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## Article

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## Incomplete markets with no Hart points

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We provide a geometric test of whether a general equilibrium incomplete markets (GEI) economy has Hart points—points at which the rank of the securities payoff matrix drops. Condition (H) says that, at each nonterminal node, there is an affine set (of appropriate dimension) that intersects all of a well-specified set of convex polyhedra. If the economy has Hart points, then Condition (H) is satisfied; consequently, if condition (H) fails, the economy has no Hart points. The shapes of the convex polyhedra are determined by the number of physical goods and the dividends of the securities, but are independent of the endowments and preferences of the agents. Condition (H) fails, and thus there are no Hart points, in interesting classes of economies with only short-lived securities, including economies obtained by discretizing an economy with a continuum of states and sufficiently diverse payoffs.

KEYWORDS. Incomplete markets, GEI model, Hart points.

JEL CLASSIFICATION. D52.

### 1. INTRODUCTION

In the general equilibrium with incomplete markets (GEI) model, the usual proofs of existence of equilibrium that apply in the Arrow–Debreu complete markets case do not hold. Moreover, existence holds generically but not universally: given the preferences of the agents, a tree describing the states, and a fixed number of securities, there is a generic set (an open set of full measure) in the space of endowments and security payoffs such that an equilibrium exists.

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The barrier to universal existence, and to the use of the usual complete markets proofs, lies in the existence of Hart points: points where the rank of the security returns matrix drops. At Hart points, budget sets fail to be upper hemicontinuous, and consequently demand need not be upper hemicontinuous. Hart (1974) showed that equilibrium may fail to exist; in his example, the natural candidate for an equilibrium price is itself a Hart point. For this reason, the proofs in the literature starting from Duffie and Shafer (1985, 1986) make use of mathematical constructs such as the Grassmanian (the space of all  $J$ -dimensional vector subspaces of  $\mathbf{R}^S$ ). The proofs work through the Grassmanian to avoid the Hart point problem and establish universal existence of pseudoequilibria. They then show that the pseudoequilibrium price is generically not a Hart point, and hence is an equilibrium. These proofs are quite difficult, and the standard text on the GEI model, Magill and Quinzii (1996), does not contain a proof.<sup>1</sup> For this reason, it is very desirable to obtain interesting classes of GEI economies in which the usual complete markets proof works.

In this paper, we provide a geometric test on the securities payoffs to determine whether the economy has Hart points. Suppose there are  $S$  states,  $J$  securities, and  $L$  physical goods. In that case, there are  $S$  convex polyhedra, each a subset of the  $J - 1$  dimensional simplex. Each convex polyhedron is the intersection of  $L$  half-spaces. Condition (H) is that there is a hyperplane in the  $J - 1$ -dimensional simplex that intersects every convex polyhedron. If  $J = 2$ , a hyperplane in the  $J - 1$ -dimensional simplex is a point, so condition (H) simply says that the convex polyhedra have nonempty intersection. We show that if an economy has Hart points, then condition (H) is satisfied; consequently, if condition (H) fails, the economy does not have any Hart points. We show also that if the economy has only short-lived securities, then condition (H) holds if and only if there are Hart points.

The geometric characterization we give is equivalent to the nonexistence of solutions of a set of polynomial equations. As a consequence, the methods of algebraic geometry give effective algorithms to determine whether the geometric characterization is satisfied for a given economy. In addition, if the characterization fails, so Hart points exist, the collection of Hart points is an algebraic variety; effective algorithms exist to describe the set of Hart points, for example by determining an upper bound on the number of isolated Hart points. See Raimondo (2003) for more details.

For any securities payoff matrix for which condition (H) fails, the usual complete markets proof suffices to establish existence of equilibrium for all possible endowment vectors, not just for endowment vectors in a generic set. Moreover, one can prove existence of equilibrium using an index-theoretic argument, and thereby establish that the number of equilibria is generically odd.

Economies with long-lived securities typically have Hart points. However, there are interesting classes of economies with short-lived securities in which condition (H) fails, and thus there are no Hart points. For example, suppose we begin with a discrete-time economy with a continuum of states in which the securities payoffs are multivariate

<sup>1</sup>Magill and Quinzii indicate the proof will be presented in a planned second volume, but we understand that this second volume is unlikely to appear in the foreseeable future.

lognormal.<sup>2</sup> Multivariate lognormal random variables are ubiquitous in finance; for example, the securities prices in the geometric Brownian motion model are multivariate lognormal. If we discretize the economy in a straightforward way, condition (H) fails in all sufficiently fine discretizations, and consequently these discretizations have no Hart points.

Discretized versions of economies with continuous distributions are important for several reasons.

- Except in quite special cases, equilibria of the continuous models cannot be computed in closed form. The only practical way to solve these models is to discretize them and apply standard algorithms for computing discrete equilibria. While there are algorithms to compute equilibria of discrete GEI economies with Hart points, these algorithms are complex because demand typically exhibits an unbounded discontinuity in the neighborhood of each Hart point. In complete markets economies, simpler and faster methods work fine. We believe that, in GEI economies without Hart points, these simpler, faster methods should also work well. In particular, we hope they will make it tractable to compute the equilibria of more complex economies than is currently possible.
- Many people regard discrete models as the appropriate models for “real” economies. For example, the prices of securities are in practice constrained to lie on a discrete grid. For those who take this view, continuous models are useful idealizations, but reality is a discrete approximation to a normal or other continuous model. Our results identify a substantial class of discrete models that are well-behaved.
- The Capital Asset Pricing Model (CAPM) and the Black–Scholes Model are the two most important models in finance. CAPM is a discrete-time GEI model with a continuum of states. Surprisingly little is known about existence of equilibrium in such models with more than two periods; see [Mas-Colell and Zame \(1996\)](#) and [Raimondo \(2002\)](#). The Black–Scholes model is a continuous-time GEI model with a continuum of states that allows pricing and replication of options in closed form. The Cox–Ross–Rubinstein model ([Cox et al. 1979](#)) is a discrete-time, discrete-state version of the Black–Scholes model that provides a mathematically simpler approach to the basic results on option pricing and replication.
- The proof of existence of equilibrium in discrete GEI models uses a fixed-point theorem on a finite-dimensional manifold, called the Grassmanian. Adapting that proof to GEI models with a continuum of states appears impossible because it is unlikely that the infinite-dimensional analogue of the Grassmanian has a fixed-

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<sup>2</sup>A family of random variables  $X_1, \dots, X_n$  is multivariate normal if there is a symmetric  $n \times n$  matrix  $\sigma$  such that the probability density of the joint distribution of  $(X_1, \dots, X_n)$  at  $x = (x_1, \dots, x_n)$  is proportional to  $e^{-x'\sigma x}$ , where  $x'$  is the transpose of the column vector  $x$ . A family of random variables  $(Z_1, \dots, Z_n)$  is multivariate lognormal if the family  $(\log Z_1, \dots, \log Z_n)$  is multivariate normal.

point property. Proofs of existence of equilibrium with complete markets and infinite-dimensional commodity spaces analyze the set of Pareto optima, a finite-dimensional manifold to which a fixed-point argument can be applied. Adapting that proof to GEI models with a continuum of states appears impossible because with incomplete markets, equilibria need not be Pareto optimal. Discretization appears to be the only possible approach to resolving the existence question with incomplete markets and a continuum of states. Discretization is used by [Monteiro \(1996\)](#) in a two-period, continuum of states GEI model. Continuous-time finance models are GEI models with a continuum of states. [Anderson and Raimondo \(2006\)](#) provide the first complete proof of the existence of equilibrium in continuous-time financial markets with multiple agents without relying on endogenous assumptions. Their theorem applies to the case in which there are enough securities to potentially dynamically complete the markets. It would be highly desirable to extend the equilibrium existence result to the case in which there are too few securities to achieve dynamic completeness. Their argument focuses on an appropriate discretization of the continuous-time economy. Getting a better understanding of the behavior of discretizations of continuous economies is a necessary first step in trying to extend their argument to the dynamically incomplete case.

## 2. THE GEOMETRIC TEST

The security market we consider is the event-tree commodity space described in Section 18 of [Magill and Quinzii \(1996\)](#), except that our securities pay off in physical goods. The set of time periods is  $\mathbf{T} = \{0, 1, \dots, T\}$ . There are  $L$  physical goods, numbered  $\ell = 1, \dots, L$ . The set of states is  $\mathbf{S} = \{1, 2, \dots, S\}$ . There is a filtration on the set of states that determines the nodes that traders can distinguish at any time period. The set of nodes is denoted  $\mathbf{D} = \{0, \dots, D\}$ , where 0 is the unique node in period 0; the set of nonterminal nodes is denoted  $\mathbf{D}^-$ . Given a nonterminal node  $\xi \in \mathbf{D}^-$ , the set of immediate successor nodes is denoted  $\xi^+$ , and  $b(\xi) = |\xi^+|$  is the *branching number* of the tree at  $\xi$ . Let  $\mathbf{D}^+(\xi)$  denote the set of all nodes, other than  $\xi$  itself, in the event subtree beginning at  $\xi$ . Given a non-initial node  $\xi$ , the immediate predecessor node is denoted by  $\xi^-$ . The set of securities available for purchase at each nonterminal node  $\xi$  is  $\mathbf{J}(\xi)$ , with  $|\mathbf{J}(\xi)| = J(\xi)$ ; let  $\mathbf{J} = \cup_{\xi \in \mathbf{D}^-} \mathbf{J}(\xi)$  and  $J = |\mathbf{J}|$ .

In each period except the terminal period  $T$ , the securities pay dividends in goods, then spot markets for goods and securities open (so securities in these markets are priced *ex dividend*). There is no spot market for securities in the terminal period  $T$  (since there is no period  $T + 1$ , they cannot pay dividends, so if there were a market, the securities would be priced at zero). A short-lived security is a security that can be bought only at some single node  $\xi \in \mathbf{D}$  and that pays dividends only in the immediate successor nodes of  $\xi$ . Securities that are not short-lived are called long-lived.

Security  $j$  pays dividend  $a^j(\xi) \in \mathbf{R}_+^L$  at node  $\xi$ . We assume that every security  $j \in \mathbf{J}(\xi)$  that is available for purchase at node  $\xi$  pays a nonzero dividend at some (not necessarily

immediate) successor node  $\xi' \in \mathbf{D}^+(\xi)$  of  $\xi$ .<sup>3</sup> We assume also that, for every  $\xi \in \mathbf{D}^-$  and every immediate successor node  $\xi' \in \xi^+$ , there is some security  $j \in \mathbf{J}(\xi)$  available for purchase at  $\xi$  and some  $\xi'' \in (\mathbf{D}^+(\xi') \cup \{\xi'\})$  such that  $a_j(\xi'') \neq 0$ .<sup>4</sup> At each node  $\xi \in \mathbf{D}$ , we denote the vector of goods prices by  $p(\xi) \in \mathbf{R}_+^L$  and the vector of securities prices by  $q(\xi) \in \mathbf{R}_+^J$ ; since the dividends of the securities are nonnegative, absence of arbitrage requires that the prices of the securities be nonnegative. As usual in GEI models, there is a free normalization at each node of the prices of the goods and securities traded at that node, so we assume that  $\sum_{\ell=1}^L p(\xi)_\ell + \sum_{j=1}^J q(\xi)_j = 1$ . Some price systems  $(p, q)$  admit arbitrage (see [Magill and Quinzii 1996](#)).

It is well known that even if preferences and endowments satisfy standard assumptions, economies of this type need not have equilibria. However, [Duffie and Shafer \(1985, 1986\)](#) prove that if preferences satisfy standard assumptions, there exists an open set  $\Omega$  of security payoffs and endowments, whose complement has Lebesgue measure zero, such that an equilibrium exists for any economy with  $(\omega, a)$  in  $\Omega$ .

The reason that existence fails for certain  $(\omega, a)$  (and the reason that the generic existence theorem is hard) is that the rank of the asset payoff matrix may fall at certain points, called Hart points. At Hart points, the budget set fails to be lower hemicontinuous, and consequently the demand may fail to be continuous.<sup>5</sup> Indeed, if the natural candidate for the equilibrium price is itself a Hart point, equilibrium may fail to exist ([Hart 1974](#)). We present a more formal definition of Hart points.

Let  $W(p, q, a)$  denote the  $(D+1) \times (\sum_{\xi \in \mathbf{D}^-} J(\xi))$  matrix (see [Magill and Quinzii 1996](#), p. 227) constructed as follows. There is one row for each node  $\xi \in \mathbf{D}$ , and  $J(\xi)$  columns for each nonterminal node  $\xi \in \mathbf{D}^-$ . For each nonterminal  $\xi \in \mathbf{D}^-$ , the entries in the columns corresponding to  $\xi$  are zero except for the row corresponding to the node  $\xi$  and the rows corresponding to the immediate successor nodes of  $\xi$ . The entries in the row corresponding to  $\xi$  are the coordinates of  $-q(\xi)$ , the negative of the vector of prices of securities in  $\mathbf{J}(\xi)$  (those securities available for purchase at  $\xi$ ). This represents the price of purchasing a portfolio of the securities in  $\mathbf{J}(\xi)$  to carry forward. For each immediate successor node  $\xi' \in \xi^+$ , the entries in the row corresponding to  $\xi'$  are the coordinates of

$$V(p, a, \xi') + q(\xi') = (p(\xi') \cdot a_1(\xi') + q_1(\xi'), \dots, p(\xi') \cdot a_{J(\xi)}(\xi') + q_{J(\xi)}(\xi'))$$

with  $q_j(\xi') = 0$  if  $j \notin \mathbf{J}(\xi')$ . This represents the value (at the spot prices  $p(\xi')$ ) of the

<sup>3</sup>This assumption is for notational convenience, and is essentially without loss of generality. If  $a_j(\xi') = 0$  for all  $\xi' \in \mathbf{D}^+(\xi)$ , then the security is a dummy at the node  $\xi$ , and any arbitrage-free pricing system must assign the security a value of zero at node  $\xi$ . One can apply our results to the economy in which the security is not available for sale at node  $\xi$ .

<sup>4</sup>This assumption is made for simplicity. Our theorems go through without this assumption, provided that one makes appropriate modifications to the definitions of conditions (H) and (H+) below.

<sup>5</sup>In financial markets, there is typically an infinite discontinuity of demand at Hart points. As prices converge to a Hart point, agents can go unboundedly long in some assets and short in other assets; since the assets are nearly linearly dependent, the risk in holding such positions is bounded. Moreover, agents typically *choose* to go unboundedly long in some assets and short in other assets in order to transfer bounded amounts of consumption among nodes.

securities at node  $\xi'$ , coming from their dividends  $(V(p, a, \xi'))$  and (if they are available for trade at  $\xi'$ ) their ex dividend prices  $q(\xi')$ .<sup>6</sup>

For a short-lived security that can be purchased at node  $\xi$  and pays dividends only at the immediate successor nodes, we have  $q(\xi') = 0$  for each immediate successor node  $\xi'$ . If there are only short-lived securities in the model, then for each nonterminal  $\xi \in \mathbf{D}^-$ , the entries in the columns corresponding to  $\xi$  are zero except for the row corresponding to the node  $\xi$  and the rows corresponding to the immediate successor nodes of  $\xi$ . The entries in the row corresponding to  $\xi$  are the coordinates of  $-q(\xi)$ , the negative of the security price vector at the node  $\xi$ ; this represents the cost of purchasing a portfolio of the  $J(\xi)$  securities to carry forward. For each immediate successor node  $\xi'$  of  $\xi$ , the entries in the row corresponding to  $\xi'$  are the coordinates of

$$V(p, a, \xi') = (p(\xi') \cdot a_1(\xi'), \dots, p(\xi') \cdot a_{J(\xi)}(\xi')).$$

This represents the value (at the spot prices  $p(\xi')$ ) of the securities at node  $\xi'$ , coming solely from their dividends  $(V(p, a, \xi'))$ .

DEFINITION 1. Fix the asset payoff matrix  $a$ . A *Hart point* is an arbitrage-free price system  $(p, q)$  such that

$$\text{rank } W(p, q, a) < \sum_{\xi \in \mathbf{D}^-} \min\{b(\xi), J(\xi)\}.$$

Let

$$\Delta^{J(\xi)-1} = \left\{ q \in \mathbf{R}_+^{J(\xi)} : \sum_{j=1}^{J(\xi)} q_j = 1 \right\}$$

denote the  $(J(\xi)-1)$ -dimensional simplex. Given a nonterminal node  $\xi \in \mathbf{D}^-$  and an immediate successor node  $\xi' \in \xi^+$ , let  $\mathbf{L}(\xi, \xi') \subset \mathbf{J}(\xi')$  denote the set of securities available for purchase at  $\xi$  which pay a nonzero dividend at some node  $\xi'' \in \mathbf{D}^+(\xi')$ .

DEFINITION 2. *a.* Let  $\xi \in \mathbf{D}^-$  be a nonterminal node and  $\xi' \in \xi^+$  an immediate successor of  $\xi$ . Let  $P_{\xi'}$  denote the convex polyhedron<sup>7</sup>

$$\{(p \cdot a_1(\xi'), \dots, p \cdot a_{J(\xi)}(\xi')) + q \in \Delta^{J(\xi)-1} : p \in \mathbf{R}_{++}^L, q \in \mathbf{R}_+^{J(\xi)}, q_j > 0 \Leftrightarrow j \in \mathbf{L}(\xi, \xi')\}.$$

By our assumptions, for every  $\xi' \in \xi^+$ , there exists  $j \in \mathbf{J}(\xi)$  such that either  $a_j(\xi') \neq 0$  or  $j \in \mathbf{L}(\xi, \xi')$ , so  $P_{\xi'} \neq \emptyset$ .

*b.* An economy satisfies condition (H) if there is a nonterminal node  $\xi \in \mathbf{D}^-$  and an affine subspace  $H \subset \Delta^{J(\xi)-1}$  with  $\dim H < \min\{b(\xi), J(\xi)\} - 1$  such that

$$H \cap P_{\xi'} \neq \emptyset$$

for every immediate successor  $\xi' \in \xi^+$ .

<sup>6</sup>This form imposes a slight restriction on the securities, namely that each security is available for trade at every node between the one in which it is first introduced and the one at which it pays its last dividend.

<sup>7</sup>The polyhedron  $\tilde{P}_{\xi'}$  is open in the affine space it generates; its closure is a convex polyhedron in the affine space.

- c. An economy satisfies condition (H+) if there is a nonterminal node  $\xi \in \mathbf{D}^-$  with  $\mathbf{L}(\xi, \xi') = \emptyset$  for all  $\xi' \in \xi^+$  and an affine subspace  $H \subset \Delta^{J(\xi)-1}$  with  $\dim H < \min\{b(\xi), J(\xi)\} - 1$  such that

$$H \cap P_{\xi'} \neq \emptyset$$

for every immediate successor  $\xi' \in \xi^+$ .

EXAMPLE 1. Let  $\mathbf{T} = \{0, 1\}$  (there are two time periods),  $\mathbf{D} = \{0, 1, 2\}$ , with  $0^+ = \{1, 2\}$  (there is one initial node with two immediate successor nodes), and  $L = 2$  (there are two physical goods). Let  $\mathbf{J}(0) = \{1, 2\}$  (there are two securities available for purchase at the initial node) with dividends

$$\begin{aligned} a_1(1) &= (1, 2) & a_1(2) &= (2, 1) \\ a_2(1) &= (1, 1) & a_2(2) &= (1, 1). \end{aligned}$$

Since  $\mathbf{L}(0, 1) = \mathbf{L}(0, 2) = \emptyset$ , we have

$$\begin{aligned} P_1 &= \{(p \cdot a_1(1), p \cdot a_2(1)) \in \Delta^1 : p \in \mathbf{R}_{++}^2\} \\ &= \{(p_1 + 2p_2, p_1 + p_2) : p \in \mathbf{R}_{++}^2, p_1 + 2p_2 + p_1 + p_2 = 1\} \\ &= \{(x, 1 - x) : x \in (\frac{1}{2}, \frac{2}{3})\} \\ P_2 &= \{(p \cdot a_1(2), p \cdot a_2(2)) \in \Delta^1 : p \in \mathbf{R}_{++}^2\} \\ &= \{(2p_1 + p_2, p_1 + p_2) : p \in \mathbf{R}_{++}^2, 2p_1 + p_2 + p_1 + p_2 = 1\} \\ &= \{(x, 1 - x) : x \in (\frac{1}{2}, \frac{2}{3})\} \\ &= P_1. \end{aligned}$$

Since  $\Delta^1$  is 1-dimensional, a 0-dimensional affine subspace of  $\Delta^1$  is just a single point. If we take any  $x_0 \in (\frac{1}{2}, \frac{2}{3})$  and set  $H = \{(x_0, 1 - x_0)\}$ , then  $H \cap P_1 = H \cap P_2 \neq \emptyset$ , so condition (H) is satisfied; since  $\mathbf{L}(0, 1) = \mathbf{L}(0, 2) = \emptyset$ , condition (H+) is also satisfied.

Fix any spot price vector  $p(1) = (p(1)_1, p(1)_2) \in \mathbf{R}_{++}^2$  for node 1. Consider the spot price vector  $p(2) = (p(1)_2, p(1)_1)$  for node 2. Then we have

$$\begin{aligned} V(p, a, 2) &= (p(2) \cdot a_1(2), p(2) \cdot a_2(2)) \\ &= (2p(2)_1 + p(2)_2, p(2)_1 + p(2)_2) \\ &= (2p(1)_2 + p(1)_1, p(1)_2 + p(1)_1) \\ &= (p(1)_1 + 2p(1)_2, p(1)_1 + p(1)_2) \\ &= (p(1) \cdot a_1(1), p(1) \cdot a_2(1)) \\ &= V(p, a, 1). \end{aligned}$$

At these spot prices, the “dollar” payouts of the two securities are the same in state 1 as in state 2; moreover, if we multiply the spot price vectors  $p(1)$  and  $p(2)$  by any two positive scalars, the vectors of “dollar” payouts of the securities in the two states are collinear. If we choose  $\alpha \in (0, 1)$  and define securities prices at time 0 by

$$\begin{aligned} q(0)_1 &= \alpha p(1) \cdot a_1(1) + (1 - \alpha) p(2) \cdot a_1(2) \\ q(0)_2 &= \alpha p(1) \cdot a_2(1) + (1 - \alpha) p(2) \cdot a_2(2), \end{aligned}$$

then  $(p, q)$  is an arbitrage-free pricing system and a Hart point.  $\diamond$



EXAMPLE 2. Let  $\mathbf{T} = \{0, 1\}$  (there are two time periods),  $\mathbf{D} = \{0, 1, 2\}$ , with  $0^+ = \{1, 2\}$  (there is one initial node with two immediate successor nodes), and  $L = 2$  (there are two physical goods). Let  $\mathbf{J}(0) = \{1, 2\}$  (there are two securities available for purchase at the initial node) with dividends

$$\begin{aligned} a_1(1) &= (1, 2) & a_1(2) &= \left(\frac{1}{2}, 1\right) \\ a_2(1) &= (1, 1) & a_2(2) &= (1, 1). \end{aligned}$$

Since  $\mathbf{L}(0, 1) = \mathbf{L}(0, 2) = \emptyset$ , we have

$$\begin{aligned} P_1 &= \{(p \cdot a_1(1), p \cdot a_2(1)) \in \Delta^1 : p \in \mathbf{R}_{++}^2\} \\ &= \{(p_1 + 2p_2, p_1 + p_2) : p \in \mathbf{R}_{++}^2, p_1 + 2p_2 + p_1 + p_2 = 1\} \\ &= \{(x, 1 - x) : x \in \left(\frac{1}{2}, \frac{2}{3}\right)\} \\ P_2 &= \{(p \cdot a_1(2), p \cdot a_2(2)) \in \Delta^1 : p \in \mathbf{R}_{++}^2\} \\ &= \{(p_1/2 + p_2, p_1 + p_2) : p \in \mathbf{R}_{++}^2, p_1/2 + p_2 + p_1 + p_2 = 1\} \\ &= \{(x, 1 - x) : x \in \left(\frac{1}{3}, \frac{1}{2}\right)\}. \end{aligned}$$

Since  $\Delta^1$  is 1-dimensional, a 0-dimensional affine subspace of  $\Delta^1$  is just a single point. If we take any  $x_0 \in (0, 1)$ , and set  $H = \{(x_0, 1 - x_0)\}$ , then either  $H \cap P_1 = \emptyset$  or  $H \cap P_2 = \emptyset$  or both, so condition (H) fails.

Fix any spot price vectors  $p(1) = (p(1)_1, p(1)_2) \in \mathbf{R}_{++}^2$  for node 1 and  $p(2) = (p(2)_1, p(2)_2) \in \mathbf{R}_{++}^2$  for node 2. Then we have

$$\begin{aligned} V(p, a, 2) &= (p(2) \cdot a_1(2), p(2) \cdot a_2(2)) \\ &= (p(2)_1/2 + p(2)_2, p(2)_1 + p(2)_2) \\ V(p, a, 1) &= (p(1) \cdot a_1(1), p(1) \cdot a_2(1)) \\ &= (2p(1)_2 + p(1)_1, p(1)_2 + p(1)_1). \end{aligned}$$

We see that  $V(p, a, 2)_1 < V(p, a, 2)_2$ , while  $V(p, a, 1)_1 > V(p, a, 1)_2$ , so  $V(p, a, 1)$  and  $V(p, a, 2)$  are not collinear; the economy has no Hart point.  $\diamond$

REMARK 1. Note that if there are only short-lived securities, the following two facts hold.

- For every nonterminal node  $\xi$  and every immediate successor node  $\xi' \in \xi^+$ ,  $\mathbf{L}(\xi, \xi') = \emptyset$ , so condition (H) is equivalent to condition (H+).
- Since  $\mathbf{L}(\xi, \xi') = \emptyset$ , our assumptions imply that  $\sum_{j \in \mathbf{J}(\xi)} a_j(\xi') \neq 0$ . The set  $P_{\xi'}$  is the interior of the closed polyhedron whose vertices are

$$\left\{ \frac{(a_1(\xi')_\ell, \dots, a_{J(\xi)}(\xi')_\ell)}{(\sum_{j \in \mathbf{J}(\xi)} a_j(\xi')_\ell)} : \left( \sum_{j \in \mathbf{J}(\xi)} a_j(\xi') \right)_\ell \neq 0 \right\}.$$

On the other hand, if there are long-lived securities, then there is a great deal of freedom in pricing the securities in an arbitrage-free pricing system, and typically there are Hart points.

THEOREM 1. Fix the event tree structure and the asset return matrix  $a$ .

- If the economy has a Hart point, then it satisfies condition (H).
- If the economy satisfies condition (H+), then it has a Hart point.

PROOF. Suppose  $(\hat{p}, \hat{q})$  is a Hart point, so  $(\hat{p}, \hat{q})$  is an arbitrage-free price system with  $\text{rank } W(\hat{p}, \hat{q}, a) < \sum_{\xi \in \mathbf{D}^-} \min\{b(\xi), J(\xi)\}$ . For each nonterminal node  $\xi \in \mathbf{D}^-$ , let

$$\rho(\xi) = \dim \text{span} \left\{ (\hat{p}(\xi') \cdot a_1(\xi'), \dots, \hat{p}(\xi') \cdot a_{J(\xi')}(\xi')) + \hat{q}(\xi') : \xi' \in \xi^+ \right\}.$$

By Proposition 22.2 of Magill and Quinzii (1996),  $\sum_{\xi \in \mathbf{D}^-} \rho(\xi) = \text{rank } W(\hat{p}, \hat{q}, a)$ . Thus there exists  $\xi \in \mathbf{D}^-$  such that  $\rho(\xi) < \min\{b(\xi), J(\xi)\}$ . If  $\xi' \in \xi^+$  and  $j \in \mathbf{J}(\xi)$ , our assumptions imply that either  $a_j(\xi') \neq 0$ , in which case  $\hat{p}(\xi') \cdot a_j(\xi') > 0$ , or  $j \in \mathbf{L}(\xi, \xi')$ , which implies that  $\hat{q}_j(\xi') > 0$  since  $(\hat{p}, \hat{q})$  is arbitrage-free. Let

$$\begin{aligned} \hat{H} &= \text{span} \left\{ (\hat{p}(\xi') \cdot a_1(\xi'), \dots, \hat{p}(\xi') \cdot a_{J(\xi')}(\xi')) + \hat{q}(\xi') : \xi' \in \xi^+ \right\} \subset \mathbf{R}^{J(\xi)} \\ H &= \hat{H} \cap \Delta^{J(\xi)-1}. \end{aligned}$$

Since  $\hat{H}$  is a vector space,  $\hat{H} \supset \{\alpha h : \alpha \in \mathbf{R}, h \in H\}$ . Therefore  $\dim H < \dim \hat{H}$ , so  $\dim H < \min\{b(\xi), J(\xi)\} - 1$ . Let

$$\begin{aligned} p(\xi') &= \frac{\hat{p}(\xi')}{\sum_{j=1}^{J(\xi)} (\hat{p}(\xi') \cdot a_j(\xi') + \hat{q}(\xi')_j)} \\ q(\xi') &= \frac{\hat{q}(\xi')}{\sum_{j=1}^{J(\xi)} (\hat{p}(\xi') \cdot a_j(\xi') + \hat{q}(\xi')_j)}. \end{aligned}$$

Then

$$(p(\xi') \cdot a_1(\xi'), \dots, p(\xi') \cdot a_{J(\xi')}(\xi')) + q(\xi') \in H \cap P_{\xi'}$$

for each  $\xi' \in \xi^+$ , so condition (H) is satisfied.

Now, suppose condition (H+) is satisfied, so there is a nonterminal node  $\xi \in \mathbf{D}^-$  with  $\mathbf{L}(\xi, \xi') = \emptyset$  for all  $\xi' \in \xi^+$  and an affine space  $H \subset \Delta^{J(\xi)-1}$  with  $\dim H < \min\{b(\xi), J(\xi)\} - 1$  such that  $H \cap P_{\xi'} \neq \emptyset$  for every  $\xi' \in \xi^+$ . Let  $x_{\xi'} \in H \cap P_{\xi'}$ , so that we have

$$x_{\xi'} = (p(\xi') \cdot a_1(\xi'), \dots, p(\xi') \cdot a_{J(\xi')}(\xi')) + q(\xi')$$

for some  $p(\xi') \in \mathbf{R}_{++}^L$  and some  $q(\xi') \in \mathbf{R}_+^{J(\xi)}$ , with  $q_j(\xi') > 0 \Leftrightarrow j \in \mathbf{L}(\xi, \xi')$ . Since  $\mathbf{L}(\xi, \xi') = \emptyset$  for all  $\xi' \in \xi^+$ , we must have  $q(\xi') = 0$ . Let  $\hat{p}(\xi') = p(\xi')$  for  $\xi' \in \xi^+$ , and choose  $\hat{p}(\xi')$  to be an arbitrary element of  $\mathbf{R}_{++}^L$  for every  $\xi' \notin \xi^+$ . Then, for every  $\xi' \in \mathbf{D}$ , let

$$\hat{q}_j(\xi') = \sum_{\xi'' \in \mathbf{D}^+(\xi')} \hat{p}(\xi'') \cdot A_j(\xi'')$$

be the expected future payoff of the  $j$ th security on all the nodes that follow  $\xi'$ . The price system  $(\hat{p}, \hat{q})$  is arbitrage-free, and  $\hat{q}(\xi') = 0$  for all  $\xi' \in \xi^+$ , so  $(\hat{p}, \hat{q})$  agrees with  $(p, q)$  on  $\xi^+$ . Then by Proposition 22.2 of Magill and Quinzii (1996),  $\text{rank } W(\hat{p}, \hat{q}, a) < \sum_{\xi \in \mathbf{D}^-} \min\{b(\xi), J(\xi)\}$ , so  $(\hat{p}, \hat{q}, a)$  is a Hart point.  $\square$

## 3. AN EXISTENCE THEOREM

On its face, the market excess demand for physical goods is a function of both the spot prices  $p$  and the security prices  $q$ . However, it is well known that existence of equilibrium can be demonstrated by writing market excess demand as a function of the spot prices alone. Given a spot price system, we set the price of each security at a node to equal the expected value of its future dividends on the subtree starting at that node. One can then compute the market excess demand function using the Cass trick: the first agent is allowed to trade using a complete set of Arrow–Debreu contingent claims, while the remaining agents must trade using only the available securities and the spot markets. One finds a zero of this market excess demand function, and shows that this zero is a no-arbitrage equilibrium. In a no-arbitrage equilibrium, the spot prices are linked across states by state prices: the state price is the shadow price of income in that state, and as a result, the equilibrium securities prices must equal the expected value of future dividends. Thus, one shows that if one takes any no-arbitrage equilibrium and defines the securities prices to equal the expected value of their future dividends, the resulting price system is an equilibrium of the GEI economy. Sections 10 and 25 of [Magill and Quinzii \(1996\)](#) give a detailed exposition of this in the case in which  $L = 1$  (there is only one physical good at each state); [Magill and Shafer \(1991\)](#) cover the case of more than one physical good.

**THEOREM 2.** *Consider a GEI economy. Assume the following:*

- (i) *all goods are perfectly divisible;*
- (ii) *each agent's preference is continuous, irreflexive, transitive, strongly monotonic, and strictly convex;*
- (iii) *the social endowment of each good is strictly positive in each state;*
- (iv) *for every nonterminal node  $\xi \in \mathbf{D}^-$  and every immediate successor node  $\xi' \in \xi^+$ , there exists a security  $j \in \mathbf{J}(\xi)$  available for purchase at  $\xi$  and a node  $\xi'' \in (\mathbf{D}^+(\xi') \cup \{\xi'\})$  such that  $a_j(\xi'') \neq 0$ ;*
- (v) *the asset return matrix  $a$  does not satisfy condition (H).*

*Then the economy has a GEI equilibrium.*

**PROOF.** The assumptions imply that the market excess demand is a function

$$Z : \Delta^{LS-1} \rightarrow \mathbf{R}^{LS}$$

where  $\Delta^{LS-1}$  is the  $LS - 1$ -dimensional price simplex, which is bounded below (by the negative of the social endowment), and satisfies Walras' Law and the boundary condition (if  $p_n \rightarrow p$  on the boundary of the price simplex, then  $|Z(p_n)| \rightarrow \infty$ ) (see [Magill and Shafer 1991](#), pages 1558–1562). The function  $Z$  is continuous except at Hart points; since there are no Hart points,  $Z$  is continuous. Hence, there exists  $p$  such that  $Z(p) = 0$ , and hence  $p$  is a pseudoequilibrium. Since there are no Hart points,  $p$  is not a Hart point, and hence  $p$  is an equilibrium price.  $\square$

## 4. AN INDEX THEOREM

**Dierker (1972)** first used index-theoretic arguments to show that in complete markets with  $C^1$  demand functions the number of equilibria is generically odd. Because the Grassmanian may or may not be orientable,<sup>8</sup> his argument does not extend to GEI markets. **Hens (1991)** establishes the index theorem and the generic oddness of equilibria for the case  $L = 1$  (one physical good). **Schmedders (1999)** establishes the index theorem for an open (but not necessarily generic) set of two-period economies using a homotopy algorithm. **Kubler and Schmedders (2000)** extend the index theorem in **Schmedders (1999)** to an open (but not necessarily generic) set of economies in a multiple-period model. **Momi (2003)** proves the index theorem for a two-period model in which the degree of market incompleteness ( $S - J$ ) is even.

If there are no Hart points, then Dierker's argument goes through unchanged. The index of an equilibrium  $p$  is defined by  $\text{index}(p) = \text{sign} | -JZ(p) |$ , where  $JZ(p)$  is the Jacobian matrix of the market excess demand function<sup>9</sup> at the price  $p$  and the absolute value sign denotes the determinant.

**THEOREM 3.** *Consider a GEI economy. Assume the following:*

- (i) *all goods are perfectly divisible;*
- (ii) *each agent's preference is defined on  $\mathbf{R}_{++}^{LS}$  and is continuous, irreflexive, transitive, strongly monotonic,  $C^2$ , differentiably strictly convex, and the closure in  $\mathbf{R}^{LS}$  of each indifference curve is contained in  $\mathbf{R}_{++}^{LS}$ ;*
- (iii) *for every nonterminal node  $\xi \in \mathbf{D}^-$  and every immediate successor node  $\xi' \in \xi^+$ , there exists a security  $j \in \mathbf{J}(\xi)$  available for purchase at  $\xi$  and a node  $\xi'' \in (\mathbf{D}^+(\xi') \cup \{\xi'\})$  such that  $a_j(\xi'') \neq 0$ ;*
- (iv) *the asset return matrix  $a$  does not satisfy condition (H).*

*Then for an open set of endowments of full measure, the economy is regular and*

$$\sum_{p \in Z^{-1}(0)} \text{index } p = +1$$

*and hence the number of equilibria is odd.*

**PROOF.** As in the previous section, the market excess demand  $Z$  satisfies Walras' Law, is bounded below, and satisfies the boundary condition. Because preferences are smooth, and there are no Hart points,  $Z$  is  $C^1$ . Then for an open set of endowments of full measure, the economy is regular, and hence satisfies the conditions of Proposition 5.6.1 of **Mas-Colell (1985)**, which proves the theorem.  $\square$

<sup>8</sup>In the two-period model, with one state in period 0 and  $S$  states in period 2, the Grassmanian is orientable if and only if  $S+1$  is even (**Dold 1972**, page 331), i.e. if and only if  $S$  is odd. In the multiperiod model, one must consider a product of Grassmanians.

<sup>9</sup>The function  $Z$  is defined as in **Section 3**, with agent 1 unconstrained.

5. ECONOMIES WITH A CONTINUUM OF STATES

In this section, we study sequences of economies obtained by discretizing an economy in which each nonterminal node has a continuum of immediate successor nodes.

Each nonterminal node  $\xi$  has a probability space of immediate successor nodes. Specifically, fix a complete separable<sup>10</sup> atomless probability space  $(\Omega_0, \mathcal{B}, P)$ . The set of states is  $\Omega = \prod_{t=0}^T (\Omega_0 \times \{t\})$ , endowed with the product measure. The set of nodes is  $\Omega \times \{0, \dots, T\}$ . A typical node is  $\xi = (((\omega_0, 0), \dots, (\omega_T, T)), t)$  where  $\omega_s \in \Omega_0$  for each  $s \in \{0, \dots, T\}$ . In situations where only  $\omega_t$  and  $t$  matter, we abuse notation by referring to this node as  $(\omega_t, t)$  with  $\omega_t \in \Omega_0$ . At each nonterminal node  $\xi = (\omega, t)$ , let

$$\xi_+ = \{(\omega', t + 1) : \omega'_s = \omega_s \text{ for } 0 \leq s \leq t\}$$

denote the set of immediate successor nodes of  $\xi$ . The filtration is given by

$$\mathcal{F}_t = \overline{\prod_{s=0}^t (\mathcal{B} \times \{s\})}.$$

There are  $J$  short-lived securities available for purchase at each nonterminal node. Although each security expires the period after it is purchased, it is convenient to identify the securities available across all nodes as  $\{1, \dots, J\}$ . Security  $j$  pays a dividend  $a_j(\omega_t, t)$  at node  $(\omega_t, t)$ ,  $t \in \{1, \dots, T\}$ ; we assume that  $a_j(\cdot, t) \in L^2(\Omega_0)_+$  is a measurable function of  $\omega_t \in \Omega_0$  and that  $a_j(\cdot, t) > 0$  except on a set of measure zero.

A vector of spot prices is a process  $p \in L^2(\Omega \times \{0, \dots, T\}, \mathbf{R}^L)_+$  that is adapted with respect to the filtration such that for every node  $(\omega, t)$ ,  $|p(\omega, t)| = 1$  (here,  $|p(\omega, t)|$  denotes the Euclidean length of  $p(\omega, t) \in \mathbf{R}^L$ ).

The specification of an economy is completed by specifying endowments in  $L^2(\Omega, \mathbf{R}^L)_+$  that are adapted with respect to the filtration, and preferences for the agents; we assume that the social endowment of each good is strictly positive in almost every state, and that the preferences are continuous, irreflexive, transitive, strongly monotonic, and strictly convex.

REMARK 2. Note that our formulation imposes a restriction on the event tree. Given two nodes  $\xi = (\omega, t)$  and  $\xi' = (\omega', t)$ , the random variable  $a_j(\cdot, t + 1)$  is the *same* random variable on  $\xi_+$  and  $\xi'_+$ , not just the same distribution. However, at time  $t$ , agents' endowments and trading strategies, as well as the security and spot prices, are allowed to depend on the whole history up to and including time  $t$ . Moreover, we do not assume that the utility function is additively separable over time.

DEFINITION 3. A *Hart point* for this economy is a vector of strictly positive spot prices  $p \in L^2(\Omega, \mathbf{R}^L)_{++}$  that is adapted with respect to the filtration such that there exists a time

<sup>10</sup> Given a family  $\mathcal{C}$  of subsets of  $\Omega$ , let  $\sigma(\mathcal{C})$  be the smallest  $\sigma$ -algebra containing  $\mathcal{C}$ . The space  $(\Omega_0, \mathcal{B}, P)$  is complete with respect to  $P$  if  $C \subset B \in \mathcal{B}$  and  $P(B) = 0 \Rightarrow C \in \mathcal{B}$ . Let  $\overline{\sigma(\mathcal{C})}$  denote the smallest complete  $\sigma$ -algebra containing  $\mathcal{C}$ . The space  $(\Omega_0, \mathcal{B}, P)$  is complete separable if there is a countable family  $\mathcal{B}_0 \subset \mathcal{B}$  such that  $\mathcal{B} = \overline{\sigma(\mathcal{B}_0)}$ . The Lebesgue probability space is complete separable; it is, for example,  $\overline{\sigma(\{(p, q) : p, q \in \mathbf{Q}\})}$ .

$t$  and a set of states  $\Omega' \subset \Omega$  of positive probability such that, for all  $\xi = (\omega, t)$  with  $\omega \in \Omega'$ , there is a subspace  $H \subset \mathbf{R}^J$  with  $\dim H < J$  such that for almost all  $\xi' \in \xi_+$ ,

$$(p(\xi') \cdot a_1(\xi'), \dots, p(\xi') \cdot a_J(\xi')) \in H.$$

A *weak Hart point* for this economy is a vector of nonnegative spot prices  $p \in L^2(\Omega, \mathbf{R}^L)_+$  that is adapted with respect to the filtration such that there exists a time  $t$  and a set of states  $\Omega' \subset \Omega$  of positive probability such that, for all  $\xi = (\omega, t)$  with  $\omega \in \Omega'$ , there is a subspace  $H \subset \mathbf{R}^J$  with  $\dim H < J$  such that for almost all  $\xi' \in \xi_+$ ,

$$(p(\xi') \cdot a_1(\xi'), \dots, p(\xi') \cdot a_J(\xi')) \in H.$$

REMARK 3. Because the random variables  $a_{j_t}(\cdot, t+1)$  are the same random variables on  $(\omega, t)_+$  as on  $(\omega', t)_+$ , there is a Hart point in the economy if and only if there exists a time  $t \in \{1, \dots, T\}$ , a strictly positive  $p \in L^2(\Omega_0, \mathbf{R}^L)_{++}$  with  $|p(\omega)| = 1$  for all  $\omega \in \Omega_0$ , and a subspace  $H \subset \mathbf{R}^J$  with  $\dim H < J$  such that for almost all  $\omega \in \Omega_0$ ,

$$(p(\omega) \cdot a_1(\omega, t), \dots, p(\omega) \cdot a_J(\omega, t)) \in H.$$

Similarly, there is a weak Hart point in the economy if and only if there exists a time  $t \in \{1, \dots, T\}$ , a nonnegative  $p \in L^2(\Omega_0, \mathbf{R}^L)_+$  with  $|p(\omega)| = 1$  for all  $\omega \in \Omega_0$ , and a subspace  $H \subset \mathbf{R}^J$  with  $\dim H < J$  such that for almost all  $\omega \in \Omega_0$ ,

$$(p(\omega) \cdot a_1(\omega, t), \dots, p(\omega) \cdot a_J(\omega, t)) \in H.$$

Because the proof of existence of equilibrium considers only strictly positive spot prices, the existence of a weak Hart point does not pose a barrier to proving existence. However, the fact that the set of Hart points is generally not closed presents technical problems in analyzing the Hart points of similar economies. Working with weak Hart points allows us to get around this problem.

We discretize the continuum model by considering finite partitions of  $\Omega_0$ . If  $\mathcal{P}$  is a finite partition of  $\Omega_0$ , the elements of  $\mathcal{P}$  are subsets of  $\Omega_0$ . Given  $\omega \in \Omega_0$ , let  $\mathcal{P}_\omega$  be the element  $B \in \mathcal{P}$  such that  $\omega \in B$ . We can form a discrete model in which, at time  $t$ , agents know that for each  $s \leq t$ ,  $(\omega, s) \in \mathcal{P}_{\omega_s}$ . In other words, agents know the partition set in which the state is located, but not the actual value of the state. The endowments of the agents and the securities payoffs in the discrete model are the expected values of the endowments and securities payoffs from the continuum model, conditional on the partition set in which the actual state lies. In particular, the securities are

$$a_j^{\mathcal{P}} = E(a_j | \mathcal{P}).$$

Since the consumption set in the discrete model is a subset of the consumption set in the continuum model, the preferences of agents are just the restrictions to the discrete consumption set. Assumptions (i)–(iv) of **Theorem 2** are inherited from the assumptions on preferences and endowments in the continuum economy.

THEOREM 4. *Suppose the (undiscretized) economy has no weak Hart points. Let  $\mathcal{P}_n$  be a sequence of finite partitions such that for each  $n$ ,  $\mathcal{P}_{n+1}$  is finer than  $\mathcal{P}_n$  and such that*

$$\mathcal{B} = \overline{\sigma(\cup_{n=1}^{\infty} \mathcal{P}_n)}.$$

Then for  $n$  sufficiently large,

- the economy discretized by  $\mathcal{P}_n$  does not satisfy condition (H), and therefore has no Hart points
- the economy discretized by  $\mathcal{P}_n$  has an equilibrium
- if the preferences in the economy discretized by  $\mathcal{P}_n$  satisfy Assumption (ii) of *Theorem 3*, then for an open set of endowments of full measure, the economy is regular and

$$\sum_{p \in Z^{-1}(0)} \text{index } p = +1$$

and hence the number of equilibria is odd.

PROOF. Fix  $t \in \{0, \dots, T\}$ . Let

$$\begin{aligned} \Delta &= \{\pi \in L^2(\Omega_0, \mathbf{R}^L)_+ : |\pi(\omega_0)| = 1 \text{ almost surely}\} \\ M &= \max\{\|a_1(\cdot, t)\|_2, \dots, \|a_J(\cdot, t)\|_2\} + 1 \\ \Gamma &= \{\alpha \in L^2(\Omega_0, \mathbf{R}^L) : \|\alpha\|_2 \leq M\} \\ S^{J-1} &= \{\theta \in \mathbf{R}^J : |\theta| = 1\} \\ \Phi &= \{\alpha \in L^2(\Omega_0, \mathbf{R}) : \|\alpha\|_2 \leq \sqrt{JM}\}. \end{aligned}$$

Define

$$Z : \Delta \times \Gamma^J \times S^{J-1} \rightarrow \Phi$$

by

$$Z(\pi, (\alpha_1, \dots, \alpha_J), \theta)(\omega_0) = (\pi(\omega_0) \cdot \alpha_1(\omega_0), \dots, \pi(\omega_0) \cdot \alpha_J(\omega_0)) \cdot \theta.$$

Since the undiscretized economy has no weak Hart points,

$$0 \notin Z(\Delta \times \{(a_1(\cdot, t), \dots, a_J(\cdot, t))\} \times S^{J-1})$$

where 0 is the zero element of  $L^2(\Omega_0, \mathbf{R})$ .

Endow  $\Delta$  and  $\Phi$  with the weak-star topology,  $\Gamma^J$  with the  $L^2$ -norm topology,  $S^{J-1}$  with the Euclidean topology, and  $\Delta \times \Gamma^J \times S^{J-1}$  with the product of the topologies on its components.

We claim that  $Z$  is continuous. Note first that since  $\Omega_0$  is a separable probability space, the weak-star topologies on  $\Delta$  and  $\Phi$  are metrizable (see [Conway 1990](#), Theorem 5.1, page 134). Hence, in showing the continuity of  $Z$ , it is sufficient to consider sequences. Consider a sequence  $(\pi_n, \alpha_n, \theta_n) \rightarrow (\pi, \alpha, \theta)$ . We need to show that

$$Z(\pi_n, \alpha_n, \theta_n) \rightarrow Z(\pi, \alpha, \theta).$$

Thus, we must show that for every  $h \in L^2(\Omega_0, \mathbf{R})$ ,

$$\int_{\Omega_0} (Z(\pi_n, \alpha_n, \theta_n) - Z(\pi, \alpha, \theta))h \, d\omega_0 \rightarrow 0.$$

Write

$$\begin{aligned} \int_{\Omega_0} (Z(\pi_n, \alpha_n, \theta_n) - Z(\pi, \alpha, \theta))h \, d\omega_0 \\ = \int_{\Omega_0} (Z(\pi_n, \alpha_n, \theta_n) - Z(\pi_n, \alpha_n, \theta))h \, d\omega_0 \end{aligned} \quad (1)$$

$$+ \int_{\Omega_0} (Z(\pi_n, \alpha_n, \theta) - Z(\pi_n, \alpha, \theta))h \, d\omega_0 \quad (2)$$

$$+ \int_{\Omega_0} (Z(\pi_n, \alpha, \theta) - Z(\pi, \alpha, \theta))h \, d\omega_0. \quad (3)$$

We consider the pieces in turn. For term (1),

$$\begin{aligned} & \left| \int_{\Omega_0} (Z(\pi_n, \alpha_n, \theta_n) - Z(\pi_n, \alpha_n, \theta))h \, d\omega_0 \right| \\ &= \left| \int_{\Omega_0} ((\pi_n(\omega_0) \cdot \alpha_{n1}(\omega_0), \dots, \pi_n(\omega_0) \cdot \alpha_{nJ}(\omega_0)) \cdot (\theta_n - \theta))h(\omega_0) \, d\omega_0 \right| \\ &\leq \int_{\Omega_0} |\theta_n - \theta| \times |(\pi_n(\omega_0) \cdot \alpha_{n1}(\omega_0), \dots, \pi_n(\omega_0) \cdot \alpha_{nJ}(\omega_0))| \times |h(\omega_0)| \, d\omega_0 \\ &\leq |\theta_n - \theta| \int_{\Omega_0} |\pi_n(\omega_0)| \times |(\alpha_{n1}(\omega_0), \dots, \alpha_{nJ}(\omega_0))| \times |h(\omega_0)| \, d\omega_0 \\ &\leq |\theta_n - \theta| \int_{\Omega_0} |(\alpha_{n1}(\omega_0), \dots, \alpha_{nJ}(\omega_0))| \times |h(\omega_0)| \, d\omega_0 \\ &\leq |\theta_n - \theta| \times \|(\alpha_{n1}, \dots, \alpha_{nJ})\|_2 \times \|h(\omega_0)\|_2 \\ &\leq |\theta_n - \theta| \times \sqrt{JM} \times \|h(\omega_0)\|_2 \\ &\rightarrow 0. \end{aligned}$$

For term (2),

$$\begin{aligned} & \left| \int_{\Omega_0} (Z(\pi_n, \alpha_n, \theta) - Z(\pi_n, \alpha, \theta))h \, d\omega_0 \right| \\ &= \left| \int_{\Omega_0} (\pi_n(\omega_0) \cdot (\alpha_{n1}(\omega_0) - \alpha_1(\omega_0)), \dots, \pi_n(\omega_0) \cdot (\alpha_{nJ}(\omega_0) - \alpha_J(\omega_0))) \cdot \theta \times h(\omega_0) \, d\omega_0 \right| \\ &\leq \int_{\Omega_0} |\theta| \times |(\pi_n(\omega_0) \cdot (\alpha_{n1}(\omega_0) - \alpha_1(\omega_0)), \dots, \pi_n(\omega_0) \cdot (\alpha_{nJ}(\omega_0) - \alpha_J(\omega_0)))| \times |h(\omega_0)| \, d\omega_0 \end{aligned}$$



$$\begin{aligned}
&\leq \|(\pi_n(\omega_0) \cdot (\alpha_{n1}(\omega_0) - \alpha_1(\omega_0)), \dots, \pi_n(\omega_0) \cdot (\alpha_{nJ}(\omega_0) - \alpha_J(\omega_0)))\|_2 \|h\|_2 \\
&\leq \|(|\pi_n(\omega_0)| \times |\alpha_{n1}(\omega_0) - \alpha_1(\omega_0)|, \dots, |\pi_n(\omega_0)| \times |\alpha_{nJ}(\omega_0) - \alpha_J(\omega_0)|)\|_2 \|h\|_2 \\
&= \|(|\alpha_{n1}(\omega_0) - \alpha_1(\omega_0)|, \dots, |\alpha_{nJ}(\omega_0) - \alpha_J(\omega_0)|)\|_2 \|h\|_2 \\
&= \|\alpha_n - \alpha\|_2 \|h\|_2 \\
&\rightarrow 0.
\end{aligned}$$

For term (3), we claim that  $\|\pi_n - \pi\|_2 \rightarrow 0$ . To see this, note that  $\pi_n \rightarrow \pi$  in the weak-star topology so

$$\langle \pi_n, \pi \rangle \rightarrow \langle \pi, \pi \rangle = 1 = \langle \pi_n, \pi_n \rangle$$

so

$$\begin{aligned}
\|\pi_n - \pi\|_2 &= \langle \pi_n - \pi, \pi_n - \pi \rangle \\
&= \langle \pi_n, \pi_n \rangle - 2\langle \pi_n, \pi \rangle + \langle \pi, \pi \rangle \\
&\rightarrow 0.
\end{aligned}$$

Therefore,  $\pi_n - \pi \rightarrow 0$  in measure, so

$$((\pi_n - \pi) \cdot \alpha_1, \dots, (\pi_n - \pi) \cdot \alpha_J) \rightarrow 0 \text{ in measure.}$$

In addition,

$$|(\pi_n(\omega_0) - \pi(\omega_0)) \cdot \alpha_j(\omega_0)| \leq \sqrt{2} |\alpha_j(\omega_0)| \in L^2(\Omega_0, \mathbf{R}) \text{ for } j = 1, \dots, J$$

so by the Lebesgue Dominated Convergence Theorem<sup>11</sup>

$$\int_{\Omega_0} |((\pi_n - \pi) \cdot \alpha_1, \dots, (\pi_n - \pi) \cdot \alpha_J)|^2 d\omega_0 \rightarrow 0.$$

Therefore

$$\begin{aligned}
&\left| \int_{\Omega_0} (Z(\pi_n, \alpha, \theta) - Z(\pi, \alpha, \theta)) h d\omega_0 \right| \\
&= \left| \int_{\Omega_0} (((\pi_n(\omega_0) - \pi(\omega_0)) \cdot \alpha_1(\omega_0), \dots, (\pi_n(\omega_0) - \pi(\omega_0)) \cdot \alpha_J(\omega_0))) \cdot \theta \times h(\omega_0) d\omega_0 \right| \\
&\leq \int_{\Omega_0} |\theta| \times |((\pi_n(\omega_0) - \pi(\omega_0)) \cdot \alpha_1(\omega_0), \dots, (\pi_n(\omega_0) - \pi(\omega_0)) \cdot \alpha_J(\omega_0))| \times |h(\omega_0)| d\omega_0 \\
&\leq \|h\|_2 \|((\pi_n - \pi) \cdot \alpha_1, \dots, (\pi_n - \pi) \cdot \alpha_J)\|_2 \\
&\rightarrow 0
\end{aligned}$$

which proves that  $Z$  is continuous.

<sup>11</sup>We use the version of the Dominated Convergence Theorem for sequences that converge in measure, rather than almost everywhere; see Halmos (1950, Theorem D, page 110).

We claim there exists  $\varepsilon > 0$  such that if  $\|\alpha - a(\cdot, t)\|_2 < \varepsilon$ , then  $0 \notin Z(\Delta \times \{\alpha\} \times S^{J-1})$ . If not, we may find a sequence  $(\pi_n, \alpha_n, \theta_n) \in \Delta \times \Gamma^J \times S^{J-1}$  with  $Z(\pi_n, \alpha_n, \theta_n) = 0$ . Since  $\Delta$  and  $S^{J-1}$  are compact and  $\alpha_n \rightarrow a$ , we can choose a subsequence such that

$$(\pi_n, \alpha_n, \theta_n) \rightarrow (\pi, a, \theta) \in \Delta \times \{a\} \times S^{J-1}.$$

Since  $Z$  is continuous,  $Z(\pi, a, \theta) = 0$ , a contradiction, which proves our claim.

By the Martingale Convergence Theorem, there is  $b \in L^2(\Omega_0, \mathbf{R}^L)$  such that  $\|a^{\mathcal{P}_n}(\cdot, t) - b(\cdot, t)\|_2 \rightarrow 0$  and  $E(b \mid \mathcal{P}_n) = a^{\mathcal{P}_n} = E(a \mid \mathcal{P}_n)$  almost everywhere. Therefore, for every  $A \in \cup_{n \in \mathbf{N}} \mathcal{P}_n$ ,  $\int_A (b - a) dP = 0$ . The set  $\{A \in \mathcal{B} : \int_A (b - a) dP = 0\}$  is a  $\sigma$ -algebra and contains  $\cup_{n \in \mathbf{N}} \mathcal{P}_n$ , so it contains  $\sigma(\cup_{n \in \mathbf{N}} \mathcal{P}_n)$ ; it is complete, so it contains  $\mathcal{B} = \overline{\sigma(\cup_{n=1}^{\infty} \mathcal{P}_n)}$ . Therefore,  $a = b$  almost everywhere, so  $\|a^{\mathcal{P}_n(\cdot, t)} - a(\cdot, t)\|_2 \rightarrow 0$ . Hence

$$0 \notin Z(\Delta \times \{a^{\mathcal{P}_n}(\cdot, t)\} \times S^{J-1})$$

for all  $t \in \{1, \dots, T\}$  and for sufficiently large  $n$ . Thus, for sufficiently large  $n$ , the discretized economy does not satisfy condition (H), and hence has no Hart point. The existence and index theorems then follow from Theorems 2 and 3.  $\square$

REMARK 4. The condition that the continuum limit economy contain no weak Hart points is essentially a condition on the dispersion of the securities payoffs. In order to show that there are no weak Hart points in the continuum limit, we need to show two things.

- There are states for which the corresponding convex polyhedra are not too large. If, for all states, the corresponding cone nearly filled the simplex, all the polyhedra would intersect, so condition (H) would necessarily be satisfied.
- The locations of the polyhedra corresponding to the various states are widely dispersed within the simplex. If all the polyhedra were located in a small region of the simplex, there would always be a hyperplane intersecting all of the polyhedra, even if the polyhedra were relatively small, so condition (H) would necessarily be satisfied.

Multivariate normal random variables are ubiquitous in economics and finance. In finance, as in our setting, basic securities typically have nonnegative payoffs, so it is customary to consider multivariate lognormal random variables, i.e. the exponentials of multivariate normal random variables; these form the basis of the geometric Brownian motion model, including the Black–Scholes model. The following corollary shows that nondegenerate multivariate lognormal random variables are sufficiently dispersed to ensure that they generate no weak Hart points in the continuum limit. Consequently, all sequences of models generated by discretizing continuum models with multivariate lognormal random variables eventually fail to satisfy condition (H), and thus have no Hart points.

COROLLARY 1. Suppose that the securities  $a_1, \dots, a_J$  satisfy  $a_{j\ell}(\cdot, t) = e^{b_{j\ell}(\cdot, t)}$ , where for each  $t = 1, \dots, T$  and each  $\ell = 1, \dots, L$ ,

$$(b_{1\ell}(\cdot, t), \dots, b_{J\ell}(\cdot, t))$$

is  $J$ -dimensional multivariate normal with arbitrary mean  $\mu_{\ell t} \in \mathbf{R}^J$  and nonsingular variance-covariance matrix  $\sigma_{\ell t}$ . Let  $\mathcal{P}_n$  be a sequence of finite partitions such that for each  $n$ ,  $\mathcal{P}_{n+1}$  is finer than  $\mathcal{P}_n$  and such that  $\mathcal{B} = \overline{\sigma(\cup_{n=1}^{\infty} \mathcal{P}_n)}$ . Then for  $n$  sufficiently large, the discretized economy has no Hart points.

PROOF. Since for each  $t$  and each  $\ell$ , the securities  $(a_{1\ell}(\cdot, t), \dots, a_{J\ell}(\cdot, t))$  are multivariate lognormal with nonsingular variance-covariance matrix, the conditional distribution of any one  $a_{k\ell}(\cdot, t)$ , given  $\{a_{j\ell}(\cdot, t) : j \neq k\}$ , has full support on  $(0, \infty)$ , and in particular has support that is unbounded above. Fix  $t \in \{1, \dots, T\}$ . Therefore, we can find  $\Omega_1, \dots, \Omega_J \subset \Omega_0$  with  $P(\Omega_k) > 0$  for each  $k = 1, \dots, J$ , such that for all  $\xi_k \in \Omega_k \times \{t\}$ ,  $a_{k\ell}(\xi_k) \geq 1$  and

$$Q_{\xi_k} = \left\{ \frac{(a_{1\ell}(\xi_k), \dots, a_{J\ell}(\xi_k))}{\sum_{j=1}^J a_{j\ell}(\xi_k)} : \ell = 1, \dots, L \right\}$$

is contained in  $R_k = \{x \in S^{J-1} : x_k \geq \frac{2}{3}\}$ . Since securities are short-lived, we have

$$\overline{P_{\xi_k}} = \{(p \cdot a_1(\xi_k), \dots, p \cdot a_J(\xi_k)) \in \Delta^{J-1} : p \in \mathbf{R}_+^L\}$$

which is just the closed polyhedron in  $\Delta^{J-1}$  with vertices in  $Q_{\xi_k}$ , so  $\overline{P_{\xi_k}}$  is contained in  $R_k$ . Since there is no affine subspace  $H \subset \Delta^{J-1}$  with  $\dim H < J-1$  that intersects each set  $R_k$ , and  $t \in \{1, \dots, T\}$  is arbitrary, **Theorem 1** implies that the continuum economy has no weak Hart points. **Theorem 4** then implies that for  $n$  sufficiently large, the discretized economy has no Hart points.  $\square$

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