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Research Report

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Notes on Endogenous Growth Models

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In the standard growth model, sustained growth is not possible without sustained improvements in technology. There is a limit on how much capital can be accumulated. There is also a steady state level of capital that the economy will converge to. It can take a lot of time to get to the steady-state, but you usually get there relatively quickly. Sustained improvement in technology over time - balanced growth path. Here we consider 4 models that take a different approach:

1. The AK model
2. Lucas model of human capital
3. Paul Romer kK model
4. Paul Romer R&D model.

All those models yield BGP.

1 Ak Model

We take the standard model, but assume a production function given by $y_t = Ak_t$ (Inada conditions are no longer satisfied, no concavity). For now, no labor choice. Single representative household with:

Preferences: $\sum_{t=0}^{\infty} \beta^t u(c_t)$

Technology:

$$y_t = Ak_t$$

$$c_t + i_t = y_t$$

$$k_{t+1} = (1 - \delta)k_t + i_t$$

Endowment: k_0 units of capital at time 0. We make the extra assumption that $u(c_t) = c_t^{1-\sigma} - 1/1 - \sigma$ for some $\sigma > 0$. We need a CRRA to have a BGP.

The social planner's problem is:

$$\begin{aligned} \max_{\{k_t\}} \quad & \sum_{t=0}^{\infty} \beta^t u(Ak_t + (1 - \delta)k_t - k_{t+1}) \\ \text{s.t.} \quad & Ak_t + (1 - \delta)k_t - k_{t+1} \geq 0 \forall t \\ & k_0 \text{ given} \\ & k_t \geq 0 \end{aligned}$$

The FOCs for an interior solution:

$$\frac{u'(c_{t-1})}{\beta u'(c_t)} = A + (1 - \delta)$$

If we repeated our earlier analysis and looked for a steady state solution to the SP problem, we would get: $1/\beta = A + (1 - \delta)$. This says that a steady state exists only if the equation holds. Therefore, in general, there does not exist an interior steady state.

But it does have a balanced growth path. Lets look for a solution $\{k_t^*\}$, $\{c_t^*\}$, with $k_t^* = (1 + g)^t k_0$, $c_t^* = (1 + g)^t c_0$ for some g . We want the solutions to satisfy:

$$\frac{u'(c_{t-1})}{\beta u'(c_t)} = \frac{1}{\beta} \left(\frac{c_t}{c_{t-1}} \right)^\sigma = A + (1 - \delta)$$

which implies,

$$\frac{1}{\beta} (1 + g)^\sigma = A + (1 - \delta)$$

This says that if we have a BGP solution to the SP problem, g must satisfy this equation:

$$g = (\beta[A + (1 - \delta)])^{1/\sigma} - 1$$

Note that we can have a BGP with a negative growth rate. Since there exists a continuum of paths that have this growth rate, there exists a continuum of BGP solutions to the SP problem. The growth rate along BGP is a function of A , β , δ , σ . Note that if the growth rate is too high – i.e. solution to the social planner's problem becomes infinite, then this model is useless. We must have $\sum_{t=0}^{\infty} \beta^t u(c_t^*) < \infty$.

An ADCE is a list of sequences $\{c_t^*\}$, $\{k_t^*\}$, $\{r_t^*\}$, $\{p_t^*\}$ s.t.

1. Consumer Maximization. Taking $\{p_t^*\}$, $\{r_t^*\}$ as given, $\{c_t^*\}$, $\{k_t^*\}$ solves:

$$\begin{aligned} & \max_{\{k_t\}, \{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & \sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) = \sum_{t=0}^{\infty} p_t^* (r_t^* k_t) \\ & c_t \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

2. Firm Maximization. Taking $\{p_t^*\}$, $\{r_t^*\}$ as given, $\forall t$, k_t^* solves:

$$\begin{aligned} & \max_{k_t} p_t^* (Ak_t - r_t^* k_t) \\ \text{s.t. } & k_t \geq 0 \end{aligned}$$

3. Market Clearing.

$$c_t^* + k_{t+1}^* - (1 - \delta)k_t^* = Ak_t^* \forall t$$

As usual, the characterization of this equilibrium is defined by:

$$\frac{u'(c_{t-1}^*)}{\beta u'(c_t^*)} = A + (1 - \delta)$$

which is the same as the social planner's problem. Note that with linear technology, as in this case, we can obtain prices directly from the firm's maximization problem: $A = r_t^*$. If we looked for a BGP ADCE we would look for g s.t. $(1 + g)^\sigma = \beta[A + (1 - \delta)]$. The first welfare theorem holds and everything is good.

With a linear technology, MPK is constant. A determines how big is MPK. Give me an initial value, and I will give you a BGP. No matter where you start, there is a BGP through this point. You continue along, no convergence, no dynamics to BGP, you are always there.

Now, take a tax on capital, τ_k . We assume revenue are rebated lump-sum to the household, and the government faces a period by period budget balance rule. We rewrite the ADCE as a list of sequences $\{c_t^*\}$, $\{k_t^*\}$, $\{r_t^*\}$, $\{p_t^*\}$, $\{T_t\}$ s.t.

1. Consumer Maximization. Taking $\{p_t^*\}$, $\{r_t^*\}$ as given, $\{c_t^*\}$, $\{k_t^*\}$ solves:

$$\begin{aligned} & \max_{\{k_t\}, \{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & \sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) = \sum_{t=0}^{\infty} p_t^* (r_t^* (1 - \tau_k)k_t + T_t^*) \\ & c_t \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

2. Firm Maximization. Taking $\{p_t^*\}$, $\{r_t^*\}$ as given, $\forall t$, k_t^* solves:

$$\begin{aligned} & \max_{k_t} p_t^* (Ak_t - r_t^* k_t) \\ \text{s.t. } & k_t \geq 0 \end{aligned}$$

3. Government. $T_t^* = \tau_k r_t^* k_t^*$

4. Market Clearing.

$$c_t^* + k_{t+1}^* - (1 - \delta)k_t^* = Ak_t^* \forall t$$

As usual, the characterization of this equilibrium is defined by:

$$\frac{u'(c_{t-1}^*)}{\beta u'(c_t^*)} = (1 - \tau_k)A + (1 - \delta)$$

Note that:

$$(1 + g)^\sigma = \beta[(1 - \tau_k)A + (1 - \delta)]$$

i.e. taxes effect BGP growth rate. Countries with different tax rates grow at different rates. The ratio of outputs will diverge to infinity.

This contrasts with the exogenous growth model. If we have a subsidy to capital accumulation, then g increases and g can be greater than the growth rate along a BGP solution to the social planner. This growth is not always good. There is an optimal growth rate. Even though it is feasible to grow at a higher rate, it might not be optimal.

2 Lucas Model

Preferences Single household.

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

Technology

$$\begin{aligned}
y_t &= F(k_t, e_t) \\
e_t &= s_t h_t \\
c_t + i_t^k + i_t^s &= y_t \\
k_{t+1} &= (1 - \delta^k)k_t + i_t^k \\
s_{t+1} &= (1 - \delta^s)s_t + i_t^s
\end{aligned}$$

Endowments: k_0 units of capital at time 0. s_0 units of skill at time 0. 1 unit of time per period t .

We can set $h_t = 1$ since the household doesn't value leisure. The social planner's problem is:

$$\begin{aligned}
\max_{\{k_t\}, \{c_t\}} \quad & \sum_{t=0}^{\infty} \beta^t u(F(k_t, s_t) + (1 - \delta^k)k_t - k_{t+1} - s_{t+1}) \\
\text{s.t.} \quad & k_t, c_t, s_t \geq 0 \forall t \\
& k_0, s_0 \text{ given}
\end{aligned}$$

Interior solution:

$$\begin{aligned}
k_t : \beta^{t-1} u'(c_{t-1}) &= \beta^t u'(c_t) (F_1(k_t, s_t) + (1 - \delta^k)) \\
s_t : \beta^{t-1} u'(c_{t-1}) &= \beta^t u'(c_t) (F_2(k_t, s_t) + (1 - \delta^s))
\end{aligned}$$

plus the TVC for each k_t, s_t . Suppose we look for a steady state solution: i.e. $k_t = k^*, c_t = c^*, s_t = s^* \forall t$:

$$\begin{aligned}
F_1(k^*, s^*) &= \frac{1}{\beta} - (1 - \delta^k) \\
F_2(k^*, s^*) &= \frac{1}{\beta} - (1 - \delta^s)
\end{aligned}$$

We know the ratio k/s from the first equation and this gives us F_2 . Then either the second equation holds or it does not. This is equivalent to the Ak model. In general, it will not hold, i.e. in a neighborhood of parameter space. So let's look for a BGP solution: $c_t^* = (1 + g)^t c_0^*, k_t^* = (1 + g)^t k_0^*, s_t^* = (1 + g)^t s_0^*$. Using the FOC and since $u(c_t) = c_t^{1-\sigma} - 1/1 - \sigma$,

$$\begin{aligned}
\frac{1}{\beta} (1 + g)^\sigma &= F_1(k_t^*, s_t^*) + (1 - \delta^k) \\
\frac{1}{\beta} (1 + g)^\sigma &= F_2(k_t^*, s_t^*) + (1 - \delta^s)
\end{aligned}$$

The LHS is independent of time, and it looks like the RHS is dependent on time. But in the BGP solution, the ratio k/S is constant, so the RHS is independent of time as well. To solve for k_t^*, s_t^* :

$$F_1(k_t^*, s_t^*) + (1 - \delta) = F_2(k_t^*, s_t^*) + (1 - \delta^s)$$

which we solve for k_t^*/s_t^* , which we can use to solve for g . This is equivalent to the Ak model. While we have capital and human capital accumulation, the model reduces to the Ak model. Holding s fixed, returns to capital die out. But since we are increasing s , returns to capital do not die out. s_t is endogenous. Accumulating more skills leads to accumulation of physical capital. This finding leads to the saying that human capital is the "engine" of growth. If we define an ADCE, we can analyze a richer set of policies.

An ADCE for this economy is a list of sequences $\{c_t^*\}, \{k_t^*\}, \{s_t^*\}, \{h_t^*\}, \{e_t^*\}, \{p_t^*\}, \{r_t^*\}, \{w_t^*\}$ s.t.

1. Consumers Maximize. Taking $\{p_t^*\}, \{r_t^*\}, \{w_t^*\}$ as given, $\{c_t^*\}, \{k_t^*\}, \{s_t^*\}, \{h_t^*\}$ solves:

$$\begin{aligned} & \max_{\{c_t\}, \{k_t\}, \{s_t\}, \{h_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & \sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta^k)k_t + s_{t+1} - (1 - \delta^s)s_t) \\ & = \sum_{t=0}^{\infty} p_t^* (r_t^* k_t + w_t^* (h_t s_t)) \\ & c_t \geq 0, 0 \leq h_t \leq 1, k_0, s_0 \text{ given} \end{aligned}$$

2. Firm Maximization. Taking $\{p_t^*\}, \{r_t^*\}, \{w_t^*\}$ as given, for all t k_t^*, e_t^* solves:

$$\begin{aligned} & \max_{k_t, e_t} p_t^* F(k_t, e_t) - w_t^* e_t - r_t^* k_t \\ \text{s.t. } & k_t, e_t \geq 0 \end{aligned}$$

3. Markets Clear.

$$\begin{aligned} c_t^* + k_{t+1}^* - (1 - \delta^k)k_t^* + s_{t+1}^* - (1 - \delta)s_t^* &= F(k_t^*, s_t^* h_t) \forall t \\ e_t^* &= h_t^* s_t^* \end{aligned}$$

Compare the three productions functions that we have encountered so far,

$$\begin{aligned} y_t &= Ak_t \\ y_t &= F^l(k_t, s_t) \\ y_t &= F^s(k_t, h_t) \end{aligned}$$

for the Ak model, the Lucas model, and the standard model, respectively. They are all CRS in the accumulable factors. The key feature of the first two equations over the last is CRS in the arguments individually. In the standard production function we can accumulate k , but not h . The standard model has DRS in the accumulable factors. This means that at some point DRS forces a stop to accumulation of factors, whereas in the first two equations returns do not die out.

3 Paul Romer's kK Model

The idea is as follows: spillovers in knowledge are critical to accumulating growth. Recall that in order for CE to exist, one needs to have CRS or DRS at the firm level. Otherwise, we can scale inputs and output and scale profits using IRS. Also if we want sustained growth we need CRS or IRS. If we have DRS at the firm level, we can have CRS overall if we have externalities. Externalities in production is one way to break the link between returns to scale at the individual firm level and the aggregate level.

Assume there is a single household with

Preferences $\sum_{t=0}^{\infty} \beta^t u(c_t)$ with $c_t = c_t^{1-\sigma} - 1/1 - \sigma$. This is required for a BGP.

Technology No time endowment as in the Ak model. We have k knowledge, N firms, k_{it} is knowledge of firm i at t , K_t is aggregate knowledge and $K_t = \sum_{i=1}^N k_{it}$, and $K_{-i,t}$ is aggregate knowledge of all firms other than firm i , $K_{-i,t} = \sum_{j \neq i}^N k_{jt}$. Each firm has a production function of the form:

$$y_{it} = f(K_{-i,t}, k_{it})$$

where your ability to produce is influenced by knowledge of others. We have DRS in k_{it} , but as $K_{-i,t}$ grows, it increases your knowledge k_{it} . Assumptions: f exhibits DRS in k_{it} individually, and satisfies the inada conditions for k_{it} , f exhibits CRS in both arguments individually and $f_{12} > 0$. Let $Y_t = \sum_{i=1}^N y_{it}$ and $c_t + \sum_{i=1}^N [k_{i,t+1} - k_{it}] = Y_t$. We also have an irreversibility assumption, $k_{i,t+1} \geq k_{it}$, i.e. can't give up knowledge for consumption.

Endowments k_{i0} given and is the same for all i .

Note that with externalities, the CE allocation is not equivalent to the SP allocation, i.e. the CE allocation is not efficient. This model is not that much different from the original Lucas model. There, spillover was in the human capital accumulation. This is similar to the situation given CE allocation does not imply the SP allocation with spillover.

The social planner's problem is not concerned with the number of firms. We write the

SP problem:

$$\begin{aligned} & \max_{\{k_{it}\}, \{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & c_t + \sum_{i=1}^N (k_{it+1} - k_{it}) = \sum_{i=1}^N f(K_{-it}, k_{it}) \\ & k_{i0} \text{ given, } k_{it+1} - k_{it} \geq 0 \end{aligned}$$

Given that the solution is symmetric (show this for homework), we can simplify the budget constraint to: $c_t + N(k_{t+1} - k_t) = Nf((N-1)k_t, k_t)$. The SPP becomes:

$$\begin{aligned} & \max_{\{k_t\}} \sum_{t=0}^{\infty} \beta^t u(Nf((N-1)k_t, k_t) - N(k_{t+1} - k_t)) \\ & \text{s.t. } k_0 \text{ given} \\ & k_{t+1} - k_t \geq 0 \end{aligned}$$

The FOCs simplify to:

$$\frac{u'(c_{t-1})}{\beta u'(c_t)} = (N-1)F_1((N-1)k_t, k_t) + f_2((N-1)k_t, k_t) + 1$$

The SP understands that what firm N does is good for everybody. When everybody accumulates more knowledge, it helps everyone. This is the definition of spillover. Note the steady state of the above equation reduces to just numbers, so it does not hold in general.

We look for a BGP solution:

$$\frac{(1+g)^\sigma}{\beta} = (N-1)f_1((N-1)(1+g)^t k_0^*, (1+g)^t k_0^*) + f_2((N-1)(1+g)^t k_0^*, (1+g)^t k_0^*) + 1$$

This is equivalent to the growth equation in the AK model. This simplifies to:

$$\frac{(1+g^{SP})^\sigma}{\beta} = (N-1)f_1(N-1, 1) + f_2(N-1, 1) + 1$$

We now turn to the competitive equilibrium in the kK model. We assume there are lots of firms. This makes each firm take what everyone else does as given. If n were small, i.e. $n = 2$, then strategic interaction would be important.

An ADCE for this economy is a list of sequences $\{c_t^*\}$, $\{k_{it}^*\}$, $\{p_t^*\}$, $\{\pi_{it}^*\}$ s.t.

1. Consumers Maximize. Taking $\{\pi_{it}^*\}$, $\{p_t^*\}$ as given, $\{c_t^*\}$ solves:

$$\begin{aligned} & \max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & \sum_{t=0}^{\infty} p_t^*(c_t) = \sum_{t=0}^{\infty} p_t^* \left(\sum_{i=1}^N \pi_{it}^* \right) \\ & c_t \geq 0 \end{aligned}$$

2. Firms Maximize. For each i , taking $\{p_t^*\}$, and $\{K_{-it}^*\}$ as given, $\{k_{it}^*\}$ solves:

$$\begin{aligned} & \max_{\{k_{it}\}} \sum_{t=0}^{\infty} p_t^*(f(K_{-it}^*, k_{it}) - k_{t+1} + k_{it}) \\ \text{s.t. } & k_{i0} \text{ given, } \quad k_{t+1} \geq k_{it} \\ & \text{and } \quad \pi_t = f(K_{-it}^*, k_{it}^*) - k_{it+1} + k_{it}^* \end{aligned}$$

3. Markets Clear.

$$c_t^* + \sum_{i=1}^N (k_{t+1}^* - k_{it}^*) = \sum_{i=1}^N f(K_{-it}^*, k_{it}^*)$$

We now characterize the ADCE. From the consumer's FOC, we have:

$$\frac{u'(c_{t-1})}{\beta u'(c_t)} = \frac{p_{t-1}^*}{p_t^*}$$

Claim: In an interior equilibrium, for each t , k_{it}^* is the same for all i . Proof: Assume wlog that $k_{21}^* > k_{11}^*$. Then $K_{-11}^* > K_{-21}^*$. By our FOC: $f_2(K_{-11}^*, k_{11}^*) > f_2(K_{-21}^*, k_{21}^*)$, which is a contradiction.

Let $k_{it}^* = k_t^*$, then the FOC becomes: $f_2((N-1)k_t^*, k_t^*) + 1 = p_{t-1}^*/p_t^*$. Combine this with the consumer FOC to get:

$$\frac{u'(c_{t-1}^*)}{\beta u'(c_t^*)} = f_2((N-1)k_t^*, k_t^*) + 1$$

Look for a BGP solution:

$$\frac{(1 + g^{CE})^\sigma}{\beta} = f_2((N-1), 1) + 1$$

Therefore $g^{CE} \neq g^{SP}$. So what doesn't get taken into account in the CE? In the CE, firms ask: What does it do for me? They care only about the benefits of knowledge for themselves. In contrast, the SP asks, what is good for the economy? The SP considers the spillover effects

of accumulating more knowledge. In fact, we can show that $g^{SP} > g^{CE}$ by the fact that $f_{21} > 0$.

Is there something we can do to generate efficiency in CE? What if we subsidize knowledge accumulation. Consider a government that subsidizes investment of knowledge, at rate s . The revenues are generated from a lump-sum tax on consumers. Assume period-by-period budget balance.

An ADCE for this economy a list of sequences $\{c_t^*\}$, $\{k_{it}^*\}$, $\{p_t^*\}$, $\{\pi_{it}^*\}$, $\{T_t^*\}$ s.t.

1. Consumers Maximize. Taking $\{\pi_{it}^*\}$, $\{p_t^*\}$, $\{T_t^*\}$ as given, $\{c_t^*\}$ solves:

$$\begin{aligned} & \max_{\{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & \sum_{t=0}^{\infty} p_t^*(c_t) = \sum_{t=0}^{\infty} p_t^* \left(\sum_{i=1}^N \pi_{it}^* - T_t^* \right) \\ & c_t \geq 0 \end{aligned}$$

2. Firms Maximize. For each i , taking $\{p_t^*\}$, and $\{K_{-it}^*\}$ as given, $\{k_{it}^*\}$ solves:

$$\begin{aligned} & \max_{\{k_{it}\}} \sum_{t=0}^{\infty} p_t^* (f(K_{-it}^*, k_{it}) - (1-s)(k_{t+1} - k_{it})) \\ \text{s.t. } & k_{i0} \text{ given, } \quad k_{t+1} \geq k_{it} \\ & \text{and } \quad \pi_t = f(K_{-it}^*, k_{it}^*) - k_{it+1} + k_{it}^* \end{aligned}$$

3. Government. $T_t^* = s \sum_{i=1}^N (k_{it+1}^* - k_{it}^*)$

4. Markets Clear.

$$c_t^* + \sum_{i=1}^N (k_{t+1}^* - k_{it}^*) = \sum_{i=1}^N f(K_{-it}^*, k_{it}^*)$$

The $(1-s)$ carries over through the FOCs and we get:

$$\frac{(1+g^{CE})^\sigma}{\beta} = \frac{f_2(N-1, 1)}{(1-s)} + 1$$

Therefore, we can raise s , such that we end up at the SP solution. That is, g would jump to the balanced growth path by choosing s appropriately. *Note: this proves that markets on their own don't always produce efficient outcomes.*

4 Models of Monopolistic Competition

Dixit-Stiglitz model - lots of producers producing goods that are not homogenous but varieties. Do not charge exactly the same price - monopolistically competitive CE. Representative consumer likes varieties. Upfront cost of investment, firms will invest only if they can realize positive profits afterwards, in order to have an equilibrium with investment. For example, sticky price models need monopolistically competitive markets. Also, growth rate should be such that total utility is finite.

Model: Assume again there is a single household with *Preferences*: N goods.

$$\left[\sum_{i=1}^N a_i c_i^\rho \right]^{1/\rho}$$

where $a_i > 0$, $0 < \rho < 1$. ρ determines substitutability between goods i and a_i are weights that determine which goods i the household likes more than others.

Technology: production function for producing good i is:

$$c_i = y_i = A_i h_i$$

where $A_i > 0$. Since this is a static model, there is no investment, so output equals consumption of good i .

Endowment: One unit of time.

To find the optimal allocation, we set up and solve the Social Planner's problem.

$$\begin{aligned} \max_{\{h_i\}} & \left(\sum_{i=1}^N a_i (A_i h_i)^\rho \right)^{1/\rho} \\ \text{s.t.} & \sum_{i=1}^N h_i = 1 \\ & h_i \geq 0 \end{aligned}$$

FOC:

$$\left(\sum_{i=1}^N a_i (A_i h_i)^\rho \right)^{\frac{1}{\rho-1}} a_i A_i^\rho h_i^{\rho-1} = \lambda$$

Diving by equation for $i = 1$ and rearranging yields:

$$h_i = \left(\frac{a_1 A_1^\rho}{a_i A_i^\rho} \right)^{\frac{1}{\rho-1}} h_1$$

where h_i is a linear multiple of h_1 . We can substitute this equation into the feasibility condition $\sum_{i=1}^N h_i = 1$. Define:

$$D_i = \left(\frac{a_i A_i^\rho}{a_1 A_1^\rho} \right)^{\frac{1}{1-\rho}}$$

then,

$$\begin{aligned} \sum_{i=1}^N D_i h_1 = 1 &\Rightarrow h_1 = \frac{1}{\sum_{i=1}^N D_i} \\ h_1 &= (a_1 A_1^\rho)^{\frac{1}{1-\rho}} \left(\sum_{i=1}^N (a_i A_i^\rho)^{\frac{1}{1-\rho}} \right)^{-1} \end{aligned}$$

A special case is when all a_i and A_i are equal. Then $h_i = 1/N \forall i$.

It is easy to show that the CE allocation is the same as the SP allocation. If N is large, then despite the fact that firms set their prices, they understand that they are competing with many other firms producing distinct, but similar substitute goods. The market power of firms is limited. What you do with your price doesn't matter too much for all other firms.

Note that each firm sets price, but takes as given all *aggregates* of the economy. More on this to follow.

Before we define equilibrium, let's investigate the nature of demand curves faced by our firms. Let's look at the consumer's problem. Taking $\{p_i^*\}$ as given, $\{c_i^*\}$ solves:

$$\begin{aligned} &\max_{\{c_i\}} \left(\sum_{i=1}^N a_i c_i^\rho \right)^{\frac{1}{\rho}} \\ \text{s.t. } &\sum_{i=1}^N p_i^* c_i = w^* + \sum_{i=1}^N \pi_I^* \\ &c_i \geq 0 \end{aligned}$$

Taking the FOCs, as before:

$$\left(\sum_{i=1}^N a_i c_i^\rho \right)^{\frac{1}{\rho-1}} a_i c_i^{\rho-1} = \lambda p_i^*$$

This gives:

$$c_i = \left(\frac{a_1}{a_i} \right)^{\frac{1}{\rho-1}} \left(\frac{p_1^*}{p_i^*} \right)^{\frac{1}{\rho-1}} c_1$$

Substitute this into the budget equation:

$$p_1^* c_1 + \sum_{i=1}^N \left(\frac{a_i}{a_1} \right)^{\frac{1}{1-\rho}} \left(\frac{p_i^*}{p_1^*} \right)^{\frac{1}{\rho-1}} p_i c_i = w^* + \sum_{i=1}^N \pi_i^*$$

$$c_i^* = a_i^{\frac{1}{1-\rho}} p_i^{*\frac{1}{\rho-1}} Y^* \left(\sum_{i=1}^N a_i^{\frac{1}{1-\rho}} p_i^{*\frac{\rho}{\rho-1}} \right)^{-1}$$

We can use this expression to determine the demand curve facing a given producer. Note, however, that when one firm changes price, Y^* remains fixed. The effect of changes in p_i^* on either Y^* or the denominator in the above equation is negligible.

Key result: Holding all other prices constant, and appealing to a large N , an individual producer faces a demand curve of the form:

$$d_i(p_i^*, \sum_{i=1}^N a_i^{\frac{1}{1-\rho}} p_i^{*\frac{\rho}{\rho-1}}, Y^*) = B p_i^{*\frac{1}{\rho-1}}$$

This equation has the nice characteristic that CES substitution makes it easy to work with. Note $1/(\rho - 1)$ is the elasticity. As $\epsilon \rightarrow -\infty$, demand goes to zero. Think about the effect of ρ .

Consider the special case of symmetry across producers, i.e. $a_i = a_j \forall i, j$ and $A_i = A_j \forall i, j$. We don't have to do this, but we want to find a symmetric equilibrium.

A symmetric monopolistic competition equilibrium for this economy is a list $c^*, h^*, p^*, w^*, d^*(p, N p^{*\frac{\rho}{\rho-1}}, w^* + N \pi^*)$ s.t.

1. Consumers Maximize: Taking p^*, w^*, π^* as given, h^*, c^* solves:

$$\max_{\{c_i\}, h} \left[\sum_{i=1}^N c_i^\rho \right]^{1/\rho}$$

$$\text{s.t. } \sum_{i=1}^N p^* c_i = w^* h + N \pi^*$$

$$c_i \geq 0, 0 \leq h \leq 1$$

and $d^*(\cdot)$ is the optimal demand function for any good as its price varies holding aggregates constant.

2. Firm Maximization. Taking $d^*(\cdot), w^*$ as given, p^*, h^* is the solution to:

$$\max_p p d^*(p, N p^{*\frac{\rho}{\rho-1}}, w^* + N \pi^*) (p - w^*/A)$$

$$\pi^* = d^*(p^*, N p^{*\frac{\rho}{\rho-1}}, w^* + n \pi^*) (p^* - w^*/A)$$

3. Market Clearing

$$h^* = \frac{Nd^*(\cdot)}{A}$$

Necessarily, c^* that solves CM is the amount produced by FM. Note that we do not impose that the equilibrium is symmetric, i.e. we cannot write the CM problem:

$$\begin{aligned} & \max_{c,h} (Nc^{1/\rho})^\rho \\ \text{s.t. } & Np^*c = w^*h + N\pi^* \end{aligned}$$

Also from this problem we cannot generate $d(\cdot)$.

We can rewrite the firm's problem as:

$$\max_p Bp^{\frac{1}{\rho-1}}(p - w^*/A)$$

Differentiating and rearranging yields the following pricing rule:

$$p = \frac{1}{\rho} \frac{w^*}{A}$$

with markup determined by ρ . Note that B doesn't matter. Price is a constant markup over marginal cost. As $\rho \rightarrow 1$, markup goes to zero (CE is the limiting case). Likewise as $\rho \rightarrow 0$, markup goes to infinity.

Note that allocations are the same as the SP allocations, but here prices are different.

4.1 Extension: Endogenous N

To make this interesting, we assume that the production function for all goods is given by:

$$c_i = A(h_i - \bar{h})$$

where $\bar{h} > 0$. If $N > 1/\bar{h}$, then there is no production, as the firm cannot even cover the fixed cost of producing. We focus on the symmetric case.

Preferences: For all N,

$$\left[\sum_{i=1}^N c_i^\rho \right]^{1/\rho}$$

Technology:

$$c_i = A(h_i - \bar{h})$$

Endowment: One unit of time.

Alternative representation assumes a continuum of goods.

Preferences:

$$\left[\int_0^N c(i)^\rho di \right]^{1/\rho}$$

Technology:

$$c(i) = A(h(i) - \bar{h})$$

and feasibility requires:

$$\int_0^N h(i) di = 1$$

In this economy, the problem of choosing N is a continuous choice problem. Given N , the Social Planner will allocate labor evenly across all firms, implying that $h = 1/N$. In this case, utility is given by:

$$N^{1/\rho}(1/N - \bar{h})$$

The optimal allocation is to choose N so as to solve:

$$\max_N N^{1/\rho}(1/N - \bar{h})$$

Taking FOC and rearranging yields:

$$N = \frac{1 - \rho}{\bar{h}}$$

Note that \bar{h} effects marginal cost. As $\rho \rightarrow 1$, $N \rightarrow 0$. The marginal benefit of an additional good is less than staying the same.

Remark: With $\bar{h} > 0$, CE does not exist. Why? Notice that once the fixed cost is paid, production technology is linear with average product normalize to one. Here production technology exhibits IRS, or decreasing marginal cost. Note that given h , $h - \bar{h}$ goes to production. Doubling inputs, $2h - \bar{h}$ goes to production. Therefore $2h - \bar{h} > 2(h - \bar{h})$.

If w is wage, to produce y units costs $w(y + \bar{h})$. Average cost is $w(y + \bar{h})/y = w + w(\bar{h}/y)$ and is decreasing. With CRS we can produce twice as much and profits are 0, but with IRS this is no longer true. Previously, we argued that with linear production technology, $y = h$, we must have $p = w$ in CE, otherwise the firm could increase profits by producing more output. But with $\bar{h} > 0$, if $p = w$, the firm makes no profit from production of output, but still pays the cost $\bar{h} + w$, which makes profits negative.

The solution is to change production to, say, $(h - \bar{h})^\theta$, which now is a CRS function.

To get firms to enter, we must set profits to 0.

The strategy is to solve the MCE for a given N , solve for profits of firm as a function of N , and then choose N such that $\pi = 0$.

Recall that given N , equilibrium will imply $1/N$ labor input per firm. Normalize w to 1, in equilibrium. Then price in equilibrium is $1/\rho \cdot 1$, i.e. constant markup over marginal cost. Profits therefore are equal to:

$$p^*y - w^*h = \frac{1}{\rho}(\frac{1}{N} - \bar{h}) - \frac{1}{N} = (\frac{1}{\rho} - 1)\frac{1}{N} - \frac{\bar{h}}{\rho}$$

Profits are decreasing in N . As $N \rightarrow 0$, profits go to ∞ . As N goes to ∞ , profits go to $-\bar{h}/\rho$. Therefore there exists a unique N , such that profits are 0.

It follows that there exists a unique N with profits equal to 0. Denote this value N^* , then N^* solves:

$$\begin{aligned} \left(\frac{1}{\rho} - 1\right) \frac{1}{N^*} &= \frac{\bar{h}}{\rho} \\ N^* &= \frac{1 - \rho}{\bar{h}} \end{aligned}$$

Profits are scaled by size of operations. N varies such that profits just equal fixed costs. If we produce twice as much, we assume demand is downward sloping, so this also helps keep profits down. Producers don't think this is optimal. Therefore, profits are zero. MCE is a way to reach equilibrium with 1) fixed costs, and 2) CRS once cost is paid.

The idea is that specialization increases output. Consider N tasks.

$$y = \left[\sum_{i=1}^N h_i^\alpha \right]^{\frac{1}{\alpha}}, \alpha \in (0, 1)$$

This function is CRS in h for a given N . New ideas for specialization are equally important as old ideas. Normalize aggregate h to 1. h_i/N is optimal. Then,

$$y = N^{\frac{1}{\alpha}-1}$$

Consider $\log(y) = (1/\alpha - 1)\log(N)$. If N grows, then y grows. This generalization allows us to increase production.

5 Romer R&D Model

Single household. *Preferences.*

$$u(c_t) = \frac{\sum_{t=0}^{\infty} u(c_t)}{c_t^{1-\sigma} - 1} \cdot (1 - \sigma)$$

Technology let N_t be the mass of specialized inputs at time t , i.e. intermediate goods.

$$y_t = \int_0^{N_t} z_{it}^\alpha h_t^{1-\alpha} di$$

where z_{it} is the input of intermediate good i .

$$\begin{aligned} z_{it} &= k_{it} \\ c_t + i_t^k + i_t^n &= y_t \\ k_{t+1} &= (1 - \delta)k_t + i_t^k \\ \int_0^{N_t} k_{it} di &= k_t \\ N_{t+1} &= N_t + i_t^N \end{aligned}$$

Endowments k_0 is capital at $t = 0$, N_0 is the set of intermediate goods at $t = 0$. Also, there is 1 unit of time for each date.

Social planner's problem. At a given date t , N_t and k_t given, SP will seek to maximize output given inputs. This will imply allocating capital evenly across intermediates, i.e. $k_{it} = k_t/N_t$ and $z_{it} = k_t/N_t$. Note that $h_t = 1$ since consumer doesn't value leisure, we get:

$$y_t = \int_0^{N_t} \left(\frac{k_t}{N_t} \right)^\alpha di = \left(\frac{k_t}{N_t} \right) N_t = N_t^{1-\alpha} k_t^\alpha$$

for all N_t , y_t is CRS. Note this is very similar to the Lucas model, except a different story line. This implies that we rewrite the social planner's problem as:

$$\begin{aligned} \max_{\{c_t\}, \{y_t\}, \{k_t\}, \{N_t\}, \{i_t^k\}, \{i_t^n\}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & y_t = N_t^{1-\alpha} k_t^\alpha \\ & c_t + i_t^k + i_t^n = y_t \\ & k_{t+1} = (1 - \delta)k_t + i_t^k \\ & N_{t+1} = N_t + i_t^n \\ & c_t \geq 0, k_t \geq 0, i_t^n \geq 0 \\ & k_0, N_0 \text{ given} \end{aligned}$$

This is similar to the Lucas problem. Instead of human capital, we have intermediate goods. So the story sounds different, but model is similar.

Think about accumulable factors like last time. Derive the FOCs like last time and look

for a BGP:

$$\begin{aligned}\frac{1}{\beta}(1+g)^\sigma &= \alpha \left(\frac{k_t}{N_t}\right)^{\alpha-1} + (1-\delta) \\ \frac{1}{\beta}(1+g)^\sigma &= (1-\alpha) \left(\frac{k_t}{N_t}\right)^\alpha + 1\end{aligned}$$

Let's turn to the decentralized equilibrium: If we think that someone is owning the new ideas, then we can't have anyone buying them if there is no profit in the model somewhere.

Suppose consumer behaves competitively in all markets. Firms that produce y behave competitively. Intermediate goods producers behave as monopolistic competitors, i.e. each intermediate good is produced by distinct firms. There is an infinite number of potential firms where profits are zero net of entry costs.

The R&D sector is special. Each new idea for an intermediate good belongs exclusively to the inventor, so they are a monopolist in terms of selling their idea (i.e. assume infinite patent rights). They sell to potential intermediate producers. We allow for free entry into the R&D sector, so that expected profits must equal zero in equilibrium.

If we have upfront costs and CRS (i.e. no one earns profits), the CE doesn't exist. In order to get people to pay an upfront cost, then there must be some profits in equilibrium. This is a generic problem with any R&D model. Central idea in CE is who gets what and who owns what. In R&D what is important is who owns the ideas.

Market structure.

1. The consumer accumulates capital as usual and owns all firms.
2. The consumer behaves competitively in output, i.e. prices are given and profits are given.
3. Final goods sector is competitive. Single firms behave competitively in input and output market.
4. If we have CRS+CE, then we can assume a single firm. We won't know anything about specialization in firm behavior.
5. Intermediate goods sector: each firm produces one good. This good is sold to firms. The firm sets price and takes everything else as given. Single firm then can set all prices. Firms rent capital to produce good and sell output. The firm behaves competitively, i.e. takes prices as given, but sets own price, i.e. "monopolist in output market".
6. R&D sector. Any new ideas for intermediate goods belong solely to the firm that produces it. They sell ideas to the new goods firm (assume that this firm creates

its own firm and then presumably earns profits?). We have free entry into this sector. Note that we could assume that there is one firm that does R&D.

We define p_t as price of output at date t . All other prices are in terms of output at t . w_t is price of labor. r_t is price of capital services. p_{it} is price of intermediate good. p_{Nt} is price of new idea at time t .

Final goods firm's problem reduces to a series of static problems:

$$\begin{aligned} \max_{z_{it}, h_t} p_t^* & \left[\int_0^{N_t} z_{it}^\alpha h_t^{1-\alpha} - \int_0^{N_t} p_{it}^* z_{it} di - w_t^* h_t \right] \\ \text{s.t. } & z_{it} \geq 0, h_t \geq 0 \end{aligned}$$

Taking FOCs and rearranging:

$$\begin{aligned} h_t &= \left[\frac{(1-\alpha)}{w_t} \int_0^{N_t} z_{it}^\alpha di \right]^{\frac{1}{\alpha}} \\ z_{it} &= p_{it}^{\frac{1}{\alpha-1}} \left[\frac{1}{\alpha} h_t^{1-\alpha} \right]^{\frac{1}{1-\alpha}} \end{aligned}$$

This gives us demand for intermediate goods: $z^*(p_z, w_t^*, p_z^*)$. Note this formula is the same as in MCE and markup is determined by α .

The intermediate producers problem takes $z_{it}(p_{it})$ as given, and solves:

$$\max_{p_{it}} z_{it}(p_{it}) [p_{it} - r_t]$$

Solving this problem yields prices: $p_{it} = 1/\alpha r_t$. Same as before.

Let π_{it} be profits in period t of interms of period t output. Then from t on, total profits are: $\sum_{s=t}^{\infty} p_s \pi_{is}$. When a new idea comes by, somebody has to buy that idea.

For the R&D sector, note that new technology is such that if you take one unit of output and use it in the R&D sector, then you can still gfet 1 new idea. This actually has a free entry condition, which means that in equilibrium, the profit from entering must be zero. Profit from entry: $(p_{Nt} - 1)p_t$ where 1 is the cost of one unit of output. $p_{Nt} - 1$ if profits are zero. Must be enough entry that is consistent with zero profits.

A symmetric equilibrium with entry for this economy is a list of sequences $\{c_t^*\}$, $\{k_t^*\}$, $\{h_t^*\}$, $\{z_t^*\}$, $\{N_t^*\}$, $\{p_t^*\}$, $\{p_{zt}^*\}$, $\{z^*(p_z, p_{zt}^*, w_t^*)\}$, $\{r_t^*\}$, $\{w_t^*\}$, $\{p_{Nt}^*\}$, $\{\pi_t^*\}$ (profits are same for all i producers in symmetric equilibrium), such that:

1. Consumers Maximize. Taking $\{r_t^*\}$, $\{w_t^*\}$, $\{p_{Nt}^*\}$, $\{N_t^*\}$, $\{\pi_t^*\}$, as given, $\{c_t^*\}$, $\{k_t^*\}$,

$\{h_t^*\}$ solves:

$$\begin{aligned}
& \max_{\{c_t\}, \{k_t\}, \{h_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
& \text{s.t.} \quad \sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) \\
& = \sum_{t=0}^{\infty} p_t^* (r_t^* k_t + w_t^* h_t + N_t^* \pi_t^* - p_{Nt}^* (N_{t+1}^* - N_t^*)) \\
& \quad c_t \geq 0, 0 \leq h_t \leq 1
\end{aligned}$$

2. Firm Maximization.

(a) Taking $\{w_t^*\}$, $\{p_t^*\}$, $\{p_z^*\}$ as given, for all t , $z_{it} = z_t^*$ and h_t^* solves:

$$\begin{aligned}
& \max_{z_{it}, h_t} p_t \left[\int_0^{N_t} z_{it}^\alpha h_t^{1-\alpha} - \int_0^{N_t} p_{it} z_{it} di - w_t h_t \right] \\
& \text{s.t.} \quad z_{it} \geq 0, h_t \geq 0
\end{aligned}$$

and $z^*(p_z, p_{zt}^*, w_t^*)$ is optimal demand for an intermediate good, holding prices of all other intermediates equal to p_{zt}^* , and wage equal to w_t^* .

(b) Intermediate Producers. Taking $\{r_t^*\}$ and $\{z^*(p_z^*, p_{zt}^*, w_t^*)\}$ as given, for all t , p_{zt}^* solves:

$$\max_{p_{zt}} z^*(p_{zt}, p_{zt}^*, w_t^*) (p_{zt} - r_t^*)$$

and

$$p_t^* p_{Nt}^* = \sum_{s=t+1}^{\infty} p_s^* \pi_s^* \forall t$$

(c) R&D Sector.

$$p_t^* (p_{Nt}^* - 1) = 0$$

3. Market Clearing.

$$\begin{aligned}
k_t^* &= N_t^* z_t^* \\
c_t^* + k_{t+1}^* - (1 - \delta)k_t^* + (N_{t+1}^* - N_t^*) &= \int_0^{N_t} z_t^* h_t^{*1-\alpha} di
\end{aligned}$$