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## The ex ante aggregation of opinions under uncertainty

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This paper presents an analysis of the problem of aggregating preference orderings under subjective uncertainty. Individual preferences, or opinions, agree on the ranking of risky prospects, but are quite general because we do not specify the perception of ambiguity or the attitude toward it. A convexity axiom for the ex ante preference characterizes a (collective) decision rule that can be interpreted as a compromise between the utilitarian and the Rawlsian criteria. The former is characterized by the independence axiom as in Harsanyi (1955). Existing results are special cases of our representation theorems, which also allow us to interpret Segal's (1987) two-stage approach to ambiguity as the ex ante aggregation of (Bayesian) future selves' opinions.

KEYWORDS. Aggregation of preferences, uncertainty.

JEL CLASSIFICATION. D70, D81.

### 1. INTRODUCTION

A number of problems in decision making reduce to the maximization of a criterion function that aggregates the different attributes of the objects of choice. For example, according to the subjective expected utility (SEU) theory and its generalizations, a decision maker (DM) chooses among acts (or bets, loosely speaking) as if maximizing a utility function that is an aggregator of the payoffs in each state.<sup>1</sup> In richer and perhaps more realistic contexts, however, DMs also take into consideration the opinions of others about those same objects, and do not solely rely on their explicit attributes. This is particularly relevant when the outcome of a course of action depends on events for which probabilities are not (completely) specified, and the DM seeks the advice of experts or specialists. In many such cases, the problem comes down to the question of how to combine different preference relations, or opinions, and the DM behaves as if maximizing a utility function that aggregates the utility functions of those experts.

Harsanyi (1955) gave a solution to the problem of aggregating a finite number of utility functions in the context of choice under risk. When preferences over lotteries satisfy the expected utility axioms and are represented by suitably normalized von Neumann–Morgenstern (vN–M) utility functions, Harsanyi concludes that the collective decision

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<sup>1</sup>See, in particular, Savage (1954), Anscombe and Aumann (1963), and Schmeidler (1989).

criterion must be a weighted average of the utility functions of the members of the group.<sup>2</sup> A mild unanimity condition is the only additional requirement for his conclusion. A similar result obtains when preferences satisfy the axioms of [Anscombe and Aumann \(1963\)](#), and the DM and the experts agree on the ranking of risky prospects (constant acts). In this case, the DM behaves as if maximizing the expected payoff of an act according to a probability measure that is a weighted average of the probability assessments of the experts.<sup>3</sup> As is well known, this representation is not descriptively accurate (see [Ellsberg 1961](#)). By contrast, the recent works of [Gilboa et al. \(2010\)](#), [Crès et al. \(2010; CGV\)](#), and [Gajdos and Vergnaud \(2010\)](#) allow for a more general specification of preferences and provide aggregation criteria that differ from Harsanyi's weighted sum. They characterize a DM who satisfies the axioms of the maxmin expected utility (MEU) theory of [Gilboa and Schmeidler \(1989\)](#) (and also [Chateauneuf 1991](#)), and whose set of priors depends on the subjective beliefs of the experts.

To gain perspective, consider the following example. Imagine a DM who plans to invest in a new firm. There is uncertainty regarding the market share, which can be small or large. For concreteness, think of the case where the market share depends on an event for which there is no objective probability.<sup>4</sup> The investment will be profitable only if the market share is large. The DM has very little knowledge about the industry and decides to consult with two experts, *A* and *B*. The DM follows their advice when *A* and *B* agree on their choice.

The standard approach (à la Harsanyi) is to specify a unique belief  $\mu_i$  for each expert ( $i = A, B$ ), a unique belief for the DM, and a common Bernoulli utility function  $u$ . We can then conclude that there exists a weight  $\lambda$  associated with expert *A* such that

$$f \mapsto \lambda \int u(f) d\mu_A + (1 - \lambda) \int u(f) d\mu_B$$

represents the DM's preferences over acts. For [Gilboa et al. \(2010\)](#), experts (or "objective rationality" in their own words) also satisfy the Anscombe–Aumann axioms, but the DM is extremely cautious and aggregates opinions according to the Rawlsian criterion

$$f \mapsto \min \left\{ \int u(f) d\mu_A, \int u(f) d\mu_B \right\}. \quad (1)$$

[Crès et al. \(2010\)](#) argue that scientific caution favors the multiplicity of priors and they model this lack of confidence in a single subjective probability with the maxmin expected utility model. The utility function of expert *i* is now  $W_{CGV}(f, i) = \min_{\mu \in M_i} \int u(f) d\mu$ , where  $M_i$  is a set of priors. They derive an aggregation rule in which the DM is reluctant to assign a single weight to expert *A*. The DM employs a set  $\Lambda$  of weights and ranks acts according to

$$f \mapsto \min_{\lambda \in \Lambda} [\lambda W_{CGV}(f, A) + (1 - \lambda) W_{CGV}(f, B)]. \quad (2)$$

<sup>2</sup>For a discussion of Harsanyi's theorem, see [Hammond \(1992\)](#), [Weymark \(2005\)](#), and [Hild et al. \(2008\)](#), and the references therein.

<sup>3</sup>See [Mongin \(1995, 1998\)](#), [Gilboa et al. \(2004\)](#), and [Gajdos et al. \(2008a\)](#) for a more detailed discussion.

<sup>4</sup>[Crès et al. \(2010\)](#) mention the event of global warming in the near future.

The criterion in (1) is a special case of (2) when the experts have unique beliefs and  $\Lambda$  is the unit interval, and yet both expressions characterize a DM who satisfies the axioms of the maxmin expected utility theory.<sup>5</sup> The minimizations in (1) and (2) also reflect the intuitive idea that the DM may be averse to dispersed opinions.

The existing models make certain assumptions about preferences and the nature of the set of experts that limit their scope. More precisely, the existing frameworks are limited to the case where the experts have either SEU or maxmin preferences, and the DM has perfect knowledge of their preferences. In this paper we provide an axiomatic characterization of a more general model of aggregation of preferences under uncertainty. It dispenses with the restrictive assumptions of Gilboa et al. (2010) and Crès et al. (2010), but still allows for a DM who is averse to situations in which experts disagree. Our model identifies an aggregation rule that can be interpreted as a compromise between Harsanyi's utilitarian aggregator and the Rawlsian criterion, and includes (1) and (2) as special cases.

We modify two assumptions made in the literature. First, we develop a framework where opinions do not necessarily conform to expected utility and need not satisfy Schmeidler's (1989) uncertainty aversion axiom. Note that the models mentioned above assume that experts are either Bayesian or have multiple priors and are fully pessimistic. The assumption of a unique subjective belief is difficult to accept. Crès et al. (2010) assume instead that experts satisfy the axioms of the maxmin expected utility theory, which they justify on the grounds of scientific caution. They do not separate the perception of ambiguity from the attitude toward it, thereby ruling out a number of plausible preferences. For instance, their model does not account for the possibility that an expert is optimistic about the assessment of the odds of the states. Scientific caution may favor the multiplicity of priors, but it does not have much to say about how an expert evaluates the uncertain payoff of a bet given a set of subjective probabilities. For example, assume that experts  $A$  and  $B$  share the same beliefs. Expert  $A$  is a standard pessimistic individual who always considers the worst-case scenario. Expert  $B$ , however, is optimistic and advises each client to take "risks"; that is, expert  $B$  evaluates an act according to  $f \mapsto \max_{\mu \in M_B} \int u(f) d\mu$ . In this case, both experts exercise scientific caution, but have different attitudes toward ambiguity. In this paper, we model a broader class of opinions in which we do not specify the perception of ambiguity or the attitude toward it. This class includes optimistic experts, and was recently studied by Ghirardato and Siniscalchi (2010).<sup>6,7</sup>

Second, we do not assume that the class of opinions is finite. Observe that the DM may face the problem of aggregating an infinite number of individual preferences even if there are finitely many experts. This is often the case when those preferences are not

<sup>5</sup>Gajdos and Vergnaud (2010) obtained the same characterization as in (2), albeit in a slightly different framework. For brevity, we focus on the work of Crès et al. (2010).

<sup>6</sup>See also Cerreia-Vioglio et al. (2011). I thank a referee for the reference.

<sup>7</sup>An additional motivation to consider this class of preferences is the recent criticism of models that assume the uncertainty aversion axiom of Schmeidler (1989). See the paradoxes of Machina (2009) and Baillon et al. (2010), and the experimental evidence of L'Haridon and Placido (2010).

completely known and the DM realizes that they may belong to a larger class.<sup>8</sup> For instance, limited knowledge may lead the DM to model the (pessimistic) opinion of expert  $A$  as an incomplete preference relation. Under some reasonable assumptions, this requires us to employ an infinite class of opinions, each one associated with a particular set of priors.<sup>9</sup> Alternatively, imagine that expert  $B$  is an optimistic individual who reports the set of priors but does not disclose his or her attitude toward ambiguity. In this situation, the DM may conceive of a set of utility functions for expert  $B$ . One possibility is that each utility is a weighted average of the pessimistic component (that is,  $\min_{\mu \in M_B} \int u(f) d\mu$ ) and the optimistic evaluation (which is  $\max_{\mu \in M_B} \int u(f) d\mu$ ). The weight on the pessimistic component is  $\alpha$  and it ranges over some nontrivial interval. The DM now faces an infinite class of opinions: the utility function of expert  $A$  and the collection of  $\alpha$ -MEU preferences (the Hurwicz criteria) just described. The model of Gilboa et al. (2010) allows for an infinite number of opinions, but all experts are Bayesian. Crès et al. (2010) consider experts with non-SEU preferences, but assume that there are finitely many utility functions. In this paper, we do not impose any restriction on the cardinality of the class of opinions. It may be finite, countable, or uncountably infinite. We require only that the preferences of experts satisfy a few regularity conditions.

Our characterization of a less restrictive model relies on a set of axioms on the domain of lotteries of acts (i.e., the original setup of Anscombe and Aumann 1963). A generalized utilitarian criterion obtains under the standard expected utility assumptions. Two conditions differentiate our more general representation from utilitarianism. One is a weakening of the independence axiom. The other is a preference for randomization that reflects the DM's willingness to hedge against the epistemic or second-order uncertainty about which expert is "right" or closest to the "truth." This idea builds on a metaphor suggested by Crès et al. (2010), who argue that the DM contemplates a collection of epistemic states. Each one corresponds to the event in which one expert is correct. The DM does not know the true epistemic state and there is no objective probability (or weight) available; that is, there is uncertainty regarding the event that one particular expert is the right one. Preference for randomization specifies how the DM reacts to this uncertainty. The axiom says that he or she prefers a course of action that is a compromise between the recommendations of the experts and for which opinions are not so dispersed when there is no consensus. The Rawlsian criterion obtains when a strong form of aversion to dispersed opinions holds.

### 1.1 Framework and interpretations

As mentioned above, we adopt the Anscombe–Aumann framework. The DM has a preference over lotteries of acts that aggregates a collection of binary relations defined on the set of acts. For this reason, we also refer to the DM's preference relation as the *ex ante preference* and refer to opinions as *ex post preferences*.

<sup>8</sup>Zhou (1997) uses a similar argument to justify his extensions of Harsanyi's aggregation theorems to general societies.

<sup>9</sup>See Nascimento and Riella (2011).

Two axiomatic models of preferences are presented in this setup. One is a generalized utilitarian aggregation, where the DM follows the expected utility axioms as in [Harsanyi \(1955\)](#). The DM behaves as if maximizing a weighted average of a strictly increasing transformation of the utility functions representing the ex post preferences. This first model is the simplest one that allows us to introduce the basic axioms. It also permits us to interpret our framework in terms of aggregation of multiple selves' opinions (see below).

The second model (our more general result) weakens the independence axiom and postulates a preference for randomization. The DM behaves as if employing the set of all (normalized) utilitarian weights. Some weights that appear implausible or poorly justified are penalized. The decision criterion can be decomposed into two parts. One is the weighted average of a strictly increasing transformation  $\phi$  of the experts' utility functions. The other is a cost function that associates a nonnegative number  $c(m)$  to each probability measure  $m$  on the set of experts. The criterion consists of minimizing the weighted sum plus the cost component; that is, the utility of an act has the form

$$W(f) = \min_m \int \phi(W(f, i)) dm(i) + c(m), \quad (3)$$

where  $W(f, i)$  is expert  $i$ 's utility of the act  $f$ . One can take the function  $\phi$  to be the identity when the DM collapses two-stage lotteries into a single stage. The representation in (3) generalizes [Crès et al. \(2010\)](#) and can be interpreted as a compromise between the standard utilitarian aggregator (namely,  $\int W(f, i) dm(i)$ ) and the Rawlsian criterion (that is,  $\min_i W(f, i)$ ). A version of [Crès et al. \(2010\)](#) obtains when there exists a closed and convex set of weights  $\mathbf{M}$  such that the utility of an act is  $\min_{m \in \mathbf{M}} \int W(f, i) dm(i)$ . This is a special instance of (3) and is characterized by a particular weakening of the independence axiom.

There are two possible interpretations of our framework. The interpretation that we emphasize is that ex post preferences are the opinions of experts, which the DM aggregates in an ex ante stage.<sup>10</sup> Another is that our model reinterprets and generalizes the original framework of [Anscombe and Aumann \(1963\)](#). It is described as follows. In an ex ante stage, the DM contemplates how an act would be ranked close to the realization of the state. Several conceivable rankings correspond to the DM's selves or ex post preferences. In the original Anscombe–Aumann framework, ex post preferences are conditional preferences over acts; that is, each self of the DM ranks an act according to the prize it delivers in a particular state. We generalize the notion of a “state” by replacing it with ex post preferences or second-stage selves. With this interpretation, our model can be read as a framework of individual decision making under uncertainty, and it includes as special cases a number of recent works. [Segal's \(1987\)](#) two-stage approach to ambiguity (as formulated in [Seo 2009](#)) is one example where ex post preferences correspond to the collection of all possible SEU preferences that agree on the ranking of risky prospects. By contrast, our framework is less restrictive, and allows for more realistic perceptions of and reactions to ambiguity in the second stage.

<sup>10</sup>Because the unanimity rule derived from the class of ex post preferences can be understood as an incomplete preference relation, we note that our framework bears some similarity to the model of [Danan \(2010\)](#) under this interpretation.

### 1.2 Outline of the paper

The paper is organized as follows. Section 2 describes the setup, and Section 3 provides an axiomatization of the class of opinions. Section 4 characterizes an ex ante preference relation that satisfies the expected utility axioms, and discusses how it relates to a branch of the literature on ambiguity. Section 5 gives a utility representation when the DM is averse to dispersed opinions, and it also presents two special cases. Section 6 contains general remarks about the limitations of our approach. While Appendix A presents the proofs of our main results, Appendix B discusses the uniqueness of the representations.

## 2. SETUP

The set  $X$  denotes a separable metric space. We represent by  $\Delta(X)$  the set of all Borel probability measures on  $X$ , and endow it with any metric that induces the topology of weak convergence. The set  $\Delta(Y)$  has similar meaning when  $Y$  is a metric space, and we endow  $\Delta(Y)$  with the topology of weak convergence. As a matter of notation, when  $Y$  is a metric space, we denote by  $\mathcal{B}(Y)$  the Borel  $\sigma$ -algebra in  $Y$ .

The set of states of the world is denoted by  $S$ , which we assume to be finite. The set of all Anscombe–Aumann acts is  $\Delta(X)^S =: \mathcal{F}$  and is endowed with the product metric. Define  $\mathcal{F}_c$  as the set of all constant acts, that is,  $\mathcal{F}_c = \{f \in \mathcal{F} : \exists p \in \Delta(X) \text{ such that } f(S) = \{p\}\}$ . We identify the set  $\mathcal{F}_c$  with  $\Delta(X)$ . In this paper, we work with lotteries of acts. For notation, when  $f, g \in \mathcal{F}$  and  $\lambda \in [0, 1]$ , we represent by  $\lambda\delta_f + (1 - \lambda)\delta_g$  the simple lottery where the probability of  $f$  is  $\lambda$  and the probability of  $g$  is  $1 - \lambda$ .

The binary relation  $\succsim$  is defined on the set of all lotteries (Borel probability measures on the space) of Anscombe–Aumann acts, that is,  $\succsim \subseteq \Delta(\mathcal{F}) \times \Delta(\mathcal{F})$ . We refer to  $\succsim$  as the ex ante preference or the DM's preference. Opinions of experts are modeled as a class  $\mathcal{P}$  of binary relations on  $\mathcal{F}$ , that is,  $\mathcal{P} \subseteq 2^{\mathcal{F} \times \mathcal{F}} \setminus \{\emptyset\}$ . A typical element of  $\mathcal{P}$  is denoted by  $\succsim^*$ . Recall that we also refer to the elements of  $\mathcal{P}$  as ex post preferences. We introduce a suitable topology for the set  $\mathcal{P}$  in the next section.

## 3. EX POST PREFERENCES: OPINIONS

The class of opinions  $\mathcal{P}$  is characterized by a set of conditions. The first five axioms are weak assumptions. They describe each ex post preference and are satisfied by most models of decision under uncertainty.

**AXIOM A1 (Weak Order).** *The binary relation  $\succsim^*$  on  $\mathcal{F}$  is complete and transitive.*

**AXIOM A2 (C-Continuity).** *For all  $f \in \mathcal{F}$ , the sets  $\{p \in \Delta(X) : p \succsim^* f\}$  and  $\{p \in \Delta(X) : f \succsim^* p\}$  are closed.*

**AXIOM A3 (Monotonicity).** *For all  $f, g \in \mathcal{F}$ , if  $f(s) \succsim^* g(s)$  for all  $s \in S$ , then  $f \succsim^* g$ .*

**AXIOM A4 (Risk Independence).** *For all  $p, q, r \in \Delta(X)$  and  $\lambda \in (0, 1)$ , if  $p \succsim^* q$ , then  $\lambda p + (1 - \lambda)r \succsim^* \lambda q + (1 - \lambda)r$ .*

AXIOM A5 (Nondegeneracy). *The relation  $\succ^*$  is nonempty.*<sup>11</sup>

The meaning of Axioms A1–A5 has been widely discussed in the literature. The next proposition, a version of which appears in Ghirardato and Siniscalchi (2010) and Cerreia-Vioglio et al. (2011), is well known and characterizes an ex post preference satisfying those properties.

PROPOSITION 1. *The following statements are equivalent.*

- (a) *The binary relation  $\succ^*$  satisfies Axioms A1–A5.*
- (b) *There exist  $u \in C_b(\Delta(X))$  nonconstant and affine, and  $I : u(\Delta(X))^S \rightarrow \mathbb{R}$  monotone, normalized, and such that  $f \mapsto I(u(f))$  represents  $\succ^*$ .*<sup>12</sup>

The next two axioms specify how the elements of  $\mathcal{P}$  are related. The first one is the easiest to state and interpret.

AXIOM A6 (Weak Agreement). *For all  $\succ_1^*, \succ_2^* \in \mathcal{P}$  and  $p, q \in \Delta(X)$ , if  $p \succ_1^* q$ , then  $p \succ_2^* q$ .*

Weak Agreement says that ex post preferences share the same “tastes” or ranking of risky prospects. Ultimately, differences in opinions are because of different perceptions of ambiguity or different attitudes toward it. Preferences over Anscombe–Aumann acts sharing the same tastes play an important role in this paper. A class  $\mathcal{Q} \subseteq 2^{\mathcal{F} \times \mathcal{F}} \setminus \{\emptyset\}$  is called *weakly compatible* when its members satisfy the condition of Axiom A6.

To formulate the next axiom we need some terminology. Fix any ex post preference  $\succ_0^*$  and let  $(f_n), (g_n) \in \mathcal{F}^\infty$ . Say that  $(f_n)$  and  $(g_n)$  are *eventually close* when, if  $p, q \in \Delta(X)$  satisfy  $p \succ_0^* q$ , then there exists  $N \in \mathbb{N}$  such that

$$\frac{1}{2}f_n(s) + \frac{1}{2}p \succ_0^* \frac{1}{2}g_n(s) + \frac{1}{2}q \quad \text{and} \quad \frac{1}{2}g_n(s) + \frac{1}{2}p \succ_0^* \frac{1}{2}f_n(s) + \frac{1}{2}q \quad (4)$$

for all  $s \in S$  and  $n \geq N$ . In view of Axiom A6, the property that two sequences of acts are eventually close does not depend on the choice of the ex post preference  $\succ_0^*$ . Therefore, to save on notation, we assume that the definition applies in the context of a weakly compatible class of ex post preferences and omit any reference to the element of  $\mathcal{P}$  that is being considered.<sup>13</sup>

For intuition, consider first two acts  $f$  and  $g$ . Suppose that the lottery  $p$  is strictly preferred to the lottery  $q$ . Assume further that for each  $s \in S$ , the ranking is preserved in the mixture with equal weights of  $f(s)$  with  $p$  and  $g(s)$  with  $q$ , or  $g(s)$  with  $p$  and  $f(s)$  with  $q$ . In this case, it is reasonable to say that, given  $p$  and  $q$ , the acts  $f$  and  $g$  are

<sup>11</sup>The symbol  $\succ^*$  stands for the asymmetric part of  $\succ^*$ .

<sup>12</sup>The function  $I$  is monotone if  $I(u(h_1)) \geq I(u(h_2))$  whenever  $u(h_1(s)) \geq u(h_2(s))$  for all  $s \in S$ . It is normalized if  $I(u(h)) = u(p)$  whenever  $h(S) = \{p\}$  for some  $p \in \Delta(X)$ .

<sup>13</sup>Ghirardato and Siniscalchi (2010) suggest a notion of convergence of acts so as to characterize uniform convergence in the set  $\{u \circ f : f \in \mathcal{F}\}$ . The sequence of acts  $(f_n)$  and the constant sequence of acts  $(f)$  are eventually close if and only if  $(f_n)$  converges to  $f$  in the sense of Ghirardato and Siniscalchi (2010). This kind of convergence alone is sufficient to describe continuity in the space of utility acts, but it is not broad enough for the notion of continuity needed in this paper for the class  $\mathcal{P}$ .



“close” to each other, as the prizes they deliver in each state are sufficiently similar in terms of taste so as not to break the original ranking between those two lotteries. The sequences of acts  $(f_n)$  and  $(g_n)$  are eventually close when, for any given pair of strictly ranked lotteries  $p$  and  $q$ , the acts  $f_n$  and  $g_n$  are close given  $p$  and  $q$  for every  $n$  sufficiently large.

We denote by  $r_{f, \succ^*}$  any element of  $\Delta(X)$  such that  $r_{f, \succ^*} \sim^* f$ ; that is,  $r_{f, \succ^*}$  is a certainty equivalent of the act  $f$  according to the binary relation  $\succ^*$ . A collection of certainty equivalents  $\{r_{f, \succ^*} : (f, \succ^*) \in \mathcal{F} \times \mathcal{P}\} \subseteq \Delta(X)$  exists when the elements of  $\mathcal{P}$  satisfy Axioms A1–A3. We use this notation to state our next continuity condition. It says that any two sequences of certainty equivalents are eventually close whenever the two sequences of acts generating them are eventually close.

**AXIOM A7 (Uniform Equicontinuity).** *For all  $(\succ_n^*) \in \mathcal{P}^\infty$  and  $(f_n), (g_n) \in \mathcal{F}^\infty$ , if  $(f_n)$  and  $(g_n)$  are eventually close, then  $(r_{f_n, \succ_n^*})$  and  $(r_{g_n, \succ_n^*})$  are eventually close.*

While Axioms A1–A5 are satisfied by the most important classes of preferences in the literature, the Uniform Equicontinuity axiom is also satisfied by their weakly compatible subclasses. Given this observation, we say that the class of ex post preferences is *admissible* if each of its elements satisfies Axioms A1–A5, and the class as a whole satisfies Axioms A6 and A7.

The next proposition characterizes any class of admissible preferences over Anscombe–Aumann acts.

**PROPOSITION 2.** *The class  $\mathcal{P}$  is admissible if and only if there exist  $u \in C_b(\Delta(X))$  nonconstant and affine, and a uniformly equicontinuous family  $\mathcal{I} \subseteq C_b(u(\Delta(X)))^S$  of monotone and normalized functions such that  $\{f \mapsto I(u(f)) : I \in \mathcal{I}\}$  is the (unique) set of utility representations of the preferences in  $\mathcal{P}$  that coincide with  $u$  in the set of constant acts.*

Given the Bernoulli utility index  $u$ , the set  $\mathcal{I}$  is unique. Let  $\mathcal{I}(u)$  denote the set of utility functions on the space of utility acts associated with  $u$ . Define the function  $\theta_u : \mathcal{P} \rightarrow \mathcal{I}(u)$ , which associates each  $\succ^* \in \mathcal{P}$  with a unique  $I \in \mathcal{I}(u)$  such that, for all  $f, g \in \mathcal{F}$ ,  $f \succ^* g$  if and only if  $I(u(f)) \geq I(u(g))$ . The family of functions  $\theta_u$ , where  $u$  ranges over all affine representations of  $\mathcal{P}|_{\Delta(X)}$  belonging to  $C_b(\Delta(X))$ , is clearly nonempty. We endow the set  $\mathcal{P}$  with the weakest topology  $\tau$  that makes that family continuous. In fact, if we fix any  $u \in C_b(\Delta(X))$  that is an affine representation of  $\mathcal{P}|_{\Delta(X)}$ , it then follows from cardinal uniqueness and the properties of the supremum norm (denoted by  $\|\cdot\|_\infty$ ) that  $\tau$  is the weakest topology that makes  $\theta_u$  continuous. This topology is induced by the metric  $d_{\theta_u} : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$  as defined by  $d_{\theta_u}(\succ_1^*, \succ_2^*) = \sup_{f \in \mathcal{F}} |\theta_u(\succ_1^*)(u(f)) - \theta_u(\succ_2^*)(u(f))|$ .

The uniformly equicontinuous set  $\mathcal{I}^*(u) := \text{cl}_{\|\cdot\|_\infty}(\mathcal{I}(u))$  (the  $\|\cdot\|_\infty$ -closure of  $\mathcal{I}(u)$ ) induces a family  $\mathcal{P}^*$  of ex post preferences that satisfies Axioms A1–A7. We identify each element of the admissible class  $\mathcal{P}^*$  with a unique element of  $\mathcal{I}^*(u)$ . Let the function  $\theta_u^* : \mathcal{P}^* \rightarrow \mathcal{I}^*(u)$  provide such an identification. The set  $\mathcal{P}^*$  has a weakest topology  $\tau^*$  that makes  $\theta_u^*$  continuous, and it is metrized by  $d_{\theta_u^*} : \mathcal{P}^* \times \mathcal{P}^* \rightarrow \mathbb{R}$ , with  $d_{\theta_u^*}(\succ_1^*, \succ_2^*) = \sup_{f \in \mathcal{F}} |\theta_u^*(\succ_1^*)(u(f)) - \theta_u^*(\succ_2^*)(u(f))|$ . Note that  $d_{\theta_u} = d_{\theta_u^*}|_{\mathcal{P} \times \mathcal{P}}$  ( $\tau$  is the relative topology induced by  $\tau^*$  on  $\mathcal{P}$ ) and  $\mathcal{P}^* = \text{cl}_{\tau^*}(\mathcal{P})$ . The following proposition completes the characterization of this enlarged class of opinions.

PROPOSITION 3. *The sets  $\mathcal{P}^*$  and  $\mathcal{I}^*(u)$  are compact metric spaces.*

In view of Proposition 2, the families  $\mathcal{P}$  and  $\mathcal{I}(u)$  can be considered to be primitives of the model. Mathematically, they induce the extended families  $\mathcal{P}^*$  and  $\mathcal{I}^*(u)$  by the closure operator with respect to a suitable topology. For intuition, assume that constant acts are monetary prizes. Let us consider for simplicity only the family  $\mathcal{P}^*$  (the intuition for  $\mathcal{I}^*(u)$  is similar). There exists a natural distance between a pair of monetary prizes: the absolute value of their difference. Recall that  $r_{f, \succ^*}$  denotes a certainty equivalent of the act  $f$  according to the ex post preference  $\succ^*$ . The topology of  $\mathcal{P}^*$  renders two ex post preferences  $\succ_1^*$  and  $\succ_2^*$  close when the set  $\{\text{distance between } r_{f, \succ_1^*} \text{ and } r_{f, \succ_2^*} : f \in \mathcal{F}\}$  is bounded above by a “small” number. Whenever some preference relation  $\succ_\infty^*$  over Anscombe–Aumann acts belongs to  $\mathcal{P}^*$ , there exists a sequence  $(\succ_n^*)$  of ex post preferences in  $\mathcal{P}$  getting arbitrarily close to it. The elements of that sequence are becoming indistinguishable from  $\succ_\infty^*$ . Because any point in  $\mathcal{P}^* \setminus \mathcal{P}$  is arbitrarily close to elements of  $\mathcal{P}$ , it is convenient to work with  $\mathcal{P}^*$ . Clearly any finite class  $\mathcal{P}$  of admissible preferences satisfies  $\mathcal{P} = \mathcal{P}^*$ , and the importance of defining  $\mathcal{P}^*$  is that we work with a not necessarily finite class of opinions.

We now mention a few examples of ex post preferences. First, any weakly compatible subclass of the variational preferences (VP) of Maccheroni et al. (2006) is admissible.<sup>14</sup> Also note that if  $I_{VP} \in C_b(u(\Delta(X))^S)$  is the functional representing the ordering on the set of utility acts  $u(\Delta(X))^S$  induced by some variational preference over  $\mathcal{F}$ , then  $I_{VP}$  is 1-Lipschitz continuous. In fact, the limit of any convergent sequence of such functionals also induces a variational preference over  $\mathcal{F}$ . Therefore, any maximal (with respect to set inclusion) and weakly compatible subclass  $\mathcal{P}_{VP}$  of variational preferences satisfies  $\mathcal{P}_{VP} = \mathcal{P}_{VP}^*$ . As a second example, one can also show that any weakly compatible subclass of  $\alpha$ -MEU preferences is admissible. Note that there is nothing particularly special about the ambiguity attitude. For instance, the union of any finite subclass of variational preferences with any finite subclass of  $\alpha$ -MEU preferences is admissible and closed when they share the same ranking over risky prospects.

Finally, as a technical requirement for the results in this paper, we state our last axiom. It applies to each element of the class of opinions, and posits the existence of a best and a worst lottery.

AXIOM A8 (Best–Worst). *If  $\succ^* \in \mathcal{P}$ , then there exist  $p^*, p_* \in \Delta(X)$  such that  $p^* \succ^* p \succ^* p_*$  for all  $p \in \Delta(X)$ .*

An admissible class  $\mathcal{P}$  that satisfies Axiom A8 also fulfills the stronger property that there exist  $p^*, p_* \in \Delta(X)$  such that  $p^* \succ^* f \succ^* p_*$  for all  $(f, \succ^*) \in \mathcal{F} \times \mathcal{P}$ . In the presence of Axioms A1–A7, the Best–Worst axiom is automatically satisfied when the set  $X$  is compact metric. Without compactness of  $X$ , the role of Axiom A8 is to allow us to prove Proposition 4 in the next section. We make no explicit use of Best–Worst anywhere else in this paper. Therefore, to some extent our last axiom is admittedly “excess baggage.”

<sup>14</sup>Certain subclasses of Chateauneuf and Faro’s (2009) preferences are also admissible.

It remains an open question whether we can obtain a version of [Proposition 4](#) without [Axiom A8](#) and under the assumption that  $X$  is a separable metric space.<sup>15</sup>

#### 4. THE EX ANTE PREFERENCE

In this section we characterize a DM who satisfies the expected utility axioms. The key axiom is [Axiom B3](#) (Independence), whose implications we discuss in [Section 5](#). We also present two additional axioms that specify how the ex ante preference relates to the class of opinions, and are also used later in this paper in the characterization of a more general model.

Our results in this section build on the version of Harsanyi's theorem given by [Zhou \(1997\)](#). They show that the DM conceives of a single probability measure on the set of utility functions of the experts, and aggregates them with a generalized utilitarian criterion. The main representation is [Theorem 1](#), which can be also viewed as an abstract version of the second-order subjective expected utility (SOSEU) model of [Seo \(2009\)](#). In this sense we replace a maximal and weakly compatible subclass of SEU preferences used in [Seo \(2009\)](#) by an admissible class of preferences over Anscombe–Aumann acts. This reformulation allows us to interpret [Segal's \(1987\)](#) two-stage approach to ambiguity as the ex ante aggregation of a collection of second-stage selves' preferences.

##### 4.1 Utilitarian aggregation

The first three properties of the ex ante preference are standard expected utility assumptions.

**AXIOM B1 (Weak Order).** *The binary relation  $\succ$  on  $\Delta(\mathcal{F})$  is complete and transitive.*

**AXIOM B2 (Continuity).** *If  $(P_n), (Q_n) \in \Delta(\mathcal{F})^\infty$  are such that  $P_n \succ Q_n$  for all  $n \in \mathbb{N}$ ,  $P_n \rightarrow P \in \Delta(\mathcal{F})$ , and  $Q_n \rightarrow Q \in \Delta(\mathcal{F})$ , then  $P \succ Q$ .*

**AXIOM B3 (Independence).** *For all  $P, Q, R \in \Delta(\mathcal{F})$  and  $\lambda \in (0, 1)$ , if  $P \succ Q$ , then  $\lambda P + (1 - \lambda)R \succ \lambda Q + (1 - \lambda)R$ .*

The next two axioms provide a connection between the binary relation  $\succ$  and the class  $\mathcal{P}$ . They are related by two intuitive conditions: one is the requirement that the ranking of constant acts be the same ([Axiom B4](#)) and the other is a consistency condition ([Axiom B5](#)).

**AXIOM B4 (C-Agreement).** *For all  $p, q \in \Delta(X)$  and  $\succ^* \in \mathcal{P}$ ,  $\delta_p \succ \delta_q$  if and only if  $p \succ^* q$ .*

C-Agreement says that the ex ante and the ex post preferences coincide on the set of constant acts. This is a mild requirement when the elements of  $\mathcal{P}$  share the same tastes.

<sup>15</sup>[Axiom A8](#) can be replaced with the assumption that  $\mathcal{P}$  satisfies the property described in [Proposition 4](#). One example is a weakly compatible collection of SEU preferences.

We now turn to the consistency condition. Intuitively, it stands for a Pareto principle: when all experts agree on the ranking of two acts, the DM follows their recommendation. Because the ex ante preference relation is defined on the set of random acts, one has to extend the notion of consistency to such a domain. The extension requires two additional objects, which we now define.

First, define a function that selects certainty equivalents. Let  $\mathcal{Q} \subseteq 2^{\mathcal{F} \times \mathcal{F}} \setminus \{\emptyset\}$ . The mapping  $\pi: \mathcal{F} \times \mathcal{Q} \rightarrow \Delta(X)$  is a *certainty equivalent selection* when, for all  $(f, \succ^*) \in \mathcal{F} \times \mathcal{Q}$ ,  $\pi(f, \succ^*) \sim^* f$ . In light of the discussion in Section 3, the existence of such a selection is not a problem. The difficulty lies in showing that it satisfies desirable properties such as measurability in its first argument and continuity in its second argument. In fact, as the next proposition demonstrates, a continuous certainty equivalent selection always exists for the class  $\mathcal{P}^*$  (and consequently for  $\mathcal{P}$  as well) when **Axiom A8** holds. The restriction to  $\mathcal{F} \times \mathcal{P}$  of the certainty equivalent selection obtained in the proposition below is denoted by  $\pi$ .

**PROPOSITION 4.** *If **Axiom A8** is satisfied, then there exists a continuous certainty equivalent selection  $\pi^*$  from  $\mathcal{F} \times \mathcal{P}^*$  to  $\Delta(X)$ .*

Second, given a certainty equivalent selection  $\pi: \mathcal{F} \times \mathcal{P} \rightarrow \Delta(X)$  with the property that the function  $f \mapsto \pi(f, \succ^*)$  is measurable for each ex post preference  $\succ^*$ , we define a *two-stage lottery reduction* as a mapping  $\Psi_\pi: \Delta(\mathcal{F}) \times \mathcal{P} \rightarrow \Delta(\Delta(X))$  satisfying

$$\Psi_\pi(P, \succ^*)(B) = P(\{f \in \mathcal{F}: \pi(f, \succ^*) \in B\}) \quad (5)$$

for all  $B \in \mathcal{B}(\Delta(X))$ . Intuitively, it extends the notion of certainty equivalent to incorporate lotteries of acts that are ranked by an expected utility functional. The definition is based on a similar concept introduced by **Seo (2009)**. He defines a two-stage lottery reduction for the special case in which  $\mathcal{P}$  is a maximal and weakly compatible subclass of SEU preferences. We extend his idea to formulate our consistency condition and show how it relates to a Pareto principle.

We now state the consistency condition.

**AXIOM B5 (Consistency).** *For all  $P, Q \in \Delta(\mathcal{F})$ , if  $\Psi_\pi(P, \succ^*) \succ \Psi_\pi(Q, \succ^*)$  for all  $\succ^* \in \mathcal{P}$ , then  $P \succ Q$ .*

In the presence of **Axiom B4**, Consistency can be interpreted as a unanimity condition. Consider for simplicity two acts  $f$  and  $g$ . The condition  $\Psi_\pi(f, \succ^*) \succ \Psi_\pi(g, \succ^*)$  expresses an ex ante preference for  $\pi(f, \succ^*)$  over  $\pi(g, \succ^*)$ . Given the C-Agreement axiom, this is equivalent to  $f \succ^* g$ . Therefore, the Consistency axiom reflects the intuitive notion that whenever each expert says that  $f$  is preferred to  $g$ , the DM also prefers  $f$  to  $g$ .

The innovation of **Axiom B5** is to extend such a notion to the domain  $\Delta(\mathcal{F})$ . The DM collapses each act in the support of  $P$  and  $Q$  to its certainty equivalent according to the ex post preference  $\succ^*$ , and then compares the resulting pair of two-stage lotteries. More concretely, the DM extends each ex post preference (originally defined on  $\mathcal{F}$ ) to the domain of random acts using expected utility. **Axiom B4** plays an important

role here because it makes the restriction of each ex post preference to  $\Delta(X)$  ordinarily equivalent to  $\succsim_{|\Delta(X)}$ . When  $u \in C_b(\Delta(X))$  is an affine representation of  $\mathcal{P}_{|\Delta(X)}$ , the DM assumes that an expert with preference  $\succsim^*$  would evaluate the lottery of acts  $P$  according to  $\int_{\mathcal{F}} \phi(u(\pi(f, \succsim^*))) dP(f)$  for some strictly increasing  $\phi \in C_b(u(\Delta(X)))$ . It is as if the DM attaches a vN–M utility

$$\omega(\cdot, \succsim^*) := \phi(u(\pi(\cdot, \succsim^*))) \tag{6}$$

to each opinion  $\succsim^*$  and extends its domain to  $\Delta(\mathcal{F})$  with the map  $P \mapsto \int_{\mathcal{F}} \omega(f, \succsim^*) dP(f)$ . This extension is implicitly embedded in **Axiom B5**. By letting  $P \succeq^* Q$  if and only if  $\Psi_{\pi}(P, \succsim^*) \succsim \Psi_{\pi}(Q, \succsim^*)$ , it is clear that  $\succeq^*$  is a continuous and complete preorder that satisfies the independence axiom and coincides (because of **Axiom B4**) with  $\succsim^*$  on  $\mathcal{F}$ . The binary relation  $\succeq^*$  is, in fact, the DM’s inference of how an expert with preference  $\succsim^*$  would rank lotteries of acts. The DM effectively aggregates the collection of all such  $\succeq^*$ . More precisely, if we denote by  $\text{EU}(\mathcal{P})$  the collection of those expected utility preferences, then Consistency can be rephrased as

$$\text{if } P \succeq^* Q \text{ for all } \succeq^* \in \text{EU}(\mathcal{P}), \text{ then } P \succsim Q. \tag{7}$$

This is a Pareto condition that also appears in **Harsanyi (1955)**.

The next theorem is the main result of this section.

**THEOREM 1.** *Let  $\mathcal{P}$  be an admissible class of ex post preferences satisfying **Axiom A8**. The following statements are equivalent.*

- (a) *The binary relation  $\succsim$  satisfies **Axioms B1–B5**.*
- (b) *There exist  $u \in C_b(\Delta(X))$  nonconstant and affine,  $\phi \in C_b(u(\Delta(X)))$  strictly increasing, and a (Borel) probability measure  $m$  on  $\mathcal{P}^*$  such that  $u$  represents the restriction of each  $\succsim^* \in \mathcal{P}^*$  to  $\Delta(X)$ , and for all  $P, Q \in \Delta(\mathcal{F})$ ,  $P \succsim Q$  if and only if*

$$\int_{\mathcal{P}^*} \left[ \int_{\mathcal{F}} \phi(u(\pi^*(f, \succsim^*))) dP(f) \right] dm(\succsim^*) \geq \int_{\mathcal{P}^*} \left[ \int_{\mathcal{F}} \phi(u(\pi^*(f, \succsim^*))) dQ(f) \right] dm(\succsim^*). \tag{8}$$

*In particular, for all  $f, g \in \mathcal{F}$ ,  $f \succsim g$  if and only if*

$$\int_{\mathcal{P}^*} \phi(u(\pi^*(f, \succsim^*))) dm(\succsim^*) \geq \int_{\mathcal{P}^*} \phi(u(\pi^*(g, \succsim^*))) dm(\succsim^*). \tag{9}$$

Regarding the interpretation of **Theorem 1**, the condition in (9) can be viewed as a generalized utilitarian criterion. Consider for simplicity the case in which  $\mathcal{P}$  is a finite set and for each  $\succsim_i^* \in \mathcal{P}$ , there exists a representation  $I_i \in C_b(u(\Delta(X)))^S$  in the induced domain of utility acts. Because (given  $u$ ) one can identify  $\mathcal{P}$  with the set  $\{I_1, \dots, I_{|\mathcal{P}|}\}$ , the criterion in (9) becomes  $\sum_{i=1}^{|\mathcal{P}|} m(i) \phi(I_i(u(f)))$ . The DM chooses among acts as if maximizing a weighted average of a strictly increasing transformation of the utility functions of the experts.

A similar interpretation applies to (8), which also shows that the DM extends each opinion to the larger domain  $\Delta(\mathcal{F})$ , and aggregates the utility representations of the extended class  $\text{EU}(\mathcal{P})$ . For finitely many opinions, the criterion in (8) is the linear aggregator of a finite collection of expected utility functionals and is given by  $\sum_{i=1}^{|\mathcal{P}|} m(i) [\int_{\mathcal{F}} \phi(I_i(u(f))) dP(f)]$ . This representation with finitely many experts is a consequence of Harsanyi’s aggregation theorem (see, for example, Domotor 1979). When the collection  $\mathcal{P}$  is not finite, one needs the more general results of Zhou (1997), which are used to prove Theorem 1.

Finally, note that the lack of reduction of compound lotteries (ROCL) introduces the distortion  $\phi$ . A standard utilitarian aggregator obtains when the function  $\phi$  is affine. As is well known, this condition is equivalent to the following axiom.

AXIOM B6 (ROCL). For all  $p, q \in \Delta(X)$  and  $\lambda \in (0, 1)$ ,  $\delta_{\lambda p + (1-\lambda)q} \sim \lambda \delta_p + (1 - \lambda) \delta_q$ .

An ex ante preference relation that satisfies ROCL and the axioms of part (a) of Theorem 1 can be represented by the aggregative criterion in (9) with  $\phi$  replaced by the identity map. In the case of a finite set of opinions, this amounts to  $f \mapsto \sum_{i=1}^{|\mathcal{P}|} m(i) I_i(u(f))$  in the subdomain of acts. One can interpret this functional form as if the DM associates a certainty equivalent with the act  $f$  that is a weighted average of the certainty equivalents according to each expert.<sup>16</sup> In general, we have the following characterization of an ex ante preference relation satisfying ROCL.

PROPOSITION 5. Let  $\mathcal{P}$  be an admissible class of ex post preferences satisfying Axiom A8. The binary relation  $\succsim$  satisfies Axioms B1–B6 if and only if there exist  $u \in C_b(\Delta(X))$  non-constant and affine, and a uniformly continuous, monotone, and normalized function  $I_o : u(\Delta(X))^S \rightarrow \mathbb{R}$  such that for all  $P, Q \in \Delta(\mathcal{F})$ ,

$$P \succsim Q \text{ if and only if } \int_{\mathcal{F}} I_o(u(f)) dP(f) \geq \int_{\mathcal{F}} I_o(u(f)) dQ(f). \tag{10}$$

Moreover, if  $(I_o, u)$  and  $(J_o, v)$  are two representations in the sense of (10), then there exists  $(\alpha, \beta) \in \mathbb{R}_{++} \times \mathbb{R}$  such that  $u = \alpha v + \beta$  and  $I_o(u(f)) = \alpha J_o(\alpha^{-1}(u(f) - \beta \mathbf{1}_S)) + \beta$  for all  $f \in \mathcal{F}$ .

Two remarks are in order. First, given the measure  $m$  obtained in Theorem 1, the construction of the function  $I_o$  is straightforward:

$$I_o(u(f)) = \int_{\mathcal{P}^*} u(\pi^*(f, \succsim^*)) dm(\succsim^*) \tag{11}$$

$$= \int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}(I), \tag{12}$$

where  $\tilde{m} \in \Delta(\mathcal{I}^*(u))$  is defined as

$$\tilde{m}(B) = m(\theta_u^{*-1}(B)) \text{ for all } B \in \mathcal{B}(\mathcal{I}^*(u));$$

<sup>16</sup>In a related setting, Chambers and Echenique (2010) study the properties of the sum of certainty equivalents as an aggregative criterion.

that is,  $I_o$  is a utilitarian aggregator. Also note that one could have written the representation (10) in terms of ex post preferences alone using (11). A similar observation applies to the results with ROCL in Section 5.

Second, Proposition 5 implies a version of Corollary 5.2 of Seo (2009). Say that the binary relation  $\succsim$  has a variational representation if its vN–M utility satisfies the properties of the utility function described in Maccheroni et al. (2006). When  $\mathcal{P}$  is a maximal and weakly compatible subclass of variational preferences, one implication of Proposition 5 is that the ex ante preference has a variational representation if and only if it satisfies Axioms B1–B6. A similar statement is valid when one replaces “variational” by maxmin expected utility or SEU (the latter as in Corollary 5.2 of Seo 2009).

#### 4.2 Segal’s two-stage approach to ambiguity

When the set  $\mathcal{P}$  is a maximal and weakly compatible subclass of SEU preferences, Theorem 1 characterizes the SOSEU model of Seo (2009). He provides an alternative axiomatization of the smooth ambiguity model of decision in Klibanoff et al. (2005). Both models are based on Segal’s (1987) explanation of the Ellsberg paradox. Segal’s approach consists of a two-stage mechanism that can be described as follows. The DM does not know the odds of the states of the world, but believes that there is a true probability law. Assume for simplicity that there are finitely many possible priors  $\mu_1, \dots, \mu_n \in \Delta(S)$ . Ambiguity gives rise to a second-order belief  $m$ , which the DM conceives of in the first stage. At the end of this stage, with probability  $m(i)$ , the DM selects a prior  $\mu_i$ . If  $\mu$  is the selected prior, in the second stage, the uncertain payoff of an act  $f$  is evaluated according to  $\int u(f) d\mu$ ; that is, the DM behaves as if maximizing an expected utility function in the second stage according to some prior. The DM chooses among acts in the first stage and uses the criterion  $f \mapsto \sum_{i=1}^n m(i) \phi(\int u(f) d\mu_i)$ , which aggregates the collection of those expected utility functions. As is well known, this utility function can explain the Ellsbergian choices when the function  $\phi$  is concave.<sup>17</sup>

This two-stage mechanism can be thought of as one in which the DM contemplates a collection of second-stage selves. They correspond to different rankings on the set of acts that the DM may use in the second stage to evaluate a vector of uncertain payoffs. The entire collection of selves induces a maximal and weakly compatible subclass of SEU preferences, which we identify with the simplex  $\Delta(S)$ . In the SOSEU model of Seo (2009), the DM chooses among lotteries of acts by aggregating that collection of rankings according to the utility function  $P \mapsto \int_{\Delta(S)} [\int_{\mathcal{F}} \phi(\int u(f) d\mu) dP(f)] dm(\mu)$ . The innovation of our model is to allow second-stage selves to have preferences other than SEU. In other words, we replace the set  $\Delta(S)$  with an admissible class of ex post preferences  $\mathcal{P}$ . The DM is not necessarily an expected utility maximizer in the second stage. In fact, we do not place any constraint on the perception or the attitude toward ambiguity.

The main axioms in Seo (2009) overlap with our Axioms B1, B2, and B3. But the connection between this paper and the SOSEU model is somewhat obscured by Seo’s dominance axiom. Its main purpose is to bring the class of opinions into the model

<sup>17</sup>This explanation requires nonreduction of compound lotteries. Segal (1990) offers some support for this hypothesis; see also Halevy (2007).

without explicitly mentioning it. This is largely accomplished by defining the map  $\Psi_\pi$ , as in (5), with each  $\succ^* \in \mathcal{P}$  identified with one element of the set of all priors,  $\Delta(S)$ . With a slight abuse of notation, the dominance axiom is equivalent to Consistency when  $\pi(f, \mu) = \sum_{s \in S} \mu(s)f(s)$ . We also note that even though Seo (2009) does not explicitly model the class  $\mathcal{P}$ , it can be recovered from the condition

$$f \succ^* g \quad \text{if and only if} \quad \delta_{\pi(f, \mu)} \succ \delta_{\pi(g, \mu)}. \quad (13)$$

An immediate consequence is that Axiom B4 is satisfied when each  $\succ^*$  is defined by (13). To ensure that  $\succ^*$  is indeed an SEU preference (with prior  $\mu$ ), he adds one additional axiom called second-stage independence. It says that independence holds in the subdomain of lotteries in the second stage: if  $\delta_p \succ \delta_q$ , then  $\delta_{\lambda p + (1-\lambda)r} \succ \delta_{\lambda q + (1-\lambda)r}$ . One can easily verify that  $\succ^*$ , as induced by (13), is a well defined SEU preference if and only if second-stage independence holds. In view of (13), the set of induced ex post preferences is a maximal and weakly compatible subclass of SEU preferences.

Therefore, the dominance axiom provides a Pareto condition as in (7). The extensions of the SEU preferences to  $\Delta(\mathcal{F})$  (using the vN–M utility  $\omega(\cdot, \mu)$  in (6)) are the opinions that the DM effectively aggregates. This is exactly the aggregation problem posed by Harsanyi (1955). It shows that both our Theorem 1 and the main representation of Seo (2009) are corollaries of Theorem 2' of Zhou (1997). As a consequence, Segal's two-stage approach to ambiguity can be interpreted as a variation of Harsanyi's utilitarian theorem.

## 5. NONINDIFFERENCE TO RANDOMIZATION

Theorem 1 and the results derived from it rely on the Independence axiom. This postulate rules out a preference for less dispersed opinions. To illustrate, consider two experts with preferences  $\succ_1^*$  and  $\succ_2^*$ , and two acts  $f$  and  $g$ . Expert 1 is indifferent between  $f$  and \$10, and between  $g$  and \$0. Expert 2 has the opposite opinion and is indifferent between  $f$  and \$0, and between  $g$  and \$10. In the face of disagreement, the DM may be indifferent between the acts  $f$  and  $g$ . Assuming for simplicity that ROCL is satisfied, the expected payoff of the even chance lottery  $\frac{1}{2}\delta_f + \frac{1}{2}\delta_g$  is \$5 according to each of the extended preferences  $\succeq_1^*$  and  $\succeq_2^*$  in  $\text{EU}(\mathcal{P})$ . This lottery can be seen as an ex ante compromise between the acts  $f$  and  $g$ . Randomization reduces the dispersion of opinions in an ex ante stage and hedges the epistemic uncertainty about which expert is right. The DM may prefer that even chance lottery to the acts  $f$  and  $g$  so as to hedge against the second-order uncertainty about which expert is correct; that is, because opinions are less dispersed when ranking the even chance lottery, it is reasonable to assume that  $\frac{1}{2}\delta_f + \frac{1}{2}\delta_g \succ f \sim g$ . The Independence axiom requires that  $\frac{1}{2}\delta_f + \frac{1}{2}\delta_g$  be indifferent to  $f$  and  $g$ , and randomization has no hedge value.

Ex post preferences are defined on the domain of acts, but the DM aggregates ex ante perceptions (the elements of  $\text{EU}(\mathcal{P})$ ) of such preferences. The example above shows that this feature of our model introduces a second layer of randomization that allows



the DM to make opinions less dispersed by mixing lotteries of acts.<sup>18</sup> In this section, we modify the expected utility axioms to accommodate preference for randomization. We weaken the Independence axiom and impose a condition that reflects the DM's aversion to dispersed opinions. The main representation is [Theorem 2](#). Slight variations of the set of axioms allow us to obtain the aggregation rules of [Crès et al. \(2010\)](#) and [Gilboa et al. \(2010\)](#) as special cases ([Propositions 7 and 8](#), respectively).

### 5.1 Axioms and representation result

We replace Independence with two axioms that are now described. The first one is a weaker version of Independence and is consistent with nonindifference to randomization. It is similar in spirit to the weak certainty independence axiom of [Maccheroni et al. \(2006\)](#).

**AXIOM B3\* (Weak Independence).** *For all  $P, Q \in \Delta(\mathcal{F})$ ,  $R_{1c}, R_{2c} \in \Delta(\mathcal{F}_c)$ , and  $\lambda \in (0, 1)$ , if  $\lambda P + (1 - \lambda)R_{1c} \succcurlyeq \lambda Q + (1 - \lambda)R_{1c}$ , then  $\lambda P + (1 - \lambda)R_{2c} \succcurlyeq \lambda Q + (1 - \lambda)R_{2c}$ .*

To interpret Weak Independence, consider two acts  $f$  and  $g$ , and two risky prospects  $p$  and  $q$ . Assume that the DM prefers  $\lambda\delta_f + (1 - \lambda)\delta_p$  to  $\lambda\delta_g + (1 - \lambda)\delta_p$ . Weak Independence requires that  $\lambda\delta_f + (1 - \lambda)\delta_q$  be preferred to  $\lambda\delta_g + (1 - \lambda)\delta_q$ . It represents invariance to unanimity: because experts agree on the ranking of constant acts, the original ranking between those lotteries of acts does not change when we replace  $p$  with  $q$ . The axiom extends this condition when lotteries of acts  $P$  and  $Q$  take the place of  $f$  and  $g$ , and the lotteries of constant acts  $R_{1c}$  and  $R_{2c}$  are used in place of  $p$  and  $q$ . The same idea applies here, since the elements of the extended class of opinions  $\text{EU}(\mathcal{P})$  agree on the ranking of lotteries of constant acts.

The second axiom is presented next.

**AXIOM B7 (Convexity).** *For all  $P, Q \in \Delta(\mathcal{F})$  and  $\lambda \in (0, 1)$ , if  $P \sim Q$ , then  $\lambda P + (1 - \lambda)Q \succcurlyeq Q$ .*

One can view Convexity as aversion to the dispersion of opinions. It is similar in spirit to the uncertainty aversion axiom of [Schmeidler \(1989\)](#) and captures the idea that the DM wants to hedge against the second-order uncertainty about which expert is right. In our example above, the DM would be indifferent between  $f$  and \$10, and between  $g$  and \$0 if expert 1 is right, but also inclined to reverse that ranking if expert 2 is right. In the face of disagreement, the ranking  $f \sim g$  appears justified. From an ex ante perspective, the lottery  $\frac{1}{2}\delta_f + \frac{1}{2}\delta_g$  always pays \$5 independently of which expert has access to the “truth” and, therefore, hedges against the epistemic uncertainty.<sup>19</sup> Convexity requires

<sup>18</sup>Mixing lotteries of acts also has hedge value in the standard models of ambiguity aversion with ROCL. They can be mapped into our framework with a maximal and weakly compatible subclass of ex post SEU preferences. This connection is discussed in [Nascimento and Riella \(forthcoming\)](#).

<sup>19</sup>Ideally, one would like to find an act  $h$  that is a compromise between  $f$  and  $g$ . One special case in which such an act exists is that in which ROCL holds and the set of opinions (sharing the same tastes) belongs to the SEU class. In general, one cannot guarantee the existence of  $h$ .

that  $\frac{1}{2}\delta_f + \frac{1}{2}\delta_g$  be weakly preferred to  $f$  and  $g$ . It also extends this idea to the domain of lotteries of acts, since the DM takes into account the extended class of preferences  $EU(\mathcal{P})$  defined on  $\Delta(\mathcal{F})$ .

The next theorem characterizes the ex ante preference relation of a DM who displays preference for randomization. It is the main result of this paper.

**THEOREM 2.** *Let  $\mathcal{P}$  be an admissible class of ex post preferences satisfying [Axiom A8](#). The following statements are equivalent.*

- (a) *The binary relation  $\succsim$  satisfies [Axioms B1, B2, B3\\*, B4, B5, and B7](#).*
- (b) *There exist  $u \in C_b(\Delta(X))$  nonconstant and affine,  $\phi \in C_b(u(\Delta(X)))$  strictly increasing, and a lower semicontinuous (l.s.c.), convex, and grounded function  $c: \Delta(\mathcal{P}^*) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  such that  $u$  represents the restriction of each  $\succsim^* \in \mathcal{P}^*$  to  $\Delta(X)$ , and  $W: \Delta(\mathcal{F}) \rightarrow \mathbb{R}$ , as defined by*

$$W(P) = \min_{m \in \Delta(\mathcal{P}^*)} \left\{ \int_{\mathcal{P}^*} \left[ \int_{\mathcal{F}} \phi(u(\pi^*(f, \succsim^*))) dP(f) \right] dm(\succsim^*) + c(m) \right\}, \quad (14)$$

*represents  $\succsim$ . Moreover, given  $\phi$  and  $u$ , there exists a (unique) minimal function  $c^*: \Delta(\mathcal{P}^*) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  that can be used in place of  $c$  and is defined as*

$$c^*(m) = \sup_{P \in \Delta(\mathcal{F})} \left\{ W(P) - \int_{\mathcal{P}^*} \left[ \int_{\mathcal{F}} \phi(u(\pi^*(f, \succsim^*))) dP(f) \right] dm(\succsim^*) \right\} \quad (15)$$

In contrast to the utilitarian aggregation in [Theorem 1](#), the DM now entertains a set of possible weights on the set of opinions. These weights are the probability measures in  $\Delta(\mathcal{P}^*)$ . Those that minimize the expression in (14) have the interpretation of local utilitarian weights. [Theorem 1](#) is a particular instance when  $c$  is the indicator function (in the sense of convex analysis) associated with some  $m \in \Delta(\mathcal{P}^*)$ . This observation is not surprising, since we derive (14) with techniques of convex duality similar to those that [Maccheroni et al. \(2006\)](#) employ to derive their generalization of SEU. There are differences, though. The elements of our set of “second-order acts” are continuous mappings from  $\mathcal{P}^*$  (the compact set of epistemic states) to  $\mathbb{R}$  and need not have the same (rich) structure as the collection of utility vectors generated by the set of Anscombe–Aumann acts. In this respect, our setup is closer to [Epstein and Segal \(1992\)](#), even though we cannot rely on their techniques (they work with a finite set of individuals and use a different form of independence).

We can express the representation in terms of the experts’ utility functions. Note that  $u(\pi^*(f, \succsim^*))$  is the utility that an expert with preference  $\succsim^*$  derives from  $f$ . If we denote it by  $W(f, \succsim^*)$ , the DM’s utility of an act  $f$  becomes

$$\min_{m \in \Delta(\mathcal{P}^*)} \left\{ \int_{\mathcal{P}^*} \phi(W(f, \succsim^*)) dm(\succsim^*) + c(m) \right\},$$

as in (3). More formally, the representation can be expressed in terms of the measures on the set  $\mathcal{I}^*(u)$  of utility functions that correspond to the set of opinions  $\mathcal{P}^*$ . To each

mapping  $c$  as given by [Theorem 2](#), we associate a convex, l.s.c. and grounded function  $e: \Delta(\mathcal{I}^*(u)) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  so that

$$e(\tilde{m}) = c(\tilde{m} \circ \theta_u^*). \tag{16}$$

Note that because  $\theta_u^*$  is bijective,  $\tilde{m} \circ \theta_u^*$  is a well defined element of  $\Delta(\mathcal{P}^*)$ . Hence, in the subdomain  $\mathcal{F}$ , the DM's utility function is precisely

$$f \mapsto \min_{\tilde{m} \in \Delta(\mathcal{I}^*(u))} \left\{ \int_{\mathcal{I}^*(u)} \phi(I(u(f))) d\tilde{m}(I) + e(\tilde{m}) \right\}. \tag{17}$$

Intuitively, because of the epistemic uncertainty about which expert is right, the DM is less confident in assigning a single vector of weights to the class of opinions. The DM contemplates the whole set  $\Delta(\mathcal{I}^*(u))$  and attaches a degree of plausibility or confidence to each of its members. The function  $e$  associates a lower cost to more plausible weights. The DM looks at the worst-case scenario: the utility of an act is the minimum of the expression that adds the cost component to the generalized utilitarian aggregator.

We now turn to a representation with reduction of compound lotteries. Each  $m \in \Delta(\mathcal{P}^*)$  induces, by [\(11\)](#), a (uniformly continuous) preference relation over Anscombe–Aumann acts that satisfies [Axioms A1–A5](#). Define

$$\mathcal{I}_o^*(u) = \left\{ I_o \in \mathcal{I}^*(u) : \exists \tilde{m} \in \Delta(\mathcal{I}^*(u)) \text{ such that } I_o(u(f)) = \int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}(I) \right\}, \tag{18}$$

the (convex and compact) set of representations on the space of utility acts of all such preferences. The next proposition is a version of [Theorem 2](#) with ROCL.

**PROPOSITION 6.** *The binary relation  $\succsim$  satisfies the axioms of part (a) of [Theorem 2](#) and ROCL if and only if there exist  $u \in C_b(\Delta(X))$  nonconstant and affine, and an l.s.c., convex, and grounded function  $c_o: \mathcal{I}_o^*(u) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  such that  $W_o: \Delta(\mathcal{F}) \rightarrow \mathbb{R}$ , as defined by*

$$W_o(P) = \min_{I_o \in \mathcal{I}_o^*(u)} \left\{ \int_{\mathcal{F}} I_o(u(f)) dP(f) + c_o(I_o) \right\},$$

represents  $\succsim$ .

The function  $W_o$  reduces to  $f \mapsto \min_{I_o \in \mathcal{I}_o^*(u)} \{I_o(u(f)) + c_o(I_o)\}$  in the subdomain of Anscombe–Aumann acts. [Proposition 5](#) is a special case of [Proposition 6](#) in which there exists  $I_o \in \mathcal{I}_o^*(u)$  such that  $c_o(I_o) = 0$  and  $c_o(\widehat{I}_o) = +\infty$  (or large enough) if  $\widehat{I}_o \neq I_o$ .

Finally, under ROCL, the restriction of  $W_o$  to  $\mathcal{F}$  can also be written as the mapping

$$f \mapsto \min_{\tilde{m} \in \Delta(\mathcal{I}^*(u))} \left\{ \int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}(I) + e(\tilde{m}) \right\}. \tag{19}$$

Both [\(17\)](#) and [\(19\)](#) can be interpreted as a modified utilitarian criterion when there is uncertainty as to which weights one should use. They represent a compromise between the utilitarian rule (which corresponds to  $\int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}(I)$ ) and the Rawlsian max–imin principle (that is,  $\min_{I \in \mathcal{I}^*(u)} I(u(f))$ ).

5.2 *Special cases*

We mention two special cases of [Theorem 2](#). One is the Rawlsian criterion. It is characterized by an extreme form of aversion to dispersed opinions. Say that the DM's preferences are represented by a Rawlsian utility function when the mapping  $W_{\text{Rawls}} : \Delta(\mathcal{F}) \rightarrow \mathbb{R}$ , as given by

$$W_{\text{Rawls}}(P) = \min_{\succ^* \in \mathcal{P}^*} \int_{\mathcal{F}} \phi(u(\pi^*(f, \succ^*))) dP(f), \quad (20)$$

represents  $\succ$ . (The utility of  $P$  can be alternatively expressed as the minimum of a collection of expected utilities,  $\min_{I \in \mathcal{I}^*(u)} \int_{\mathcal{F}} \phi(I(u(f))) dP(f)$ .)

The following axiom characterizes a Rawlsian ex ante preference.

**AXIOM B8 (Certainty Bias).** *For all  $P \in \Delta(\mathcal{F})$ , and  $P_c \in \Delta(\mathcal{F}_c)$ , if  $P_c \succ \Psi_{\pi}(P, \succ^*)$  for some  $\succ^* \in \mathcal{P}$ , then  $P_c \succ P$ .*

Certainty Bias is similar to the “default to certainty” condition of [Gilboa et al. \(2010\)](#) and has the following interpretation. Consider the case of an act  $f$  and a constant act  $p$ . In view of [Axioms B4](#) and [B5](#), the existence of an ex post preference such that  $\delta_p \succ \Psi_{\pi}(\delta_f, \succ^*) = \delta_{\pi(f, \succ^*)}$  means that it is not true that  $f$  is weakly preferred to  $p$  according to each element of  $\mathcal{P}$ . Hence, there exists some potential disagreement about the ranking of the acts  $f$  and  $p$ . [Axiom B8](#) says that, in this situation, the DM strictly prefers the constant act  $p$  to  $f$ . Put differently, whenever an act is not weakly preferred to a constant act according to each opinion, the DM chooses the constant act. This is an extreme form of aversion to dispersed opinions that the DM may display when a unanimous recommendation of a bet against a sure prize is not obtained. Certainty Bias extends this idea to the domain of lotteries of acts.

The next proposition characterizes an ex ante preference relation that satisfies Certainty Bias.

**PROPOSITION 7.** *Let  $\mathcal{P}$  be an admissible class of ex post preferences satisfying [Axiom A8](#). The following statements are equivalent.*

- (a) *The binary relation  $\succ$  satisfies [Axioms B1](#), [B2](#), [B3\\*](#), [B4](#), [B5](#), and [B8](#).*
- (b) *The utility function  $W_{\text{Rawls}}$  in (20) represents  $\succ$ .*

If we assume in [Proposition 7](#) that  $\succ$  also satisfies ROCL, then one can take the function  $\phi$  in (20) to be the identity.

A version of [Proposition 7](#) with ROCL appears in [Gilboa et al. \(2010, GMMS\)](#). They consider a class of opinions that is interpreted as “objective rationality.” It is a subclass of weakly compatible SEU preferences axiomatized by [Bewley \(2002\)](#), and identified with a closed and convex subset  $\tilde{M}_{\text{GMMS}}$  of priors. They characterize a preference relation on the domain of Anscombe–Aumann acts by the Rawlsian criterion that minimizes the expectation  $\int u(f) d\mu$  over  $\mu \in \tilde{M}_{\text{GMMS}}$ . This is a particular example of the representation in [Proposition 7](#) under ROCL. The only difference between the representations is

that Gilboa et al. (2010) do not assume the outer layer of randomization as we do in this paper.

The second special case consists in strengthening Weak Independence to the following condition.

**AXIOM B3\*\* (C-Independence).** For all  $P, Q \in \Delta(\mathcal{F})$ ,  $R_c \in \Delta(\mathcal{F}_c)$ , and  $\lambda \in (0, 1)$ ,  $P \succcurlyeq Q$  if and only if  $\lambda P + (1 - \lambda)R_c \succcurlyeq \lambda Q + (1 - \lambda)R_c$ .

The C-Independence axiom is similar to the independence condition that characterizes the maxmin expected utility model, and has a similar interpretation as Weak Independence. C-Independence is weaker than Independence and compatible with preference for randomization. An ex ante preference that satisfies the axioms of Theorem 2 with Weak Independence replaced with C-Independence admits a utility representation that is presented next.

**PROPOSITION 8.** Let  $\mathcal{P}$  be an admissible class of ex post preferences satisfying Axiom A8. The following statements are equivalent.

- (a) The binary relation  $\succcurlyeq$  satisfies Axioms B1, B2, B3\*\*, B4, B5, and B7.
- (b) There exist  $u \in C_b(\Delta(X))$  nonconstant and affine,  $\phi \in C_b(u(\Delta(X)))$  strictly increasing, and a closed and convex set  $\mathbf{M} \subseteq \Delta(\mathcal{P}^*)$  such that  $u$  represents the restriction of each  $\succcurlyeq^* \in \mathcal{P}^*$  to  $\Delta(X)$ , and

$$P \mapsto \min_{m \in \mathbf{M}} \int_{\mathcal{P}^*} \left[ \int_{\mathcal{F}} \phi(u(\pi^*(f, \succcurlyeq^*))) dP(f) \right] dm(\succcurlyeq^*) \quad (21)$$

represents  $\succcurlyeq$ .

The set  $\mathbf{M}$  induces a set of probability measures  $\tilde{\mathbf{M}} \subseteq \Delta(\mathcal{I}^*(u))$  in a way that the criterion in (21) can be written as  $P \mapsto \min_{\tilde{m} \in \tilde{\mathbf{M}}} \int_{\mathcal{I}^*(u)} [\int_{\mathcal{F}} \phi(I(u(f))) dP(f)] d\tilde{m}(I)$ . The DM behaves as if he or she employs a set of utilitarian weights for the utility representations of the elements of  $\text{EU}(\mathcal{P}^*)$ , and looks at the worst-case scenario. Compared with Theorem 2, the DM behaves as if conceiving of a set of utilitarian weights that are uniformly penalized (that is,  $e(\tilde{m}) = 0$  for all  $\tilde{m} \in \tilde{\mathbf{M}}$ ), and every element of  $\Delta(\mathcal{I}^*(u))$  outside of that set has a very large cost. This representation resembles the maxmin expected utility model. Their derivations are based on similar techniques. For the criterion in (21), we employ a version of Zhou's aggregation theorem with a weaker form of the independence axiom proved in Nascimento (2011).

When we add ROCL to the axioms in part (a) of Proposition 8, we obtain a characterization along the lines of Proposition 6; that is, there exists a closed and convex set  $\mathcal{J}_o^* \subseteq \mathcal{I}_o^*(u)$ , which depends on  $\tilde{\mathbf{M}}$  and such that  $P \mapsto \min_{I_o \in \mathcal{J}_o^*} \int_{\mathcal{F}} I_o(u(f)) dP(f)$  represents  $\succcurlyeq$ . The set  $\mathcal{J}_o^*$  is constructed as in (18) when one replaces  $\Delta(\mathcal{I}^*(u))$  with  $\tilde{\mathbf{M}}$ . Note that the utility function in the subdomain of acts can also be expressed as

$$f \mapsto \min_{\tilde{m} \in \tilde{\mathbf{M}}} \int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}(I). \quad (22)$$

A particular instance of this representation appears in Crès et al. (2010). They consider a finite class of opinions in which every ex post preference satisfies the axioms of the maxmin expected utility theory of Gilboa and Schmeidler (1989). Each element  $\succsim_i^* \in \mathcal{P} = \mathcal{P}^*$  is identified with a closed and convex subset of priors  $M_i \subseteq \Delta(S)$ . Their axioms are necessary and sufficient for the existence of a closed and convex set  $\tilde{\mathbf{M}}_{\text{CGV}} \subseteq \Delta(\{1, \dots, |\mathcal{P}|\})$  such that the DM aggregates opinions according to

$$f \mapsto \min_{\tilde{\mathbf{m}} \in \tilde{\mathbf{M}}_{\text{CGV}}} \sum_{i=1}^{|\mathcal{P}|} \tilde{m}_i \left[ \min_{\mu \in M_i} \int u(f) d\mu \right]. \quad (23)$$

The mapping in (23) is a special case of (22) when  $\mathcal{P}$  is a finite set of maxmin preferences. Crès et al. (2010) show that the expression in (23) can be simplified and rewritten as in the standard maxmin expected utility model with a set of priors  $M_{\text{CGV}}$  that ultimately depends not only on the collection  $\{M_1, \dots, M_{|\mathcal{P}|}\}$ , but also on the set  $\tilde{\mathbf{M}}_{\text{CGV}}$ ; that is, the DM satisfies the axioms of the maxmin expected utility theory and employs the closed and convex set of priors  $M_{\text{CGV}} = \bigcup \{\sum_{i=1}^{|\mathcal{P}|} \tilde{m}_i M_i : \tilde{\mathbf{m}} \in \tilde{\mathbf{M}}_{\text{CGV}}\}$ . This result also follows from our construction of  $\mathcal{J}_o^*$  and the properties of support functions.

## 6. FINAL REMARKS

While we are mainly interested in the restriction of the ex ante preference to the set of acts, the axioms that characterize our models strongly rely on the extra layer of randomization originally employed by Anscombe and Aumann (1963). It allows us to make the domain of second-order acts convex and place fewer assumptions on the set of opinions. The alternative models that dispense with lotteries of acts place stronger assumptions on the set of opinions and on the DM's preferences. At this point, it is not clear whether one can still work with a general set of preferences  $\mathcal{P}$  and dispense with the extra layer of randomization.

This paper offers two interpretations of the set  $\mathcal{P}$ . One suggests that the DM takes into account the opinions of a group of experts or specialists when choosing a course of action. In this respect, our framework is close to the social choice literature, especially the single-profile approach. A natural extension is to consider a version of the multi-profile approach, where the set of opinions ranges over a suitably defined domain. It will most likely be useful to obtain uniqueness results that are better than those discussed in Appendix B, but the technical difficulties involved are beyond the scope of this paper.

Alternatively, we can view the set  $\mathcal{P}$  as a collection of future selves that the DM conceives of in an ex ante stage. This is not entirely satisfactory, as the set  $\mathcal{P}$  is exogenously given (i.e., it is not derived from the ex ante preference). In this respect, perhaps the set  $\mathcal{P}$  could be understood as objective information. The framework of Gajdos et al. (2008b) is more suitable to study objective information, which they identify with a set of priors. The connection between their framework and the aggregation of opinions of experts is implicit in the work of Gajdos and Vergnaud (2010), who generalize and reinterpret the setup of Gajdos et al. (2008b) as a model of aggregation of preferences. Our paper suggests that objective information can be identified with a collection of rankings over Anscombe–Aumann acts and need not necessarily correspond to a set of probabilities.

APPENDIX A: PROOFS

A.1 Proof of Proposition 1

We only show that the axioms are sufficient. The existence of an affine function  $u \in C_b(\Delta(X))$  that represents  $\succ^*|_{\Delta(X)}$  follows from Axioms A1, A2, and A4, and the expected utility theorem (Grandmont 1972, Theorem 2). The function  $u$  is nonconstant because of Axioms A3 and A5. Using Monotonicity and C-Continuity, one can show by standard arguments that each act admits a certainty equivalent, and that the function  $I: u(\Delta(X))^S \rightarrow \mathbb{R}$ , as given by  $I(u(f)) = u(p_f)$  for any certainty equivalent  $p_f$  of  $f$ , is well defined and monotone. Moreover,  $f \mapsto I(u(f))$  represents  $\succ^*$ .

A.2 Proof of Proposition 2

In view of Proposition 1, for the “if” part, we need to show only that the Uniform Equicontinuity axiom is satisfied. The sequences of acts  $(f_n)$  and  $(g_n)$  are eventually close if and only if  $\max_{s \in S} |u(f_n(s)) - u(g_n(s))| \rightarrow 0$  (compare with (4)). Let  $(\succ_n^*) \in \mathcal{P}^\infty$  and denote by  $(I_n) \in \mathcal{I}^\infty$  the corresponding sequence of representations. By uniform equicontinuity of the set  $\mathcal{I}$ , if  $\varepsilon > 0$ , then there exists  $\delta > 0$  such that  $\max_{s \in S} |u(f(s)) - u(g(s))| < \delta$  implies  $|I_n(u(f)) - I_n(u(g))| < \varepsilon$  for all  $n$ . For some  $N$  sufficiently large, we obtain  $\max_{s \in S} |u(f_n(s)) - u(g_n(s))| < \delta$  for all  $n \geq N$ , and thus  $|I_n(u(f_n)) - I_n(u(g_n))| < \varepsilon$  for all  $n \geq N$ . Hence  $|I_n(u(f_n)) - I_n(u(g_n))| \rightarrow 0$ , which is equivalent to  $(r_{f_n, \succ_n^*})$  and  $(r_{g_n, \succ_n^*})$  being eventually close.

We now show the “only if” part. Fix any  $\succ^* \in \mathcal{P}$ . Since  $\mathcal{P}$  is admissible, it follows from Proposition 1 that there exist  $u_{\succ^*} \in C_b(\Delta(X))$  nonconstant and affine, and  $I_{\succ^*}: u(\Delta(X))^S \rightarrow \mathbb{R}$  monotone and normalized such that the mapping  $f \mapsto I_{\succ^*}(u_{\succ^*}(f))$  represents  $\succ^*$ . Using Axiom A6, if  $\succ_1^*$  is any element of  $\mathcal{P}$ , normalize  $u_{\succ_1^*}$  so that  $u_{\succ_1^*} = u_{\succ^*} =: u$ . (This entails a normalization for  $I_{\succ_1^*}$ , which is also without loss of generality (w.l.o.g.)) Uniqueness of the function  $I_{\succ^*}$  given  $u$  is well known. Let  $\mathcal{I} := \{I_{\succ^*} : \succ^* \in \mathcal{P}\}$ . Claim 1 completes the proof.

CLAIM 1. *The set  $\mathcal{I}$  is a uniformly equicontinuous subset of  $C_b(u(\Delta(X))^S)$ .*

PROOF. If  $\mathcal{I}$  is not uniformly equicontinuous, then there exists  $\varepsilon > 0$  such that, for each  $n \in \mathbb{N}$ , one can find acts  $f_n$  and  $g_n$  satisfying  $\max_{s \in S} |u(f_n(s)) - u(g_n(s))| < 1/n$  and  $|I_n(u(f_n)) - I_n(u(g_n))| \geq \varepsilon$  for some  $I_n \in \mathcal{I}$ . Let  $\succ_n^*$  denote the ex post preference that  $f \mapsto I_n(u(f))$  represents. Then the sequences  $(f_n)$  and  $(g_n)$  are eventually close, but  $(r_{f_n, \succ_n^*})$  and  $(r_{g_n, \succ_n^*})$  are not, thus violating Axiom A7.  $\square$

A.3 Proof of Proposition 3

The function  $\theta_u^*$  is a bijection. Let  $(I_n) \in \mathcal{I}^*(u)^\infty$  and  $I \in \mathcal{I}^*(u)$ . If  $I_n \rightarrow I$ , then  $d_{\theta_u^*}(\theta_u^{*-1}(I_n), \theta_u^{*-1}(I)) = \sup_{f \in \mathcal{F}} |I_n(u(f)) - I(u(f))| \rightarrow 0$ . Therefore,  $\theta_u$  is a homeomorphism and it suffices to show that  $\mathcal{I}^*(u)$  is a compact subset of  $C_b(u(\Delta(X))^S)$ . We will use the following theorem. Its proof relies on the property that  $\mathcal{I}(u)$  is bounded, and it can be found in Dunford and Schwartz (1988, Theorem IV.6.5).

**THEOREM 3.** *The set  $\mathcal{I}^*(u)$  is compact if and only if, for all  $\varepsilon > 0$ , there exist a finite collection  $\{A_1, \dots, A_N\}$  of subsets of  $\mathcal{F}$  and a list  $(f_1, \dots, f_N) \in A_1 \times \dots \times A_N$  such that  $\sup_{I \in \mathcal{I}(u)} \sup_{f \in A_i} |I(u(f)) - I(u(f_i))| < \varepsilon$  for all  $i = 1, \dots, N$ .*

Let  $\varepsilon > 0$ . Because  $\mathcal{I}(u)$  is uniformly equicontinuous, there exists  $\gamma > 0$  such that, for all  $f, g \in \mathcal{F}$ , if  $\max_{s \in S} |u(f(s)) - u(g(s))| < \gamma$ , then  $|I(u(f)) - I(u(g))| < \varepsilon/2$  for all  $I \in \mathcal{I}(u)$ . The set  $u(\Delta(X))^S$  is a bounded subset of  $\mathbb{R}^S$ . Therefore, there exist finitely many nonempty subsets  $\widehat{A}_1, \dots, \widehat{A}_N$  of  $u(\Delta(X))^S$ , each with diameter less than  $\gamma$  and satisfying  $u(\Delta(X))^S = \bigcup_{i=1}^N \widehat{A}_i$ . Let  $A_i := \{f \in \mathcal{F} : u(f) \in \widehat{A}_i\}$  ( $i = 1, \dots, N$ ), and note that compactness of  $\mathcal{I}^*(u)$  now becomes a consequence of **Theorem 3** and the fact that each element of the collection  $\{A_1, \dots, A_N\}$  is nonempty.

#### A.4 Proof of Proposition 4

Fix any  $\succ^* \in \mathcal{P}^*$ . In view of Axioms **A1–A8**, there exist  $p^*, p_* \in \Delta(X)$  such that  $p^* \succ^* p_*$  and  $p^* \succ^* f \succ^* p_*$  for all  $f \in \mathcal{F}$ . Hence, there exists a function  $\rho : \mathcal{F} \times \mathcal{P}^* \rightarrow [0, 1]$  such that  $f \sim^* \rho(f, \succ^*)p^* + (1 - \rho(f, \succ^*))p_*$  for all  $(f, \succ^*) \in \mathcal{F} \times \mathcal{P}^*$ . Alternatively, given a common affine and continuous representation  $u$  of  $\mathcal{P}^*|_{\Delta(X)}$ , one can write

$$\rho(f, \succ^*) = \frac{\theta_u^*(\succ^*)(u(f)) - u(p_*)}{u(p^*) - u(p_*)}. \quad (24)$$

Define  $\pi^* : \mathcal{F} \times \mathcal{P}^* \rightarrow \Delta(X)$  so that  $\pi^*(f, \succ^*) = \rho(f, \succ^*)p^* + (1 - \rho(f, \succ^*))p_*$ . Because of (24) and the discussion in **Section 3**, the mapping  $\rho$  is continuous. Therefore,  $\pi^*$  is continuous.

#### A.5 Proof of Theorem 2

*Necessity.* The Weak Order axiom is clearly satisfied, while Continuity follows from the observation that  $W$  is continuous, a consequence of the maximum theorem (e.g., **Aliprantis and Border 1999**, Theorem 16.31). Given  $p \in \Delta(X)$ , one can easily check that  $u(\pi^*(p, \succ_1^*)) = u(\pi^*(p, \succ_2^*)) = u(p)$  for all  $\succ_1^*, \succ_2^* \in \mathcal{P}^*$ . Now one can readily show that Weak Independence holds. Because  $\phi \circ u$  represents  $\succ|_{\Delta(X)}$  and  $\phi$  is strictly increasing, the C-Agreement axiom is satisfied.

For  $R \in \Delta(\mathcal{F})$  and  $\succ^*, \succ_0^* \in \mathcal{P}^*$ , using the change of variables theorem (**Aliprantis and Border 1999**, Theorem 12.46), we have

$$\int_{\Delta(X)} \phi(u(\pi^*(p, \succ^*))) d\Psi_{\pi^*}(R, \succ_0^*)(p) = \int_{\mathcal{F}} \phi(u(\pi^*(\pi^*(f, \succ_0^*), \succ^*))) dR(f).$$

Because  $u$  is a Bernoulli index that represents the restriction of each element of  $\mathcal{P}^*$  to  $\Delta(X)$ , we obtain  $u(\pi^*(\pi^*(f, \succ_0^*), \succ^*)) = u(\pi^*(\pi^*(f, \succ_0^*), \succ_0^*)) = u(\pi^*(f, \succ_0^*))$ , where the last equality is a consequence of  $\pi^*(\pi^*(f, \succ_0^*), \succ_0^*) \sim_0^* \pi^*(f, \succ_0^*)$ . Hence

$$\int_{\Delta(X)} \phi(u(\pi^*(p, \succ^*))) d\Psi_{\pi^*}(R, \succ_0^*)(p) = \int_{\mathcal{F}} \phi(u(\pi^*(f, \succ_0^*))) dR(f),$$



and thus

$$W(\Psi_{\pi^*}(R, \succ^*)) = \int_{\mathcal{F}} \phi(u(\pi^*(f, \succ^*))) dR(f), \tag{25}$$

which is continuous in  $\succ^*$ . If  $P, Q \in \Delta(\mathcal{F})$  are such that  $\Psi_{\pi^*}(P, \succ^*) \succ \Psi_{\pi^*}(Q, \succ^*)$  for all  $\succ^* \in \mathcal{P}$ , then  $W(\Psi_{\pi^*}(P, \succ^*)) \geq W(\Psi_{\pi^*}(Q, \succ^*))$  for all  $\succ^* \in \mathcal{P}^*$  (note that  $\mathcal{P}^*$  is the  $\tau$ -closure of  $\mathcal{P}$ , and  $\succ^* \mapsto W(\Psi_{\pi^*}(R, \succ^*))$ , as induced by (25), is continuous). The Consistency axiom now follows from the representation and a close inspection of the function

$$C(\mathcal{P}^*) \ni \xi \mapsto \min_{m \in \Delta(\mathcal{P}^*)} \left\{ \int_{\mathcal{P}^*} \xi(\succ^*) dm(\succ^*) + c(m) \right\} \tag{26}$$

when we replace  $\xi$  with  $\succ^* \mapsto W(\Psi_{\pi^*}(R, \succ^*))$ .

Finally, Convexity follows from the observation that the function in (26) is concave. *Sufficiency.* Let  $P_c, Q_c, R_c \in \Delta(\mathcal{F}_c)$ . Using Axioms B1 and B3\*, one can show that  $P_c \sim Q_c$  implies  $\frac{1}{2}P_c + \frac{1}{2}R_c \sim \frac{1}{2}Q_c + \frac{1}{2}R_c$ . Hence, it follows from the expected utility theorem (Grandmont 1972, Theorem 3), and Axioms B1, B2, and B3\* that there exists  $w \in C_b(\Delta(X))$  such that the function  $\tilde{W} : \Delta(\mathcal{F}_c) \rightarrow \mathbb{R}$ , as given by  $\tilde{W}(P_c) = \int w dP_c$ , represents  $\succ|_{\Delta(\mathcal{F}_c)}$ . If  $u \in C_b(\Delta(X))$  denotes the common nonconstant and affine representation of the restriction of each  $\succ^* \in \mathcal{P}^*$  to  $\Delta(X)$ , then (using C-Agreement) there exists  $\phi \in C_b(u(\Delta(X)))$  strictly increasing such that  $w = \phi \circ u$  (Lemma B.9 of Seo 2009). Because  $w$  is unique up to a positive affine transformation, we can assume w.l.o.g. that  $[-1, 1] \subseteq \phi(u(\Delta(X)))$ .

CLAIM 2. *If  $P \in \Delta(\mathcal{F})$ , then there exists  $P_c \in \Delta(\mathcal{F}_c)$  such that  $P \sim P_c$ .*

PROOF. It follows from the measurable maximum theorem (e.g., Aliprantis and Border 1999, Theorem 17.18) that the argmax correspondence associated with the program  $\max_{\succ^* \in \mathcal{P}^*} \phi(u(\pi^*(f, \succ^*)))$  admits a measurable selection; that is, there exists a measurable function  $\bar{\varphi} : \mathcal{F} \rightarrow \mathcal{P}^*$  satisfying, for any given  $f \in \mathcal{F}$ ,

$$\phi(u(\pi^*(f, \bar{\varphi}(f)))) \geq \phi(u(\pi^*(f, \succ^*))) \quad \text{for all } \succ^* \in \mathcal{P}^*. \tag{27}$$

Define  $\bar{P}_c \in \Delta(\mathcal{F}_c)$  so that  $\bar{P}_c(B) = P(\{f \in \mathcal{F} : \pi^*(f, \bar{\varphi}(f)) \in B\})$  for all  $B \in \mathcal{B}(\Delta(X))$ . Measurability of  $\bar{\varphi}$  and continuity of  $\pi^*$  ensure that  $f \mapsto \pi^*(f, \bar{\varphi}(f))$  is measurable and hence  $\bar{P}_c$  is well defined. For all  $\succ^* \in \mathcal{P}$ , the function  $f \mapsto \phi(u(\pi^*(f, \succ^*)))$  is  $P$ -integrable. The function  $f \mapsto \phi(u(\pi^*(f, \bar{\varphi}(f))))$  is  $P$ -integrable as well. Hence it follows from (27) that

$$\int_{\mathcal{F}} \phi(u(\pi^*(f, \bar{\varphi}(f)))) dP(f) \geq \int_{\mathcal{F}} \phi(u(\pi^*(f, \succ^*))) dP(f) \quad \text{for all } \succ^* \in \mathcal{P}. \tag{28}$$

Fix any  $\succ^* \in \mathcal{P}$ . Consider the transformations  $T_1, T_{2, \succ^*} : (\Delta(X), \mathcal{B}(\Delta(X))) \rightarrow (\mathcal{F}, \mathcal{B}(\mathcal{F}))$  as defined by  $T_1(B) = \{f \in \mathcal{F} : \pi^*(f, \bar{\varphi}(f)) \in B\}$  and  $T_{2, \succ^*}(B) = \{f \in \mathcal{F} : \pi^*(f, \succ^*) \in B\}$  for all  $B \in \mathcal{B}(\Delta(X))$ . Continuity of  $\pi^*$  and measurability of  $\bar{\varphi}$  imply that the transformations  $T_1$  and  $T_{2, \succ^*}$  are measurable. Note that  $P = \bar{P}_c \circ T_1^{-1}$  and  $P = \Psi_{\pi^*}(P, \succ^*) \circ T_{2, \succ^*}^{-1}$ . Therefore, it follows from the change of variables theorem (Aliprantis and Border 1999, Theorem 12.46) that  $\int_{\mathcal{F}} \phi(u(\pi^*(f, \bar{\varphi}(f)))) dP(f) = \int_{\Delta(X)} \phi(u(p)) d\bar{P}_c(p) = \tilde{W}(\bar{P}_c)$

and  $\int_{\mathcal{F}} \phi(u(\pi^*(f, \succ^*))) dP(f) = \int_{\Delta(X)} \phi(u(p)) d\Psi_{\pi^*}(P, \succ^*)(p) = \tilde{W}(\Psi_{\pi^*}(P, \succ^*))$ . Using (28), we obtain that  $\bar{P}_c \succ \Psi_{\pi^*}(P, \succ^*)$  for all  $\succ^* \in \mathcal{P}$ . The Consistency axiom now implies that  $\bar{P}_c \succ P$ . Similarly, by considering the program  $\max_{\succ^* \in \mathcal{P}^*} -\phi(u(\pi^*(f, \succ^*)))$ , one can construct  $\underline{P}_c \in \Delta(\mathcal{F}_c)$  such that  $P \succ \underline{P}_c$ . The existence of some  $\lambda^* \in [0, 1]$  such that  $P \sim \lambda^* \bar{P}_c + (1 - \lambda^*) \underline{P}_c \in \Delta(\mathcal{F}_c)$  now becomes a consequence of Axioms B1 and B2 and standard arguments.  $\square$

Define  $W: \Delta(\mathcal{F}) \rightarrow \mathbb{R}$  such that, for all  $P \in \mathcal{F}$ ,  $W(P) = \tilde{W}(P_c)$  if  $P_c \in \Delta(\mathcal{F}_c)$  satisfies  $P \sim P_c$ . It follows from Claim 2 and the definition of  $\tilde{W}$  that the function  $W$  is well defined. An easy observation is that even though the implicit state space of preferences is  $\mathcal{P}$ , the relevant one is  $\mathcal{P}^*$ . We now define the set of utility acts  $\Phi \subseteq C(\mathcal{P}^*)$  as

$$\Phi = \left\{ \xi \in C(\mathcal{P}^*): \right. \\ \left. \exists P \in \Delta(\mathcal{F}) \text{ such that } \xi(\succ^*) = \int_{\mathcal{F}} \phi(u(\pi^*(f, \succ^*))) dP(f) \text{ for all } \succ^* \in \mathcal{P}^* \right\}.$$

As a matter of notation, we denote by  $\xi_P$  the element of  $\Phi$  that is induced by  $P \in \Delta(\mathcal{F})$ . We identify any  $a \in \mathbb{R}$  with  $a\mathbf{1}_{\mathcal{P}^*} \in C(\mathcal{P}^*)$ . Note that  $\Phi$  is convex and  $\phi(u(\Delta(X))) \subseteq \Phi$ . Let  $J: \Phi \rightarrow \mathbb{R}$  be defined so that  $J(\xi) = W(P)$  if  $P \in \Delta(\mathcal{F})$  satisfies  $\xi(\succ^*) = \int_{\mathcal{F}} \phi(u(\pi^*(f, \succ^*))) dP(f)$  for all  $\succ^* \in \mathcal{P}^*$ . The function  $J$  is well defined. It is also monotone ( $\xi \geq \zeta$  implies  $J(\xi) \geq J(\zeta)$ ) and normalized ( $J(a) = a$  for all  $a \in \phi(u(\Delta(X)))$ ).

CLAIM 3. For all  $\xi, \zeta \in \Phi$ ,  $J(\xi) - J(\zeta) \leq \max_{\succ^* \in \mathcal{P}^*} [\xi(\succ^*) - \zeta(\succ^*)]$ .

PROOF. Let  $p_0, p_d \in \Delta(X)$  be such that  $\phi(u(p_0)) = 0$  and  $\phi(u(p_d)) = d$ , where  $d \in \phi(u(\Delta(X)))$ . Fix  $\gamma \in (0, 1)$  and  $P \in \Delta(\mathcal{F})$ . Using the same argument as in the proof of Claim 2 to construct  $\bar{P}_c$  and  $\underline{P}_c$ , one can find  $\lambda \in [0, 1]$  such that

$$\begin{aligned} \gamma P + (1 - \gamma)\delta_{p_0} &\sim \lambda(\gamma \bar{P}_c + (1 - \gamma)\delta_{p_0}) + (1 - \lambda)(\gamma \underline{P}_c + (1 - \gamma)\delta_{p_0}) \\ &= \gamma[\lambda \bar{P}_c + (1 - \lambda)\underline{P}_c] + (1 - \gamma)\delta_{p_0} \end{aligned} \quad (29)$$

Hence, using Weak Independence and (29), we obtain

$$\gamma P + (1 - \gamma)\delta_{p_d} \sim \gamma[\lambda \bar{P}_c + (1 - \lambda)\underline{P}_c] + (1 - \gamma)\delta_{p_d},$$

and (a similar argument appears in Maccheroni et al. 2006) then

$$J(\gamma \xi_P + (1 - \gamma)d) = J(\gamma \xi_P) + (1 - \gamma)d. \quad (30)$$

Define  $\tilde{\Phi}$  as the (convex) set  $\bigcup_{\lambda \in (0,1)} \lambda \Phi + (1 - \lambda)\{0\}$ , and note that its sup norm closure is  $\Phi$ . Moreover, one can check that if  $\sigma \in \tilde{\Phi}$ , then there exists a strictly positive number  $\bar{a} \in \phi(u(\Delta(X)))$  such that  $\sigma + a \in \Phi$  whenever  $|a| \leq \bar{a}$ . In fact, for  $\sigma = \lambda \xi$  with  $\lambda \in (0, 1)$ , there exists  $\bar{a} \in \phi(u(\Delta(X)))$  sufficiently small such that  $a/(1 - \lambda) \in \phi(u(\Delta(X)))$  when  $|a| \leq \bar{a}$ . Because  $\Phi$  is convex, we have  $\sigma + a = \lambda \xi + (1 - \lambda)(a/(1 - \lambda)) \in \Phi$ .

Take any  $\sigma, \sigma' \in \tilde{\Phi}$ , and let  $\bar{a} > 0$  in  $\phi(u(\Delta(X)))$  be such that  $\sigma + a, \sigma' + a \in \Phi$  when  $|a| \leq \bar{a}$ , as constructed in the previous paragraph. Let  $d := \max_{\succ^* \in \mathcal{P}^*} [\sigma(\succ^*) - \sigma'(\succ^*)]$ . If  $d = 0$ , then  $\sigma \leq \sigma' + d = \sigma'$ , and by monotonicity we obtain

$$J(\sigma) - J(\sigma') \leq \max_{\succ^* \in \mathcal{P}^*} [\sigma(\succ^*) - \sigma'(\succ^*)]. \tag{31}$$

Now assume w.l.o.g. that  $d$  is strictly positive (the analysis of the case  $d < 0$  is similar). If  $d \leq \bar{a}$ , then it follows from (30) and monotonicity of  $J$  that (31) holds. In case  $d > \bar{a}$ , there exists  $N \in \mathbb{N}$  such that  $N\bar{a} \leq d < (N + 1)\bar{a}$ . Find  $\alpha_1 > \alpha_2 > \dots > \alpha_N \in (0, 1)$  and  $\sigma_1, \dots, \sigma_N \in \tilde{\Phi}$  that fulfill  $(1 - \alpha_1)d = (\alpha_1 - \alpha_2)d = \dots = (\alpha_{N-1} - \alpha_N)d = \bar{a}$ ,  $\alpha_N d < \bar{a}$ , and  $\sigma_i = \alpha_i \sigma + (1 - \alpha_i)\sigma'$  ( $i = 1, \dots, N$ ). Observe that for some  $\lambda, \gamma \in (0, 1)$  and  $\xi, \zeta \in \Phi$ ,

$$\begin{aligned} \sigma_i + \bar{a} &= \alpha_i \sigma + (1 - \alpha_i)\sigma' + \bar{a} \\ &= \alpha_i \lambda \xi + (1 - \alpha_i)\gamma \zeta + \bar{a} \\ &= \kappa \left( \frac{\alpha_i \lambda}{\kappa} \xi + \frac{(1 - \alpha_i)\gamma}{\kappa} \zeta \right) + (1 - \kappa) \frac{\bar{a}}{1 - \kappa} \in \Phi, \end{aligned}$$

where  $\kappa = \alpha_i \lambda + (1 - \alpha_i)\gamma$ , and  $\bar{a}/(1 - \kappa)$  is between  $\bar{a}/(1 - \lambda)$  and  $\bar{a}/(1 - \gamma)$  (both in  $\phi(u(\Delta(X)))$ ). Moreover,  $\sigma' + \alpha_N d = \gamma \zeta + (1 - \gamma)\alpha_N d/(1 - \gamma)$ , with  $\alpha_N d/(1 - \gamma) \in \phi(u(\Delta(X)))$ . Hence, using (30) and monotonicity,

$$J(\sigma) \leq J(\sigma_1 + (1 - \alpha_1)d) = J(\sigma_1) + \bar{a} \tag{32}$$

$$J(\sigma_i) \leq J(\sigma_{i+1} + (\alpha_i - \alpha_{i+1})d) = J(\sigma_{i+1}) + \bar{a} \quad (i = 1, \dots, N - 1) \tag{33}$$

$$J(\sigma_N) \leq J(\sigma' + \alpha_N d) = J(\sigma') + \alpha_N d. \tag{34}$$

Therefore, using (32), (33), and (34), we obtain

$$\begin{aligned} J(\sigma) - J(\sigma') &= J(\sigma) - J(\sigma_1) + J(\sigma_1) - J(\sigma_2) + \dots + J(\sigma_{N-1}) - J(\sigma_N) + J(\sigma_N) - J(\sigma') \\ &\leq (1 - \alpha_1)d + (\alpha_1 - \alpha_2)d + \dots + (\alpha_{N-1} - \alpha_N)d + \alpha_N d = d, \end{aligned}$$

so that (31) holds in  $\tilde{\Phi}$ .<sup>20</sup>

The function  $J|_{\tilde{\Phi}}$  is uniformly continuous, and hence admits a unique continuous extension to  $\text{cl}_{\|\cdot\|_\infty}(\tilde{\Phi}) = \Phi$  (e.g., Royden 1988, p. 149), which we denote by  $J^\bullet$ . We now show that such an extension coincides with  $J$ . Let  $P \in \Delta(\mathcal{F})$  and  $P_n := (1 - 1/n)P + (1/n)\delta_{p_0}$ . Let  $\bar{P}_c, \underline{P}_c \in \Delta(\mathcal{F}_c)$  be defined as in the proof of Claim 2. For  $\bar{P}_{c,n} := (1 - 1/n)\bar{P}_c + (1/n)\delta_{p_0}$  and  $\underline{P}_{c,n} := (1 - 1/n)\underline{P}_c + (1/n)\delta_{p_0}$ , we have  $P_n \sim \lambda_n \bar{P}_{c,n} + (1 - \lambda_n)\underline{P}_{c,n}$  for some  $\lambda_n \in [0, 1]$ . Note that  $(\xi_{P_n}) \in \tilde{\Phi}^\infty$  and  $\xi_{P_n} \rightarrow \xi_P$ . Moreover,  $\bar{P}_{c,n} \rightarrow \bar{P}_c$  and  $\underline{P}_{c,n} \rightarrow \underline{P}_c$ . Take a convergent subsequence  $(\lambda_{n_k})$  with limit  $\lambda^*$ . Therefore,

$$\begin{aligned} J^\bullet(\xi_{P_{n_k}}) &= W(P_{n_k}) \\ &= \lambda_{n_k} W(\bar{P}_{c,n_k}) + (1 - \lambda_{n_k})W(\underline{P}_{c,n_k}) \\ &\rightarrow \lambda^* W(\bar{P}_c) + (1 - \lambda^*)W(\underline{P}_c). \end{aligned}$$

<sup>20</sup>Dekel et al. (2007) use a similar procedure to show Lipschitz continuity of an aggregator.

If  $\bar{P}_c \sim \underline{P}_c \sim P$ , then  $\lim_{n_k} J^\bullet(\xi_{P_{n_k}}) = W(P) = J(\xi_P)$ . Otherwise,  $\bar{P}_c \succ \underline{P}_c$  and there exists a unique  $\lambda \in [0, 1]$  such that  $P \sim \lambda \bar{P}_c + (1 - \lambda) \underline{P}_c$ . We must have  $\lambda^* = \lambda$ , for otherwise  $P \approx \lambda^* \bar{P}_c + (1 - \lambda^*) \underline{P}_c$ , which contradicts **Axiom B2** and the fact  $P \leftarrow P_{n_k} \sim \lambda_{n_k} \bar{P}_{c, n_k} + (1 - \lambda_{n_k}) \underline{P}_{c, n_k} \rightarrow \lambda^* \bar{P}_c + (1 - \lambda^*) \underline{P}_c$ . By uniqueness of the limit, we have  $\lim_n J^\bullet(\xi_{P_n}) = J(\xi_P)$ , which shows that  $J$  agrees with the unique continuous extension of  $J|_{\tilde{\Phi}}$  to  $\Phi$ .

Finally, let  $\sigma, \sigma' \in \Phi$ , and let  $(\sigma_n), (\sigma'_n) \in \tilde{\Phi}^\infty$  be such that  $\sigma_n \rightarrow \sigma$  and  $\sigma'_n \rightarrow \sigma'$ . Then

$$\begin{aligned} J(\sigma_n) - J(\sigma'_n) &\leq \max_{\succ^* \in \mathcal{P}^*} [\sigma_n(\succ^*) - \sigma'_n(\succ^*)] \\ &= \max_{\succ^* \in \mathcal{P}^*} [\sigma(\succ^*) - \sigma'(\succ^*) + \sigma_n(\succ^*) - \sigma(\succ^*) + \sigma'(\succ^*) - \sigma'_n(\succ^*)] \\ &\leq \max_{\succ^* \in \mathcal{P}^*} [\sigma(\succ^*) - \sigma'(\succ^*)] + \max_{\succ^* \in \mathcal{P}^*} [\sigma_n(\succ^*) - \sigma(\succ^*)] \\ &\quad + \max_{\succ^* \in \mathcal{P}^*} [\sigma'(\succ^*) - \sigma'_n(\succ^*)] \\ &\leq \max_{\succ^* \in \mathcal{P}^*} [\sigma(\succ^*) - \sigma'(\succ^*)] + \|\sigma_n - \sigma\|_\infty + \|\sigma'_n - \sigma'\|_\infty \end{aligned}$$

for all  $n$ . Since  $\lim_n [J(\sigma_n) - J(\sigma'_n)] = J(\sigma) - J(\sigma')$ , we conclude from the above that (31) holds in  $\Phi$ .  $\square$

The next claim establishes that  $J$  is concave when restricted to the dense subset of  $\Phi$  defined in the proof of **Claim 3**.

**CLAIM 4.** *The function  $J|_{\tilde{\Phi}}$  is concave.*

**PROOF.** Using **Axiom B7**, follow the arguments in **Nascimento and Riella (forthcoming, Claim A.1.4)**.  $\square$

Since  $J$  is continuous,  $J|_{\tilde{\Phi}}$  is concave, and  $\tilde{\Phi}$  is dense in  $\Phi$ , the function  $J$  is itself concave. We now extend  $J$  to the space  $C(\mathcal{P}^*)$ . The extension we use is the same one proposed by **Dolecki and Greco (1995)**; that is, we define  $\hat{J}: C(\mathcal{P}^*) \rightarrow \mathbb{R}$  so that  $\hat{J}(\xi) = \sup_{\xi' \in \Phi} \{J(\xi') + \min_{\succ^* \in \mathcal{P}^*} [\xi(\succ^*) - \xi'(\succ^*)]\}$ .

**CLAIM 5.** *The function  $\hat{J}$  is normalized, concave, and satisfies  $\hat{J}(\xi) - \hat{J}(\zeta) \leq \max_{\succ^* \in \mathcal{P}^*} [\xi(\succ^*) - \zeta(\succ^*)]$  for all  $\xi, \zeta \in C(\mathcal{P}^*)$ .*

**PROOF.** If  $a \in \mathbb{R}$ , then  $\hat{J}(a) = a + \sup_{\xi' \in \Phi} \{J(\xi') - \max_{\succ^* \in \mathcal{P}^*} \xi'(\succ^*)\} \leq a$ , where the last inequality follows from monotonicity of  $J$ . Since  $0 \in \Phi$ , we obtain  $\hat{J}(a) \geq a$  and, hence,  $\hat{J}(a) = a$ . Now let  $\xi, \zeta \in C(\mathcal{P}^*)$  and  $\lambda \in (0, 1)$ . Let  $(\xi_n), (\zeta_n) \in \Phi^\infty$  be such that  $J(\xi_n) + \min_{\succ^* \in \mathcal{P}^*} [\xi(\succ^*) - \xi_n(\succ^*)] \uparrow \hat{J}(\xi)$  and  $J(\zeta_n) + \min_{\succ^* \in \mathcal{P}^*} [\zeta(\succ^*) - \zeta_n(\succ^*)] \uparrow \hat{J}(\zeta)$ . Let  $\sigma := \lambda \xi + (1 - \lambda) \zeta$  and  $\sigma_n := \lambda \xi_n + (1 - \lambda) \zeta_n$ . Using the concavity of  $J$ ,

$$\begin{aligned} \hat{J}(\sigma) &\geq J(\sigma_n) + \min_{\succ^* \in \mathcal{P}^*} [\sigma(\succ^*) - \sigma_n(\succ^*)] \\ &\geq \lambda \left\{ J(\xi_n) + \min_{\succ^* \in \mathcal{P}^*} [\xi(\succ^*) - \xi_n(\succ^*)] \right\} + (1 - \lambda) \left\{ J(\zeta_n) + \min_{\succ^* \in \mathcal{P}^*} [\zeta(\succ^*) - \zeta_n(\succ^*)] \right\} \end{aligned}$$

for all  $n \in \mathbb{N}$ . Therefore,  $\widehat{J}(\lambda\xi + (1 - \lambda)\zeta) \geq \lambda\widehat{J}(\xi) + (1 - \lambda)\widehat{J}(\zeta)$  after taking the limit as  $n \rightarrow \infty$ . Finally, the fact that  $\widehat{J}(\xi) - \widehat{J}(\zeta) \leq \max_{\succ^* \in \mathcal{P}^*} [\xi(\succ^*) - \zeta(\succ^*)]$  for all  $\xi, \zeta \in C(\mathcal{P}^*)$  is shown by Dolecki and Greco (1995).  $\square$

Because the set  $\mathcal{P}^*$  is a compact metric space, the norm dual of  $C(\mathcal{P}^*)$  is isometrically isomorphic to the space of finite signed Borel measures on  $\mathcal{P}^*$ ,  $\text{ca}(\mathcal{B}(\mathcal{P}^*))$  (e.g., Aliprantis and Border 1999, Corollary 13.15). Using a standard result of duality theory (e.g., Ekeland and Turnbull 1983, Proposition 1, p. 97), we obtain  $\widehat{J}(\xi) = \inf_{m \in \text{ca}(\mathcal{B}(\mathcal{P}^*))} \{ \int \xi \, dm - \widehat{J}^*(m) \}$ , where  $\widehat{J}^*(m) = \inf_{\xi \in C(\mathcal{P}^*)} \{ \int \xi \, dm - \widehat{J}(\xi) \}$ .

The function  $\widetilde{c} := -\widehat{J}^*$  is l.s.c. and convex, and satisfies  $0 = \widehat{J}(0) = \inf_{m \in \text{ca}(\mathcal{B}(\mathcal{P}^*))} \{ \widetilde{c}(m) \}$ . Hence  $\widetilde{c}$  takes values on  $\mathbb{R}_+ \cup \{+\infty\}$  and is grounded. The proof of the theorem is complete once we show that

$$\inf_{m \in \text{ca}(\mathcal{B}(\mathcal{P}^*))} \left\{ \int \xi \, dm + \widetilde{c}(m) \right\} = \min_{m \in \Delta(\mathcal{P}^*)} \left\{ \int \xi \, dm + c(m) \right\}, \tag{35}$$

where  $c = \widetilde{c}|_{\Delta(\mathcal{P}^*)}$ . If  $m_0 \in \text{ca}(\mathcal{B}(\mathcal{P}^*))$  does not induce a positive linear functional in the norm dual of  $C(\mathcal{P}^*)$ , then there exists  $\zeta \in C(\mathcal{P}^*)$  satisfying  $\zeta \geq 0$  and  $\int \zeta \, dm_0 < 0$ . Therefore,  $\widetilde{c}(m_0) = \sup_{\xi \in C(\mathcal{P}^*)} \{ \widehat{J}(\xi) - \int \xi \, dm_0 \} \geq \widehat{J}(n\zeta) - n \int \zeta \, dm_0 \geq n | \int \zeta \, dm_0 |$  for all  $n \in \mathbb{N}$ . As a consequence,  $\widetilde{c}(m_0) = +\infty$  if  $m_0$  is not a positive measure. In fact, if  $\widetilde{c}(m_0) < \infty$ , then for all  $n \in \mathbb{N}$ ,

$$\begin{aligned} n = \widehat{J}(n) &= \inf_{m \in \text{ca}(\mathcal{B}(\mathcal{P}^*))} \left[ n \int \mathbf{1}_{\mathcal{P}^*} \, dm + \widetilde{c}(m) \right] \leq n \times m_0(\mathcal{P}^*) + \widetilde{c}(m_0) \quad \text{and} \\ n = -\widehat{J}(-n) &= \sup_{m \in \text{ca}(\mathcal{B}(\mathcal{P}^*))} \left[ n \int \mathbf{1}_{\mathcal{P}^*} \, dm - \widetilde{c}(m) \right] \geq n \times m_0(\mathcal{P}^*) - \widetilde{c}(m_0), \end{aligned}$$

so that  $m_0(\mathcal{P}^*) = 1$  after taking the limit as  $n \rightarrow \infty$ . (35) follows from the fact that the superdifferential of  $\widehat{J}$  satisfies  $\partial\widehat{J}(\xi) = \partial J(\xi) = \arg \min_{m \in \Delta(\mathcal{P}^*)} \{ \int \xi \, dm + c(m) \} \neq \emptyset$  for all  $\xi \in \Phi$  (e.g., Ekeland and Turnbull 1983, Proposition 2, p. 112).

Define  $c^* : \Delta(\mathcal{P}^*) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  so that

$$c^*(m) = \sup_{\xi \in \Phi} \left\{ J(\xi) - \int \xi \, dm \right\}$$

for all  $m \in \Delta(\mathcal{P}^*)$ . Using the definition of  $c^*$ , if  $c^{**}$  is any other cost function satisfying (35), then  $c^* \leq c^{**}$ . Finally, we observe that, given  $\xi \in \Phi$ , for any fixed  $m \in \partial J(\xi)$ , we obtain  $J(\xi) \geq \sup_{\zeta \in \Phi} [J(\zeta) + \int (\xi - \zeta) \, dm] \geq \min_{m \in \Delta(\mathcal{P}^*)} [\int \xi \, dm + c^*(m)] \geq J(\xi)$ , so that (35) is satisfied for all  $\xi \in \Phi$  when we replace  $c$  with  $c^*$ .

### A.6 Proof of Theorem 1

The proof that the axioms are necessary follows the same steps as in the proof of Theorem 2. We show only that the axioms are sufficient for the representation. The axioms of part (a) of Theorem 1 imply the axioms of part (a) of Theorem 2. Hence, assume that the utility representation is given as in (14), and the induced monotone functional  $J$  on the

convex set  $\Phi \subseteq C(\mathcal{P}^*)$  is as defined in the proof of [Theorem 2](#). Using the Independence axiom and [Claim 2](#), we note that  $J(\lambda\xi + (1-\lambda)\zeta) = \lambda J(\xi) + (1-\lambda)J(\zeta)$  for all  $\xi, \zeta \in \Phi$  and  $\lambda \in (0, 1)$ . Moreover, one can assume w.l.o.g. that for some  $p_*$  and  $p^* \in \Delta(X)$ ,  $W(\delta_{p_*}) = 0$  and  $W(\delta_{p^*}) = 1$ , so that  $J(0) = 0$  and  $J(1) = 1$ . Therefore, it follows from [Theorem 2'](#) of [Zhou \(1997\)](#) that  $J(\xi) = \int_{\mathcal{P}^*} \xi(\succ^*) dm(\succ^*)$  for some Borel measure  $m$  on  $\mathcal{P}^*$ . Because  $J(1) = 1$ ,  $m$  is a probability measure.

### A.7 Proof of [Proposition 5](#)

The necessity of the axioms is clear and the uniqueness part is well known. We show only that the axioms are sufficient. As a consequence of [Theorem 1](#) and the discussion in [Section 3](#), there exists a probability measure  $\tilde{m} \in \Delta(\mathcal{I}^*(u))$  such that  $\succ$  is represented by  $P \mapsto \int_{\mathcal{I}^*(u)} [\int_{\mathcal{F}} \phi(I(u(f))) dP(f)] d\tilde{m}(I)$ . By [Axiom B6](#),  $\phi$  is also affine. Hence the representation simplifies to  $P \mapsto \int_{\mathcal{I}^*(u)} [\int_{\mathcal{F}} I(u(f)) dP(f)] d\tilde{m}(I) = \int_{\mathcal{F}} [\int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}(I)] dP(f)$  (by Fubini's theorem, the order of integration is immaterial in our setup). Now let  $I_o : u(\Delta(X))^S \rightarrow \mathbb{R}$  be defined so that  $I_o(u(f)) = \int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}(I)$ . The set  $\mathcal{I}^*(u)$  is uniformly bounded and, hence,  $I_o$  is bounded. Each element of  $\mathcal{I}(u)$  is monotone, normalized, and uniformly continuous, and these three properties are also preserved in  $\mathcal{I}^*(u) = \text{cl}_{\|\cdot\|_\infty}(\mathcal{I}(u))$ . The function  $I_o$  is clearly normalized. Given  $f, g \in \mathcal{F}$  such that  $u(f) - u(g) \geq 0$ , we obtain  $I(u(f)) - I(u(g)) \geq 0$  for all  $I \in \mathcal{I}^*(u)$  and, hence,  $I_o(u(f)) \geq I_o(u(g))$ . Finally, since  $\mathcal{I}^*(u)$  is also uniformly equicontinuous, given  $\varepsilon > 0$  there exists  $\gamma > 0$  such that  $\max_{s \in S} |u(f(s)) - u(g(s))| < \gamma$  implies  $|I(u(f)) - I(u(g))| < \varepsilon/2$  for all  $I \in \mathcal{I}^*(u)$  and, hence,  $|I_o(u(f)) - I_o(u(g))| \leq \int_{\mathcal{I}^*(u)} |I(u(f)) - I(u(g))| d\tilde{m}(I) < \varepsilon$ .

### A.8 Proof of [Proposition 6](#)

We only show sufficiency of the axioms. It follows from [Theorem 2](#) that a representation is given by the functional defined in (14). Under ROCL, after two normalizations if necessary, the criterion in (14) becomes

$$\begin{aligned} W(P) &= \min_{m \in \Delta(\mathcal{P}^*)} \left\{ \int_{\mathcal{P}^*} \left[ \int_{\mathcal{F}} u(\pi^*(f, \succ^*)) dR(f) \right] dm(\succ^*) + c(m) \right\} \\ &= \min_{m \in \Delta(\mathcal{P}^*)} \left\{ \int_{\mathcal{F}} \left[ \int_{\mathcal{P}^*} u(\pi^*(f, \succ^*)) dm(\succ^*) \right] dR(f) + c(m) \right\}, \end{aligned} \tag{36}$$

where the last equality is a consequence of Fubini's theorem. Using (11) and (12), each  $m \in \Delta(\mathcal{P}^*)$  is associated with an element of (18). For  $I_o \in \mathcal{I}_o^*(u)$ , let

$$D_{I_o} := \left\{ \tilde{m} \in \Delta(\mathcal{I}^*(u)) : I_o(u(f)) = \int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}(I) \ (f \in \mathcal{F}) \right\}.$$

The set  $D_{I_o}$  is nonempty (the set of extreme points of  $\Delta(\mathcal{P}^*)$  can be identified with  $\mathcal{I}^*(u)$ ) and convex. Moreover, it is a closed subset of a compact metric space and, hence,  $D_{I_o}$  is itself compact. Now let  $c_o : \mathcal{I}_o^*(u) \rightarrow \mathbb{R}_+ \cup \{+\infty\}$  be such that

$$c_o(I_o) = \min_{\tilde{m} \in D_{I_o}} e(\tilde{m}), \tag{37}$$

where  $e$  is defined in (16).

CLAIM 6. *The set  $\mathcal{I}_o^*(u)$  is a nonempty, compact, and convex subset of  $C_b(u(\Delta(X))^S)$ .*

PROOF. By definition (see (18)), the set  $\mathcal{I}_o^*(u)$  is nonempty and convex. As for compactness, using Theorem 5.2 of Aliprantis and Border (1999) and Proposition 3 of this paper, it suffices to show that  $\mathcal{I}_o^*(u) = \overline{\text{co}}(\mathcal{I}^*(u))$  (the closed convex hull of  $\mathcal{I}^*(u)$ ). The inclusion  $\subseteq$  follows from the density theorem (e.g., Aliprantis and Border 1999, Theorem 14.10), and the fact that each element  $u(f)$  of  $u(\Delta(X))^S$  induces a continuous and bounded function on  $\mathcal{I}^*(u)$  (via the evaluation functional of the family  $\mathcal{I}^*(u)$  at  $u(f)$ ). Now let  $I_o^n := \sum_{i=1}^{N_n} \alpha_i^n I_{o,i}^n \in \text{co}(\mathcal{I}^*(u))$  and  $C_b(u(\Delta(X))^S) \ni I_o^\infty \leftarrow I_o^n$ . Each  $\tilde{m}_n := \sum_{i=1}^{N_n} \alpha_i^n \delta_{I_{o,i}^n}$  is a simple measure on  $\mathcal{I}^*(u)$ . Because  $\Delta(\mathcal{I}^*(u))$  is sequentially compact, there exists a convergent subsequence  $(\tilde{m}_{n_k})$ . Then  $\tilde{m}_{n_k} \rightarrow \tilde{m}_\infty$  for some  $\tilde{m}_\infty \in \Delta(\mathcal{I}^*(u))$ . Using again the fact that each  $u(f)$  of  $u(\Delta(X))^S$  induces an element of  $C(\mathcal{I}^*(u))$  via the evaluation functional, we obtain  $I_o^\infty(u(f)) = \lim_{n_k} \int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}_{n_k}(I) = \int_{\mathcal{I}^*(u)} I(u(f)) d\tilde{m}_\infty(I)$  (the last equality follows from the definition of weak\*-convergence). Hence  $I_o^\infty \in \mathcal{I}_o^*(u)$  and this proves the inclusion  $\supseteq$ .  $\square$

CLAIM 7. *The function  $e$  is convex, l.s.c., and grounded.*

PROOF. If  $\tilde{m}_1, \tilde{m}_2 \in \Delta(\mathcal{I}_o^*(u))$  and  $\beta \in (0, 1)$ , then

$$\begin{aligned} e(\beta\tilde{m}_1 + (1 - \beta)\tilde{m}_2) &= c((\beta\tilde{m}_1 + (1 - \beta)\tilde{m}_2) \circ \theta_u^*) \\ &= c(\beta\tilde{m}_1 \circ \theta_u^* + (1 - \beta)\tilde{m}_2 \circ \theta_u^*) \\ &\leq \beta c(\tilde{m}_1 \circ \theta_u^*) + (1 - \beta)c(\tilde{m}_2 \circ \theta_u^*) \\ &= \beta e(\tilde{m}_1) + (1 - \beta)e(\tilde{m}_2), \end{aligned} \tag{38}$$

where (38) follows from the fact that  $c$  is convex. Therefore,  $e$  is convex. Let  $(\tilde{m}_n) \in \Delta(\mathcal{I}^*(u))^\infty$  be such that  $\tilde{m}_n \rightarrow \tilde{m} \in \Delta(\mathcal{I}^*(u))$ . Then  $\int \tilde{G} d\tilde{m}_n \rightarrow \int \tilde{G} d\tilde{m}$  for every  $\tilde{G} \in C(\mathcal{I}^*(u))$ . Now note that for each  $G \in C(\mathcal{P}^*)$  and  $m \in \Delta(\mathcal{P}^*)$ ,

$$\begin{aligned} \int_{\mathcal{P}^*} G dm &= \int_{\mathcal{P}^*} (G \circ \theta_u^{*-1} \circ \theta_u^*) dm \\ &= \int_{\mathcal{P}^*} (\tilde{G} \circ \theta_u^*) dm \\ &= \int_{\mathcal{I}^*(u)} \tilde{G} d(m \circ \theta_u^{*-1}), \end{aligned} \tag{39}$$

where  $\tilde{G} = G \circ \theta_u^{*-1}$  and the last equality follows from the change of variables theorem (e.g., Aliprantis and Border 1999, Theorem 12.46). Hence, for  $(\tilde{m}_n \circ \theta_u^*) \in \Delta(\mathcal{P}^*)^\infty$  and any  $G \in C(\mathcal{P}^*)$ , we have

$$\int_{\mathcal{P}^*} G d(\tilde{m}_n \circ \theta_u^*) = \int_{\mathcal{I}^*(u)} \tilde{G} d\tilde{m}_n \rightarrow \int_{\mathcal{I}^*(u)} \tilde{G} d\tilde{m} = \int_{\mathcal{P}^*} G d(\tilde{m} \circ \theta_u^*),$$

so that  $\tilde{m}_n \circ \theta_u^* \rightarrow \tilde{m} \circ \theta_u^*$ . Now using the fact  $c$  is l.s.c., we obtain

$$\liminf_n e(\tilde{m}_n) = \liminf_n c(\tilde{m}_n \circ \theta_u^*) \geq c(\tilde{m} \circ \theta_u^*) = e(\tilde{m})$$

and, thus,  $e$  is l.s.c. as well. The proof that  $e$  is also grounded is straightforward and, thus, omitted.  $\square$

CLAIM 8. *The function  $c_o$  is convex, l.s.c., and grounded.*

PROOF. Because the set  $D_{I_o}$  is a closed subset of a compact metric space (this is easy to verify), and  $c$  is convex and l.s.c. (Claim 7), the minimum is attained in (37). Convexity of the function  $e$  now follows from the observations just made and standard arguments (which we omit). Now define the correspondence  $H : \mathcal{I}^*(u) \rightrightarrows \Delta(\mathcal{I}^*(u))$  as  $H(I_o) = D_{I_o}$ . It is clearly upper hemicontinuous and compact-valued. Using a version of the maximum theorem (e.g., Aliprantis and Border 1999, Lemma 16.30), we conclude that  $c_o$  is l.s.c. Because  $e$  is itself grounded and the minimum is attained in (37),  $c_o$  is grounded as well.  $\square$

It remains to show that (36) equals  $\min_{I_o \in \mathcal{I}_o^*(u)} \left\{ \int_{\mathcal{F}} I_o(u(f)) dR(f) + c_o(I_o) \right\}$ . Note that

$$\begin{aligned} \int_{\mathcal{P}^*} u(\pi^*(f, \succ^*)) dm(\succ^*) + c(m) &= \int_{\mathcal{P}^*} \theta_u^*(\succ^*)(u(f)) dm(\succ^*) + e(m \circ \theta_u^{*-1}) \\ &= \int_{\mathcal{I}^*(u)} I(u(f)) d(m \circ \theta_u^{*-1})(I) + e(m \circ \theta_u^{*-1}) \quad (40) \\ &=: \Xi(f, m \circ \theta_u^{*-1}), \end{aligned}$$

where (40) follows from (39). Therefore, it suffices to show that

$$\min_{\tilde{m} \in \Delta(\mathcal{I}^*(u))} \int_{\mathcal{F}} \Xi(f, \tilde{m}) dR(f) = \min_{I_o \in \mathcal{I}_o^*(u)} \left\{ \int_{\mathcal{F}} I_o(u(f)) dR(f) + c_o(I_o) \right\}. \quad (41)$$

Let  $\tilde{m}_R \in \arg \min_{\tilde{m} \in \Delta(\mathcal{I}^*(u))} \int_{\mathcal{F}} \Xi(f, \tilde{m}) dR(f)$  and let  $I_{o,R} \in C_b(u(\Delta(X))^S)$  be such that  $I_{o,R}(u(f)) = \int I(u(f)) d\tilde{m}_R$  for all  $f \in \mathcal{F}$ . Note that  $\tilde{m}_R \in \arg \min_{\tilde{m} \in D_{I_{o,R}}} e(\tilde{m})$ . Hence

$$\begin{aligned} \min_{\tilde{m} \in \Delta(\mathcal{I}^*(u))} \int_{\mathcal{F}} \Xi(f, \tilde{m}) dR(f) &= I_{o,R}(u(f)) + e(\tilde{m}_R) \\ &\geq \min_{I_o \in \mathcal{I}_o^*(u)} \left\{ \int_{\mathcal{F}} I_o(u(f)) dR(f) + c_o(I_o) \right\} \quad (42) \end{aligned}$$

The inequality in (42) cannot be strict as that would contradict the definition of  $\tilde{m}_R$ . Therefore, (41) holds.

#### A.9 Proof of Proposition 7

We show only that (a) implies (b). The proof adapts the arguments of Theorem 4 of Gilboa et al. (2010).



If  $\succeq^*$  belongs to  $\text{EU}(\mathcal{P}^*)$ , then  $\succeq^*$  is represented by  $P \mapsto \int_{\mathcal{F}} w(\pi^*(f, \succ^*)) dP(f)$  for some  $w \in C_b(\Delta(X))$  such that  $w(p) = \phi(u(p))$ . By adapting [Claim 2](#), we can construct  $W_{\text{Rawls}}: \Delta(\mathcal{F}) \rightarrow \mathbb{R}$ , which represents  $\succ$  and agrees with  $R_c \mapsto \int_{\Delta(X)} \phi(u(p)) dR_c(p)$  in the set  $\Delta(\mathcal{F}_c)$ . It also satisfies the following property: if  $P \in \Delta(\mathcal{F})$ , then there exists  $P_c \in \Delta(\mathcal{F}_c)$  such that  $W_{\text{Rawls}}(P) = W_{\text{Rawls}}(P_c)$ . Moreover, because  $\succ^* \mapsto \int_{\mathcal{F}} \phi(u(\pi^*(f, \succ^*))) dP(f)$  is continuous, using [Axiom B5](#) we obtain  $W_{\text{Rawls}}(P) \geq \min_{\succ^* \in \mathcal{P}^*} \int_{\mathcal{F}} \phi(u(\pi^*(f, \succ^*))) dP(f)$ . If the inequality is strict, then  $P_c \succ \Psi_{\pi^*}(P, \succ^*_0)$  for some  $\succ^*_0 \in \mathcal{P}$ , where  $\Delta(\mathcal{F}_c) \ni P_c \sim P$ . Therefore,  $P_c \succ P$  by [Axiom B8](#), a contradiction.

A.10 Proof of Proposition 8

The proof of the necessity of the axioms is as in the proof of [Theorem 2](#). We now show that the postulates are sufficient for the representation. The axioms of part (a) of [Proposition 8](#) imply the axioms of part (a) of [Theorem 2](#). Consider the function  $J: \Phi \rightarrow \mathbb{R}$  as constructed in the proof of [Theorem 2](#). Using the same arguments as in the proof of [Theorem 2](#), one can show that it is monotone and concave. Moreover, it follows from [Axiom B3\\*\\*](#) that it satisfies  $J(\lambda\xi + (1 - \lambda)a) = \lambda J(\xi) + (1 - \lambda)a$  for all  $\xi \in \Phi$ ,  $a \in \phi(u(\Delta(X)))$ , and  $\lambda \in (0, 1)$ . Also, we can assume w.l.o.g. that  $J(0) = 0$  and  $J(1) = 1$ . Therefore, it follows from Theorem C of [Nascimento \(2011\)](#) that there exists a set  $\widehat{\mathbf{M}}$  of Borel measures on  $\mathcal{P}^*$  such that  $J(\xi) = \min_{m \in \widehat{\mathbf{M}}} \int_{\mathcal{P}^*} \xi(\succ^*) dm(\succ^*)$  and  $m(\mathcal{P}^*) = m'(\mathcal{P}^*) = J(1)$  for all  $m, m' \in \widehat{\mathbf{M}}$ . Hence  $\widehat{\mathbf{M}} \subseteq \Delta(\mathcal{P}^*)$ . Now let  $\mathbf{M} := \overline{\text{co}}(\widehat{\mathbf{M}})$  and note that  $J(\xi) = \min_{m \in \mathbf{M}} \int_{\mathcal{P}^*} \xi(\succ^*) dm(\succ^*)$ .

APPENDIX B: UNIQUENESS

The main representation in [Section 4](#) faces the same problems of uniqueness of the utilitarian weights as in [Harsanyi’s \(1955\)](#) theorem. In particular, the measure in the generalized utilitarian representation in (8) need not be unique. In fact, given any two representations  $(u_0, \phi_0, m_0)$  and  $(u_1, \phi_1, m_1)$  in the sense of [Theorem 1](#), the measures  $m_0, m_1 \in \Delta(\mathcal{P}^*)$  are related by

$$\begin{aligned} \int_{\mathcal{P}^*} \left[ \int_{\mathcal{F}} \phi_0(u_0(\pi^*(f, \succ^*))) dP(f) \right] dm_0(\succ^*) \\ = \int_{\mathcal{P}^*} \left[ \int_{\mathcal{F}} \phi_0(u_0(\pi^*(f, \succ^*))) dP(f) \right] dm_1(\succ^*) \end{aligned} \tag{43}$$

for all  $P \in \Delta(\mathcal{F})$ . Uniqueness of the probability measure  $m \in \Delta(\mathcal{P}^*)$  in our framework is associated with the “richness” of the set  $\Phi$  of all mappings of the form  $\succ^* \mapsto \int_{\mathcal{F}} \phi(u(\pi^*(f, \succ^*))) dP(f)$  ( $P \in \Delta(\mathcal{F})$ ). This is discussed further in [Seo \(2009\)](#) for the case in which  $\mathcal{P}$  is a maximal and weakly compatible subclass of SEU preferences.

In a finite setting, [Weymark \(1991\)](#) mentions an assumption called *independent prospects* that allows one to pin down a unique utilitarian weight. For a finite class of opinions, that condition applied to the domain  $\mathcal{F}$  is sufficient for the uniqueness of the utilitarian weight. An extension of this condition to our framework with a not necessarily finite set of ex post preferences is not well understood at this point. We observe

only that the probability measures  $m_0$  and  $m_1$  fulfilling the condition in (43) are necessarily identical if and only if the set  $\Phi$  satisfies the property  $\text{cl}_{\|\cdot\|_\infty}(\text{span}(\Phi)) = C(\mathcal{P}^*)$  (cf. McAfee and Reny 1992, Theorem 3).

For the model in Section 5, note that the representation in (14) is completely characterized by a triple  $(u, \phi, c)$ . Say that the triple  $(u, \phi, c^*)$  is a “minimal representation” if it induces a representation in the sense of Theorem 2, where the function  $c^*$  is given as in (15). A uniqueness result for Theorem 2 reads as follows: if  $(u_0, \phi_0, c_0^*)$  and  $(u_1, \phi_1, c_1^*)$  induce two minimal representations for  $\succsim$  in the sense of Theorem 2, then there exist  $(\tilde{\alpha}, \tilde{\beta}), (\hat{\alpha}, \hat{\beta}) \in \mathbb{R}_{++} \times \mathbb{R}$  such that  $u_0 = \tilde{\alpha}u_1 + \tilde{\beta}$ ,  $\phi_0 \circ u_0 = \hat{\alpha}\phi_1 \circ u_1 + \hat{\beta}$ , and  $c_0^* = \hat{\alpha}c_1^*$ . This is similar to the uniqueness of the variational preferences when the set of acts is not necessarily order unbounded. It is reminiscent of cardinal uniqueness in the standard expected utility theory.

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