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## Article

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Theoretical Economics

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*Suggested Citation:* Hart, Sergiu; Foster, Dean P. (2013) : A wealth-requirement axiomatization of riskiness, Theoretical Economics, ISSN 1555-7561, The Econometric Society, New Haven, CT, Vol. 8, Iss. 2, pp. 591-620,  
<https://doi.org/10.3982/TE1150>

This Version is available at:

<https://hdl.handle.net/10419/150201>

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## A wealth-requirement axiomatization of riskiness

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We provide an axiomatic characterization of the measure of riskiness of gambles (risky assets) introduced by Foster and Hart (2009). The axioms are based on the concept of “wealth requirement.”

KEYWORDS. Riskiness, gamble, risky asset, reserve, wealth.

JEL CLASSIFICATION. D81, G00, G32.

### 1. INTRODUCTION

How are risks evaluated? Here *risk* is meant in the simplest sense: facing certain gains or losses, with given probabilities. The “subjective” approach considers each individual decision-maker separately, and proceeds according to that decision-maker’s preference and utility. But can risks be evaluated in an “*objective*” manner—depending only on the risks themselves and not on the specific decision-maker’s attitude? While at first sight this may appear to be a tall order, remember that objective measures do exist, as, for example, the return of the gamble (its expectation) and the spread of the gamble (its standard deviation). While the standard deviation is at times used also to measure riskiness, in general it is not a good measure of it. This is so, in particular, since the standard deviation is not monotonic: one may increase the gains and decrease the losses—which clearly lowers the risks—in such a way that the standard deviation actually increases.

New objective measures of riskiness have recently been developed by Aumann and Serrano (2008) and Foster and Hart (2009). While the approach of Aumann and Serrano is axiomatic (it is based mainly on their “duality” axiom), the approach of Foster

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Previous versions of this article, starting from December 2007, were titled “A Reserve-Based Axiomatization of the Measure of Riskiness.” Research of the second author was partially supported by grants from the Israel Science Foundation and the European Research Council. The authors thank Robert Aumann, Moti Michaeli, Aldo Rustichini, Ran Shorrer, Benjamin Weiss, the anonymous referees, and the co-editor for useful comments. A presentation is available on Hart’s website.

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DOI: 10.3982/TE1150

and Hart is constructive, in providing for each gamble the critical wealth level that separates “bad” investments (such as those leading to bankruptcy) from “good” ones (such as those leading to increasing wealth).<sup>1</sup>

In the present paper, we provide an *axiomatic approach to the Foster–Hart measure of riskiness*: we propose four basic axioms for a riskiness measure and show that the minimal function satisfying these axioms is precisely the Foster–Hart measure.

Since “riskiness” does not appear to be a straightforward and obvious concept, one needs to have in mind a certain viewpoint and interpretation. The leading one that we propose is that of *wealth requirement*: the minimal wealth required to engage in a risky activity. Wealth requirements are common, for instance, in high-risk investments (such as hedge funds, in which one should not invest more than a certain proportion of one’s wealth), in risky endeavors (such as getting a license for exploration of natural resources—e.g., oil and gas—or for building a large project—e.g., a new transportation system). After all, risks have to do with possible changes in wealth—whether gains or losses—and so it is natural to use the “wealth effects” to measure the riskiness.

The first two axioms that we propose are standard (and are satisfied by the return, the spread, and many other objective measures): the **Distribution** axiom, which says that only the outcomes and their probabilities matter, and the **Scaling** axiom, which says, for instance, that doubling the gamble doubles its riskiness (which is measured in the same units as the outcomes). The fact that we are dealing with riskiness and wealth requirements is expressed in the other two axioms: the **Monotonicity** axiom, which says that decreasing some gains or increasing some losses must increase the wealth requirement, and the **Compound Gamble** axiom, which says that once the wealth effect is taken into account, the way the gamble is presented does not matter.

Our result is that these axioms characterize the “critical wealth” for a certain class of von Neumann–Morgenstern utility functions, i.e., that wealth level where the decision-maker is indifferent between accepting and rejecting the gamble.<sup>2</sup> Perhaps surprisingly, one of these functions turns out to be *minimal* for all gambles; that is, it bounds from below *all* wealth requirements for *all* gambles (one may thus refer to it as the *critical critical wealth*). This minimal wealth requirement is precisely the Foster–Hart measure of riskiness. In other words, any wealth requirement (that satisfies the axioms) must be at least as conservative as the one given by the Foster–Hart measure.<sup>3</sup>

The paper is organized as follows. In **Section 2**, we present the formal model and the four axioms. The main result that the Foster–Hart measure is the minimal function satisfying the axioms is stated in **Section 3**. Our axioms characterize a family of riskiness measures, which turn out to be the critical wealth levels for a certain one-parameter family of utility functions (specifically: CRRA- $\gamma$  with  $\gamma \geq 1$ ); see **Section 4**. In **Section 5**,

<sup>1</sup>An alternative, “ordinal,” approach—comparing gambles according to how often they are rejected by risk-averse decision-makers—is provided by Hart (2011): it yields precisely the two orders generated by the Aumann–Serrano and the Foster–Hart measures.

<sup>2</sup>We emphasize that our axioms characterize not only these expected utility functions, but also the resulting “fixed point” where the certainty equivalent of the final wealth (which is the current wealth plus the gamble outcome) equals the current wealth.

<sup>3</sup>For generalizations to sets of gambles and non-expected-utility models, see Michaeli (2012).

we show that dropping the wealth effect from the **Compound Gamble** axiom yields the Aumann–Serrano index (which may be viewed, in a certain sense, as the *maximal* riskiness function; see [Section 7\(f\)](#)); this is a new axiomatic characterization of the Aumann–Serrano index. An outline of the proofs, together with some additional results, is provided in [Section 6](#). We conclude in [Section 7](#) with discussions and comments on further issues. The proofs and additional material are relegated to the [Appendices](#).

## 2. THE SETUP

Following [Aumann and Serrano \(2008\)](#) and [Foster and Hart \(2009\)](#), a *gamble*—or “risky asset”—is a real-valued random variable  $g$  that represents net changes to the wealth, such that losses are possible and the expected return is positive; i.e.,<sup>4</sup>  $\mathbf{P}[g < 0] > 0$  and  $\mathbf{E}[g] > 0$ . For simplicity, we assume that each gamble  $g$  takes only finitely many values, say  $x_1, x_2, \dots, x_m$ , with respective probabilities  $p_1, p_2, \dots, p_m$  (where  $p_i > 0$  and  $\sum_{i=1}^m p_i = 1$ ); we denote this as  $(x_1, p_1; x_2, p_2; \dots; x_m, p_m)$ , or  $(x_i, p_i)_{i=1, \dots, m}$  for short. We denote by  $\mathcal{G}$  the collection of all such gambles. For each  $g$  in  $\mathcal{G}$ , let  $L(g) := -\min g \equiv -\min_{1 \leq i \leq m} x_i > 0$  be the *maximal loss* of  $g$ .

To each gamble  $g$  in  $\mathcal{G}$ , we want to associate a positive number  $Q(g)$  that measures its *riskiness*. As stated in the [Introduction](#), an interpretation the reader may want to keep in mind is that  $Q(g)$  represents a certain kind of “cushion” or “reserve” needed for  $g$ : the *wealth requirement* of  $g$ . Let thus<sup>5</sup>  $Q: \mathcal{G} \rightarrow \mathbb{R}_+$ , where the positive number  $Q(g) > 0$  is measured in the same units as the outcomes of  $g$ .

### 2.1 The axioms

We propose four axioms that a riskiness or wealth requirement should satisfy. In [Section 7\(b\)](#), we provide some further comments on the rationale behind these postulates; let us already say here that a leading concern is that one wants to avoid manipulations that do not affect the gamble but do affect the wealth requirement. From now on,  $g, h, \dots$  always denote gambles in  $\mathcal{G}$ .

**DISTRIBUTION.** *If  $g$  and  $h$  have the same distribution, then  $Q(g) = Q(h)$ .*

That is, only the outcomes and their probabilities matter. We thus no longer distinguish between a gamble and its distribution, and write  $g = (x_1, p_1; \dots; x_m, p_m)$ .

**SCALING.**  *$Q(\lambda g) = \lambda Q(g)$  for every  $\lambda > 0$ .*

That is,  $Q(\lambda x_1, p_1; \dots; \lambda x_m, p_m) = \lambda Q(x_1, p_1; \dots; x_m, p_m)$ . Thus, the wealth requirement does not depend on the unit in which the outcomes are measured: rescaling all outcomes by a factor  $\lambda > 0$  rescales it by the same  $\lambda$ .

<sup>4</sup>We write  $\mathbf{P}$  for probability and  $\mathbf{E}$  for expectation. Since, as we will see shortly, only the distribution of the random variables matter, there is no need to specify the underlying probability spaces. Alternatively, take a (large enough) probability space over which all the gambles are defined.

<sup>5</sup>We write  $\mathbb{R} = (-\infty, \infty)$  and  $\mathbb{R}_+ = (0, \infty)$  for the set of reals and the set of positive reals, respectively.

MONOTONICITY. If<sup>6</sup>  $g \geq h$  and  $g \neq h$ , then  $Q(g) < Q(h)$ .

Thus, if the outcomes of  $h$  are less than or equal to the outcomes of  $g$  (i.e., the gains in  $h$  can only be less than those in  $g$  and the losses can only be greater)—with some inequalities being strict—then the wealth requirement for  $h$  must be strictly greater than the wealth requirement for  $g$ . In terms of distributions, one can write this as  $Q(x_1 + \delta, p_1; x_2, p_2; \dots; x_m, p_m) < Q(x_1, p_1; x_2, p_2; \dots; x_m, p_m)$  for any  $\delta > 0$  (iterating this condition yields, together with the **Distribution** axiom, the same condition for any  $g \gneq h$ ). Clearly, **Monotonicity** together with **Distribution** imply that the function  $Q$  is monotonically decreasing with respect to first-order stochastic domination (see **Proposition 14** in **Appendix A.3.1**).

While the axioms up to now are standard, the final one—the **Compound Gamble** axiom—is the main one that embodies the idea of “wealth requirement.” We illustrate it with a simple example.

Let  $g$  be a gamble with two possible outcomes, \$200 and  $-\$100$  (i.e., a gain of \$200 or a loss of \$100), with probabilities  $p_1$  and  $p_2 = 1 - p_1$ , respectively, and assume that its wealth requirement is, say,  $Q(g) = \$500$ . Consider two other gambles,  $h_1$  and  $h_2$ , that are independent of  $g$ , and let  $f$  be the compound gamble consisting of  $g$  followed by  $h_1$  when  $g$  has resulted in a gain of \$200, and  $g$  followed by  $h_2$  when  $g$  has resulted in a loss of \$100 (see **Figure 1**, top). Assume that the wealth requirement of  $h_1$  is  $Q(h_1) = \$700$ , which equals the original wealth requirement of  $g$  of  $\$500 = Q(g)$  plus the gain of  $g$  of \$200 realized before  $h_1$  is taken; assume also that the wealth requirement of  $h_2$  is  $Q(h_2) = \$400$ , which equals the wealth requirement of  $g$  minus the \$100 loss realized before  $h_2$  is taken. Thus in each case, the “new” wealth requirement—which equals the “old” wealth requirement  $Q(g)$  of  $g$  plus the outcome  $x$  of  $g$ —is precisely the correct wealth requirement for the continuation,  $h_1$  or  $h_2$ . What the **Compound Gamble** axiom says is that the original wealth requirement of  $g$  of \$500 is appropriate also for the compound gamble  $f$ . This may be viewed as a kind of *sure-thing principle*: if no matter what the outcome of  $g$  will be—i.e., ex post—the resulting wealth requirement will be just right to continue, then the wealth requirement is the right one also overall—i.e., ex ante. In other words, one does not need to know what the outcome of  $g$  will be, since one will always have the correct wealth requirement to continue (with either  $h_1$  or  $h_2$ , respectively), no matter what the outcome of  $g$  will be.

The **Compound Gamble** axiom is slightly more general: it allows for no continuation gamble in some instances. For example, consider the compound gamble  $f'$  where the same  $h_1$  is taken after the \$200 gain as in  $f$ , but there is no further gamble after the \$100 loss (see **Figure 1**, bottom). Since, again, whenever a gamble is taken there is always the right wealth requirement, it follows that the wealth requirement for  $f'$  is also the same<sup>7</sup>  $\$500 = Q(g)$ .

To state the axiom formally, let  $\mathbf{1}_A$  denote the indicator of an event  $A$  (i.e.,  $\mathbf{1}_A$  is a random variable that equals 1 if  $A$  occurs and 0 otherwise) and let  $[g = x]$  denote the

<sup>6</sup>This presumes that  $g$  and  $h$  are defined on the same probability space  $\Omega$ . The condition is thus  $g(\omega) \geq h(\omega)$  for every  $\omega \in \Omega$ , with strict inequality for some  $\omega \in \Omega$ .

<sup>7</sup>In the extreme case where there are no continuation gambles at all,  $f = g$  and so, of course,  $Q(f) = Q(g)$ .

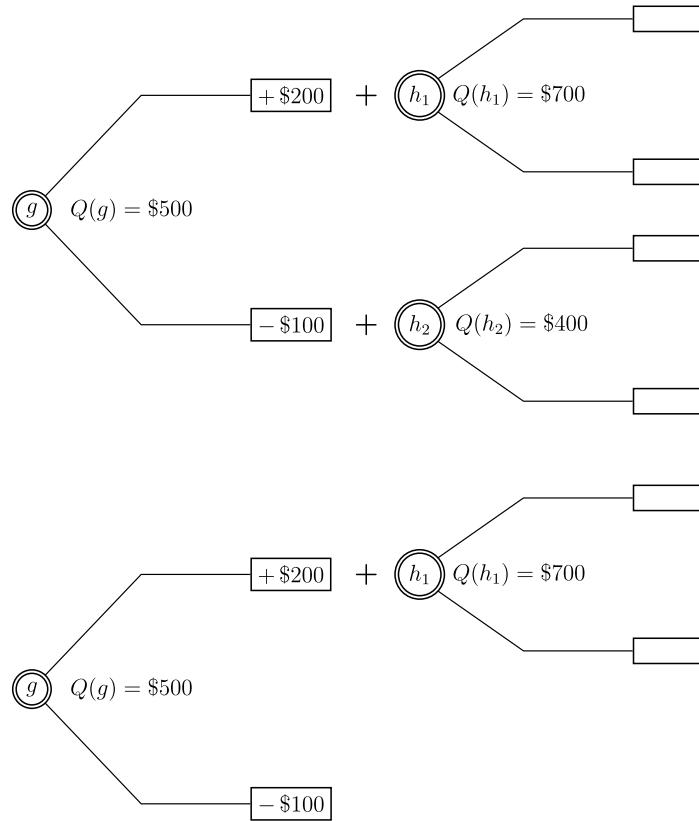


FIGURE 1. The compound gambles  $f$  (top) and  $f'$  (bottom).

event  $\{\omega : g(\omega) = x\}$  that the value of  $g$  is  $x$ . The compound gambles above can thus be written as  $f = g + \mathbf{1}_{[g=\$200]}h_1 + \mathbf{1}_{[g=-\$100]}h_2$  and  $f' = g + \mathbf{1}_{[g=\$200]}h_1$ . Write  $g|_A$  for the restriction of  $g$  on  $A$ ; thus  $g|_A \equiv x$  means that  $g(\omega) = x$  for all  $\omega \in A$ , or  $A \subset [g = x]$ . Finally,  $h$  is independent of  $A$  if the random variables  $h$  and  $\mathbf{1}_A$  are independent; that is,<sup>8</sup>  $\mathbf{P}[h = y|A] = \mathbf{P}[h = y]$  for every value  $y$  of  $h$ . Since one can apply the axiom repeatedly, we state it in its simplest form.

**COMPOUND GAMBLE.** Let  $f = g + \mathbf{1}_A h$  be a compound gamble, where  $g, h \in \mathcal{G}$  and  $A$  is an event such that  $g$  is constant on  $A$ , i.e.,  $g|_A \equiv x$  for some  $x$ , and  $h$  is independent of  $A$ . If  $Q(h) = Q(g) + x$ , then<sup>9</sup>  $Q(f) = Q(g)$ .

For example, if  $g = (x_1, p_1; x_2, p_2; \dots; x_m, p_m)$ ,  $h = (y_1, q_1; y_2, q_2; \dots; y_k, q_k)$ , and  $A = [g = x_1]$  is the event that  $g$  takes the value  $x_1$ , then  $f = (x_1 + y_1, p_1q_1; x_1 + y_2, p_1q_2; \dots; x_1 + y_k, p_1q_k; x_2, p_2; \dots; x_m, p_m)$ . In the example of Figure 1, bottom, the Compound Gamble axiom is applied once; in Figure 1, top, it is applied twice.

<sup>8</sup>The event  $A$  is always assumed to have positive probability, i.e.,  $\mathbf{P}[A] > 0$ , and so the conditional probabilities are well defined.

<sup>9</sup>To see that  $f$  is in  $\mathcal{G}$ : first,  $\mathbf{E}[f] = \mathbf{E}[g] + \mathbf{P}[A]\mathbf{E}[h] > 0$ ; and second, a negative value of  $f$  is  $-L(g)$  when  $x \neq -L(g)$  and  $-L(g) - L(h)$  when  $x = -L(g)$ .

## 3. THE MAIN RESULT

Let  $\mathbf{R}$  be the *measure of riskiness*, introduced by Foster and Hart (2009): for every gamble  $g \in \mathcal{G}$ , the riskiness  $\mathbf{R}(g)$  of  $g$  is given by<sup>10</sup>

$$\mathbf{E}\left[\log\left(1 + \frac{1}{\mathbf{R}(g)}g\right)\right] = 0. \quad (1)$$

Our main result is the following theorem.

**THEOREM 1 (Main Theorem).** *The minimal function that satisfies the four axioms the Distribution, Scaling, Monotonicity, and Compound Gamble axioms is the measure of riskiness  $\mathbf{R}$ . More precisely: (i) the function  $\mathbf{R}$  satisfies these four axioms; and (ii) if a function  $Q$  satisfies these four axioms, then either  $Q(g) = \mathbf{R}(g)$  for all  $g \in \mathcal{G}$ , or  $Q(g) > \mathbf{R}(g)$  for all  $g \in \mathcal{G}$ .*

The Main Theorem says that the *minimal* wealth requirement is precisely our riskiness measure  $\mathbf{R}$ . As we see in the next section, there are other possible wealth requirement functions  $Q$  that satisfy the axioms; however, each one yields *strictly higher* wealth requirement levels for *all* gambles.

We may now combine this result with that of Foster and Hart (2009). A function  $Q$  that associates a positive number with each gamble may be used to decide at which wealth levels to accept or reject gambles. For instance, consider a decision-maker who accepts gambles whenever his wealth is no less than the wealth requirement given by  $Q$ ; i.e., he accepts a gamble  $g$  at wealth  $w$  if and only if  $w \geq Q(g)$ ; call him a *Q-decision-maker*. As in Foster and Hart (2009), a decision-maker is *guaranteed no bankruptcy* if, for any sequence of gambles, his acceptance and rejection decisions make his wealth never go to zero (with probability 1). The result of Theorem 1 in Foster and Hart (2009) together with Theorem 1 above immediately yield a corollary.

**COROLLARY 2.** *If  $Q$  satisfies the four axioms, then a  $Q$ -decision-maker is guaranteed no bankruptcy.*

## 4. THE OTHER RISKINESS MEASURES

What happens if we drop the *minimality* requirement? It turns out that our four axioms characterize a one-parameter family of functions (the minimal one being the Foster–Hart measure  $\mathbf{R}$ ). Rewriting equation (1), which determines  $\mathbf{R}(g)$ , as

$$\mathbf{E}[\log(\mathbf{R}(g) + g)] = \log(\mathbf{R}(g))$$

shows that  $\mathbf{R}(g)$  is that wealth level where a decision-maker with utility function  $\log(x)$  is indifferent between accepting and rejecting  $g$  (at all higher wealths, he accepts  $g$ ; at all lower wealths, he rejects  $g$ ). Call this the *critical wealth* of the log utility for  $g$ .

<sup>10</sup>More precisely, the equation  $\mathbf{E}[\log(1 + (1/r)g)] = 0$  (for  $r > L(g)$ , since  $\log x$  is only defined for  $x > 0$ ) has a unique solution  $r \equiv \mathbf{R}(g)$ ; see Foster and Hart (2009, Lemma 9). When  $g = (x_1, p_1; \dots; x_m, p_m)$ , (1) becomes  $\sum_{i=1}^m p_i \log(1 + x_i/\mathbf{R}(g)) = 0$ . We emphasize that while (1) (as well as (3) and (4) below) may appear at first sight to be a sort of expected utility representation of  $\mathbf{R}(g)$ , this is *not* the case; (1) is an *implicit* equation defining it (we thank Moti Michaeli for this observation).

Now  $\log(x)$  is precisely the utility function  $u(x)$  with *Constant Relative Risk Aversion* (CRRA) equal to 1; i.e., its Arrow–Pratt coefficient of Relative Risk Aversion,  $\text{RRA}_u(x) := -xu''(x)/u'(x)$ , satisfies  $\text{RRA}_u(x) = 1$  for all  $x > 0$ ; see Arrow (1965, 1971) and Pratt (1964). More generally, the CRRA- $\gamma$  utility function, which satisfies  $\text{RRA}_u(x) = \gamma$  for all  $x > 0$ , is<sup>11</sup>  $u_\gamma(x) = (1 - \gamma)x^{1-\gamma}$  for  $\gamma \neq 1$  and  $u_1(x) = \log(x)$  for  $\gamma = 1$ . The *critical wealth* of CRRA- $\gamma$  for a gamble  $g$ , which we denote<sup>12</sup>  $R_\gamma(g)$ , is that wealth level where CRRA- $\gamma$  is indifferent between accepting and rejecting  $g$ ; it is thus given by the equation

$$\mathbf{E}[u_\gamma(R_\gamma(g) + g)] = u_\gamma(R_\gamma(g)). \quad (2)$$

As we see below (Lemmata 5 and 8), for each  $\gamma \geq 1$  and gamble  $g$  in  $\mathcal{G}$ , the positive number  $R_\gamma(g)$  is well defined by (2) and, moreover,  $R_\gamma(g)$  is strictly increasing in  $\gamma$  (this is due to CRRA- $\gamma$  becoming more risk-averse—i.e., more “conservative”—as  $\gamma$  increases, and so the corresponding critical wealth is getting higher). We call the function  $R_\gamma$  the  $\gamma$ -*riskiness measure*: for  $\gamma = 1$ , it is equivalently given by (1)—and so  $R_1 \equiv \mathbf{R}$ , the Foster–Hart measure—and for  $\gamma > 1$ , it is given by

$$\mathbf{E}\left[\left(1 + \frac{1}{R_\gamma(g)}g\right)^{1-\gamma}\right] = 1. \quad (3)$$

Our result is that the four axioms precisely characterize the family  $R_\gamma$ , the critical wealth levels of CRRA- $\gamma$ , for  $\gamma \geq 1$ .

**THEOREM 3.** *A function  $Q$  satisfies the four axioms of the Main Theorem if and only if there exists  $\gamma \geq 1$  such that  $Q(g) = R_\gamma(g)$  for all  $g \in \mathcal{G}$ .*

Thus, our axioms imply that each wealth requirement function is determined by a von Neumann–Morgenstern utility that has a constant coefficient of relative risk aversion of at least 1; the wealth requirement is then the minimal wealth level where the gamble is accepted by this utility.

To illustrate the connection between (2) and the **Compound Gamble** axiom, recall the example of Figure 1 in Section 2.1. Using (2) for  $g$  and for  $h$  (in the first and third equalities below, respectively), we get (put  $u \equiv u_\gamma$ )

$$\begin{aligned} u(500) &= \mathbf{E}[u(500 + g)] \\ &= p_1 u(700) + (1 - p_1) u(400) \\ &= p_1 \mathbf{E}[u(700 + h_1)] + (1 - p_1) u(400) \\ &= \mathbf{E}[u(500 + f')], \end{aligned}$$

i.e., (2) for the compound gamble  $f'$ .

<sup>11</sup>Up to positive linear transformations, which do not affect what we do (see (2)); in the proofs in Appendix A, it will be convenient to use a different version of  $u_\gamma$  (see (6)).

<sup>12</sup>In Aumann and Serrano (2008, Section IV.C),  $R_\gamma(g)$  is denoted  $w_\gamma(g)$ .



## 5. WEALTH INDEPENDENCE AND THE AUMANN–SERRANO INDEX

To understand the role of “wealth,” note that it appears only in the **Compound Gamble** axiom. There the change in wealth due to the outcome  $x$  of the first gamble  $g$  affects the  $Q$ -requirement for the second gamble  $h$ , i.e.,  $Q(h) = Q(g) + x$ ; if, for instance,  $g$  resulted in a gain (i.e.,  $x > 0$ ), then one can afford an  $h$  with a higher  $Q$ -requirement. What happens if we make the measure  $Q$  “wealth independent” instead? That is, replace  $Q(h) = Q(g) + x$  with  $Q(h) = Q(g)$ , which yields the following axiom.

**WEALTH-INDEPENDENT COMPOUND GAMBLE.** *Let  $f = g + \mathbf{1}_A h$  be a compound gamble, where  $g, h \in \mathcal{G}$  and  $A$  is an event such that  $g$  is constant on  $A$ , i.e.,  $g|_A \equiv x$  for some  $x$ , and  $h$  is independent of  $A$ . If  $Q(h) = Q(g)$ , then  $Q(f) = Q(g)$ .*

Using this axiom instead of the **Compound Gamble** axiom characterizes the Aumann–Serrano index<sup>13</sup>  $R^{\text{AS}}$ :

**THEOREM 4.** *A function  $Q$  satisfies the **Distribution**, **Monotonicity**, **Scaling**, **Wealth-Independent Compound Gamble** axioms if and only if  $Q$  is proportional to the Aumann–Serrano index of riskiness  $R^{\text{AS}}$  (i.e., there exists a constant  $c > 0$  such that  $Q(g) = cR^{\text{AS}}(g)$  for all  $g \in \mathcal{G}$ ).*

The *index of riskiness* developed by Aumann and Serrano (2008)<sup>14</sup> is uniquely determined by the equation<sup>15</sup>

$$\mathbf{E} \left[ \exp \left( -\frac{1}{R^{\text{AS}}(g)} g \right) \right] = 1. \quad (4)$$

How can one interpret the **Wealth-Independent Compound Gamble** axiom? One can show that it implies  $Q(\sum_{t=1}^T g_t) = Q(g_1)$ , where the  $g_t$  are independent and identically distributed gambles (see **Proposition 22** in **Appendix B** for a general statement). This suggests<sup>16</sup> that such a  $Q$  may represent a certain “acceptable level of riskiness” that remains fixed over time, regardless of the gambles already taken, their realizations, and the resulting changes in wealth (see also Aumann and Serrano 2008, Section V.H). For instance, this level of riskiness could correspond to acceptance of  $g$  at *all* wealth levels (cf. Hart 2011)—again, a “wealth independence” requirement.

Returning to the class of  $\gamma$ -riskiness functions  $R_\gamma$  (for  $\gamma \geq 1$ ) that are given by **Theorem 3**, the minimal one is  $R_1 \equiv \mathbf{R}$ . What is the *maximal* one—the *most conservative* riskiness measure?

For every gamble  $g$ , the sequence  $R_\gamma(g)$  increases with  $\gamma$  (since a higher  $\gamma$  corresponds to higher risk aversion and, thus, to a higher critical wealth level); moreover, it

<sup>13</sup>Up to rescaling by a constant  $c > 0$  (minimality does not help here).

<sup>14</sup>This index was used in the technical report of Palacios-Huerta, Serrano, and Volij (2004); see the footnote on page 810 of Aumann and Serrano (2008). It is the inverse of the “adjustment coefficient” of the insurance risk literature; see Meilijson (2009).

<sup>15</sup>We write  $\exp(x)$  for  $e^x$ .

<sup>16</sup>Proposed by Robert Aumann (personal communication).

can be shown that  $R_\gamma(g) \rightarrow \infty$  as  $\gamma \rightarrow \infty$  and so the maximal riskiness is infinite. Nevertheless, the functions  $R_\gamma$  have a well-defined limit behavior as  $\gamma \rightarrow \infty$ , which turns out to be related to the Aumann–Serrano index. Specifically, for every  $g \in \mathcal{G}$ , we have

$$\lim_{\gamma \rightarrow \infty} \frac{1}{\gamma} R_\gamma(g) = R^{\text{AS}}(g) \quad (5)$$

(this can be easily shown by comparing (3) with (4); see also Aumann and Serrano 2008, Theorem C).

Equivalently,

$$\lim_{\gamma \rightarrow \infty} \frac{R_\gamma(g)}{R_\gamma(h)} = \frac{R^{\text{AS}}(g)}{R^{\text{AS}}(h)} = \frac{Q^{\text{AS}}(g)}{Q^{\text{AS}}(h)}$$

for any  $Q^{\text{AS}} = cR^{\text{AS}}$  with  $c > 0$ . Thus any  $Q^{\text{AS}}$  that satisfies the axioms of Theorem 4 yields *riskiness comparisons* that are the same as those of the most conservative  $Q$  that satisfies the axioms of the Main Theorem; i.e.,

$$R^{\text{AS}}(g) > R^{\text{AS}}(h) \quad \text{if and only if} \quad R_\gamma(g) > R_\gamma(h) \quad \text{for all } \gamma \text{ large enough.}$$

The Aumann–Serrano index may thus be viewed as the *maximal* riskiness in the family of measures  $R_\gamma$  of Theorem 3, whereas the Foster–Hart measure is the *minimal* one (Theorem 1). See also Section 7(f).

## 6. AN OUTLINE OF THE PROOFS

This section is devoted to an outline of our proofs: we state a number of intermediate results that may be of interest on their own, and provide an informal tour of the proofs. The formal proofs are relegated to the [Appendices](#).

### 6.1 Proof outline for Theorems 1 and 3

We start with Theorem 3, which says that  $Q$  satisfies the four axioms of Section 2.1 if and only if  $Q = R_\gamma$  for some  $\gamma \geq 1$ . This is the combination of the following three results:

LEMMA 5. *For each  $\gamma \geq 1$ , the function  $R_\gamma$  is well defined by (2).*

PROPOSITION 6. *For each  $\gamma \geq 1$ , the function  $R_\gamma$  satisfies the four axioms.*

PROPOSITION 7. *If  $Q$  satisfies the four axioms, then there exists  $\gamma \geq 1$  such that  $Q = R_\gamma$ .*

While the proofs of Lemma 5 and Proposition 6 are essentially straightforward, the proof of Proposition 7 is not. We describe the latter in Section 6.2.

The next lemma (whose proof is also easy) shows that the  $R_\gamma$  functions are “well ordered.”

LEMMA 8.  $R_\gamma(g)$  strictly increases with  $\gamma$ , i.e.,  $R_\beta(g) < R_\gamma(g)$  for every  $g \in \mathcal{G}$  and  $1 \leq \beta < \gamma$ .

So  $R_1 \equiv \mathbf{R}$  is the minimal  $R_\gamma$  for  $\gamma \geq 1$ ; hence, by Theorem 3, it is the minimal  $Q$  that satisfies the four axioms—which proves our Main Theorem.

## 6.2 Proof outline for Proposition 7

The main difficulty lies in the proof of Proposition 7, which we outline now. We first provide a number of consequences of our axioms.

The first one says that  $Q$  is always greater than the maximal loss.

PROPOSITION 9. *If  $Q$  satisfies the four axioms, then  $Q(g) > L(g)$  for every  $g$ .*

The proof here is somewhat tricky.<sup>17</sup>

Second, take a sequence of independent gambles  $g_1, g_2, \dots, g_t, \dots$  with  $Q(g_t) = 1$  for all  $t$ . Let  $x_1, \dots, x_m$  be the distinct values of  $g_1$  and let  $f_2$  be the compound gamble consisting of  $g_1$  followed, after each outcome  $x_j$ , by the  $(1 + x_j)$ -multiple of  $g_2$ . Thus,

$$\begin{aligned} f_2 &= g_1 + \sum_{j=1}^m \mathbf{1}_{[g_1=x_j]}(1 + x_j)g_2 = g_1 + (1 + g_1)g_2 \\ &= (1 + g_1)(1 + g_2) - 1. \end{aligned}$$

Since  $Q((1 + x_j)g_2) = (1 + x_j)Q(g_2) = 1 + x_j$  by the Scaling axiom, applying the Compound Gamble axiom  $m$  times yields  $Q(f_2) = 1$ . In the same way, from  $f_2$  and  $g_3$ , we get

$$f_3 = (1 + f_2)(1 + g_3) - 1 = (1 + g_1)(1 + g_2)(1 + g_3) - 1$$

with  $Q(f_3) = 1$  and so on, leading to the following condition.

- If  $1 + f_N = \prod_{t=1}^N (1 + g_t)$ , where  $(g_t)_{t=1,2,\dots}$  is a sequence of independent gambles with  $Q(g_t) = 1$  for all  $t$  and  $N$  is a positive integer, then  $Q(f_N) = 1$ .

Since the Compound Gamble axiom allows for no gamble after some outcomes, we can get a more general property. To state it formally, it is convenient to use the notion of a *stopping time*  $T$ , which is a random variable whose values are positive integers that specify how many gambles we take. The requirement is that  $T$  be *adapted to the sequence*  $(g_t)_t$ , which means that for each integer  $n$ , the event  $[T = n]$  that we stop immediately after  $g_n$  is determined by the outcomes of  $g_1, \dots, g_n$  only (i.e., by the past and not the future). We get the following condition.

MULTIPLICATIVE COMPOUNDING. *If  $1 + f_T = \prod_{t=1}^T (1 + g_t)$ , where  $(g_t)_{t=1,2,\dots}$  is a sequence of independent gambles with  $Q(g_t) = 1$  for all  $t$  and  $T$  is a bounded stopping time adapted to the sequence  $(g_t)_t$ , then  $Q(f_T) = 1$ .*

<sup>17</sup>In fact, it was a stumbling block for a long time, and, in previous versions of the paper, the statement of Proposition 9 appeared as an additional axiom.

PROPOSITION 10. *If  $Q$  satisfies the four axioms, then  $Q$  satisfies the **Multiplicative Compounding condition**.*

Third, recall that  $R_\gamma(g)$  is that wealth level where CRRA- $\gamma$  is indifferent between accepting and rejecting  $g$ ; see (2). When  $\gamma \geq 1$ , this critical wealth is well defined (see Lemma 5), since each gamble is rejected at low enough wealth and accepted at high enough wealth. This is, however, no longer true when  $0 < \gamma < 1$ : for each such  $\gamma$ , there are gambles in  $\mathcal{G}$  that CRRA- $\gamma$  *accepts at all wealth levels* (and so in this case, (2) has no solution). For example, the gamble  $g = (4, 1/2; -1, 1/2)$  is always accepted by the CRRA-1/2 utility  $u_{1/2}(x) = \sqrt{x}$ , since  $(1/2)\sqrt{w+4} + (1/2)\sqrt{w-1} > \sqrt{w}$  for all  $w \geq 1$ . However, if a critical wealth does exist, it is necessarily unique, and so we extend the definition of  $R_\gamma(g)$  to all  $\gamma > 0$ : it is given by (2) whenever it has a solution (and thus for  $0 < \gamma < 1$ , it is defined only for some<sup>18</sup>  $g$ ).

From (2), it can be shown that  $R_\gamma$  satisfies the **Multiplicative Compounding condition** also for  $0 < \gamma < 1$  (for  $\gamma \geq 1$ , recall Propositions 6 and 10). Moreover, as  $\gamma$  increases,  $R_\gamma(g)$  increases continuously from  $L(g)$  to  $\infty$  (this generalizes Lemma 8) and we have the following lemma.

LEMMA 11. *For every  $g \in G$  with  $L(g) < 1$ , there exists a unique  $\gamma \equiv \gamma_g > 0$  such that  $R_\gamma(g) = 1$ .*

(This is stated as Lemma 16 in Appendix A.3.4.) We emphasize that this  $\gamma$  need not satisfy  $\gamma \geq 1$ .

We are now ready to present the basic argument that proves Proposition 7. Fix a function  $Q$  that satisfies the four axioms. Let  $g$  be a gamble with  $Q(g) = 1$ , and apply Lemma 11 (recall Proposition 9) to get  $\gamma > 0$  such that  $R_\gamma(g) = 1$ . Take a sequence  $(g_t)_t$  of independent and identically distributed (i.i.d.) gambles, all with the same distribution as  $g$ . For every bounded stopping time  $T$ , the gamble  $f \equiv f_T = \prod_{t=1}^T (1 + g_t) - 1$  satisfies  $Q(f) = 1$  and also  $R_\gamma(f) = 1$  (see Proposition 10 and the paragraph just before Lemma 11 above). Now take  $z_1$  very large, and  $z_2 > -1$  and very close to  $-1$ ; then there is a stopping time  $T$  for which the resulting  $f$  has, with high probability, values that are close to either  $z_1$  or  $z_2$  (the sequence  $f_n$  is a multiplicative random walk, and we put two absorbing barriers at  $z_1$  and  $z_2$ ). For simplicity, ignore the various technical approximation issues,<sup>19</sup> and assume that  $T$  is bounded and that  $f$  takes only these two values  $z_1$  and  $z_2$ .

Next, take another  $g'$  gamble with  $Q(g') = 1$ , and apply Lemma 11 to get  $\gamma'$  such that  $R_{\gamma'}(g') = 1$ . Proceed as in the previous paragraph: starting with an i.i.d. sequence  $(g'_t)_t$  with  $g'_t$  having the same distribution as  $g'$ , we get a stopping time  $T'$  and a resulting  $f' \equiv f'_{T'}$  that takes only the two values  $z_1$  and  $z_2$ , the same ones as  $f$  (we again ignore the approximation issues) for which we have  $Q(f') = 1$  and  $R_{\gamma'}(f') = 1$ . Since  $f$  and  $f'$  take

<sup>18</sup>All statements below regarding  $R_\gamma$  for  $\gamma < 1$  should thus be understood to hold whenever  $R_\gamma$  is defined.

<sup>19</sup>The **Continuity** requirement of  $Q$  could be of help here; however, we did not require it as an axiom (and it does not hold in full generality; cf. Foster and Hart 2009, Section V.B). We use the **Monotonicity** axiom instead, by bounding from above and from below as needed.

the same two values, and  $Q(f) = 1 = Q(f')$ , it follows that  $f$  and  $f'$  must have the same distribution (otherwise one would dominate the other<sup>20</sup> and then, by the **Distribution** and **Monotonicity** axioms, we would have either  $Q(f) < Q(f')$  or  $Q(f') < Q(f)$ ), and so  $R_{\gamma'}(f) = R_{\gamma'}(f') = 1$ . But we also have  $R_{\gamma}(f) = 1$ , which implies, by **Lemma 11**, that  $\gamma' = \gamma$ .

What we have thus shown is that there is a unique  $\gamma^*$  such that  $Q(g) = 1$  implies that  $R_{\gamma^*}(g) = 1$ . From this, it follows (by the **Scaling** axiom applied to both functions) that  $Q \equiv R_{\gamma^*}$ . But  $Q$  must be defined for all gambles  $g$ , whereas, as we have seen, this is not the case with  $R_{\gamma}$  when  $0 < \gamma < 1$ ; therefore  $\gamma^* \geq 1$ , completing the proof of **Proposition 7**.

### 6.3 Proof outline for Theorem 4

The proof of **Theorem 4** is similar to the above proof (and in some ways simpler). One change is that when we iterate the **Wealth-Independent Compound Gamble** axiom, we get the following condition.

**ADDITIVE COMPOUNDING.** *If  $f_T = \sum_{t=1}^T g_t$ , where  $(g_t)_{t=1,2,\dots}$  is a sequence of independent gambles with  $Q(g_t) = 1$  for all  $t$  and  $T$  is a bounded stopping time adapted to the sequence  $(g_t)_t$ , then  $Q(f_T) = 1$ .*

**PROPOSITION 12.** *If  $Q$  satisfies the **Distribution** and **Wealth-Independent Compound Gamble** axioms, then  $Q$  satisfies the **Additive Compounding** condition.*

The proof of **Proposition 12** is parallel to the proof of **Proposition 10** (see **Section 6.2** above). Using the **Additive Compounding** condition together with the appropriate stopping times then shows that if  $Q(g) = 1 = cR^{\text{AS}}(g)$  and  $Q(g') = 1 = c'R^{\text{AS}}(g')$ , then necessarily  $c = c'$  (this is the counterpart of showing that  $\gamma = \gamma'$  at the end of **Section 6.2**), which proves **Theorem 4**.

## 7. DISCUSSION

In this section we provide a number of additional comments.

(a) *Nonmanipulability interpretations of the axioms.* The axioms may be viewed as certain conditions that make the wealth requirement immune to “manipulations.”

Take the **Scaling** axiom. If, for instance, there is a gamble  $g$  such that the wealth requirement  $Q(g/2)$  for  $g/2$  is less than  $1/2$  the requirement for  $g$ , i.e.,  $Q(g/2) < Q(g)/2$ , then one can lower the requirement for  $g$  by splitting it into two parts,<sup>21</sup>  $g_1 = g_2 = g/2$ , and then  $Q(g_1) + Q(g_2) < Q(g_1 + g_2) = Q(g)$ . If, instead,  $Q(g/2) > Q(g)/2$ , then two institutions each holding  $g/2$  can lower their wealth requirement from  $Q(g/2)$  to  $Q(g)/2$  by presenting a “merged” exposure to  $g$  and splitting the wealth requirement. More generally, dividing  $g$  into  $\lambda g$  and  $(1 - \lambda)g$  should not affect the wealth requirement, so we

<sup>20</sup>In the sense of first-order stochastic dominance.

<sup>21</sup>Thus,  $g_1$  and  $g_2$  are identical random variables: in each state of the world, both of them lose or gain exactly the same amount; the risks of  $g_1$  and  $g_2$  are thus fully correlated.

must have  $Q(g) = Q(\lambda g) + Q((1 - \lambda)g)$  for every  $0 < \lambda < 1$ , which yields the **Scaling** axiom. Since  $\lambda g$  and  $(1 - \lambda)g$  are fully correlated risks, **Scaling** essentially says that the wealth requirement is additive over such fully correlated risks.

Next, the **Compound Gamble** axiom may be viewed as saying that the wealth requirement should not be affected by the way the gamble is presented, whether as a two-step gamble ( $g$  followed by  $h$ ) or a one-step gamble<sup>22</sup> ( $f$ ).

Finally, consider the **Monotonicity** axiom. If it is not satisfied, then one can keep the wealth requirement the same, or even lower it, while making the gamble worse by decreasing gains and/or increasing losses. Most existing riskiness measures—such as the standard deviation or the Value-at-Risk (VaR)—suffer from this very significant drawback, which allows easy manipulations of the risks involved without affecting the required wealth or the needed reserves.<sup>23</sup> We emphasize that these manipulations are possible even when there is **Weak Monotonicity** (i.e.,  $Q(g) \leq Q(h)$  instead of  $Q(g) < Q(h)$ ); see **Appendix A.3.2** and **Appendix C**. Note that VaR does satisfy **Weak Monotonicity**.

(b) *The axioms are indispensable.* In **Appendix C**, we show that the axioms are indispensable for our result: dropping any one of them while keeping the others allows for additional functions  $Q$ . In particular, **Proposition 23** in **Appendix C** shows what happens when one replaces **Monotonicity** with **Weak Monotonicity**. Moreover, in **Appendix D**, we prove that if we replace the **Scaling** axiom by a certain the **Continuity** requirement, then the axioms characterize the critical wealth level for a class of utility functions that is larger than CRRA.

(c) *Relative returns.* In view of the **Scaling** axiom, one may restate everything in terms of *relative* rather than *absolute* returns. Indeed, the net absolute returns described by a gamble  $g$  at wealth level  $w$  yield net relative returns of  $g/w$  (and gross relative returns of  $1 + g/w$ ). Then, instead of dealing with the “right” wealth for the gamble  $g$ , one deals with the “right” *proportion* of  $g$  per unit of wealth. The two approaches are thus equivalent.<sup>24,25</sup>

(d) *Maximal growth rate (the “Kelly criterion”).* While our axioms lead to the log utility, we emphasize that a decision-maker with log utility does *not* behave according to **R**. A log-utility maximizer takes the gamble  $g$  when his wealth equals  $K \equiv K(g)$ , which maximizes  $\mathbf{E}[\log(1 + g/K)]$  (and not when his wealth is  $\mathbf{R}(g)$ , where he is indifferent between accepting and rejecting  $g$ ). Following **Kelly (1956)** and the subsequent extensive literature (see **Foster and Hart 2009**, Section IV.E, point 5), the rule induced by this function  $K(g)$  (called the “Kelly function”) *maximizes the growth rate* of one’s wealth.

Now the first-order condition that determines  $K \equiv K(g)$  is  $\mathbf{E}[(1 + g/K)^{-1}(-g/K^2)] = 0$ , which is equivalent to  $\mathbf{E}[(1 + g/K)^{-1}] = 1$  (multiply by  $K$  and subtract from 1). Comparing this equation with (3) shows that it is precisely the equation for  $R_2$  (i.e., when

<sup>22</sup>At first sight, one may ask why do we not take  $f = g + h$ ? The reason is that after  $g$  has been realized, the wealth has changed and so different gambles can now be afforded (cf. the discussion in **Section 5** above).

<sup>23</sup>For instance, any losses that are greater than VaR can be arbitrarily increased, which yields a significantly riskier gamble, but does not affect VaR.

<sup>24</sup>In fact, **Multiplicative Compounding** can be interpreted as compounding the *relative* returns.

<sup>25</sup>However, they are not equivalent in the Aumann–Serrano framework; see **Schreiber (2012)**.

$\gamma = 2$ ) and so  $K(g) = R_2(g)$  for all  $g$ ; i.e.,  $K$  is the *critical* wealth for the CRRA-2 utility  $u_2(x) = -1/x$ . The Kelly function is thus one of our riskiness functions—albeit not the minimal one  $\mathbf{R}$ ; in particular, it satisfies the four axioms and all their consequences (see (e) below).<sup>26</sup>

(e) *Additional properties.* It is straightforward to check that all the properties of  $\mathbf{R}$  in Section V of Foster and Hart (2009)—such as subadditivity, convexity, dilution, independent gambles, and continuity—are also satisfied by all the  $R_\gamma$  with  $\gamma \geq 1$ ; indeed, all the proofs there work mutatis mutandis with  $u_\gamma$  instead of  $u_1 \equiv \log$ . Theorem 3, therefore, implies that all these properties follow from our four axioms **Distribution**, **Monotonicity**, **Scaling**, and **Compound Gamble**.

(f) *The Aumann–Serrano index as the maximal riskiness.* As we saw in Section 5, the Aumann–Serrano index may be viewed as the “asymptotic maximal riskiness.” In terms of axioms, consider the functions  $\tilde{R}_\gamma := R_\gamma/\gamma$ ; these rescaled versions of the  $R_\gamma$  functions satisfy **Distribution**, **Scaling**, and **Monotonicity**, and the following appropriately rescaled version of **Compound Gamble**:

$\delta$ -COMPOUND GAMBLE. *Let  $f = g + \mathbf{1}_A h$  be a compound gamble, where  $g, h \in \mathcal{G}$  and  $A$  is an event such that  $g$  is constant on  $A$ , i.e.,  $g|_A \equiv x$  for some  $x$ , and  $h$  is independent of  $A$ . If  $Q(h) = Q(g) + \delta x$ , then  $Q(f) = Q(g)$ .*

(The change is in the condition  $Q(h) = Q(g) + \delta x$ .) Clearly,  $\tilde{R}_\gamma$  satisfies this axiom for  $\delta = 1/\gamma$ .

As  $\delta$  decreases to 0, so does the wealth effect (i.e., the outcome  $x$  of  $g$  multiplied by  $\delta$ ), leading in the limit to the **Wealth-Independent Compound Gamble** axiom (which is nothing but the **0-Compound Gamble** axiom). This is the axiomatic counterpart of (5):  $\tilde{R}_\gamma = R_\gamma/\gamma$  converges to the Aumann–Serrano index  $R^{\text{AS}}$ , which is indeed independent of wealth.

(g) *Bounded gambles.* As in Aumann and Serrano (2008) and Foster and Hart (2009), we assume that our gambles have finite support. We can easily accommodate general distributions, provided they are bounded. Let  $\mathcal{B}$  be the collection of all *bounded gambles*: a random variable  $g \in \mathcal{B}$  if  $\mathbf{E}[g] > 0$ ,  $\mathbf{P}[g < 0] > 0$ , and there is a constant  $B$  such that  $|g| \leq B$ . Our results continue to hold on the larger domain  $\mathcal{B}$ . An easy way to see this is to take, for every  $g$  in  $\mathcal{B}$ , gambles  $g_1$  and  $g_2$  in  $\mathcal{G}$  such that  $g_1 \leq g \leq g_2$ , and  $g_2 - g_1$  and  $L(g_1) - L(g_2)$  are arbitrarily small (use **Monotonicity**).

(h) *CRRA( $\gamma$ ) for  $\gamma < 1$ .* Consider the CRRA( $\gamma$ ) utility function  $u(x) = x^{1-\gamma}$  for  $0 \leq \gamma < 1$ . Can one define its critical wealth  $R_\gamma(g)$ ? Not for all gambles, since for each such  $u_\gamma$ , there are gambles  $g$  for which (2) has no solution. The reason is that these gambles are always accepted by these utilities (which are not sufficiently risk averse; cf. Hart 2011). See the discussion following Proposition 10 in Section 6.2 and Lemma 20 in Appendix A.

<sup>26</sup>While the rule that one always takes the proportion  $w/K(g) \equiv w/R_2(g)$  of the gamble  $g$  at wealth  $w$  is the rule that maximizes expected log utility, a similar statement does *not* hold when maximizing a CRRA- $\gamma$  utility function  $u_\gamma$  with  $\gamma \neq 1$ . Indeed, for such  $u_\gamma$ , the optimal proportion of  $g$  taken at  $w$  changes when  $g$  may be followed by another gamble  $h$  (the additive separability of log is critical here).

## APPENDICES

Here we provide the formal proofs and some additional results: Theorems 1 and 3 are proved in [Appendix A](#), [Theorem 4](#) is proved in [Appendix B](#), and [Appendix C](#) deals with the indispensability of the axioms. Finally, [Appendix D](#) provides a characterization of those functions  $Q$  that satisfy our axioms except [Scaling](#) (but with an additional continuity axiom).

## APPENDIX A: PROOF OF THEOREMS 1 AND 3

We prove here our main result; for an outline of the proof, see [Sections 6.1](#) and [6.2](#).

The proof is divided into three parts. In [Section A.1](#), we show that the functions  $R_\gamma$  for  $\gamma \geq 1$  are well defined ([Lemma 5](#)) and strictly increasing in  $\gamma$  ([Lemma 8](#)); in [Section A.2](#), that these functions satisfy the four axioms ([Proposition 6](#)); and in [Section A.3](#), that these are the only functions to satisfy the four axioms ([Proposition 7](#)). Several consequences of the axioms are provided in [Section A.3](#).

Since applying increasing linear transformations to the  $u_\gamma$  functions does not affect (2) and (3), in the proofs, we use the following convenient version of  $u_\gamma$  (see [Lemma 13](#) below):

$$u_\gamma(x) := \begin{cases} \frac{1}{1-\gamma}(x^{1-\gamma} - 1) & \text{for } 0 \leq \gamma < 1 \\ \log(x) & \text{for } \gamma = 1 \\ \frac{1}{\gamma-1}(1 - x^{-(\gamma-1)}) & \text{for } \gamma > 1. \end{cases} \quad (6)$$

For every  $\gamma \geq 0$ , the utility function  $u_\gamma: \mathbb{R}_+ \rightarrow \mathbb{R}$  has constant relative risk aversion (CRRA) equal to  $\gamma$ , i.e.,  $-xu''_\gamma(x)/u'_\gamma(x) = \gamma$  for every  $x > 0$ .

A.1 The functions  $R_\gamma$ 

In the next lemma, we collect several useful properties of the functions  $u_\gamma$ ; we omit the proofs as they are immediate.

LEMMA 13. (i) For each  $\gamma \geq 0$ , the function  $u_\gamma$  is strictly increasing,  $u_\gamma(1) = 0$ , and  $u'_\gamma(1) = 1$ .

(ii) For each  $\gamma > 0$ , the function  $u_\gamma$  is strictly concave.

(iii) If  $0 \leq \beta < \gamma$ , then  $u_\beta(x) > u_\gamma(x)$  for every  $x \neq 1$ , and there exists a strictly increasing and strictly concave function  $\psi \equiv \psi_{\beta,\gamma}$  with  $\psi(0) = 0$  such that  $u_\gamma(x) = \psi(u_\beta(x))$  for every  $x > 0$ .

(iv)  $u_\gamma(x)$  is continuous in  $\gamma$  for every  $x > 0$ .

With  $u_\gamma$  given by (6), the equality (2) that defines  $R_\gamma(g)$  can be rewritten as

$$\mathbf{E} \left[ u_\gamma \left( 1 + \frac{1}{R_\gamma(g)} g \right) \right] = u_\gamma(1) = 0 \quad (7)$$



(cf. (1) and (3)). We now prove Lemma 5 (that  $R_\gamma(g)$  is well defined) and Lemma 8 (that  $R_\gamma(g)$  increases in  $\gamma$ ) together.

PROOF OF LEMMATA 5 AND 8. Fix the gamble  $g = (x_i, p_i)_{i=1}^m$ . The function  $\phi_\gamma(\lambda) := \mathbf{E}[u_\gamma(1 + \lambda g)]$  satisfies the following conditions (see Lemma 13(i) and (ii)): it is strictly concave,  $\phi_\gamma(0) = 0$ ,  $\lim_{\lambda \rightarrow (1/L(g))^-} \phi_\gamma(\lambda) = -\infty$  (this is where  $\gamma \geq 1$  is used),  $\phi'_\gamma(\lambda) = \sum_{i=1}^m p_i x_i (1 + \lambda x_i)^{-\gamma}$ , and  $\phi'_\gamma(0) = \mathbf{E}[g] > 0$ ; therefore  $\phi_\gamma(\lambda)$  has a unique root  $\bar{\lambda}_\gamma$  in the open interval<sup>27</sup>  $(0, 1/L(g))$ . By (7),  $R_\gamma(g) = 1/\bar{\lambda}_\gamma$ , proving Lemma 5.

Moreover,  $\beta < \gamma$  implies by Lemma 13(iii) that

$$\phi_\gamma(\bar{\lambda}_\beta) = \mathbf{E}[\psi(u_\beta(1 + \bar{\lambda}_\beta g))] < \psi(\mathbf{E}[u_\beta(1 + \bar{\lambda}_\beta g)]) = \psi(0) = 0$$

(the strict inequality follows from the strict concavity of  $\psi$ ). Now  $\phi_\gamma(\lambda) \geq 0$  for every  $\lambda \in [0, \bar{\lambda}_\gamma]$  (since  $\phi_\gamma$  is concave and  $\phi_\gamma(0) = \phi_\gamma(\bar{\lambda}_\gamma) = 0$ ); hence,  $\bar{\lambda}_\beta$  cannot lie in this interval and so  $\bar{\lambda}_\beta > \bar{\lambda}_\gamma$ , proving Lemma 8.  $\square$

### A.2 The functions $R_\gamma$ for $\gamma \geq 1$ satisfy the axioms

In this section, we prove Proposition 6 (that  $R_\gamma$  satisfies the four axioms).

PROOF OF PROPOSITION 6. Distribution and Scaling are immediate from the definition of  $R_\gamma$  and Lemma 5 (cf. Lemma 10 in Foster and Hart 2009). Monotonicity holds since  $u_\gamma$  is a strictly increasing function (cf. the proof of Proposition 6 in Foster and Hart 2009).

To prove that Compound Gamble holds, take  $g, h, A$ , and  $x$  to be as in the statement of the axiom; put  $r := R_\gamma(g)$ , and so  $R_\gamma(h) = r + x$ . Conditioning on the event  $A$ , where  $g = x$  and  $f = x + h$ , we get

$$\begin{aligned} \mathbf{E}[u_\gamma(r + f)|A] &= \mathbf{E}[u_\gamma(r + x + h)|A] = \mathbf{E}[u_\gamma(r + x + h)] \\ &= u_\gamma(r + x) = \mathbf{E}[u_\gamma(r + g)|A] \end{aligned} \tag{8}$$

(the second equality follows from the independence of  $h$  and  $A$ , and the third follows from (2) since  $R_\gamma(h) = r + x$ ). On the complementary event  $A^C$ , where  $f = g$ , we get

$$\mathbf{E}[u_\gamma(r + f)|A^C] = \mathbf{E}[u_\gamma(r + g)|A^C]. \tag{9}$$

Combining (8) and (9), and then using (2) for  $g$  yields

$$\mathbf{E}[u_\gamma(r + f)] = \mathbf{E}[u_\gamma(r + g)] = u_\gamma(r).$$

But this says, again by (2) (and Lemma 5), that  $R_\gamma(f) = r = R_\gamma(g)$ .  $\square$

### A.3 Only the functions $R_\gamma$ for $\gamma \geq 1$ satisfy the axioms

We start with three consequences of the axioms, concerning first-order stochastic dominance, the relation to the maximal loss, and multiplicative compounding (see Propositions 14, 9, and 5, respectively). Throughout this section,  $Q$  is a fixed wealth-requirement function that satisfies the four axioms of Section 2.1.

<sup>27</sup>Compare Figure 1 and the proof of Lemma 9 in Foster and Hart (2009).

**A.3.1 First-order stochastic dominance** Let  $g, h \in \mathcal{G}$ . The gamble  $g$  *first-order stochastically dominates* the gamble  $h$ , which we write as  $g \text{ SD}_1 h$ , if  $\mathbf{P}[g \geq c] \geq \mathbf{P}[h \geq c]$  for every real constant  $c$ , with strict inequality for some  $c$ ; informally,  $g$  gets higher values than  $h$ . As is well known,  $g \text{ SD}_1 h$  if and only if there exist  $g'$  and  $h'$  defined on the same probability space such that  $g$  and  $g'$  have the same distribution,  $h$  and  $h'$  have the same distribution,  $g' \geq h'$ , and  $g' \neq h'$ .

**PROPOSITION 14.** *If  $g \text{ SD}_1 h$ , then  $Q(g) < Q(h)$ .*

**PROOF.** Taking  $g'$  and  $h'$  as above, we have  $Q(g) = Q(g') < Q(h') = Q(h)$  by **Distribution**, **Monotonicity**, and again **Distribution**.  $\square$

**A.3.2 Maximal loss and wealth requirement** Here we prove **Proposition 9**: the wealth requirement must always be strictly greater than any possible loss. We first prove a weak inequality.

**LEMMA 15.**  *$Q(g) \geq L(g)$  for every  $g$ .*

**PROOF.** By way of contradiction, let  $h = (x_1, p_1; \dots; x_m, p_m)$  satisfy  $Q(h) < L(h)$ . Without loss of generality (use the **Scaling** axiom), assume that  $Q(h) = 1 < L(h)$ .

For every  $n \geq 2$ , let  $g_n := (n, 1/2; -1, 1/2) \in \mathcal{G}$  and put  $q_n := Q(g_n) > 0$ . By **Monotonicity**, we have  $q_n \leq q_2$ . Let  $f_n$  be the compound gamble

$$\begin{aligned} f_n &:= g_n + \mathbf{1}_{\{g_n=n\}}(q_n + n)h \\ &= \left( n + (q_n + n)x_1, \frac{1}{2}p_1; \dots; n + (q_n + n)x_m, \frac{1}{2}p_m; -1, \frac{1}{2} \right). \end{aligned}$$

Since  $Q((q_n + n)h) = q_n + n = Q(g_n) + n$  (by **Scaling**), it follows that  $Q(f_n) = Q(g_n) = q_n$  (by **Compound Gamble**).

Let  $h'$  be the gamble

$$h' := \left( x_1 + 1, \frac{1}{2}p_1; \dots; x_m + 1, \frac{1}{2}p_m; 0, \frac{1}{2} \right)$$

(it is the so-called 1/2-dilution of  $h + 1$ ). Note that  $h'$  has negative values since  $L(h) > 1$ , and is thus a gamble in  $\mathcal{G}$ . Comparison with  $f_n$  shows that  $(q_n + n)h'$  first-order stochastically dominates  $f_n$ , since  $(q_n + n)(x_i + 1) > n + (q_n + n)x_i$  for every  $i = 1, \dots, m$ , and  $0 > -1$ . Therefore,  $Q((q_n + n)h') \leq Q(f_n) = q_n$  by **Monotonicity** (more precisely, **Proposition 14**) and so

$$Q(h') \leq \frac{q_n}{q_n + n}$$

by **Scaling**. This holds for each  $n \geq 2$ , and the limit as  $n \rightarrow \infty$  yields  $Q(h') = 0$  (recall that the  $q_n$ 's are bounded:  $0 < q_n \leq q_2$ ), contradicting the fact that  $Q(h')$  must be positive.  $\square$

**REMARK.** Only a weak form of **Monotonicity**—and, henceforth, of stochastic dominance—was used in the proof of **Lemma 15** above, namely, the following condition.

WEAK MONOTONICITY. If  $g \geq h$ , then  $Q(g) \leq Q(h)$ .

See [Appendix C](#) for further discussion and results.

PROOF OF [PROPOSITION 9](#). By [Lemma 15](#), we have  $Q(g) \geq L(g)$ . If there is a gamble  $h \in \mathcal{G}$  with  $Q(h) = L(h)$ , then increasing one of the positive values of  $h$  yields another gamble  $h' \in \mathcal{G}$  with  $Q(h') < Q(h)$  (by [Monotonicity](#)). But  $Q(h) = L(h) = L(h')$  (the first equality is by assumption; the second, by construction), and so  $Q(h') < L(h')$ , contradicting [Lemma 15](#) for  $h'$ .  $\square$

**A.3.3 Multiplicative compounding** Given a sequence of random variables  $(g_t)_{t=1,2,\dots}$ , a *stopping time* (adapted to the sequence  $(g_t)_t$ ) is a random variable  $T$  with values in  $\mathbb{N} \cup \{\infty\}$ , where  $\mathbb{N}$  is the set of positive integers  $\{1, 2, \dots\}$ , such that for each finite  $n \in \mathbb{N}$ , the event  $[T = n]$  is determined by  $\{g_1, g_2, \dots, g_n\}$  only; that is, whether one stops after  $n$  is determined by the past—the realizations of the first  $n$  random variables in the sequence—and not the future. As usual,  $T$  is *bounded* if  $T \leq N$  for some finite  $N < \infty$ , and it is *almost surely* (a.s.) *finite* if  $\mathbf{P}[T < \infty] = 1$ . For every finite  $n \in \mathbb{N}$ , put  $f_n := \prod_{t=1}^n (1 + g_t) - 1$ . The random variable  $f_T$  is then given by  $(f_T)(\omega) := f_{T(\omega)}(\omega)$  for each  $\omega \in \Omega$ , where  $\Omega$  denotes the probability space over which the  $g_t$ —and thus also  $T$ —are all defined.

We now prove [Proposition 10](#), which says that the [Multiplicative Compounding](#) condition follows from our axioms.

PROOF OF [PROPOSITION 10](#). Since  $T$  is bounded, there is an integer  $N$  such that  $T \leq N$ ; we use induction on  $N$ . For  $N = 1$ , we have  $T \equiv 1$  and thus  $f_T = g_1$ . Assume that the result holds for all stopping times  $T' \leq N - 1$  and take  $T$  such that  $T \leq N$ . Let  $x_1, x_2, \dots, x_k$  be those values of  $g_1$  where  $T > 1$  (recall that the event  $[T = 1]$ , and so its complement  $[T > 1]$ , depend only on  $g_1$ ); assume that  $x_1, x_2, \dots, x_k$  are *distinct*.

For each  $j = 1, \dots, k$ , let  $T_j$  denote the stopping time  $T$  when  $g_1 = x_j$ ; i.e.,  $T_j$  is determined by the sequence  $g_2, g_3, \dots$  and takes the value that  $T$  takes on the sequence  $g_1, g_2, g_3, \dots$  with  $g_1 = x_j$  (formally,  $T_j(g_2, g_3, \dots) := T(x_j, g_2, g_3, \dots)$ ); thus  $2 \leq T_j \leq N$ . Put  $h_j := \prod_{t=2}^{T_j} (1 + g_t) - 1$ . Since the product has  $T_j - 1 \leq N - 1$  terms, the induction hypothesis implies that  $Q(h_j) = 1$ . We have

$$\begin{aligned} f_T &= (1 + g_1) + (1 + g_1) \left( \prod_{t=2}^T (1 + g_t) - 1 \right) - 1 \\ &= g_1 + \sum_{j=1}^k \mathbf{1}_{[g_1=x_j]} (1 + x_j) h_j. \end{aligned}$$

Let  $f^{(0)} := g_1$ , and, for each  $j = 1, \dots, k$ , put  $A_j := [g_1 = x_j]$  and

$$f^{(j)} := f^{(j-1)} + \mathbf{1}_{A_j} (1 + x_j) h_j = g_1 + \sum_{\ell=1}^j \mathbf{1}_{[g_1=x_\ell]} (1 + x_\ell) h_\ell.$$

Now  $f^{(j-1)}|_{A_j} = g_1|_{A_j} \equiv x_j$  (here we use the fact that  $x_j$  and  $x_1, \dots, x_{j-1}$  are distinct),  $(1+x_j)h_j$  is independent of  $A_j$  (the former depends on  $g_t$  for  $t \geq 2$  and the latter depends on  $g_1$ ), and  $Q((1+x_j)h_j) = (1+x_j)Q(h_j) = 1+x_j$  (by **Scaling**). Therefore, by **Compound Gamble**,<sup>28</sup> if  $Q(f^{(j-1)}) = 1$ , then  $Q(f^{(j)}) = 1$ ; starting with  $Q(f^{(0)}) \equiv Q(g_1) = 1$ , after  $k$  such applications we get  $Q(f^{(k)}) = 1$ , completing the proof<sup>29</sup> since  $f^{(k)} \equiv f_T$ .  $\square$

REMARKS. (i) The **Monotonicity** axiom was not used in the proof of **Proposition 10**.

(ii) The **Compound Gamble** axiom is a special case (with  $T \leq 2$ ) of the **Multiplicative Compounding** condition.

(iii) When  $Q(g_t) = q$  for every  $t$ , multiplicative compounding yields  $Q(f_T) = q$ , where  $f_T = \prod_{t=1}^T (q + g_t) - q$  (take  $g'_t = g_t/q$  and use the **Scaling** axiom).

(iv) A more general version of the **Multiplicative Compounding** condition is the following:

*If  $1 + f/q = \prod_{t=1}^T (1 + g_t/Q(g_t))$ , where  $(g_t)_{t=1,2,\dots}$  is a sequence of independent gambles,  $q > 0$ , and  $T$  is a bounded stopping time adapted to the sequence  $(g_t)_t$ , then  $Q(f) = q$ .*

Interestingly, when  $Q$  satisfies **Distribution**, the above requirement is equivalent to **Scaling** and **Compound Gamble** together (indeed,  $T = 1$  gives **Scaling** and  $T \leq 2$  gives **Compound Gamble**).

A.3.4 *Proof of Proposition 7* We are now ready for the main proof here, namely, the proof of **Proposition 7**: if  $Q$  satisfies the four axioms, then  $Q$  coincides with one of the  $R_\gamma$  functions for some  $\gamma \geq 1$ . We start by associating with every gamble  $g$  such that  $Q(g) = 1$  (and thus, by **Proposition 9**,  $L(g) < 1$ ) a  $\tilde{\gamma}$  such that (essentially)  $R_{\tilde{\gamma}}(g) = 1$ ; this is stated as **Lemma 11** in **Section 6.2**.

LEMMA 16. *For every  $g$  with  $L(g) < 1$ , there exists a unique  $\tilde{\gamma} \equiv \tilde{\gamma}(g) > 0$  such that  $\mathbf{E}[u_{\tilde{\gamma}}(1 + g)] = 0$ .*

PROOF. The function  $\gamma \rightarrow \mathbf{E}[u_\gamma(1 + g)]$  is strictly decreasing and continuous (by **Lemma 13**(iii) and (iv); recall that  $g$  has finitely many values). Moreover,  $\mathbf{E}[u_0(1 + g)] = \mathbf{E}[g] > 0$ , and  $\mathbf{E}[u_M(1 + g)] < 0$  for all  $M > 1$  large enough (let  $p > 0$  be the probability that  $g$  takes the value  $L \equiv L(g) \in (0, 1)$ ; then  $(M - 1)\mathbf{E}[u_M(1 + g)] = 1 - \mathbf{E}[(1 + g)^{-M+1}] < 1 - p(1 - L)^{-M+1} \rightarrow_{M \rightarrow \infty} -\infty$ ).  $\square$

When  $\tilde{\gamma} \geq 1$ , this indeed says that  $R_{\tilde{\gamma}}(g) = 1$ ; however, at this point it is conceivable that we have  $\tilde{\gamma} < 1$  for some gambles. Next we show that  $\tilde{\gamma}$  is not affected by multiplicative compounding (compare **Proposition 10**).

<sup>28</sup>This is a little more subtle than it may seem at first sight, since, for instance, in the second application of **Compound Gamble**,  $h_2$  is *not* independent of  $f^{(1)} := g_1 + \mathbf{1}_{[g_1=x_1]}(1+x_1)h_1$  (since  $h_1$  and  $h_2$  are not necessarily independent). However,  $h_2$  is independent of the event  $[f^{(1)} = x_2]$ , which is the same as  $[g_1 = x_2]$ , since  $x_1$  and  $x_2$  are distinct values of  $g_1$ .

<sup>29</sup>In particular,  $f_T \in \mathcal{G}$ .

LEMMA 17. Let  $(g_t)_{t=1,2,\dots}$  be a sequence of independent gambles with  $L(g_t) < 1$  and  $\tilde{\gamma}(g_t) = \gamma$  for all  $t$ . Let  $T$  be a stopping time and let  $f_T := \prod_{t=1}^T (1 + g_t) - 1$ . If either (i)  $T$  is bounded or (ii)  $T$  is a.s. finite and  $f_n$  for  $n \leq T$  are uniformly bounded, then<sup>30</sup>  $\mathbf{E}[u_\gamma(1 + f_T)] = 0$ .

PROOF. When  $\gamma \neq 1$ , for every integer  $n$ , we have  $\mathbf{E}[(1 + g_n)^{1-\gamma}] = 1$  (since  $\tilde{\gamma}(g_n) = \gamma$ ), which implies that the sequence  $X_n := (1 + f_n)^{1-\gamma} = \prod_{t=1}^n (1 + g_t)^{1-\gamma}$  is a martingale:

$$\mathbf{E}[X_n | X_1, X_2, \dots, X_{n-1}] = X_{n-1} \mathbf{E}[(1 + g_n)^{1-\gamma}] = X_{n-1}$$

(the  $g_t$  are independent). Therefore,  $\mathbf{E}[X_n] = \mathbf{E}[X_0] = 1$  for every  $n$  and so, given the assumptions on the stopping time  $T$ , it follows that  $\mathbf{E}[X_T] = 1$  or  $\mathbf{E}[u_\gamma(1 + f_T)] = 0$ .

When  $\gamma = 1$ , the sequence  $X_n := \log(1 + f_n) = \sum_{t=1}^n \log(1 + g_t)$  is a martingale (since  $\mathbf{E}[X_n] = 0$ ) and, in the same manner, we get  $\mathbf{E}[X_T] = \mathbf{E}[X_0] = 0$  or  $\mathbf{E}[u_1(1 + f_T)] = 0$ .  $\square$

Note that for  $\gamma \geq 1$ , this result follows from Proposition 10 applied to  $R_\gamma$  (recall Proposition 6); however, here we also prove it for  $0 < \gamma < 1$ .

For every  $\beta > 0$  and  $z > 1$ , let  $h_{\beta,z}$  be the two-valued gamble that takes the positive value  $z - 1$  and the negative value  $1/z - 1$  with appropriate probabilities so that  $R_\beta(h_{\beta,z}) = 1$ ; specifically,

$$h_{\beta,z} = \left( z - 1, 1 - p; \frac{1}{z} - 1, p \right),$$

where

$$p = \frac{1}{z^{\beta-1} + 1}. \quad (10)$$

Clearly  $\tilde{\gamma}(h_{\beta,z}) = \beta$  (when  $\beta \neq 1$ , we have  $\mathbf{E}[(1 + h_{\beta,z})^{1-\beta}] = 1$ , and when  $\beta = 1$ , we have  $\mathbf{E}[\log(1 + h_{1,z})] = 0$  since then  $p = 1/2$  by (10)); and  $h \equiv h_{\beta,z}$  is a gamble in  $\mathcal{G}$  (since  $1/z - 1 < 0$  and  $\mathbf{E}[h] = \mathbf{E}[1 + h] - 1 > \mathbf{E}[(1 + h)^{1-\beta}] = 1 - 1 > 0$ ).

LEMMA 18. Let  $g \in \mathcal{G}$  satisfy  $Q(g) = 1$  and  $R_\gamma(g) = 1$ .

- (i) For each  $\beta > \gamma$ , there exists  $z_0$  large enough such that  $Q(h_{\beta,z}) < 1$  for all  $z > z_0$ .
- (ii) For each  $\beta < \gamma$ , there exists  $z_0$  large enough such that  $Q(h_{\beta,z}) > 1$  for all  $z > z_0$ .

PROOF. (i) Put  $b := \beta - 1$  and  $c := \gamma - 1$ , and so  $b > c$ .

We have  $1 + g > 0$  (since  $L(g) < Q(g) = 1$  by Proposition 9) and so there is  $d > 1$  large enough such that  $1/d \leq 1 + g \leq d$ . Consider the sequence of gambles  $f_n := \prod_{t=1}^n (1 + g_t) - 1$ ,

<sup>30</sup>Unlike case (i), where it is easy to check that  $f_T \in \mathcal{G}$  (cf. Proposition 10), in case (ii) the random variable  $f_T$  may have infinitely many values and thus need not be in  $\mathcal{G}$  (we make use of case (ii) in the proof of Lemma 18).

where the  $g_i$  are independent gambles all with the same distribution as  $g$ , and define, for  $z \geq d$ ,

$$T := \inf \left\{ n : f_n \leq \frac{1}{z} - 1 \text{ or } f_n \geq \frac{z}{d} - 1 \right\}.$$

Then  $T$  is an a.s. finite stopping time (but in general not bounded).<sup>31</sup> When  $T$  is finite, if  $1 + f_T \leq 1/z$ , then  $1 + f_T > 1/(zd)$  (since  $1 + f_{T-1} > 1/z$  and  $1 + g_T \geq 1/d$ ), and if  $1 + f_T \geq z/d$ , then  $1 + f_T < z$  (since  $1 + f_{T-1} < z/d$  and  $1 + g_T \leq d$ ). Therefore, a.s.,

$$1 + f_T \in \left( \frac{1}{zd}, \frac{1}{z} \right] \cup \left[ \frac{z}{d}, z \right). \tag{11}$$

Since  $T$  is a.s. finite and  $f_n$  for  $n \leq T$  are uniformly bounded,  $\tilde{\gamma}(g) = \gamma$  implies by Lemma 17(ii) that  $\mathbf{E}[u_\gamma(1 + f_T)] = 0$ . Letting  $q := \mathbf{P}[f_T \leq 1/z - 1] = 1 - \mathbf{P}[f_T \geq z/d - 1]$ , when  $c = \gamma - 1 \neq 0$ , we get from (11) that

$$1 = \mathbf{E}[(1 + f_T)^{-c}] \geq q \left( \frac{1}{zd} \right)^{-c} + (1 - q) \left( \frac{z}{d} \right)^{-c};$$

hence

$$q \geq \frac{D - z^{-c}}{z^c - z^{-c}} \sim \begin{cases} Dz^{-c} & \text{for } c > 0 \\ 1 - Dz^c & \text{for } c < 0, \end{cases} \tag{12}$$

where  $D := d^{-c}$ , and we write “ $A \sim B$ ” for “ $A/B \rightarrow 1$  as  $z \rightarrow \infty$ .” Similarly, when  $c = \gamma - 1 = 0$ , we get

$$0 = \mathbf{E}[\log(1 + f_T)] \geq q \log \left( \frac{1}{zd} \right) + (1 - q) \log \left( \frac{z}{d} \right);$$

hence

$$q \geq \frac{\log z - \log d}{2 \log z} \sim \frac{1}{2} \text{ for } c = 0. \tag{13}$$

From (10) we get

$$p \sim \begin{cases} z^{-b} & \text{for } b > 0 \\ \frac{1}{2} & \text{for } b = 0 \\ 1 - z^b & \text{for } b < 0. \end{cases}$$

Comparing this with (12) and (13) and recalling that  $b > c$  shows that  $q > p$  for all  $z$  large enough, say  $z > z_0$ .

Writing  $T \wedge n$  for  $\min\{T, n\}$ , we have  $\mathbf{P}[f_{T \wedge n} \leq 1/z - 1] \rightarrow_{n \rightarrow \infty} \mathbf{P}[f_T \leq 1/z - 1] = q > p$ , and so for each  $z > z_0$ , there is  $N$  large enough such that  $\mathbf{P}[f_{T \wedge N} \leq 1/z - 1] > p$ . This implies that  $h_{\beta, z}$  first-order stochastically dominates  $f_{T \wedge N}$ , since  $\mathbf{P}[f_{T \wedge N} \leq z - 1] = 1 = \mathbf{P}[h_{\beta, z} \leq z - 1]$  and  $\mathbf{P}[f_{T \wedge N} \leq 1/z - 1] > p = \mathbf{P}[h_{\beta, z} \leq 1/z - 1]$  (recall that  $h_{\beta, z}$  has only

<sup>31</sup>As usual, the infimum of an empty set is  $\infty$ . The fact that  $T = \infty$  has probability 0 is standard: let  $x > 0$  be a value of  $g$  and let  $p > 0$  be its probability; then for  $K$  large enough so that  $(1 + x)^K \geq z^2/d$  holds, the sequence  $(1 + f_{N+n})_{n \geq 1}$  starting with  $1 + f_N \in (1/z, z/d)$  reaches the upper bound  $z/d$  in  $K$  steps with a probability of at least  $p^K > 0$ .

two values,  $z - 1$  and  $1/z - 1$ ). Now  $Q(f_{T \wedge N}) = 1$  (by [Proposition 10](#)), and so  $Q(h_{\beta, z}) < 1$  (by [Proposition 14](#)).

(ii) The proof here is similar: letting  $T := \inf\{n : f_n \leq d/z - 1 \text{ or } f_n \geq z - 1\}$ , one shows that  $\mathbf{P}[f_{T \wedge N} \geq z - 1] > \mathbf{P}[h_{\beta, z} \geq z - 1]$  for all  $z$  and  $N$  large enough.  $\square$

LEMMA 19. *There exists  $\gamma^* > 0$  such that  $\tilde{\gamma}(g) = \gamma^*$  for every  $g$  with  $Q(g) = 1$ .*

PROOF. If  $g$  and  $g'$  are two gambles with  $Q(g) = Q(g') = 1$  but, say,  $\tilde{\gamma}(g) < \tilde{\gamma}(g')$ , then take  $\beta$  such that  $\tilde{\gamma}(g) < \beta < \tilde{\gamma}(g')$ . [Lemma 18](#) implies that for all  $z$  large enough, we have  $Q(h_{\beta, z}) < 1$  (since  $\beta > \tilde{\gamma}(g)$ ) and also  $Q(h_{\beta, z}) > 1$  (since  $\beta < \tilde{\gamma}(g')$ ), a contradiction.  $\square$

LEMMA 20.  $\gamma^* \geq 1$ .

PROOF. Assume that  $0 < \gamma^* < 1$ . Let  $z > 1/(1 - \gamma^*)$  and consider the gamble  $g = (2^z - 1, 1/2; -1, 1/2) \in \mathcal{G}$ . Let  $q := Q(g)$ ; then  $q \geq L(g) = 1$  by [Lemma 15](#) and  $Q((1/q)g) = 1$  by the [Scaling](#) axiom, and so  $\mathbf{E}[u_{\gamma^*}(1 + (1/q)g)] = 0$  by [Lemma 19](#). Put  $\phi(\lambda) := \mathbf{E}[u_{\gamma^*}(1 + \lambda g)]$ ; then  $\phi$  is a concave function with  $\phi(0) = 0$  and

$$\phi(1) = \mathbf{E}[u_{\gamma^*}(1 + g)] = \frac{1}{1 - \gamma^*} \left( \frac{1}{2}(2^z)^{1 - \gamma^*} + \frac{1}{2}(0)^{1 - \gamma^*} - 1 \right) > 0.$$

Therefore,  $\phi(\lambda) > 0$  for every  $0 < \lambda \leq 1$ , contradicting  $\phi(1/q) = 0$  (recall that  $q \geq 1$ ).  $\square$

COROLLARY 21.  $Q(g) = R_{\gamma^*}(g)$  for every  $g \in \mathcal{G}$ .

PROOF. Let  $q := Q(g)$  and  $h := (1/q)g$ . Then  $Q(h) = 1$  (by [Scaling](#)), and so  $\mathbf{E}[u_{\gamma^*}(1 + (1/q)g)] = 0$  (by [Lemma 19](#)), which says that  $R_{\gamma^*}(g) = q$  (by [Lemma 20](#)).  $\square$

PROOF OF [PROPOSITION 7](#). This follows from [Corollary 21](#) and [Lemma 20](#).  $\square$

#### A.4 Conclusion

PROOF OF [THEOREM 3](#). This follows from [Lemma 5](#) and [Propositions 6](#) and [7](#).  $\square$

PROOF OF [THEOREM 1](#). This follows from [Theorem 3](#) and [Lemma 8](#).  $\square$

#### APPENDIX B: PROOF OF [THEOREM 4](#)

In this Appendix, we prove [Theorem 4](#) that characterizes the Aumann–Serrano index. We start with the counterpart of [Proposition 10](#), using the [Wealth-Independent Compound Gamble](#) axiom instead of the [Compound Gamble](#) axiom.

PROPOSITION 22. *Let  $Q$  satisfy the [Wealth-Independent Compound Gamble](#) axiom and let  $(g_t)_{t=1,2,\dots}$  be a sequence of independent gambles with  $Q(g_t) = 1$  for all  $t$ . Let  $T$  be a bounded stopping time and put*

$$f_T := \sum_{t=1}^T g_t.$$

*Then  $f_T \in \mathcal{G}$  and  $Q(f_T) = 1$ .*

It is noteworthy how the two “compound gamble” axioms lead to multiplicative compounding and additive compounding, respectively. The proof of [Proposition 22](#) is similar to that of [Proposition 10](#) and is omitted.<sup>32</sup>

**PROOF OF THEOREM 4.** The proof is similar to the proofs in [Appendix A](#), and in some places is simpler; we are thus brief here.

First, note that  $Q$  satisfies the axioms if and only if  $cQ$  satisfies the axioms for any  $c > 0$ . To see that  $R^{\text{AS}}$  satisfies the [Wealth-Independent Compound Gamble](#) axiom (the other axioms are immediate), take  $g, h, A$ , and  $f$  as in the statement of the axiom. Put  $\lambda := 1/R^{\text{AS}}(g) = 1/R^{\text{AS}}(h)$ . Then  $\mathbf{E}[\exp(-\lambda g)] = \mathbf{E}[\exp(-\lambda h)] = 1$ . On  $A$ , we have  $f = x + h$  and so

$$\begin{aligned} \mathbf{E}[\exp(-\lambda f)|A] &= \mathbf{E}[\exp(-\lambda x - \lambda h)] \\ &= \exp(-\lambda x) \cdot \mathbf{E}[\exp(-\lambda h)] = \exp(-\lambda x) \\ &= \mathbf{E}[\exp(-\lambda g)|A]. \end{aligned}$$

On  $A^C$  we have  $f = g$ , and so

$$\mathbf{E}[\exp(-\lambda f)|A^C] = \mathbf{E}[\exp(-\lambda g)|A^C].$$

Altogether,

$$\mathbf{E}[\exp(-\lambda f)] = \mathbf{E}[\exp(-\lambda g)] = 1,$$

and so  $R^{\text{AS}}(f) = 1/\lambda = R^{\text{AS}}(g)$ .

Conversely, let  $Q$  satisfy the four axioms. Take  $g \in \mathcal{G}$  with  $Q(g) = 1$ , and put  $c := 1/R^{\text{AS}}(g) > 0$ . Given a sequence  $g_t$  of independent gambles, all with the same distribution as  $g$ , let  $f_n := \sum_{t=1}^n g_t$ . Then  $\exp(-cf_n)$  is a martingale (since  $\mathbf{E}[\exp(-cg_t)] = \mathbf{E}[\exp(-cg)] = 1$ ) and so  $\mathbf{E}[\exp(-cf_n)] = 1$  for every  $n \geq 1$ .

Next, for  $b > 0$  and  $z > 1$ , let  $h_{b,z} \in \mathcal{G}$  take the values  $\log z$  and  $-\log z$  with probabilities  $1 - p$  and  $p$ , respectively, where  $p = 1/(z^b + 1)$ . Then  $\mathbf{E}[\exp(-bh_{b,z})] = 1$ , and so  $1/R^{\text{AS}}(h_{b,z}) = b$ . We claim that if  $b > c$ , then  $Q(h_{b,z}) < 1$  for all  $z$  large enough, and if  $b < c$ , then  $Q(h_{b,z}) > 1$  for all  $z$  large enough. The proof, similar to that of [Lemma 18](#), is as follows: let  $d := \max|g| > 0$ , and take  $z > e^d$ . When  $b > c$ , put  $T := \inf\{n: f_n \leq -\log z \text{ or } f_n \geq \log z - d\}$ . Then  $f_T \in (-\log z - d, -\log z] \cup [\log z - d, \log z)$ , and also  $\mathbf{E}[\exp(-cf_T)] = 1$  (since  $\exp(-cf_n)$  is a martingale and  $f_T$  is bounded). This implies that  $q := \mathbf{P}[f_T \leq -\log z] \geq (D - z^{-c})/(z^c - z^{-c}) \sim Dz^{-c}$  as  $z \rightarrow \infty$ , where  $D := \exp(-cd)$ . Comparing this with  $p \sim z^{-b}$  shows that there exists  $z_0$  such that  $q > p$  for all  $z > z_0$ ; hence for all  $N$  large enough,  $h_{b,z}$  first-order stochastically dominates  $f_{T \wedge N}$ . [Monotonicity](#) ([Proposition 14](#)) implies that  $Q(h_{b,z}) < Q(f_{T \wedge N}) = 1$ , where we use [Proposition 22](#) for the bounded stopping time  $T \wedge N$ . The case  $b < c$  is similar, with  $T := \inf\{n: f_n \leq -\log z + d \text{ or } f_n \geq \log z\}$ .

<sup>32</sup>[Scaling](#) is no longer needed here. Also,  $Q(g_t) = 1$  for all  $t$  may be replaced by  $Q(g_t) = q$  for all  $t$ , with the conclusion that  $Q(f_T) = q$ .



Therefore, if there is another gamble  $g' \in \mathcal{G}$  with  $Q(g') = 1$  but  $1/R^{\text{AS}}(g') = c' \neq c$ , say  $c < c'$ , then take  $b$  such that  $c < b < c'$ . For large enough  $z$ , we get  $Q(h_{b,z}) < 1$  (since  $b > c$ ) and also  $Q(h_{b,z}) > 1$  (since  $b < c'$ ), a contradiction. This shows that there exists a constant  $c^* > 0$  such that  $Q(g) = 1$  implies that  $R^{\text{AS}}(g) = 1/c^*$ , or  $Q(g) = c^*R^{\text{AS}}(g)$  for any  $g$  with  $Q(g) = 1$ . The **Scaling** axiom applied to  $Q$  and  $R^{\text{AS}}$  implies that  $Q(g) = c^*R^{\text{AS}}(g)$  for any  $g \in \mathcal{G}$ .  $\square$

REMARKS. (i) Since  $\alpha := 1/R^{\text{AS}}(g)$  (the “adjustment coefficient” of  $g$ ) satisfies  $E[\exp(-\alpha g)] = 1$ , the **Additive Compounding** condition (and its special case, the **Wealth-Independent Compound Gamble** axiom) immediately follows (for instance,  $\exp(-\alpha \sum_{t=1}^N g_t)$  is a martingale when the  $g_t$  are i.i.d. and distributed like  $g$ ). Interestingly, **Theorem 4** shows that in the presence of the other axioms, this is the *only* way to get the **Additive Compounding** condition.

(ii) The **Scaling** axiom is used only in the last line of the proof. Without **Scaling**, we get the conclusion that  $Q$  is ordinally equivalent to  $R^{\text{AS}}$ , i.e.,  $Q$  is a monotonic transformation of  $R^{\text{AS}}$ . (Indeed, we have shown that there exists a constant  $c^* > 0$  such that  $Q(g) = 1$  implies that  $R^{\text{AS}}(g) = 1/c^*$ ; in the same way, one shows that for every  $q > 0$ , there exists  $c^*(q) > 0$  such that  $Q(g) = q$  implies that  $R^{\text{AS}}(g) = 1/c^*(q)$ .)

#### APPENDIX C: THE AXIOMS ARE INDISPENSABLE

We show here that the axioms of **Section 2.1—Monotonicity, Scaling, and Compound Gamble**—are *indispensable*: for each one we provide a function  $Q$  with  $Q(g) < \mathbf{R}(g)$  for some  $g$  (or even all  $g$ ) in  $\mathcal{G}$  that satisfies all the other axioms. We conclude with a discussion of the **Distribution** axiom.

##### *The Monotonicity axiom*

The function  $L$  that associates to each gamble  $g$  its maximal loss  $L(g)$  clearly satisfies **Distribution** and **Scaling**. To see that  $L$  also satisfies **Compound Gamble**, take  $g$ ,  $h$ ,  $A$ , and  $f$  as in the statement of the axiom; thus  $g|_A \equiv x$  and  $L(h) = L(g) + x$ . Then  $\min f|_A = \min(x + h) = x - L(h) = -L(g)$  and  $\min f|_{A^c} = \min g|_{A^c} \geq \min g = -L(g)$ , and so  $\min f = -L(g)$ , i.e.,  $L(f) = L(g)$ . But  $L$  does not satisfy **Monotonicity**: just increase any of the outcomes of  $g$  that are above  $L(g)$  and then  $L$  does not decrease. Of course,  $L(g) < \mathbf{R}(g)$  for all  $g \in \mathcal{G}$ .

Clearly,  $L$  satisfies the **Weak Monotonicity** axiom, with  $Q(g) \leq Q(h)$  instead of  $Q(g) < Q(h)$ ; see the Remark following the proof of **Lemma 15**. Interestingly, replacing the **Monotonicity** axiom with this weaker axiom yields the following result (compare our main result **Theorem 1**).

**PROPOSITION 23.** *The minimal function that satisfies the four axioms **Distribution, Scaling, Weak Monotonicity, and Compound Gamble** is the maximal loss function  $L$ .*

**PROOF.** We have just seen that  $L$  satisfies these axioms. **Lemma 15** (see the Remark following its proof) shows that  $Q \geq L$  for any  $Q$  satisfying these four axioms, so  $L$  is indeed the minimal function.  $\square$

### The *Scaling* axiom

Take the utility function  $u(x) = x + \log(x)$  for all  $x > 0$  (note that it is *not* a CRRA utility), and define  $Q(g) := r$  as the unique solution  $r > L(g)$  of the equation  $\mathbf{E}[u(r + g)] = u(r)$ .

We claim that  $Q$  is well defined: for every  $x > 0$ , we have  $u'(x) > 0$ ,  $u''(x) < 0$ ,  $\rho(x) \equiv -u''(x)/u'(x) > 0$ ,  $\rho'(x) < 0$ ; thus  $u$  is a Decreasing Absolute Risk Aversion (DARA) utility function. An agent with such a utility is monotonic in his decisions: for each  $g$ , there is a unique critical wealth level  $w_0 \equiv w_0(g)$  with  $0 \leq w_0 \leq \infty$  such that at all wealth levels  $w < w_0$ , the agent rejects  $g$  (i.e.,  $\mathbf{E}[u(w + g)] < u(w)$ ), and at all wealth levels  $w > w_0$ , it accepts  $g$  (i.e.,  $\mathbf{E}[u(w + g)] > u(w)$ ); see Pratt (1964), Yaari (1969), Dybvig and Lippman (1983), Hart (2011). Put  $\phi(r) := \mathbf{E}[u(r + g)] - u(r)$  (for our specific  $u$ ); then  $\lim_{r \rightarrow L(g)^+} \phi(r) = -\infty$  (since  $\lim_{x \rightarrow 0^+} u(x) = -\infty$ ), and  $\phi(\mathbf{R}(g)) = \mathbf{E}[g] > 0$  (since  $\mathbf{E}[\log(\mathbf{R}(g) + g)] = \log(\mathbf{R}(g))$ ). Therefore, the equation  $\phi(r) = 0$  has a solution  $r$  in the interval  $(L(g), \mathbf{R}(g))$ , which is unique since that is precisely the critical  $w_0$ .

Moreover,  $Q$  satisfies **Distribution**, **Monotonicity** (if  $g \geq h$  and  $g \neq h$ , then  $\mathbf{E}[u(Q(h) + g)] > \mathbf{E}[u(Q(h) + h)] = u(Q(h))$  since  $u$  is strictly increasing; recalling the function  $\phi$  of the previous paragraph,  $Q(h)$  thus lies in the region where  $\phi$  is positive, and so it is larger than the root  $Q(g)$  of  $\phi$ ), and **Compound Gamble** (the proof of Proposition 6 did not use any special property of the  $u_\gamma$  function beyond the existence and uniqueness of the solution of (2), and so it applies to our  $u$  too). However,  $Q$  does not satisfy **Scaling**, since  $u$  is *not* homogeneous of degree 1; e.g., for  $g = (2, 1/2; -1, 1/2)$ , we get  $Q(g) \approx 1.155$  and  $Q(2g) \approx 2.098 \neq 2Q(g)$ .

As we have seen above, this function  $Q$  satisfies  $Q(g) < \mathbf{R}(g)$  for every  $g$ .

This example suggests that without **Scaling**, the other axioms may characterize the critical wealth functions for a wider class of utilities than CRRA. In Appendix D we show that is indeed the case, once we add a **Continuity** axiom (which is needed to provide a certain degree of regularity; in the results of our Theorems 1 and 3, regularity is implicitly provided by **Scaling**).

### The *Compound Gamble* axiom

Take  $Q(g) = \delta \mathbf{R}(g) < \mathbf{R}(g)$  for some fixed  $0 < \delta < 1$ : it satisfies all the axioms (as well as the  $\delta$ -Compound Gamble axiom; see Section 7(f)) except the **Compound Gamble** axiom.<sup>33</sup>

### The *Distribution* axiom

The **Distribution** axiom is needed since it is more convenient to work with random variables than with probability distribution. Without it, one can easily get a counterexample to the result of Theorem 3: fix a probability space  $\Omega_0$ , and put  $Q(g) = R_2(g)$  for all  $g$  defined on  $\Omega_0$  and  $Q(g) = R_1(g)$  otherwise. However, we conjecture that one cannot go beyond this: a function  $Q$  that satisfies the other three axioms must equal on each

<sup>33</sup>Another example that satisfies the first three axioms but not the **Compound Gamble** axiom is  $R^{\text{AS}}$  (see Theorem 4); while  $R^{\text{AS}}$  differs from all the  $R_\gamma$  of Theorem 3, it is not always less than  $\mathbf{R}$ .

probability space one of the  $R_\gamma$  functions (and thus the result of the Main Theorem may well hold without the [Distribution](#) axiom).<sup>34</sup>

#### APPENDIX D: WITHOUT SCALING

In the previous appendix, we saw a critical wealth function that does not satisfy the [Scaling](#) axiom: it is obtained from a non-CRRA utility. This suggests<sup>35</sup> that one may characterize the class of functions  $Q$  that satisfy all axioms except [Scaling](#). We show here that the other axioms, together with a certain [Continuity](#) axiom,<sup>36</sup> characterize the critical wealth functions for a class of DARA (Decreasing Absolute Risk Aversion) utilities. Moreover, we show that the infimum of all these functions  $Q$  is the maximal loss function  $L$ . The proofs here are quite different from those of our main results.

The continuity requirement is stated as follows.

**CONTINUITY.** *Let  $g_n$  be a sequence of gambles in  $\mathcal{G}$ , all with the same finite support  $S$ , such that  $g_n \rightarrow g$  in distribution<sup>37</sup> (note that  $g$  need not be a gamble in  $\mathcal{G}$ ).*

(i) *If  $g$  is a gamble (i.e.,  $g \in \mathcal{G}$ ) and its support is also<sup>38</sup>  $S$ , then  $Q(g_n) \rightarrow Q(g)$ .*

(ii) *If  $\mathbf{E}[g] = 0$  and  $g$  is not identically 0, then  $Q(g_n) \rightarrow \infty$ .*

(iii) *If  $g \geq 0$  and  $g$  is not identically 0, then  $Q(g_n) \rightarrow L(S)$ , where  $L(S) := -\min S$  (which equals  $L(g_n)$  for all  $n$ ).*

An immediate consequence of [Continuity](#) is the next lemma.

**LEMMA 24.** *Let  $a, b > 0$ . For every  $c > b$ , there exists  $q$  such that  $g_{a,b;q} := (a, q; -b, 1 - q) \in \mathcal{G}$  and  $Q(g_{a,b;q}) = c$ .*

**PROOF.** The function  $\theta(q) := Q(g_{a,b;q})$  satisfies the following conditions: it is continuous for  $0 < q < b/(a + b)$  (by (i));  $\theta(q) \rightarrow 0$  when  $q \rightarrow b/(a + b)$  (by (ii)); and  $\theta(q) \rightarrow b$  when  $q \rightarrow 0$  (by (iii)).  $\square$

Let  $\mathcal{U}_0$  be the set of utility functions  $u: \mathbb{R}_+ \rightarrow \mathbb{R}$  that are increasing, concave, continuously differentiable (i.e.,  $C^1$ ), and satisfy the following conditions<sup>39</sup>:  $\lim_{x \rightarrow 0^+} u(x) = -\infty$ ; the absolute risk-aversion coefficient  $\rho_u(x) \equiv -u''(x)/u'(x)$  (which exists almost everywhere) is decreasing; and  $\lim_{x \rightarrow \infty} \rho_u(x) = 0$ . Our characterization result is the following proposition.

<sup>34</sup>The proof, based on arguments of the kind used in [Section A.3](#), may be quite complex.

<sup>35</sup>We thank the referees for this suggestion.

<sup>36</sup>[Continuity](#) is needed to provide a certain degree of “regularity” for  $Q$ ; in the results of our [Theorems 1 and 3](#), this regularity is implicitly provided by [Scaling](#). In addition, we can no longer prove that  $Q(g) > L(g)$  for all  $g$  (cf. [Proposition 9](#)), and so it becomes part of [Continuity](#).

<sup>37</sup>The support  $S$  of all  $g_n$  is fixed and finite, say of size  $m$ , so convergence in distribution is the same as convergence of the vectors of probabilities (as points in the unit simplex of  $\mathbb{R}^m$ ).

<sup>38</sup>This means that there is no value in  $S$  whose probability vanishes in the limit  $g$ .

<sup>39</sup>See [Hart \(2011\)](#) for a discussion of the first two conditions ( $u(0^+) = -\infty$  and DARA). Note that  $u \in \mathcal{U}_0$  need not be twice continuously differentiable (i.e.,  $C^2$ ): take, for instance,  $u(x) = 1 - 1/x$  for  $x \leq 1$  and  $u(x) = \log(x)$  for  $x \geq 1$  (thus  $\rho_u(x) = 2$  for  $x < 1$  and  $\rho_u(x) = 1$  for  $x > 1$ ).

PROPOSITION 25. *The function  $Q$  satisfies the **Distribution**, **Monotonicity**, **Compound Gamble**, and **Continuity** axioms if and only if there exists a utility function  $u \in \mathcal{U}_0$  such that for every  $g \in \mathcal{G}$ ,*

$$Q(g) = w \quad \text{if and only if} \quad \mathbf{E}[u(w + g)] = u(w). \quad (14)$$

The proof is split into two parts: [Lemma 26](#) and [Lemma 27](#).

LEMMA 26. *Let  $u \in \mathcal{U}_0$  and let  $Q$  be given by (14). Then  $Q$  is well defined and satisfies the four axioms of [Proposition 25](#).*

PROOF. First, we claim that for every  $g \in \mathcal{G}$  and  $u \in \mathcal{U}_0$  the equation

$$\mathbf{E}[u(w + g)] = u(w) \quad (15)$$

has a unique solution  $w > L(g)$ . Indeed, for every  $d > 0$ , we have  $\rho_u(x) > \rho_u(x + d)$  for all  $x$ , and so  $u(x) = \psi(u(x + d))$  with  $\psi$  strictly increasing and strictly concave. From this it follows that  $\mathbf{E}[u(w + d + g)] \leq u(w + d)$  implies  $\mathbf{E}[u(w + g)] < u(w)$  (using the same arguments as in the proof of [Proposition 2](#) in<sup>40</sup> [Hart 2011](#), with  $u_1(x) = u(x)$  and  $u_2(x) = u(x + d)$ ). Therefore, there can be at most one  $w = w_0$  satisfying (15), and  $\mathbf{E}[u(w + g)] < u(w)$  for all  $w < w_0$ , and  $\mathbf{E}[u(w + g)] > u(w)$  for all  $w > w_0$ . To see that such  $w_0$  exists, note that  $\mathbf{E}[u(w + g)] < u(w)$  as  $w$  decreases to  $L(g)$  (the left-hand side goes to  $-\infty$ ) and  $\mathbf{E}[u(w + g)] > u(w)$  as  $w$  goes to  $\infty$  (cf. [Proposition 4\(iv\)](#) in [Hart 2011](#)). In summary,  $Q(g) = w_0$  is well defined and, moreover,

$$\text{sign}(\mathbf{E}[u(w + g)] - u(w)) = \text{sign}(w - Q(g)). \quad (16)$$

Second, we show that the axioms are satisfied. The **Distribution** axiom is immediate. For **Monotonicity**, note that  $g' \succeq g$  implies  $\mathbf{E}[u(Q(g) + g')] > \mathbf{E}[u(Q(g) + g)] = u(Q(g))$ , and so  $Q(g') < Q(g)$  by (16). For **Compound Gamble**, use the proof of [Proposition 6](#). For **Continuity**, (i) is immediate; for (ii), if  $Q(g_n)$  converges to some finite  $w \geq L(S) > 0$ , then  $\mathbf{E}[u(w + g)] = u(w)$ , contradicting the strict concavity of  $u$  (recall that  $\rho(x) > 0$  and thus  $u''(x) > 0$  for a.e.  $x$ ) since  $\mathbf{E}[g] = 0$  and  $g$  is not identically 0; for (iii), if  $Q(g_n)$  converges to some  $w > L(S)$ , then  $\mathbf{E}[u(w + g)] = u(w)$ , contradicting the strict monotonicity of  $u$ , since  $g \succeq 0$ .  $\square$

LEMMA 27. *Let  $Q$  satisfy the four axioms of [Proposition 25](#). Then there exists  $u \in \mathcal{U}_0$  such that  $Q$  is given by (14).*

We first prove a simpler result.

LEMMA 28. *Let  $Q$  satisfy the four axioms of [Proposition 25](#) and let  $0 < s < t$ . Then there exists a strictly increasing function  $u: [s, t] \rightarrow \mathbb{R}$  such that  $\mathbf{E}[u(Q(g) + g)] = u(Q(g))$  holds for all  $g \in \mathcal{G}$  with  $Q(g) + g \in [s, t]$ . Moreover,  $u$  is unique up to positive affine transformations.*

<sup>40</sup>It can be verified that the proofs in [Hart \(2011\)](#) apply also when the nonincreasing function  $\rho_u(x)$  is defined only a.e., since  $\rho_u(x)$  is the derivative of the concave function  $-\log u'(x)$ .

PROOF. Put  $u(s) := 0$  and  $u(t) := 1$ . For every  $w$  in the open interval  $(s, t)$ , put  $u(w) := q_w$ , where  $q_w$  is given by Lemma 24 for  $a = t - w$  and  $b = w - s$ , i.e., so that  $Q(g_{t-w, s-w; q_w}) = w$ . Since  $Q(g_{t-w, s-w; q})$  is strictly decreasing in  $q$  and in  $w$  by **Monicity**, it follows that  $q_w = u(w)$  is unique and strictly increasing in  $w$ .

To show that  $\mathbf{E}[u(Q(g) + g)] = u(Q(g))$ , consider first  $g$  such that  $Q(g) + g \in \{s, t\}$ . Thus  $g = (t - w, p; s - w, 1 - p)$  for some  $p$ , where  $w := Q(g)$ . But then  $p = q_w$  by the uniqueness of  $q_w$ , and so  $\mathbf{E}[u(Q(g) + g)] = q_w u(w + t - w) + (1 - q_w)u(w + s - w) = q_w = u(w) = u(Q(g))$  (recall that  $u(t) = 1$  and  $u(s) = 0$ ).

Next, take any  $g$  such that  $Q(g) + g \in (s, t)$ , say  $g = (x_1, p_1; \dots; x_m, p_m)$ , and let  $w := Q(g)$ . For each  $i$ , let  $h_i$  be a two-point gamble  $h_i = (t - w - x_i, r_i; s - w - x_i, 1 - r_i)$ , where  $r_i := u(w + x_i)$ , or, equivalently,  $Q(h_i) = w + x_i$ ; assume that  $g$  and all  $h_i$  are independent. Then, by **Compound Gamble**,  $f := g + \sum_i \mathbf{1}_{g=x_i} h_i$  satisfies  $Q(f) = Q(g) = w$ . But  $f$  has only two values,  $t - w$  and  $s - w$ , and so by the argument in the previous paragraph,  $\mathbf{E}[u(w + f)] = u(w)$ . Now  $\mathbf{E}[u(w + g)] = \sum_i p_i u(w + x_i) = \sum_i p_i \mathbf{P}[h_i = t - w - x_i] = \sum_i p_i \mathbf{P}[f = t - w | g = x_i] = \mathbf{P}[f = t - w] = \mathbf{E}[u(w + f)] = u(w)$ . Finally, if  $g$  takes values  $x_i$  such that  $Q(g) + x_i$  equals either  $s$  or  $t$ , then we need no further gamble  $h_i$  after such an  $x_i$ .

The uniqueness of  $u$  up to positive affine transformations follows by considering again the two-point gambles  $g_{t-w, s-w; q_w}$ , which yield  $u(w) = q_w u(t) + (1 - q_w)u(s)$ .  $\square$

PROOF OF LEMMA 27. For each integer  $n \geq 2$ , let  $u_n$  be given by Lemma 28 for  $s = 1/n$  and  $t = n$ , normalized by  $u_n(1) = 1$  and  $u_n(2) = 2$ . For  $n' > n$ , the function  $u_{n'}$  restricted to  $[1/n, n]$  must be a positive affine transformation of  $u_n$ ; since  $u_n$  and  $u_{n'}$  coincide at 1 and 2, they coincide on all of  $[1/n, n]$ . Thus  $u_{n'}$  is an extension of  $u_n$ , and so the function  $u: \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by  $u(w) := u_n(w)$  for any  $n$  such that  $w \in [1/n, n]$  satisfies  $\mathbf{E}[u(Q(g) + g)] = u(Q(g))$  for every  $g$  (just take  $n$  large enough so that  $Q(g) + g \in [1/n, n]$ ). Thus  $\mathbf{E}[u(w + g)] = u(w)$  when  $w = Q(g)$ , for all  $g \in \mathcal{G}$ .

Conversely, let  $w := Q(g)$  and assume that  $\mathbf{E}[u(w' + g)] = u(w')$  for some  $w' \neq w$  (where  $w, w' > L(g)$ ). Consider first the case  $w' > w$ . By moving probability mass from the positive values of  $g$  to its lowest value  $-L(g)$ , we get  $g'$  that satisfies  $Q(g') = w'$  and has the same support as  $g$  (such a  $g'$  exists since the intermediate value  $w < w' < \infty$  is attained by **Continuity**). Therefore,  $\mathbf{E}[u(w' + g')] = \mathbf{E}[u(w' + g)]$  (they both equal  $u(w')$ : the first by the previous paragraph for  $g'$  and the second by our assumption), which contradicts the construction of  $g'$  as  $u$  is strictly increasing and thus  $u(w' + x) > u(w' - L(g))$  for every  $x > 0$ . The case  $w' < w$  is handled in a similar manner, moving probability mass from the negative values of  $g$  to its highest value  $M(g)$ .

Thus  $u$  is an increasing function, and the equation  $\mathbf{E}[u(w + g)] = u(w)$  has a unique solution  $w > L(g)$  for every  $g \in \mathcal{G}$ .

To show that  $u$  is a concave function, let  $0 < \delta < x$ . By Lemma 24, there is  $1/2 < p < 1$  such that  $Q(g_{\delta, -\delta; p}) = x$ , and so  $pu(x + \delta) + (1 - p)u(x - \delta) = u(x)$ . Since  $p > 1/2$  and  $u$  is increasing, it follows that  $(1/2)u(x + \delta) + (1/2)u(x - \delta) < u(x)$ , or  $u(x + \delta) - u(x) < u(x) - u(x - \delta)$ , which implies, in particular, that  $u$  is concave on the  $\delta$ -grid (i.e., when restricted to  $x = \delta, 2\delta, \dots, n\delta, \dots$ ). Since this holds for all  $\delta > 0$ , it follows that  $u$  is concave.

Next, we claim that for every  $\varepsilon > 0$  and  $\varepsilon < s < t$ , we have

$$\frac{u(s + \varepsilon) - u(s)}{u(t + \varepsilon) - u(t)} < \frac{u(s) - u(s - \varepsilon)}{u(t) - u(t - \varepsilon)}. \tag{17}$$

Indeed, let  $p, p'$  be such that  $Q(g_{\varepsilon, -\varepsilon; p}) = s$  and  $Q(g_{\varepsilon, -\varepsilon; p'}) = t$ , and so  $s < t$  implies  $p > p'$  by **Monotonicity**. From (14), we get

$$\frac{u(t) - u(t - \varepsilon)}{u(t + \varepsilon) - u(t)} = \frac{p'}{1 - p'} < \frac{p}{1 - p} = \frac{u(s) - u(s - \varepsilon)}{u(s + \varepsilon) - u(s)},$$

which yields (17).

Let  $v(x) := \log u'_+(x)$ , where  $u'_+(x) = \lim_{\varepsilon \rightarrow 0^+} (u(x + \varepsilon) - u(x))/\varepsilon$  is the right-derivative of  $u$  at  $x$ . We claim that  $v$  is a convex function: for every  $\delta > 0$  and  $0 < x < y$ , we have  $v(x + \delta) - v(x) \leq v(y + \delta) - v(y)$  or, equivalently,  $u'_+(x + \delta)/u'_+(x) \leq u'_+(y + \delta)/u'_+(y)$ . Indeed, otherwise we would have for all small enough  $\varepsilon > 0$ ,

$$\frac{u(x + \delta + \varepsilon) - u(x + \delta)}{u(x + \varepsilon) - u(x)} > \frac{u(y + \delta + \varepsilon) - u(y + \delta)}{u(y + \varepsilon) - u(y)}. \tag{18}$$

Taking a large enough integer  $m$  so that (18) holds for  $\varepsilon = \delta/m$ , and then using (17) with  $s = x + i\varepsilon$  and  $t = x + i\varepsilon$  for  $i = m, m - 1, \dots, 1$  yields the chain of inequalities

$$\begin{aligned} \frac{u(x + (m + 1)\varepsilon) - u(x + m\varepsilon)}{u(y + (m + 1)\varepsilon) - u(y + m\varepsilon)} &< \frac{u(x + m\varepsilon) - u(x + (m - 1)\varepsilon)}{u(y + m\varepsilon) - u(y + (m - 1)\varepsilon)} \\ &< \dots < \frac{u(x + \varepsilon) - u(x)}{u(y + \varepsilon) - u(y)}, \end{aligned}$$

which contradicts (18). Thus  $v$  is a convex function as claimed. Therefore,  $v$  is continuous and so  $u'_+$  is continuous; hence  $u'$  exists everywhere and is continuous (since  $u'_-(x) = \lim_{y \rightarrow x^-} u'_+(y)$ ), i.e.,  $u \in C^1$ . Moreover,  $-v'(x) = -u''(x)/u'(x) = \rho(x)$  exists at all points  $x$  except at most countably many, and so the absolute risk-aversion coefficient of  $u$  is a nonincreasing function.

If  $\lim_{x \rightarrow 0^+} u(x) > -\infty$ , then there exist gambles  $g$  such that  $\mathbf{E}[u(w + g)] > u(w)$  for all  $w > L(g)$ , and thus (15) has no solution for such  $g$  (see Proposition 6(ii) in Hart 2011; take for instance  $g = (1, 1 - \varepsilon; -1, \varepsilon)$  for small enough  $\varepsilon > 0$ ). Similarly, if  $\lim_{x \rightarrow \infty} \rho(x) > 0$ , then there exist gambles  $g$  such that  $\mathbf{E}[u(w + g)] < u(w)$  for all  $w > L(g)$ , and thus (15) has no solution for such  $g$  (see Proposition 4(i) in Hart 2011; take  $g$  with  $R^{\text{AS}}(g) < 1/\lim_{x \rightarrow \infty} \rho(x)$ ). Finally,  $\rho(x)$  must be strictly decreasing, since otherwise there is an interval where  $\rho$  is constant, and then there exist gambles for which (15) has multiple solutions (if  $\rho(x) = \alpha$  for all  $x$  in some interval, then  $u$  is CARA- $\alpha$  in that interval; take  $g$  with small values and  $R^{\text{AS}}(g) = 1/\alpha$ ).

Thus  $u \in \mathcal{U}_0$ . □

For  $u \in \mathcal{U}_0$ , let  $Q_u : \mathcal{G} \rightarrow \mathbb{R}_+$  denote the function given by (14).

**PROPOSITION 29.** *For every  $g \in \mathcal{G}$ ,*

$$\inf_{u \in \mathcal{U}_0} Q_u(g) = L(g).$$

PROOF. For every  $m > 0$ , take  $u_m(x) = x + (1/m) \log x$ . It is easy to see that  $u_m \in \mathcal{U}_0$ , and that (14) becomes  $\mathbf{E}[\log(1 + g/Q_{u_m}(g))] = -m\mathbf{E}[g]$ . As  $m \rightarrow \infty$ , the right-hand side converges to  $-\infty$ , and therefore,  $Q_{u_m}(g) \rightarrow L(g)$  (see the proof of Lemma 1 and Figure A1 in Appendix A of Foster and Hart 2009).  $\square$

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