coordinate, and denote by  $(\mu_i)_h$  the marginal of  $\mu_i$  on the coordinate for information set  $h$ . Note that the belief has sufficient information to calculate conditional expected payoffs at each information set.

We allow  $b(\mu_i)$  to be any distribution on  $\Pi_{-i}$  (with finite support), as opposed to a product of independent mixed strategies. [Example 8](#) explains why this is desirable. We allow assessments to be correlated with the beliefs over opponents' strategies; [Example 9](#) explains why.

**DEFINITION 1.** Belief  $\mu_i$  is an *independent belief* if the following conditions hold:

- (i) For each  $\hat{\pi}_{-i}$  in the support of  $b(\mu_i)$ , we require

$$b(\mu_i)(\hat{\pi}_{-i}) = \prod_{j \neq i} \left( \sum_{\pi_{-i,j} \text{ s.t. } (\pi_j, \pi_{-i,j}) \in \text{supp}(b(\mu_i))} b(\mu_i)(\hat{\pi}_j, \pi_{-i,j}) \right).$$

- (ii) For each  $h$  and each  $(\hat{a}_i, \hat{\pi}_{-i})$  in the support of  $(\mu_i)_h$ , we require

$$\begin{aligned} & (\mu_i)_h(\hat{a}_i, \hat{\pi}_{-i}) \\ &= \left( \sum_{\pi_{-i} \text{ s.t. } \exists a_i \text{ s.t. } (a_i, \pi_{-i}) \in \text{supp}((\mu_i)_h)} (\mu_i)_h(\hat{a}_i, \pi_{-i}) \right) \\ & \quad \cdot \prod_{j \neq i} \left( \sum_{(a_i, \pi_{-i,j}) \text{ s.t. } \exists \pi_j \text{ s.t. } (a_i, (\pi_j, \pi_{-i,j})) \in \text{supp}((\mu_i)_h)} (\mu_i)_h(a_i, (\hat{\pi}_j, \pi_{-i,j})) \right). \end{aligned}$$



That is,  $\mu_i$  is independent if  $b(\mu_i)$  and the  $(\mu_i)_h$  are all product measures. We allow  $i$ 's belief to vary with  $i$ 's information sets, because the posterior belief about which element in the support of  $b(\mu_i)$  has been used may be different from the prior belief. [Example 10](#) explains why such variability is desirable.

**DEFINITION 2.** A belief  $\mu_i$  satisfies *accordance* if it satisfies the following conditions.

- (i) Belief  $(\mu_i)_h$  is derived by Bayes rule if there exists  $\pi_{-i}$  in the support of  $b(\mu_i)$  such that  $h$  is reachable under  $\pi_{-i}$ .<sup>5</sup>
- (ii) For all  $h \in H_i$ , if  $(\mu_i)_h$  assigns positive probability to  $\hat{\pi}_{-i}$ , then there exists  $\tilde{\pi}_{-i} \in \text{supp}(b(\mu_i))$  such that  $\hat{\pi}_{-i}(h') = \tilde{\pi}_{-i}(h')$  for each  $h'$  after  $h$ .

The first part of the definition restricts the belief at on-path information sets, and the second part does so for off-path information sets. Given our restriction to one-move games, the information sets  $h'$  referred to in part (ii) belong to players who did not move before  $h$ ; part (ii) imposes a form of consistency between player  $i$ 's “initial” beliefs  $b(\mu_i)$  about what these players will do and player  $i$ 's beliefs conditional on unexpectedly arriving at  $h$ .

There are other reasonable alternatives for off-path restrictions on beliefs, both to weaker conditions that allow for the sort of correlation we discuss in the next example and to conditions that impose additional restrictions in games where some players can act multiple times on a path of play. We do not examine these alternatives here, because refining off-path beliefs is not our focus. Instead, we assume accordance throughout the paper. Accordance is an easy-to-check condition, and, in particular, implies that if  $b(\mu_i)$  has a singleton support, then  $b(\mu_i)$  and  $(\mu_i)_h$  coincide. For example, consider the extensive form in [Figure 2](#).

If  $b(\mu_2)$  assigns probability 1 to  $(u, U)$ , then under accordance, player 2's belief  $(\mu_2)_{h_2}$  at his information set  $h_2$  must assign probability 1 to player 3 playing  $U$ . Note, though, that if we did not impose part (ii) of [Definition 2](#), then when player 2 unexpectedly sees player 1 play  $r$ , she could change her belief about the future play of player 3 from  $U$  to  $D$ , which would make her want to play  $b$ . This sort of change in beliefs can arise if deviations occur as the result of correlated trembles or payoff shocks. In particular, the outcome  $(u, U)$  is a c-perfect equilibrium ([Fudenberg, Kreps, and Levine \(1988\)](#)), because  $b$  for player 2 is a best response to the correlated distribution  $((1 - \epsilon)(u, U), \epsilon(r, D))$ . However, this cannot occur when the trembles are required to be independent across players as in trembling-hand perfection and sequential equilibrium, and  $(u, U)$  is not a sequential equilibrium outcome. We implicitly impose “independent trembles” in the accordance condition for simplicity, but as noted above, alternative conditions may be reasonable as well.

<sup>5</sup>For each  $\pi_{-i}$  in the support of  $b(\mu_i)$  such that  $h$  is reachable under  $\pi_{-i}$ , let  $a(x|h, \pi_{-i})$  be the probability that node  $x \in H_i$  is reached conditional on the event that  $\pi_{-i}$  is used and  $h$  is reached. Since  $h$  is reachable under  $\pi_{-i}$ , this conditional probability is well defined. If  $\pi_{-i}$  is in the support of  $b(\mu_i)$ , define

$$(\mu_i)_h(a, \pi_{-i}) = \frac{b(\mu_i)(\pi_{-i}) \cdot \text{Prob}(h|\pi_{-i})}{\sum_{\pi'_{-i} \in \text{supp}(b(\mu_i))} b(\mu_i)(\pi'_{-i}) \cdot \text{Prob}(h|\pi'_{-i})},$$

where  $a = a(\cdot|h, \pi_{-i})$ ; otherwise we define  $(\mu_i)_h(a, \pi_{-i}) = 0$ .



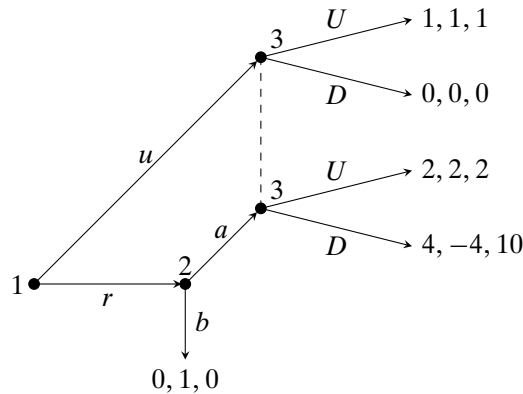


FIGURE 2. Example for accordance.

The following result is immediate and is stated without a proof, and together with [Theorem 2](#) below will establish that a RPCE exists.

**THEOREM 1.** *Suppose that an assessment–strategy pair  $(\tilde{a}, \tilde{\pi})$  satisfies [Kreps and Wilson’s \(1982\) consistency](#),  $b(\mu_i)(\tilde{\pi}_{-i}) = 1$  for all  $i$  and that  $(\mu_i)_h(\tilde{a}, \tilde{\pi}) = 1$  for each  $h$ . Then  $\mu_i$  satisfies accordance.*

We say that  $\pi_i \in \Pi_i$  is a *best response to a belief  $\mu_i$  at  $h \in H_i$*  if the restriction of  $\pi_i$  to the subtree starting at  $h$  maximizes player  $i$ ’s expected payoff against  $(\mu_i)_h$  in that subtree.<sup>6</sup>

### 2.3 Versions, conjectures, and belief models

To facilitate comparison with DFL, we model the beliefs of the players about the beliefs and play of others—their “interactive beliefs”—in the same way as DFL, using the idea of versions  $v_i$  of each player  $i$ . Only one of these versions represents the way player  $i$  actually behaves; the other versions  $v_i$  of player  $i$  are descriptions of player  $i$  that some player  $j$  thinks are possible.<sup>7</sup> In DFL,  $v_i$  specifies player  $i$ ’s strategy, her assessment, and her belief about the opponents’ play. The definition of a version in our context will be

<sup>6</sup>Note that for the purpose of computing this best response, the relevant part of  $(\mu_i)_h$  is the distribution of play at successors of  $h$ . Formally, suppose that there are  $K$  points in the support of  $(\mu_i)_h$ , and index them by superscript  $k$  to write  $(a_i^k, \pi_{-i}^k)$ . An assessment  $a_i$  at  $h$  and a strategy profile  $\pi$  together induce a unique probability distribution over terminal nodes, denoted  $f(a_i(h), \pi)$ . The restriction of a strategy  $\pi_i^*$  to the subtree starting at  $h$  maximizes player  $i$ ’s expected payoff against  $(\mu_i)_h$  if

$$\pi_i^* \in \arg \max_{\pi_i \in \Pi_i} \sum_{k=1}^K \left( (\mu_i)_h(a_i^k, \pi_{-i}^k) \cdot \sum_{z \in Z} [f(a_i^k, (\pi_i, \pi_{-i}^k))(z) \cdot u_i(z)] \right).$$

<sup>7</sup>An alternative approach would be to use the notion of an “epistemic structure” as in [Ben-Porath \(1997\)](#), [Battigalli and Siniscalchi \(2002\)](#), and [Battigalli and Friedenberg \(2012\)](#). That approach would facilitate comparison with some of the literature on rationalizability, but complicate the comparison with DFL.

slightly different, as instead of specifying beliefs, we associate with each version a probability distribution over opponents' versions that we call a *conjecture*. We use these conjectures below to formalize an analog of the usual belief-closed condition—the idea that the play that player  $i$  expects to see is generated by the versions he expects are present. To introduce the notion of conjectures formally, we first need to specify a profile of sets of versions.

A *belief model* is a collection  $V = (V_1, \dots, V_n)$ , where each  $V_i$  is a finite set of player  $i$ 's versions. In our setting, version  $v_i$  of player  $i$  is denoted by  $v_i = (\pi_i, p_i)$ , where the first element is version  $v_i$ 's strategy  $\pi_i \in \Pi_i$ , and the second element is her conjecture  $p_i \in \Delta(\times_{j \neq i} V_j)$ .

Notice that the specification of conjectures allows correlated beliefs, as otherwise  $p_i$  must lie in the space  $\times_{j \neq i} \Delta(V_j)$ . We do not require that  $p_i$  assigns probability 1 to a single version profile of the opponents: Even if player  $i$  is sure that there is only a single agent in player  $j$ 's player role, she may not be sure whether this single agent is of version  $v'_j$  or  $v''_j$ .

Finally, we will associate with each  $V_i$  in a belief model an *actual version*  $v_i^* \in V_i$ , which is the version that is objectively present. Any other versions of player  $i$  are called *hypothetical versions*, as they exist only in the minds of the other players.

## 2.4 Rationalizable partition-confirmed equilibrium

For notational simplicity, let  $\pi_j(v_j)$ ,  $\pi(v)$ , and  $\pi_{-i}(v_{-i})$  denote the strategy (profile) generated by  $v_j \in V_j$ ,  $v \in \times_{j \in I} V_j$ , and  $v_{-i} \in \times_{j \neq i} V_j$ , respectively.

**DEFINITION 3.** A belief  $\mu_i$  is *coherent* with a conjecture  $p_i$  if  $b(\mu_i)$  assigns probability  $\sum_{\pi_{-i}(v_{-i}) = \tilde{\pi}_{-i}} p_i(v_{-i})$  to each  $\tilde{\pi}_{-i} \in \Pi_{-i}$ .

In the definition of RPCE, we require that all versions in a belief model have a coherent belief; this is analogous to requiring the belief model be *belief-closed*, as defined in DFL.

**DEFINITION 4.** Given a belief model  $V$ , version  $v_i = (\pi_i, p_i) \in V_i$  is *self-confirming with respect to*  $\pi^*$  if for all  $v_{-i}$  in the support of  $p_i$ ,  $D_i(\pi_i, \pi_{-i}(v_{-i})) = D_i(\pi_i, \pi_{-i}^*)$ .<sup>8</sup>

In the defining equality, the left hand side is the distribution over  $i$ 's terminal node partition generated by version  $v_i$ 's strategy and the belief about the opponents' play that is induced by  $v_i$ 's conjecture. The right hand side is the distribution that version  $v_i$  observes if the actual distribution of the play is  $\pi^*$ . That is, this equality says that  $v_i$ 's observation (the left hand side) is equal to the actual play (the right hand side).

**DEFINITION 5.** Given a belief model  $V$ ,  $\pi^* \in \Pi$  is *generated by* a version profile  $v = (v_i)_{i \in I} = (\pi_i, p_i)_{i \in I} \in \times_{j \in I} V_j$  if for each  $i$ ,  $\pi_i = \pi_i^*$ .

<sup>8</sup>Because each version  $v_i = (\pi_i, p_i)$  views  $\pi_{-i}(v_{-i})$  as possible for each  $v_{-i}$  in the support of  $p_i$ , every such  $\pi_{-i}(v_{-i})$  must be consistent with the version's observations.

DEFINITION 6. Given a belief model  $V$ ,  $v_i = (\pi_i, p_i)$  is *observationally consistent* if  $p_i(\tilde{v}_{-i}) > 0$  implies, for each  $j \neq i$ , that  $\tilde{v}_j$  is self-confirming with respect to  $\pi(v_i, \tilde{v}_{-i})$ .

REMARK 1. (a) If  $\tilde{v}_j$  is self-confirming with respect to  $\pi(v_i, \tilde{v}_{-i})$ , then by Definition 4, for all  $\hat{v}_{-j}$  in the support of  $\tilde{p}_j$ ,  $D_j(\pi_j(\tilde{v}_j), \pi_{-j}(\hat{v}_{-j})) = D_j(\pi_j(\tilde{v}_j), \pi_{-j}(v_i, \tilde{v}_{-i,j})) = D_j(\pi_i, \pi_{-i}(\tilde{v}_{-i}))$ . Hence Definition 6 is equivalent to the statement, “Given a belief model  $V$ ,  $v_i = (\pi_i, p_i)$  is observationally consistent if  $p_i(\tilde{v}_{-i}) > 0$  implies, for each  $j \neq i$ ,  $D_j(\pi_j(\tilde{v}_j), \pi_{-j}(\hat{v}_{-j})) = D_j(\pi_i, \pi_{-i}(\tilde{v}_{-i}))$  for all  $\hat{v}_{-j}$  in the support of  $\tilde{p}_j$ .” The left hand side in this equality is what  $\tilde{v}_j$  expects to observe given his belief under the partition given by  $D_j$ . The right hand side describes what  $v_i$  thinks  $\tilde{v}_j$  is observing under the partition given by  $D_j$ . Thus the equality requires that  $v_i$  believes that  $\tilde{v}_j$ ’s belief is consistent with what  $\tilde{v}_j$  observes. Thus this definition incorporates the idea that players know (i) the terminal node partitions of other players and (ii) that the opponents satisfy the self-confirming condition.

- (b) To better understand observational consistency, consider the following example: Suppose that  $v'_1$  believes that  $(v'_2, v'_3)$  and  $(v''_2, v''_3)$  are possible, and that no other profiles are possible. Then we require that  $v'_1$  thinks what  $v'_2$  would be observing is consistent with  $v'_2$ ’s play,  $v'_1$ ’s play, and also  $v'_3$ ’s play. It is important to note that we do not require that  $v'_1$  thinks  $v'_2$ ’s belief is consistent with  $v'_3$ ’s play. This is because, even though  $v'_1$  thinks each of  $v'_2$  and  $v''_3$  is possible, she thinks  $(v'_2, v''_3)$  is impossible.
- (c) Note that the condition in Definition 6 only needs to hold when  $v_i$  thinks the profile  $\tilde{v}_{-i}$  has positive probability; otherwise,  $v_i$  need not believe that  $\tilde{v}_j$ ’s observation is consistent with her belief. Note also that, even if  $v_i$  thinks  $\tilde{v}_j$  has positive probability and  $\tilde{v}_j$  thinks version  $v_k$  has positive probability,  $v_i$ ’s belief need not be consistent with what  $v_k$  observes. This is because  $v_i$  might think that  $\tilde{v}_j$  has positive probability, and  $\tilde{v}_j$  incorrectly conjectures that  $v_k$  has positive probability. Finally, if  $\tilde{v}_j$  is self-confirming with respect to  $\pi^*$ , then in the left hand side of the equation of the alternative definition in Remark 1(a),  $D_j(\pi_j(\tilde{v}_j), \pi(\hat{v}_{-j}))$  can be replaced with  $D_j(\pi_j(\tilde{v}_j), \pi^*_{-j})$ .
- (d) The rationale for requiring observational consistency is that player  $i$  knows  $j$ ’s terminal node partition and knows that  $j$ ’s belief is consistent with what  $j$  observes. In the model developed so far, this knowledge is informal. In Appendix E (in the supplement), we make this interpretation precise by constructing an epistemic model.

DEFINITION 7. The strategy profile  $\pi^*$  is a *rationalizable partition-confirmed equilibrium (RPCE)* if there exist a belief model  $V$  and an actual version profile  $v^*$  such that the following conditions hold:

- (i) Strategy profile  $\pi^*$  is generated by  $v^*$ .

- (ii) For each  $i$  and  $v_i = (\pi_i, p_i)$ , there exists  $\mu_i$  such that (a)  $\mu_i$  is coherent with  $p_i$  and (b)  $\pi_i$  is a best response to  $\mu_i$  at all  $h \in H_i$ .
- (iii) For all  $i$ ,  $v_i^*$  is self-confirming with respect to  $\pi^*$ .
- (iv) For all  $i$  and  $v_i$ ,  $v_i$  is observationally consistent.

One consequence of the definition of RPCE is that the set of RPCE shrinks if terminal node partitions become coarser.<sup>9</sup> Since the belief model that supports a strategy profile as a RPCE under finer partitions can support the same strategy profile under coarser partitions, the following comparative statics is immediate.

**THEOREM 2.** *If the terminal node partitions  $\mathbf{P}$  are coarser than  $\mathbf{P}'$ , then any strategy profile that is RPCE under partition  $\mathbf{P}'$  is also a RPCE under  $\mathbf{P}$ .*

[Theorem 1](#) implies that with discrete terminal node partitions for each player, every sequential equilibrium is a RPCE with a single version for each player and correct beliefs.<sup>10</sup> Hence, all sequential equilibria are RPCE under the discrete terminal node partitions. Combining this observation with [Theorem 2](#) and the fact that every game has a sequential equilibrium [Kreps and Wilson \(1982\)](#) yields the following result.

**COROLLARY 1.** *In any extensive-form game, a RPCE exists.*

### 3. IMPLICATIONS OF RPCE

In this section, we consider several examples to illustrate the implications of RPCE. One theme will be the difference between situations where player 1 (say) prevents other players from acting (and thus from learning) and situations where the other players do act but player 1 does not observe their play. First, we revisit [Example 1](#) to show how the RPCE definition delivers the desired conclusion there. [Example 2](#) adds a player to game B to study the assumption of higher order knowledge of rationality. In [Example 3](#), RPCE implies that belief about unobservable play should assign probability 1 to actions that are not only rationalizable but also Nash. [Appendix B](#) generalizes this result to a class of “participation games.” [Example 4](#) provides an example that shows that some RPCE outcomes can only be sustained with belief models in which multiple versions of a given player play the same strategy. In that example, player 1 and player 2 each has a single version, and their beliefs involve differing implicit models of the beliefs of player 4. However, there is a strong restriction on the versions to which actual versions can assign positive probability: [Lemma 1](#) in [Appendix A](#) shows that in a RPCE, any version profile to

<sup>9</sup>The supplement provides further examples to illustrate the effects of changes of terminal node partitions.

<sup>10</sup>Note that this applies even to sequential equilibria that are ruled out by strategic stability ([Kohlberg and Mertens 1986](#)). Thus RPCE corresponds to sequential equilibrium's assumption that players view deviations as trembles, as opposed to the “forward induction” view that, whenever possible, deviations should be viewed as a deliberate choice.

which an actual version assigns positive probability is the actual version profile of some RPCE.

Terminal node partitions have various effects on the set of strategies that a player can play in a RPCE. [Example 5](#) demonstrates that a player need not expect unobservable play by the opponents to resemble a Nash equilibrium if their terminal node partitions are not discrete. [Example 6](#) shows how giving a player a more refined terminal node partition can change his RPCE play even though that player's beliefs were correct in the RPCE for the coarser partition: The effect comes from the fact that with the finer partition, other players know that the player's beliefs are correct. The supplement also provides examples to illustrate how the terminal node partitions change the set of strategies in RPCE.

**EXAMPLE 1 (Revisited).** Here we show that in game A, it is possible for player 1 to play *Out* in RPCE, but this is not possible in game B.

Consider game A, in which players 2 and 3 play matching pennies if and only if player 1 plays *In*. We argue that player 1 can play *Out* in a RPCE with the belief model and actual versions<sup>11</sup>

$$\begin{aligned} V_1 &= \{v'_1\}, & v'_1 &= (Out, (v'_2, v'_3)) \\ V_2 &= \{v'_2, v''_2\}, & v'_2 &= (H_2, (v'_1, v'_3)), v''_2 = (T_2, (v'_1, v'_3)) \\ V_3 &= \{v'_3, v''_3\}, & v'_3 &= (T_3, (v'_1, v'_2)), v''_3 = (H_3, (v'_1, v'_2)) \end{aligned}$$

The actual version profile is  $(v'_1, v'_2, v'_3)$ .

Here,  $v'_2$  can believe that player 3 plays  $H_3$  because she never gets to observe player 3's play, while  $v''_3$  plays  $H_3$  because he believes that player 2 plays  $T_2$ , which again is justified by the fact that he is not observing player 2's play. Since  $v'_1$  never observes player 2's and player 3's play, and she knows that they do not get to play on the path so do not observe each other's play, she can believe that they can have such mutually inconsistent beliefs, hence can entertain a belief that the opponents play  $(H_2, T_3)$ , which is consistent with the self-confirming condition.

Now we turn to game B, where players 2 and 3 play matching pennies regardless of player 1's action but player 1 only observes their play when she chooses *In*. Fix a RPCE  $\pi^*$ , with an associated belief model  $V$ . Suppose that some version of player 1's conjecture assigns positive probability to a version profile  $(\tilde{v}_2, \tilde{v}_3)$  such that  $\pi(\tilde{v}_2)$  and  $\pi(\tilde{v}_3)$  are not best responses to each other. Suppose without loss of generality that  $\pi(\tilde{v}_2)$  is not a best response to  $\pi(\tilde{v}_3)$ . Notice that by the observational consistency condition, we have  $D_2(\tilde{\pi}_2, \pi_{-2}(v_{-2})) = D_2(\tilde{\pi}_2, \cdot, \pi(\tilde{v}_3))$  for all  $v_{-2}$  in the support of  $\tilde{p}_2$ . Since player 2 observes the exact terminal node reached, this implies that  $\tilde{p}_2$  assigns probability 1 to  $v_3$  such that  $\pi_3(v_3) = \pi_3(\tilde{v}_3)$ . But this means that any belief  $\tilde{\mu}_2$  coherent with  $\tilde{p}_2$

<sup>11</sup>The notation that we use when presenting belief models in examples involves a slight abuse of notation. In particular, when a player's conjecture is a point mass on a particular version profile  $v_{-i}$ , we write that profile in place of the Dirac measure concentrated on  $v_{-i}$ .

has a property that  $b(\mu_2)$  assigns probability 1 to  $\pi_3(\tilde{v}_3)$ , so the best response condition is violated for player 2.

Therefore, it must be the case that for any  $v_1 = (\pi_1, p_1)$ , any belief  $\mu_1$  coherent with  $p_1$  assigns probability  $\frac{1}{2}$  to each of  $H_2$  and  $H_3$ . The best response condition then implies that  $\pi_1$  assigns probability 1 to *In*, as playing *In* gives her the expected payoff of 0.1, while playing *Out* gives her 0. Because this is true for any version  $v_1$  of player 1 and  $\pi^*$  is generated by the actual versions, we conclude that  $\pi_1^*$  assigns probability 1 to *In*, that is, player 1 plays *In* with probability 1.  $\diamond$

**EXAMPLE 2.** Consider a modification of game B, where we add “player 0” at the top of the extensive-form game. Specifically, player 0 moves first, choosing between *In* and *Out*. Whatever action is played, the game goes on and game B is played, where only player 1 knows the action taken by player 0. The map from the action profile for players 1, 2, and 3 to their payoffs are exactly the same as in game B, while player 0 gets 0 if he plays *Out*, 1 if he plays *In* and player 1 also plays *In*, and  $-1$  if he plays *In* and player 1 plays *Out*. The terminal node partitions are the same as in game B, where everyone knows the move by player 0, and player 0 observes everything if he plays *In* and does not observe anything if he plays *Out*.

In any RPCE of this game, player 0 must play *In*, because player 0 must infer that player 1 plays *In*. Remember that in game B of [Example 1](#), all versions of player 1 must play *In*; the coherent belief condition ensures that player 0 believes that 1 plays *In* with probability 1.

This example shows that RPCE assumes that a player not only believes that the play by the opponents has converged, but she also believes that an opponent believes that the play by these opponents has converged.  $\diamond$

**EXAMPLE 3.** Consider the game in [Figure 3](#). Everyone observes the exact terminal node reached, except that player 1 cannot distinguish between the opponents’ action profiles if she plays *Out*.

Player 1’s action has no effect on the information or payoffs of players 2 and 3, so it makes sense to talk of the subgame involving just those two players. Notice that  $H_2$  is a best response to  $H_3$ , which is a best response to  $T_2$ , which is a best response to  $T_3$ , which in turn is a best response to  $H_2$ , so all actions in the subgame are rationalizable, but it has a unique Nash equilibrium, namely  $(N_2, N_3)$ .

In this game, RPCE requires not only that player 1 expects players 2 and 3 to play rationalizable actions, but also that she expects their play to be a Nash equilibrium of the subgame. Hence player 1 should expect the payoff of 1 from playing *In*, so player 1 should play *In*. The proof of this is exactly the same as in [Example 1](#): if player 1’s conjecture assigns a positive probability to a version profile such that player 2 is not best responding to player 3, the observational consistency condition for player 1 implies that the best response condition for player 2 should be violated.

It is important here that players 2 and 3 do not observe player 1’s action before they move, as otherwise player 1 can play *Out*, believing that players 2 and 3 play  $H_i$  or  $T_i$  after *In*. This example shows that in RPCE, beliefs about unobserved actions on the

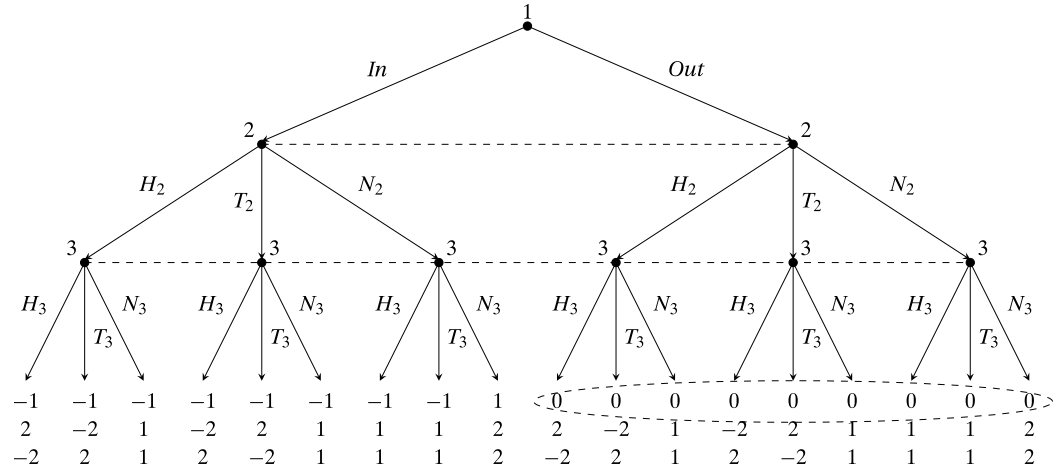


FIGURE 3. Example 3.

path of play should assign probability 1 to actions that are not only rationalizable but also Nash. We generalize this in [Theorem 5](#) in [Appendix B](#).  $\diamond$

**EXAMPLE 4** (Need for duplicate versions). The game depicted in [Figure 4](#) is a modification of the “horse” example in [Fudenberg and Kreps \(1988\)](#). Instead of having player 3 move only when player 1 or player 2 plays down, we now suppose that player 3 moves whenever player 4 plays a dominant action at the root node, and players 1 and 2 do not know player 3’s play as long as both play “across.” The terminal node partitions are such that everyone observes the terminal node reached, except that if  $(A_1, A_2)$  is taken, then player 1’s and players 2’s partitions do not reveal player 3’s choice.

This game has a RPCE in which  $(A_1, A_2)$  is chosen. Specifically, consider the belief model and actual versions

$$V_1 = \{v'_1\}, \quad v'_1 = (A_1, (v'_2, v'_3, v'_4))$$

$$V_2 = \{v'_2\}, \quad v'_2 = (A_2, (v'_1, v'_3, v'_4))$$

$$V_3 = \{v'_3, v''_3\}, \quad v'_3 = (R_3, (v'_1, v'_2, v'_4)), v''_3 = (L_3, (v'_1, v'_2, v'_4))$$

$$V_4 = \{v'_4, v''_4\}, \quad v'_4 = (R_4, (v'_1, v'_2, v'_3)), v''_4 = (R_4, (v'_1, v'_2, v'_3))$$

The actual version profile is  $(v'_1, v'_2, v'_3, v'_4)$ .

Notice that  $V_4$  has two versions, both of which play the same strategy. This is a necessary feature of any belief model that supports the outcome involving  $(A_1, A_2)$ . This is because this  $(A_1, A_2)$  can happen only when players 1 and 2 disagree about player 3’s play, and know that player 4 observes player 3’s play. This means players 1 and 2 must also disagree about what player 4 believes, which requires there be (at least) two versions of player 4, and both versions need to play  $R_4$  as it is a dominant action.<sup>12</sup>

<sup>12</sup>A formal proof is as follows: Suppose that there is only one version  $\hat{v}_4$  in  $V_4$ , and that  $\hat{v}_4$  believes that  $L_3$  is played with probability  $p \in [0, 1]$ . By coherency, all versions of players 1 and 2 must have a conjecture



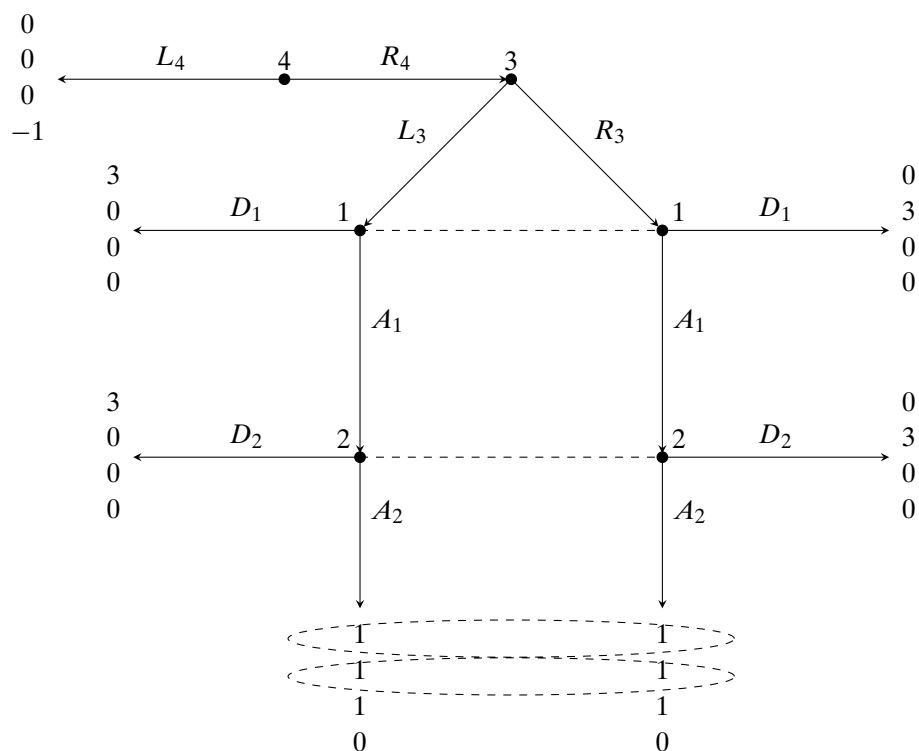


FIGURE 4. Example 4.

This need for two versions that play the same strategy is a new feature that arises with nondiscrete terminal node partitions; such duplicate versions do not enlarge the set of RSCE, because in RSCE, players can only disagree about play off of the equilibrium path.<sup>13</sup>  $\diamond$

**EXAMPLE 5** (Participation game with unobservable actions). Consider the game in Figure 5. Player 1 does not observe the exact terminal node if she plays  $Out_1$ , and she observes the exact terminal node reached if she plays  $In_1$ . The other players' terminal node partitions always reveal player 1's move but only reveal the exact terminal node if they play  $In_i$ .

that assigns probability 1 to  $\hat{v}_4$ . Then observational consistency implies that all versions of players 1 and 2 must believe that  $L_3$  is played with probability  $p \in [0, 1]$ . But since  $p > \frac{1}{3}$  implies that  $D_1$  is strictly better than  $A_1$  and  $p < \frac{2}{3}$  implies that  $D_2$  is strictly better than  $A_2$ ,  $(A_1, A_2)$  cannot be played.

<sup>13</sup>Fix a belief model used to justify a RSCE  $\pi^*$  in the DFL model, and suppose that it has  $m$  versions  $(v_i^{(1)}, \dots, v_i^{(m)})$  that use the same strategy in a single player role  $i$ . Now consider a new belief model formed by eliminating  $(v_i^{(2)}, \dots, v_i^{(m)})$ . If a version of some opponent player role  $j$  has a mixture over  $(v_i^{(2)}, \dots, v_i^{(m)})$  in DFL's belief-closed condition, then the belief-closed condition will be satisfied in the new belief model by assigning the sum of probabilities on  $(v_i^{(1)}, \dots, v_i^{(m)})$  in the original mixture to  $v_i^{(1)}$  in the new mixture. Thus the new belief model supports the RPCE  $\pi^*$ .

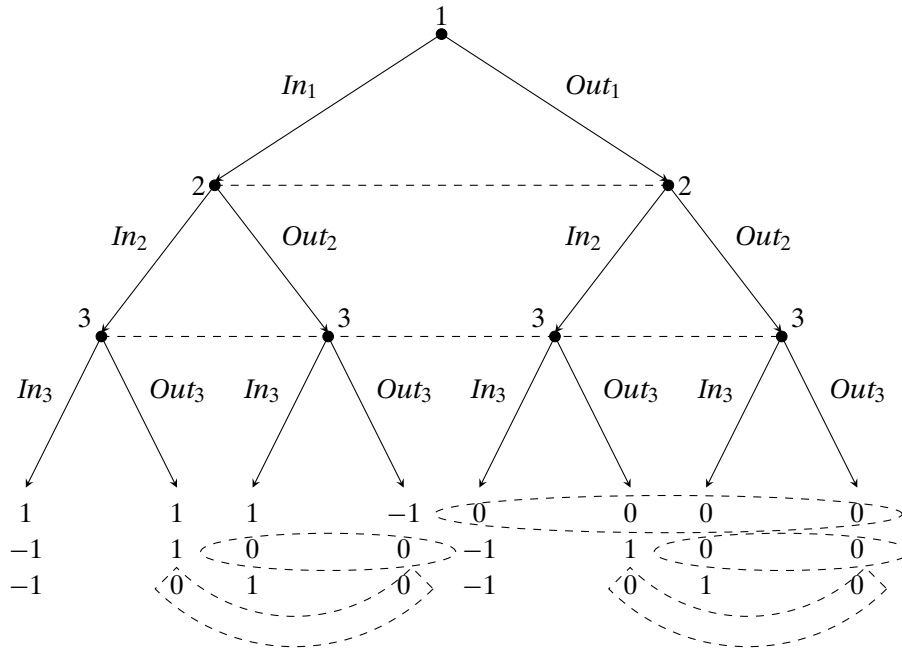


FIGURE 5. Example 5.

Notice that for any Nash equilibrium of player 2's and player 3's simultaneous move game, player 1 expects a payoff of at least  $\frac{1}{2}$  from playing  $In_1$ . Thus if player 1 believes that players 2 and 3 play a Nash equilibrium of the subgame, she must play  $In_1$ . We argue, however, that in RPCE it is possible for player 1 to play  $Out_1$ . Specifically, consider the belief model and actual versions

$$V_1 = \{v'_1\}, \quad v'_1 = (Out_1, (v'_2, v'_3))$$

$$V_2 = \{v'_2, v''_2\}, \quad v'_2 = (Out_2, (v'_1, v'_3)), v''_2 = (In_2, (v'_1, v'_3))$$

$$V_3 = \{v'_3, v''_3\}, \quad v'_3 = (Out_3, (v'_1, v'_2)), v''_3 = (In_3, (v'_1, v'_2))$$

The actual version profile is  $(v'_1, v'_2, v'_3)$ .

In this belief model, player 1 believes that both players 2 and 3 play  $Out_i$ . Although  $Out_2$  is not a best response against  $Out_3$ , player 2 does not observe player 3's play when he is playing  $Out_2$ , and so he can believe that player 3 plays  $In_3$ . Likewise, player 3 can play  $Out_3$ , believing that player 2 plays  $In_2$ . Player 1 plays  $Out_1$  because she believes that  $(Out_2, Out_3)$  is played as a result of such mutually inconsistent beliefs.

We note that  $Out_1$  could not be played in any RPCE if the terminal node partitions for players 2 and 3 were discrete. This is because player 1's payoff is  $\frac{1}{2}$  in every Nash equilibrium of the game between players 2 and 3, so by Theorem 5 in Appendix B, she should play  $In$ . Hence, nondiscrete terminal node partitions allow an action to be played even if the action is outside the support of equilibria under finer partitions. In other

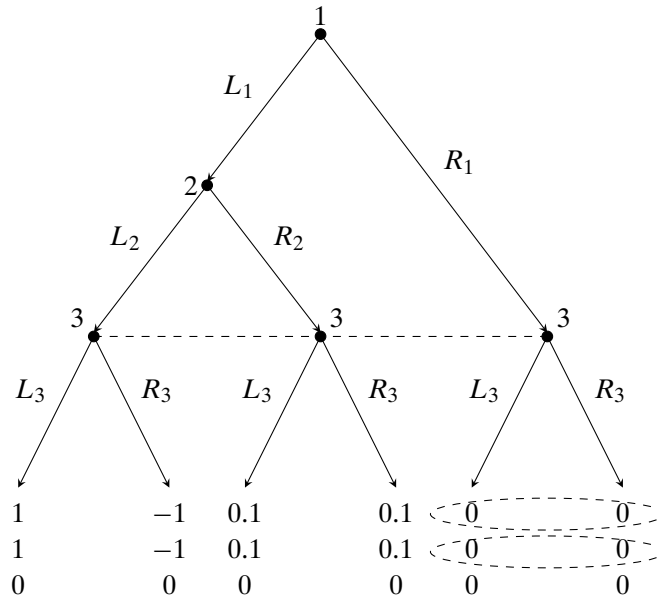


FIGURE 6. Example 6.

words, the conclusion of [Theorem 5](#) may fail if the hypothesis that player 1's opponents have discrete partitions is weakened.

To sum up, this example shows that a player need not expect unobserved play to be a Nash equilibrium if these opponents do not observe the exact terminal nodes and, as a consequence, she may play an action that she would not play otherwise.  $\diamond$

**EXAMPLE 6** (Terminal node partitions and learning one player's actions from those of another). Here we provide an example in which the difference in terminal node partitions affects learning. If one player's terminal node partition is finer than another player's, then the latter can learn by observing the play of the former and can respond accordingly, while if the partitions are the same, then there is nothing to learn.

In the game in [Figure 6](#), all players observe the exact terminal node reached, except that player 1's and player 2's partitions do not reveal player 3's action if 1 plays  $R_1$ .

First, we show that player 1 can play  $R_1$  in a RPCE. To see this, consider the belief model and actual versions

$$V_1 = \{v'_1\}, \quad v'_1 = (R_1, (v'_2, v'_3))$$

$$V_2 = \{v'_2\}, \quad v'_2 = (L_2, (v'_1, v''_3))$$

$$V_3 = \{v'_3, v''_3\}, \quad v'_3 = (R_3, (v'_1, v'_2)), v''_3 = (L_3, (v'_1, v'_2))$$

The actual version profile is  $(v'_1, v'_2, v'_3)$ .

Notice that players 1 and 2 disagree about player 3's action, which neither of them observe when player 1 plays  $R_1$ , which is why player 2 can play  $L_2$  even though player 1 is playing  $R_1$ .

Now we show that if player 1's partition is discrete, she can no longer play  $R_1$  in a RPCE. In such a situation, because player 1 has a discrete terminal node partition and player 2 does not, player 2 can learn player 3's play by observing player 1's play. For this reason, player 1 cannot play  $R_1$  in a RPCE, while player 1 could play  $R_1$  if players are not required to believe that other players act rationally.<sup>14</sup> To see that  $R_1$  cannot be played, suppose the contrary. The best response condition for player 1 and observational consistency applied to player 2 imply that  $b(\mu_2)$  assigns probability at least  $\frac{1}{2}$  to  $R_3$ . By accordance, player 2's belief  $(\mu_2)_{h_2}$  at his information set  $h_2$  assigns probability at least  $\frac{1}{2}$  to  $R_3$ . Then the best response for player 2 is to play  $R_2$  with probability 1. However, this implies that player 1's payoff from playing  $L_1$  is  $0.1 > 0$ , so she cannot play  $R_1$ . Alternatively, if player 1 does not know player 2's payoff function, the fact that player 2's behavior reflects her belief about player 3's play does not convey any information to player 1. So player 1 can believe  $(L_2, R_3)$  is played with probability 1, making  $R_1$  possible. The key is the observational consistency condition: player 2 knows that player 1 observes player 3's play, so player 2's belief about player 3's play must match with what player 2 thinks player 1 is best-responding against.

Notice that player 1's belief in the RPCE we constructed for the original terminal node partitions is, in fact, correct. However, when player 1's terminal node partition is discrete, player 1 can no longer play  $R_1$ : With a discrete terminal node partition for player 1, player 1 knows player 2 can and should learn player 3's play by observing player 1's play. But this is impossible when player 1's and player 2's terminal node partitions coincide.  $\diamond$

#### 4. JUSTIFICATION OF THE RPCE DEFINITION

In this section, we discuss several examples of a game and a RPCE outcome that we think is a plausible consequence of rational learning, and study whether the outcome would still be a RPCE under alternative definitions that might seem natural to some readers.

Specifically, [Example 7](#) explains why the self-confirming condition should not be imposed on hypothetical versions, [Example 8](#) argues that we should allow for correlated beliefs in our model, [Example 9](#) justifies our specification of the space of beliefs, and [Example 10](#) discusses the role of accordance.

**EXAMPLE 7** (Self-confirming condition for hypothetical versions). Consider the game in [Figure 7](#). The terminal node partitions are such that everyone observes the exact terminal node reached, except that player 1 does not observe player 2's and player 3's play if she plays *Out*.

Intuitively, if player 1 thinks that players 2 and 3 coordinate on the  $(R_2, R_3)$  equilibrium, she has an incentive to play *Out*, which makes her unable to observe how players 2 and 3 play. Indeed, the outcome  $(Out, L_2, L_3)$  is possible in RPCE. To see this, consider

<sup>14</sup>The supplementary Appendix develops the concept of *partition-confirmed equilibrium* (PCE), which extends SCE to games with nondiscrete terminal node partitions. Roughly speaking, PCE weakens condition (ii)(b) of RPCE ([Definition 7](#)) and also drops the coherency condition in condition (ii)(a).

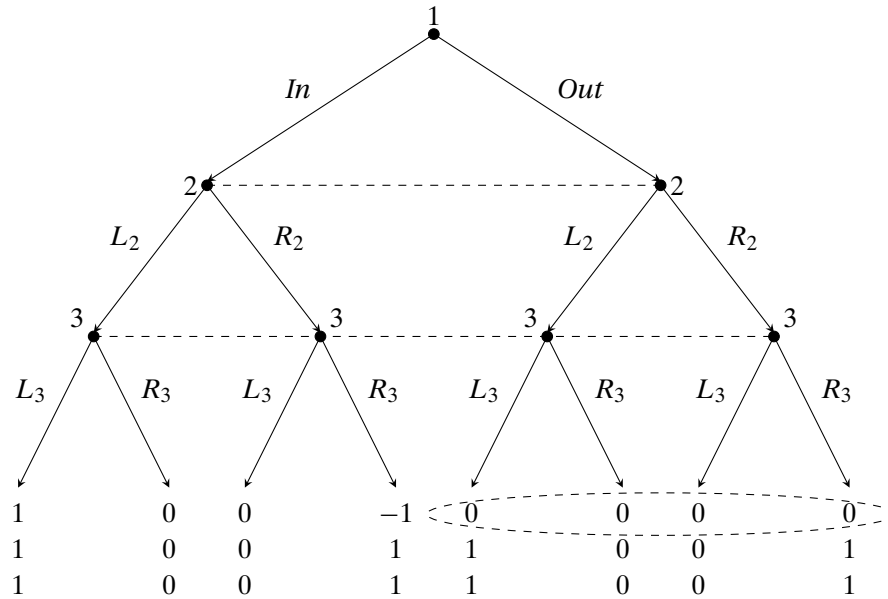


FIGURE 7. Example 7.

the belief model and actual versions

$$V_1 = \{v'_1\}, \quad v'_1 = (Out, (v''_2, v''_3))$$

$$V_2 = \{v'_2, v''_2\}, \quad v'_2 = (L_2, (v'_1, v'_3)), v''_2 = (R_2, (v'_1, v'_3))$$

$$V_3 = \{v'_3, v''_3\}, \quad v'_3 = (L_3, (v'_1, v'_2)), v''_3 = (R_3, (v'_1, v'_2))$$

The actual version profile is  $(v'_1, v'_2, v'_3)$ .

Notice that  $v''_2$  and  $v''_3$  are hypothetical versions, and they do not satisfy the self-confirming condition. Version  $v'_1$  plays *Out* because she conjectures that these hypothetical versions exist, and her conjecture is never falsified because she plays *Out*.

Now we show that the outcome  $(Out, L_2, L_3)$  is impossible if we require the self-confirming condition with respect to the equilibrium strategy profile for hypothetical versions. To see this, suppose that we strengthen Definition 7 by replacing condition (iii) with the condition that for all  $i$  and  $v_i$ ,  $v_i$  is self-confirming with respect to  $\pi^*$ . If  $(Out, L_2, L_3)$  is a RPCE under this stronger condition, the best response condition implies that all versions of player 2 should play  $L_2$  and that all versions of player 3 should play  $L_3$ , so player 1 must believe that players 2 and 3 play  $(L_2, L_3)$ . But then by the best response condition, player 1 must play *In*.<sup>15</sup>  $\diamond$

<sup>15</sup>Without knowledge of opponents' payoff functions (as in the partition-confirmed equilibrium concept in the supplementary Appendix), player 1 may still play *Out*, believing that players 2 and 3 play  $R_2$  and  $R_3$ , respectively.

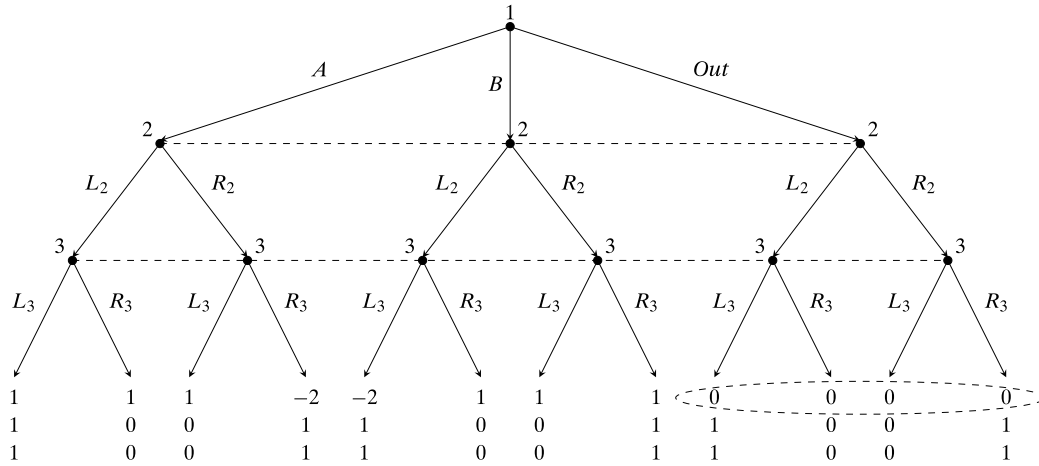


FIGURE 8. Example 8.

EXAMPLE 8 (Correlated beliefs). Our formulation of beliefs is more complicated than DFL, because we allow for correlated beliefs, while DFL restricted attention to independent beliefs. In this example, player 1 can play an action only when she has correlated beliefs about the play at information sets that she does not observe.

Consider the game depicted in Figure 8. This game is similar to Example 7, but player 1 has two actions that make the terminal nodes observable for her: her decision amounts either to betting on the action that players 2 and 3 will coordinate on or to declining to bet. The terminal node partitions are such that everyone observes the exact terminal node reached except that player 1 cannot distinguish among four terminal nodes that are caused by the action *Out*.

To capture the long-run consequences of rational learning, RPCE should allow for the possibility that 1 plays *Out*. Intuitively, since players 2 and 3 get to play on the path, they should play as in a Nash equilibrium of their coordination game. Hence, it makes sense for player 1 to believe that players 2 and 3 coordinate on either  $(L_2, L_3)$  or  $(R_2, R_3)$ , and each is equally likely.<sup>16</sup> Given this belief, the expected payoff from playing action *A* is the average of 1 and  $-2$ , which is  $-\frac{1}{2}$ , and the payoff for action *B* is also  $-\frac{1}{2}$  in the same way. Hence, with this belief, playing *Out* is optimal, as it leads to the payoff of 0.

Player 1 can play *Out* in RPCE as shown by the system<sup>17</sup>

$$\begin{aligned} V_1 &= \{v'_1\}, & v'_1 &= \left( \text{Out}, \left( \frac{1}{2}(v'_2, v'_3), \frac{1}{2}(v''_2, v''_3) \right) \right) \\ V_2 &= \{v'_2, v''_2\}, & v'_2 &= (L_2, (v'_1, v'_3)), v''_2 = (R_2, (v'_1, v'_3)) \end{aligned}$$

<sup>16</sup>Note that with this belief, player 1 is certain that the actual play of players 2 and 3 is deterministic and, hence, independent; the correlation here is in player 1's subjective uncertainty about which pure strategies players 2 and 3 are using. This is the same sort of subjective correlation that SCE allows for in beliefs about off-path play.

<sup>17</sup>Version  $v_i$ 's conjecture  $(pv'_{-i}, (1-p)v''_{-i})$  denotes the probability distribution with probability  $p$  on  $v'_{-i}$  and  $1-p$  on  $v''_{-i}$ .

$$V_3 = \{v'_3, v''_3\}, \quad v'_3 = (L_3, (v'_1, v'_2)), v''_3 = (R_3, (v'_1, v'_2))$$

The actual version profile is  $(v'_1, v'_2, v'_3)$ .

Just as with SCE, one can refine the set of RPCE by requiring independent beliefs. In some cases, this might be viewed as an innocuous simplifying assumption, but we think the restriction would be problematic here, because the fact that players 2 and 3 observe each other's play means that the extensive form and terminal node partitions provide them with a particular sort of correlating device.<sup>18</sup>

Moreover, if player 1 is restricted to hold an independent belief, the action *Out* cannot be played in a RPCE. To see this, notice that for *Out* to be at least as good as playing *A* for a version of player 1, her belief has to assign probability at least  $\frac{1}{3}$  to  $(R_2, R_3)$ . In the same way, for *Out* to be at least as good as playing *B* for a version of player 1, her belief has to assign probability at least  $\frac{1}{3}$  to  $(L_2, L_3)$ . However, any independent randomization by players 2 and 3 leads to the situation where the minimum of the probabilities assigned to  $(L_2, L_3)$  and  $(R_2, R_3)$  is no more than  $\frac{1}{4}$ . Hence, for any independent beliefs, *Out* cannot be a best response.

We note that, as in [Example 5](#), if the terminal node partitions were discrete, player 1 could not play *Out*. However, the reason behind this effect of terminal node partitions is different: Here it is that player 1 can entertain a correlated belief, which she would be unable to have if she actually observes player 2's and player 3's play.<sup>19</sup>  $\diamond$

**EXAMPLE 9** (Assessment–strategies correlation). We have allowed  $v_i$ 's belief at  $h$ ,  $(\mu_i)_h$ , to lie in the space  $\Delta(\Delta(h) \times \Pi_{-i})$  and not necessarily in  $\Delta(h) \times \Delta(\Pi_{-i})$ . Here we provide an example that justifies this specification.

Consider the extensive-form game depicted in [Figure 9](#). All players observe the exact terminal node reached, except that players 1 and 4 do not distinguish among those terminal nodes that are caused by  $R_4$ .

We first show that  $R_4$  is compatible with RPCE. For example, consider the belief model and actual versions

$$\begin{aligned} V_1 &= \{v'_1\}, \quad v'_1 = \left( Out, \left( \frac{1}{2}(v'_2, v'_3, v'_4), \frac{1}{2}(v''_2, v''_3, v'_4) \right) \right) \\ V_2 &= \{v'_2, v''_2\}, \quad v'_2 = (L_2, (v'_1, v'_3, v'_4)), v''_2 = (R_2, (v'_1, v''_3, v'_4)) \\ V_3 &= \{v'_3, v''_3\}, \quad v'_3 = (L_3, (v'_1, v'_2, v'_4)), v''_3 = (R_3, (v'_1, v'_2, v'_4)) \\ V_4 &= \{v'_4\}, \quad v'_4 = (R_4, (v'_1, v'_2, v'_3)) \end{aligned}$$

The actual version profile is  $(v'_1, v'_2, v'_3, v'_4)$ .

<sup>18</sup>If players 2 and 3 have trivial terminal node partitions (and so do not observe their own ex post payoffs), then there is no reason for player 1 to think their play has converged. In this case too, RPCE would allow player 1 to have correlated beliefs about the actions of players 2 and 3, but absent the explicit correlating device of own past moves, the restriction to independent beliefs strikes us as less problematic.

<sup>19</sup>A similar argument can be made in [Example 9](#) below to show that, with the discrete terminal node partition, player 1 cannot play *Out* and so player 4 cannot play  $R_4$ .



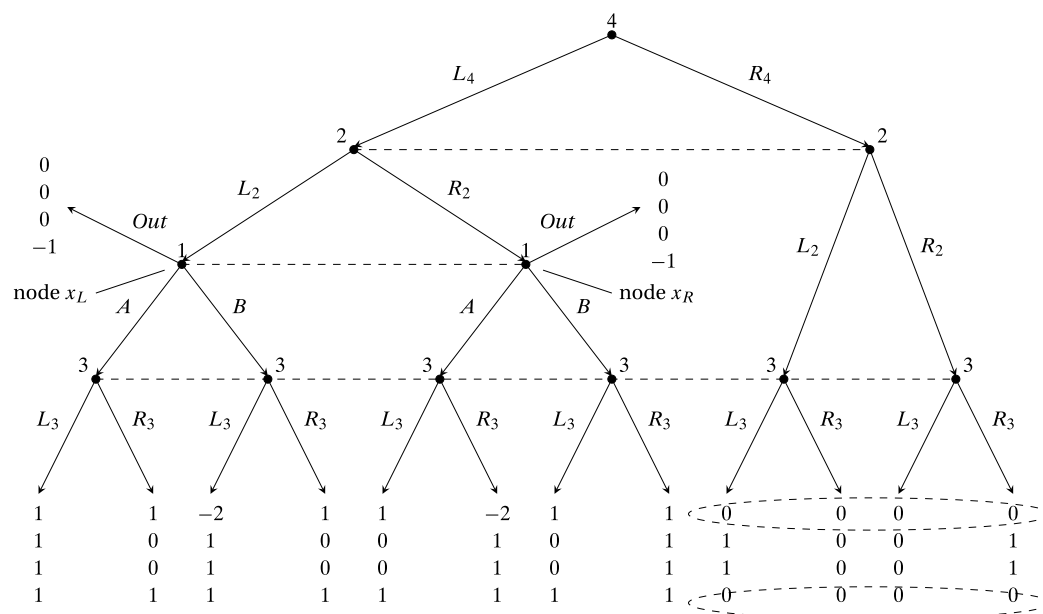


FIGURE 9. Example 9.

To support  $R_4$ , it must be possible that player 4 believes player 1 plays *Out* once her information set is reached. For this play to satisfy the best response condition at this information set, we should allow for player 1 to believe that player 2's and player 3's play is correlated, just as in [Example 8](#). Specifically, suppose a belief  $\mu'_1$  of  $v'_1$  is such that  $b(\mu'_1)$  assigns equal probabilities to  $(L_2, L_3, R_4)$  and  $(R_2, R_3, R_4)$ , and  $(\mu'_1)_h$  assigns equal probabilities to  $(x_L, (L_2, L_3, R_4))$  and  $(x_R, (R_2, R_3, R_4))$ , where  $h$  is player 1's information set. This belief satisfies coherency and accordance, and it makes player 1 playing *Out* a best response. Notice that player 1 knows that players 2 and 3 are actually playing the coordination game on the path of play because player 4 plays  $R_4$ , thus this correlated belief seems plausible, and it is possible in RPCE when each profile of opponents' strategies is associated with a different assessment. However, it is impossible if only a single assessment is used for a distribution of the opponents' strategies. Indeed, for any single assessment at player 1's information set, player 1's expected payoff from playing either *A* or *B* is at least  $\frac{1}{4}$ , so playing *Out* can never be a best response. Hence, player 4 should expect the payoff of 1 by playing  $L_4$ , which means that player 4 cannot play  $R_4$ .

Because the belief model underlying the play of  $R_4$  seems sensible, we would not want to refine the set of RPCE by insisting that each version has a point distribution on assessments. The definition of RPCE allows each version to have a non-degenerate distribution on assessments and, in particular, it enables player 4 to play  $R_4$  in this example.  $\diamond$

EXAMPLE 10 (Accordance). We use Figure 10 to explain why the accordance condition allows for a belief  $(\mu_i)_h$  to have different marginals over continuation strategies across

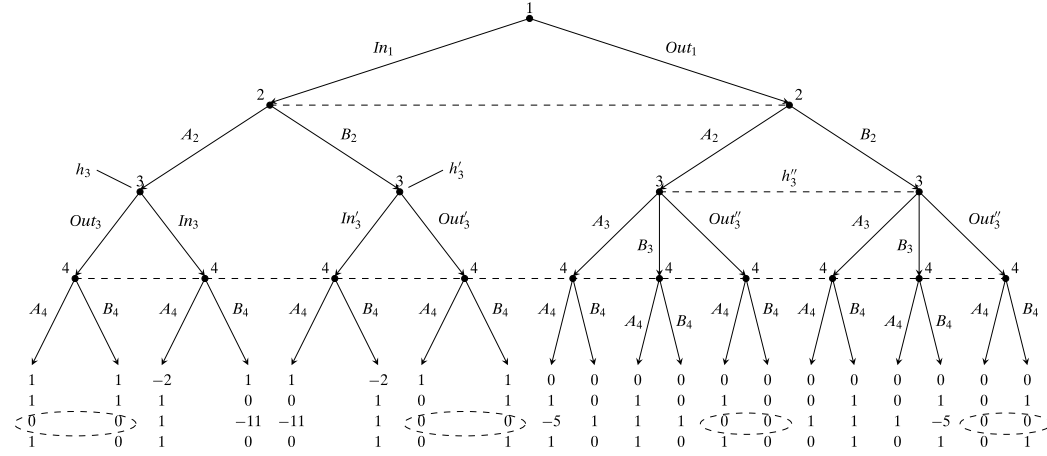


FIGURE 10. Example 10.

different  $h$ 's. All players observe the exact terminal node reached, except that player 3 does not observe player 4's choice when player 3 plays *Out*. Intuitively, if player 1 plays *Out*<sub>1</sub> and player 3 thinks that players 2 and 4 play either the  $(A_2, A_4)$  or  $(B_2, B_4)$  equilibria regardless of player 1's play, then player 1's deviation to *In*<sub>1</sub> would inform player 3 of which equilibrium players 2 and 4 are coordinating on. To model this inference, player 3's belief about the continuation play has to vary across information sets, which the definition of accordance allows.

We show here that  $(Out_1, A_2, Out''_3, A_4)$  is a RPCE outcome while it would not be if we strengthened the definition by replacing part (ii) by the following condition: For all  $h \in H_i$ ,

$$(\mu_i)_h(\hat{\pi}_{-i}) = \sum_{\tilde{\pi}_{-i}(h') = \hat{\pi}_{-i}(h') \text{ for all } h' \text{ after } h} b(\mu_i)(\tilde{\pi}_{-i}). \quad (1)$$

This stronger condition requires that the continuation play has to agree with  $b(\mu)$  as opposed to merely having a weakly smaller support.

First we show that  $(Out_1, A_2, Out''_3, A_4)$  is a RPCE. In particular, it satisfies accordance. To see this, consider the belief model and actual versions

$$\begin{aligned} V_1 &= \{v'_1, v''_1\}, & v'_1 &= (Out_1, (v'_2, v'_3, v'_4)), & v''_1 &= (Out_1, (v''_2, v''_3, v''_4)) \\ V_2 &= \{v'_2, v''_2\}, & v'_2 &= (A_2, (v'_1, v'_3, v'_4)), & v''_2 &= (B_2, (v''_1, v''_3, v''_4)) \\ V_3 &= \{v'_3\}, & v'_3 &= \left( (In_3, In'_3, Out''_3), \left( \frac{1}{2}(v'_1, v'_2, v'_4), \frac{1}{2}(v''_1, v''_2, v''_4) \right) \right) \\ V_4 &= \{v'_4, v''_4\}, & v'_4 &= (A_4, (v'_1, v'_2, v'_3)), & v''_4 &= (B_4, (v''_1, v''_2, v''_3)) \end{aligned}$$

The actual version profile is  $(v'_1, v'_2, v'_3, v'_4)$ .

In this belief model, player 3 assigns probability  $\frac{1}{2}$  to each of  $(A_2, A_4)$  and  $(B_2, B_4)$ . Then, when player 1 plays *Out*<sub>1</sub>, RPCE allows for a belief in which (i) player 3 at  $h_3$  thinks that player 4 will play  $A_4$ , (ii) player 3 at  $h'_3$  thinks that player 4 will play  $B_4$ , and

(iii) player 3 at  $h_3''$  thinks that player 4 will play  $A_4$  after  $A_2$  and  $B_4$  after  $B_2$ . Given this belief,  $(In_3, In_3', Out_3'')$  is a best response for player 3.

If player 1 thinks that player 3's version is as above, then player 1 would expect payoff  $-2$  from playing  $In_1$  and  $0$  from playing  $Out_1$ . So  $Out_1$  is a best response.

Now we show that  $(Out_1, A_2, Out_3'', A_4)$  is not a RPCE outcome with the stronger version of accordance that imposes condition (1).

To see this, note that player 3's belief about player 2's and player 4's play can only assign positive probability to the strategy profiles  $(A_2, A_4)$ ,  $(B_2, B_4)$ , or  $((\frac{1}{2}A_2, \frac{1}{2}A_4), (\frac{1}{2}B_2, \frac{1}{2}B_4))$ . Some observations about player 3's incentives are in order. First, for  $A_3''$  not to be strictly better than  $Out_3''$  at  $h_3''$ , player 3 cannot assign probability more than  $\frac{5}{6}$  to  $(B_2, B_4)$ , while for  $B_3''$  not to be strictly better than  $Out_3''$  at  $h_3'$ , player 3 cannot assign probability more than  $\frac{5}{6}$  to  $(A_2, A_4)$ .<sup>20</sup>

Next, for  $Out_1$  to be a best response for player 1, player 1 has to think that player 3 plays  $In_3$  or  $In_3'$  with a positive probability. This means that player 1 thinks that either player 3 at  $h_3$  thinks that player 4 would play  $A_4$  with probability at least  $\frac{11}{12}$ ,<sup>21</sup> or player 3 at  $h_3'$  thinks that player 4 would play  $B_4$  with probability at least  $\frac{11}{12}$ . But this is impossible under the strengthened definition of accordance because, given the conclusion above, player 3's belief about player 4's continuation strategy can assign probability at most  $\max_p[\frac{5}{6} \cdot (1-p) + \frac{1}{2} \cdot p] = \frac{5}{6}$  to each of  $A_4$  and  $B_4$ , where  $p$  in the maximand denotes the probability that player 3 attaches to the mixed equilibrium play by players 2 and 4.  $\diamond$

## 5. RPCE, RSCE, AND RCE

In this section, we compare RPCE with other concepts from the literature. In Section 5.1, we compare RPCE with RSCE, and show that RPCE “reduces” to RSCE if the terminal node partitions are discrete and beliefs are independent. In Section 5.2, we compare RPCE with RCE Rubinstein and Wolinsky (1994), and show that when the signal function specified in the definition of RCE gives the same information as the partitions of the terminal nodes, RPCE is equivalent to RCE if moves are simultaneous.

### 5.1 Rationalizable self-confirming equilibrium

In this subsection, we show that RPCE is implied by RSCE if we require independent beliefs. One part of this argument is that any independent beliefs can be reduced to a single behavior strategy profile for the opponents, as shown by Fudenberg and Kreps (1995); the idea is that Kuhn's theorem allows us to associate a behavior strategy to any probability distribution on strategies, and that with independence, the profile of these associated behavior strategies is equivalent to the original belief.

To see this formally, let us first define RSCE (notations are adjusted to accord with ours). This concept is defined for games with discrete terminal node partitions.

<sup>20</sup>If  $(B_2, B_4)$  is assigned probability  $\frac{5}{6}$ , then playing  $A_3''$  ensures the payoff of  $1 \times \frac{5}{6} + (-5) \times \frac{1}{6} = 0$  because  $-2$  is the worst payoff that player 3 can get given  $Out_1$ . A similar computation applies to the play of  $B_3''$  at  $h_3'$ .

<sup>21</sup>If  $A_4$  is assigned probability  $\frac{11}{12}$ , then playing  $In_3$  ensures the payoff of  $1 \times \frac{11}{12} + (-11) \times \frac{1}{12} = 0$ . A similar computation applies to the expected payoff of player 3 at  $h_3'$ .

DEFINITION 8. The strategy profile  $\pi^*$  is a *rationalizable self-confirming equilibrium* if there exist a belief model  $V$  and an actual version profile  $v^*$  such that the following three conditions hold:<sup>22</sup>

- (i) Strategy profile  $\pi^*$  is generated by  $v^*$ .
- (ii') For each  $i$  and  $v_i = (\pi_i, p_i)$ , there exists  $\mu_i$  such that (a)  $\mu_i$  is coherent with  $p_i$ , (b)  $\pi_i$  is a best response to  $\mu_i$  at all  $h \in H_i$ , and (c)  $\mu_i$  is an independent belief.<sup>23</sup>
- (iii') For all  $i$  and  $v_i = (\pi_i, p_i)$ ,  $d(\pi_i, \pi_{-i}(v_{-i})) = d(\pi^*)$  for all  $v_{-i}$  in the support of  $p_i$ .

There are two main differences between this definition and that of RPCE, namely that Definition 8(iii') (every version expects the same distribution over terminal nodes) is stronger than Definition 7(iii), and that observational consistency Definition 7(iv) is not directly imposed in RSCE. Even with a discrete terminal node partition, the way Definition 7(iii) is stated is somewhat different than Definition 8(iii'),<sup>24</sup> but as the next result shows, this difference is irrelevant.

THEOREM 3. *Fix a game with discrete terminal node partitions.*

- (i) *If an actual version profile  $v^*$  and a belief model  $V$  satisfy Definition 7(iii) and Definition 7(iv), then there exists a belief model  $\hat{V}$  such that  $v^*$  and  $\hat{V}$  satisfy Definition 8(iii') and Definition 7(iv).*
- (ii) *Definition 8(iii') implies Definition 7(iv).*

The proof of this result is given in the Appendix C. Part (ii) is not surprising: Since the terminal node partitions are discrete, Definition 8(iii') essentially requires that the terminal node reached is common knowledge, so observational consistency holds. Part (i) says that in the presence of the observational consistency condition, requiring the self-confirming condition for hypothetical versions does not further restrict the set of equilibria. Notice that this conclusion was not true when we considered RPCE with nondiscrete terminal node partitions (see Example 7).

COROLLARY 2. *In games with discrete terminal node partitions, any outcome of a RSCE is the outcome of a RPCE with independent beliefs.*

In the next example, which is taken from DFL's Example 3.2, we show that the set of possible outcomes can expand if we relax the definition of RSCE equilibrium by replacing Definition 8(iii') with Definition 7(iii).

<sup>22</sup>DFL allow all  $\hat{\pi}$  that have the same distribution over terminal nodes as  $\pi^*$  to be RSCE, but this difference is not important for our purpose.

<sup>23</sup>DFL required optimality only at the information sets that have positive probability under  $\pi_i$ , but the difference is immaterial in one-move games.

<sup>24</sup>If  $v_i$  is self-confirming, then  $d(\pi_i, \pi_{-i}(v_{-i}))$  equals  $d(\pi_i, \pi_{-i}^*)$  for all  $v_{-i}$  in the support of  $p_i$ , but Definition 8(iii') states that it is equal to  $d(\pi_i^*, \pi_{-i}^*)$ .

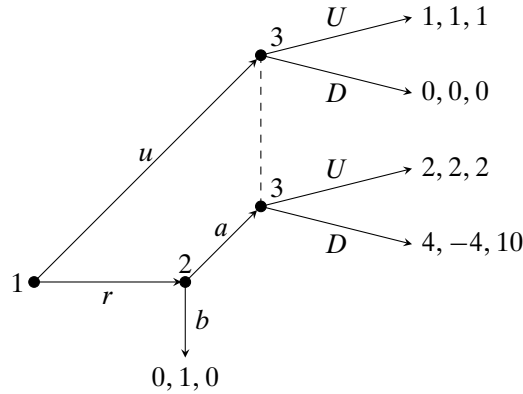


FIGURE 11. Example 11.

EXAMPLE 11 (DFL). Consider the game depicted in Figure 11, where all players' terminal node partitions are discrete. DFL argue that the outcome  $(u, U)$  is impossible in RSCE, because if player 3 chooses  $U$ , then player 2 should play  $a$  since he observes the terminal node, and then player 1 should take  $r$ . However, if we replace Definition 8(iii') by Definition 7(iii) in Definition 8, where observational consistency is not imposed, this outcome becomes possible. To see this, consider the belief model and actual versions

$$\begin{aligned} V_1 &= \{v'_1, v''_1\}, & v'_1 &= (u, (v''_2, v'_3)), & v''_1 &= (r, (v'_2, v''_3)) \\ V_2 &= \{v'_2, v''_2\}, & v'_2 &= (a, (v'_1, v'_3)), & v''_2 &= (b, (v'_1, v'_3)) \\ V_3 &= \{v'_3, v''_3\}, & v'_3 &= (U, (v'_1, v'_2)), & v''_3 &= (D, (v'_1, v'_2)) \end{aligned}$$

The actual version profile is  $(v'_1, v'_2, v'_3)$ .

Here all the conditions in the definition of RSCE other than Definition 8(iii') hold, as does Definition 7(iii). Notice that  $v''_1$ ,  $v''_2$ , and  $v''_3$  are not self-confirming with respect to the actual distribution  $\pi(v'_1, v'_2, v'_3)$ , and they are hypothetical versions and not actual ones.

The key is that the actual version of player 1,  $v'_1$ , conjectures that player 2 believes that player 3 plays  $D$ , and this conjecture is ruled out by observational consistency: The equation in Remark 1(a) of observational consistency applied to  $v'_1$ 's belief is  $d(b, (u, D)) = d(b, (u, U))$ . But this equation is false.  $\diamond$

Notice that the set of SCE is the same with Definition 7(iii) or Definition 8(iii'), thus requiring optimality at off-path information sets is the key to this example.

Finally, we have shown in Section 2 that Kreps and Wilson's (1982) consistency implies our restriction on beliefs. Since our restrictions do not imply consistency, the converse of Corollary 2 need not hold.

### 5.2 Rationalizable conjectural equilibrium

The main difference between RPCE and RCE is that RPCE, like RSCE, requires players believe others will play rationally (maximize the presumed payoff functions) as long as they have not behaved irrationally in the past, while RCE is designed to model normal-form games and places no restrictions on play at off-path information sets.<sup>25,26</sup> Because of this difference, RPCE makes stronger predictions than RCE in most extensive-form games. If all information sets are on every path, this distinction becomes moot and the two concepts become equivalent. In particular, in one-shot simultaneous-move games, we can state the precise connection between RCE and RPCE. To do so, we first define RCE.

Consider a normal-form game with players  $I = \{1, \dots, n\}$ , the action set  $A_i$ ,  $A = \times_{i \in I} A_i$ , and  $A_{-i} = \times_{j \neq i} A_j$ , and the payoff function  $u_i : A \rightarrow \mathbb{R}$ . The set of mixed strategies is  $M_i = \Delta(A_i)$ ,  $M = \times_{i \in I} M_i$ , and  $M_{-i} = \times_{j \neq i} M_j$ . There is a set of private signals  $S_i$ , and a signal function  $g_i : A \rightarrow S_i$ .  $g_i(a)$  is the signal that  $i$  privately observes when the action profile is  $a \in A$ . With an abuse of notation, we write  $g_i(m)$  for a probability distribution over  $S_i$  given the mixed profile  $m \in M$ , called a random signal. Let  $\sigma_i \in \Delta(S_i)$  be the general element of the set of random signals.

The strategy-signal pair  $(m_i, \sigma_i)$  is said to be *g-rationalized* by  $\gamma \in \Delta(M_{-i})$  if (i)  $g_i(m_i, m_{-i}) = \sigma_i$  for all  $m_{-i} \in \text{supp}(\gamma)$  and (ii)  $m_i$  is a best response against  $\gamma$ .

The sets of strategy-signal pairs  $B_1, \dots, B_n$  are *g-rationalizable* if for all  $i$ , every  $(m_i, \sigma_i) \in B_i$  is *g-rationalized* by some  $\gamma$  such that for all  $m_{-i} \in \text{supp}(\gamma)$  and all  $j$ ,  $(m_j, g_j(m_i, m_{-i})) \in B_j$ .

An RCE is  $m^* \in M$  such that there exists *g-rationalizable* sets  $B_1, \dots, B_n$  such that  $(m_i^*, g_i(m^*)) \in B_i$  for each  $i$ .

For an extensive-form game  $\Gamma$  with terminal node partitions  $\mathbf{P} = (\mathbf{P}_1, \dots, \mathbf{P}_n)$ , let  $(A^\Gamma, g^\mathbf{P})$  be the pair of normal-form representation of  $\Gamma$  and the profile of signal functions (denoted by  $g^\mathbf{P} := (g_1^\mathbf{P}, \dots, g_n^\mathbf{P})$ ) such that  $g_i^\mathbf{P}(a) = \mathbf{P}_i(a)$  for each action profile  $a \in A^\Gamma$ . Conversely, given any  $(A, g)$  with  $g = (g_1, \dots, g_n)$  such that  $g_i(m) = g_i(m')$  implies  $m_i = m'_i$  (so that the (extended notion of) perfect recall assumption is satisfied), we define the related simultaneous-move extensive-form game  $\Gamma^A$  and endow it with the terminal node partition  $\mathbf{P}^g$  such that  $\mathbf{P}_i^g(a) = g_i(a)$  for each action profile  $a \in A^\Gamma$ .

Finally, we say that a behavioral strategy  $\pi$  is equivalent to a mixed strategy profile  $m$  or a mixed strategy profile  $m$  is equivalent to a behavioral strategy  $\pi$  if  $\pi$  is generated by  $m$  according to the Kuhn's theorem.

Now we are ready to state the formal connection between the two concepts. We omit the proof.

**THEOREM 4.** (i) Any RPCE in  $(\Gamma, \mathbf{P})$  is equivalent to some RCE in  $(A^\Gamma, g^\mathbf{P})$ .

(ii) Any RCE in  $(A, g)$  is equivalent to some RPCE in  $(\Gamma^A, \mathbf{P}^g)$ .

<sup>25</sup>See the example in Figure 2.1 of DFL.

<sup>26</sup>Gilli (1999) proposes a related solution concept; Battigalli (1999) shows it is equivalent to RCE.

One consequence of this equivalence is that RCE, like RPCE, requires that in games like [Example 3](#) when player 1 plays  $Out_1$ , she believes the play of the others is a Nash equilibrium of the subgame.<sup>27</sup> In particular this is true even in a three-player game where players 2 and 3 play the game of [Shapley \(1964\)](#), where fictitious play and smooth fictitious play do not converge.<sup>28</sup> Because the long-run joint distribution over actions in the Shapley cycle is a correlated equilibrium, this example may suggest an alternative equilibrium concept in which players expect that the empirical distribution of unobserved on-path play is a correlated equilibrium in the subgame. We do not define this alternative here because it is typically too inclusive.

## 6. CONCLUSION

Like RCE and RSCE, RPCE combines the idea that players have partial but objective information about equilibrium play with the idea that players reason about the observations and incentives of others. RSCE applies to extensive-form games where players see the realized terminal node at the end of each play of the game; RPCE generalizes this to situations where players see only a partition of the terminal nodes. In addition, RPCE relaxes the independent-beliefs condition of RSCE to allow for correlation.

The examples show that (i) under RPCE, a player's belief about the actions of others can depend on whether those others get to act along the equilibrium path, (ii) unobserved on-path play provides a natural form of correlating device, (iii) a player can learn about the unobserved actions of a second player from the actions of a third, and, finally, (iv) the precise implications of all of the above depend on the nature of the terminal node partitions. In general, coarsening a player's terminal node partition cannot restrict the set of that player's RPCE strategies, but it can enlarge it. We identified four reasons that this enlargement can occur, and provided a sufficient condition under which coarsening a player's terminal partition has no effect on his RPCE strategies. We also showed how RPCE reduces to RCE and RSCE in the appropriate special cases.

The supplementary Appendix discusses three additional topics: The definition of partition-confirmed equilibrium (PCE), the epistemic interpretation of observational consistency, and the effect of changes in terminal node partitions on the outcomes under RPCE.

## APPENDIX A: [LEMMA 1](#)

The next lemma shows that in a RPCE, any version profile to which an actual version assigns positive probability is the actual version profile of some RPCE.

<sup>27</sup>[Appendix B](#) gives a formal definition of the class of “player-1 participation games” and proves this claim.

<sup>28</sup>[Brown \(1951\)](#) introduced fictitious play as a way to compute Nash equilibria. [Fudenberg and Kreps \(1993\)](#) give fictitious play a descriptive interpretation in strategic form games, and point out some problems with that interpretation when the process cycles instead of converging to constant play of a fixed pure action profile.



LEMMA 1. *Fix a RPCE  $\pi^*$ , a belief model  $V$ , and actual versions  $v^*$  that support it. For every player  $i$ , if  $v_i^*$ 's conjecture assigns positive probability to  $\tilde{v}_{-i}$ , then  $\pi(v_i^*, \tilde{v}_{-i})$  is also a RPCE.*

PROOF. Pick player  $i$  and  $\tilde{v}_{-i}$  to which  $v_i^*$  assigns positive probability. We will use the belief model  $V$  to support a  $\pi(v_i^*, \tilde{v}_{-i})$  as a RPCE. Conditions (ii) and (iv) of the definition of RPCE (Definition 7) hold for all versions in  $V$ , so they hold for strategy  $\pi(v_i^*, \tilde{v}_{-i})$  and the belief model  $V$  as well. So it remains to show that if  $(v_i^*, \tilde{v}_{-i})$  is the actual version profile, then  $v_i^*$  satisfies the self-confirming condition with respect to  $\pi(v_i^*, \tilde{v}_{-i})$  and that  $\tilde{v}_j$  for each player  $j \neq i$  does as well. First, since  $v_i^*$  satisfies the self-confirming condition in the original RPCE and since  $\tilde{v}_{-i}$  is in the support of the conjecture of  $v_i^*$  by assumption,  $v_i^*$  is self-confirming with respect to  $\pi(v_i^*, \tilde{v}_{-i})$ . Second, since  $v_i^*$  satisfies observational consistency in the original RPCE,  $D_j(\pi_j(\tilde{v}_j), \pi_{-j}(\hat{v}_{-j})) = D_j(\pi(v_i^*, \tilde{v}_{-i}))$  for all  $\hat{v}_{-j}$  in the support of the conjecture of  $\tilde{v}_j$ . Thus  $\tilde{v}_j$  satisfies the self-confirming condition with respect to  $\pi(v_i^*, \tilde{v}_{-i})$ .  $\square$

REMARK 2. The proof of Lemma 1 shows that even the hypothetical version  $\tilde{v}_j$  must satisfy the self-confirming condition with respect to  $\pi(v_i^*, \tilde{v}_{-i})$  if  $v_i^*$  assigns positive probability to  $\tilde{v}_{-i}$ . This does not imply that  $\tilde{v}_j$  is self-confirming with respect to  $\pi^*$ . Indeed, imposing that condition (i.e., imposing Definition 7(iii) for all versions and not just the actual ones) would be unduly restrictive, as we show in Example 7.

## APPENDIX B: A THEOREM FOR PARTICIPATION GAMES

The class of participation games generalizes some of the examples from the text. Intuitively, this is a game in which player 1 has an option to play *Out* at the root node that prevents her from observing the consequence of the opponents' actions at the terminal nodes, and other players play a game, not knowing player 1's action. Formally, a *player-1 participation game*  $\Gamma$  (with a payoff function  $u$  and the set of players  $I$ ) is an extensive-form game with the following properties: Fix player 1's set of actions  $A_1$  such that one of its element is *Out*, and fix another extensive-form game  $\Gamma'$  with a payoff function  $v$  and the set of players  $I \setminus \{1\}$ . Denote by  $n(x, a)$  the node in  $\Gamma$  that corresponds to  $x$  in  $\Gamma'$  after action  $a \in A_1$  is taken.

- At the root node, player 1 moves, choosing between *In* and *Out*.
- Whichever action is taken,  $\Gamma'$  is played after player 1's decision.
- Nodes  $n(x, a)$  and  $n(x', a')$  for  $x, x' \in X \setminus Z$  are in the same information set in  $\Gamma$  if and only if  $x$  and  $x'$  are in the same information set in  $\Gamma'$ .
- Terminal nodes  $n(z, a)$  and  $n(z', a')$  for  $z, z' \in Z$  are in the same cell of the terminal node partition of player  $i$  if and only if  $z$  and  $z'$  are in the same cell for  $i$  and  $a = a'$ , except for the following exception.

- Terminal nodes  $n(z, Out)$  and  $n(z', Out)$  for  $z, z' \in Z$  are in the same cell of the terminal node partition of player 1.
- For all  $i$ ,  $u_i(n(z, In)) = v_i(z)$ ,  $u_i(n(z, Out)) = v_i(z)$  for all  $i \neq 1$ , and  $u_1(n(z, Out)) = 0$ .

**THEOREM 5.** *Fix a player-1 participation game  $\Gamma$ . If player 1 plays Out with probability 1 in a RPCE, then there is a convex combination of RPCE of  $\Gamma'$  such that no action of player 1 has a strictly positive payoff.*

**PROOF.** Fix a RPCE in which player 1 plays Out with probability 1. Pick a version profile for player 1's opponents  $\tilde{v}_{-1}$  to which the conjecture of  $v_1^*$  assigns positive probability. By Lemma 1,  $\pi(v_1^*, \tilde{v}_{-1})$  is a RPCE of  $\Gamma$ , and it is clear from the proof of the lemma that the same belief model  $V$  that supports the original RPCE can be used to support  $\pi(v_1^*, \tilde{v}_{-1})$  as a RPCE, where the actual version profile is  $(v_1^*, \tilde{v}_{-1})$ . By the definition of a player-1 participation game, player 1's action does not affect any opponent's payoff or observation. Thus  $\pi_{-1}(\tilde{v}_{-1})$  is trivially a RPCE of  $\Gamma'$ , with the belief model simply deleting player 1. Since this is true for any  $\tilde{v}_{-1}$  in the support of the conjecture of  $v_1^*$ , and the strategy of the actual version of player 1 is a best response to her belief in the original RPCE, the proof is complete.  $\square$

**COROLLARY 3.** *Fix a player-1 participation game  $\Gamma$  such that  $\Gamma'$  is a simultaneous-move game with discrete terminal node partitions and a unique Nash equilibrium. If player 1 plays Out with probability 1 in a RPCE of  $\Gamma$ , then no action of player 1 gives her a positive payoff against this unique Nash equilibrium.*

#### APPENDIX C: PROOF OF THEOREM 3

**PROOF.** Part (i). Fix an actual version  $v^*$  and a belief model  $V$  that satisfies Definition 7(iii) and Definition 7(iv). Construct a new belief model  $\hat{V}$  that is identical to the original one, except that all versions that do not satisfy the equality in Definition 8(iii') in  $V$  are eliminated and each version's conjecture assigns the same weight to the versions that are still in  $\hat{V}$ . Specify the same actual version profile as in  $V$  (such versions are not eliminated because of Definition 7(iii)). By construction, Definition 8(iii') holds. Hence by part (ii) that we prove below, Definition 7(iv) holds as well. Finally, we check that the sum of probabilities assigned by the conjecture of any remaining version is unity. To see this, note first that Definition 7(iii) implies that the actual version  $v_i^* = (\pi_i^*, p_i^*)$  must satisfy  $(\pi_i^*, \pi_{-i}(v_{-i})) = d(\pi(v^*))$  for all  $v_{-i}$  in the support of  $p_i^*$ . Also, for  $\tilde{v}_j = (\tilde{\pi}_j, \tilde{p}_j)$ , whenever  $d(\tilde{\pi}_j, \pi_{-j}(v_{-j})) = d(\pi(v^*))$  for all  $v_{-j}$  in the support of  $\tilde{p}_j$ , observational consistency implies that for any version of  $j$ 's opponent  $\hat{v}_k = (\hat{\pi}_k, \hat{p}_k)$  in the support of  $\tilde{p}_j$ ,  $d(\hat{\pi}_k, \pi_{-k}(v_{-k})) = d(\pi(v^*))$  for all  $v_{-k}$  in the support of  $\hat{v}_k$ . This means that no version who is assigned a positive probability by any remaining version is eliminated, implying that the sum of probabilities is still unity.

Part (ii). Since terminal node partitions are discrete, the observational consistency condition for version  $v_i = (\pi_i, p_i)$  reduces to the requirement that  $p_i(v_{-i}) > 0$  implies, for each  $j \neq i$ ,  $d(\pi_j(v_j), \pi_{-j}(v_{-j})) = d(\pi_i, \pi_{-i}(v_{-i}))$  for all  $v_{-j}$  in the support of  $v_j$ 's conjecture. But the conclusion of this requirement is implied by Definition 8(iii'), as Definition 8(iii') implies that both sides of the equality are equal to  $d(\pi^*)$ .  $\square$

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