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Research Report

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Notes on Exogenous Growth Models

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1 Exogenous Growth Models

Note: Macroeconomics was invented by John Maynard Keynes, particularly to describe one event - the Great Depression in the US in the late 1920s, early 1930s. He wanted to understand aggregate behavior.

Nowadays, there are three important ingredients of the analysis when thinking about modern macroeconomics. These are:

- (1) The analysis should be conducted within a general equilibrium framework - not only what happens in one sector of the economy, but also in all the others.
- (2) Dynamics is important - e.g. the investment process. It is important for the process of economic growth (and business cycle analysis). In addition, firms solve forward-looking dynamic optimization problems.
- (3) Uncertainty is important - economies are subject to shocks. [In this course, however, we will stick mostly to deterministic setups.]

Note: In other words, modern macroeconomics is concerned with dynamic stochastic general equilibrium models. Used in growth literature, business cycles studies, labor economics. MATLAB, GAUSS programming needed to solve numerically this class of models. The methodology, the tools needed to address those ingredients/issues were not developed at the time of Keynes. What Keynes did was to try to take a shortcut. In 60 – 70 those shortcuts failed. Economists in the 70s went back to the micro-foundations.

When thinking about an economy, one should always think in terms of a *resource allo-*

cation problem. A full description of an economy should imply resource allocation. The following classic macroeconomics problems are expressed in terms of resource allocation:

(i) *Growth* - Why is it that per capita consumption in the U.S. is 10 times higher than it was 100 years ago? (time series analysis)

A follow-up question is: How much is produced, consumed, allocation of free time?

(ii) *Business Cycles* - Why (and by what magnitude) does aggregate output fluctuate over time?

(iii) *Development* - Why is per capita consumption in the U.S. 15 times larger than, say, India? (cross-section issue)

(iv) *Labor* - Why is time devoted to market work in the U.S. 25 % higher than in Europe?

In turn, the question of allocations leads to the issue of welfare. Note that prices are not part of the fundamentals of the economy - but they influence resource allocation through the relative valuation of the commodities. In addition, macroeconomists are interested in the quantitative nature of the issue, not the qualitative one. A good model should be able to explain the magnitudes (by resorting to MATLAB).

To begin, we consider only the first ingredient listed above: general equilibrium. The following section presents the simplest exposition of a model in general equilibrium without dynamics and uncertainty.

1.1 Static General Equilibrium Models: Abstract Theoretical Setup

1.1.1 The Endowment Economy

An economy, E , is described as a list, $\{(x_i, u_i, w_i), i = 1, \dots, I\}$, where:

$$\begin{aligned} X_i &\subset \mathbb{R}_+^K \\ u_i &: X_i \rightarrow \mathbb{R} \\ w_i &\in X_i. \end{aligned}$$

That is, the list describes the consumption set for agent i , a utility function for agent i , and an endowment for each agent i , where there are I individuals and K goods.

The above is a clearly defined economy; with the information we can tell i) what allocations are feasible, and ii) given a choice of anything that is feasible, we can say what is the utility out of those choices.

Definition An *allocation* for economy E is a vector $\{x_1, x_2, \dots, x_I\}$ with $x_i \in X_i$ for $i = 1, \dots, I$. That is, each individual is described by the vector of goods in their consumption set. Note that at this point there are no constraints on feasibility.

Definition A *feasible allocation* for this economy is an allocation $\{x_1, x_2, \dots, x_I\}$ with $x_i \in X_i$ for $i = 1, \dots, I$ and,

$$\sum_{i=1}^I x_i = \sum_{i=1}^I w_i.$$

The individual is described by a vector of goods in their consumption set that satisfy the total resources available in the economy E .

Note each x_i is a vector and we could also write the constraint as: $\sum_{i=1}^I x_{ik} = \sum_{i=1}^I w_{ik}$, $k = 1, \dots, K$.

Note that there may be commodities that are not good (radioactive waste, pollution, environmental degradation). In this case we will have strict inequality.

The analysis is judgement-free, because we are not discussing issues like fairness.

Pareto efficient allocation - we cannot change it so that we can make somebody better off, without making someone else worse off.

Definition An allocation is *Pareto efficient* if,

1. $\{x_1, x_2, \dots, x_I\}$ is feasible, AND
2. there does not exist another feasible allocation $\{x'_1, x'_2, \dots, x'_I\}$ with $u_i(x'_i) \geq u_i(x_i)$, for $i = 1, \dots, I$, with strict inequality for at least one i .

Consider $I = 2$, $K = 1$, $u_i(x_i) = x_i$, $x_i \in \mathbb{R}_+$, and $w_i = 1$, $i = 1, 2$. Here all allocations are Pareto efficient - we cannot make consumer 1 better off, without making consumer 2 worse off.

Definition A *competitive equilibrium* for economy E is a price vector $p^* \in \mathbb{R}^K$ and an allocation $\{x_1^*, x_2^*, \dots, x_I^*\} \in X_1 \times X_2 \times \dots \times X_I$ (dimensionality of the price vector) subject to:

1. (Consumer Maximization)

For each $i = 1, \dots, I$ taking p^* as given, x_i^* is a solution to:

$$\begin{aligned} \max_{x_i} \quad & u_i(x_i) \\ \text{s.t.} \quad & p^* \cdot x_i \leq p^* \cdot w_i \\ & x_i \in X_i \end{aligned}$$

2. (Market Clearing) $\sum_{i=1}^I x_i^* = \sum_{i=1}^I w_i$

Note in 1, the agent does not do what is *feasible* in the economy as a whole - the agent will go out and purchase goods that are not there, i.e. do not satisfy market clearing, even if they satisfy the budget constraint. Market clearing ensures that agents act according to what is feasible in the economy E .

Assuming preferences and consumption sets are well behaved (i.e. continuous, monotone and convex), then a competitive equilibrium exists. Existence can be shown to exist given certain weak conditions. Later the setup can be extended to consider markets with monopoly power and monopolistic competition, bargaining between workers and firms (search), and strategic equilibria in industrial organization (IO).

Two other questions are also of great interest for macroeconomists:

- (1) If $(p^*, x_1^*, x_2^*, \dots, x_I^*)$ represents a competitive equilibrium, under what conditions is $(x_1^*, x_2^*, \dots, x_I^*)$ Pareto efficient? (First Welfare Theorem)
- weaker conditions than those for existence are required.
- (2) If $(x_1^*, x_2^*, \dots, x_I^*)$ is a Pareto efficient allocation, is it possible to achieve it as part of a competitive equilibrium, subject to allowing redistribution of endowments (Second Welfare Theorem)?

These questions comprise the two welfare theorems. Again, note that equilibrium allocations do not imply fairness.

Next, we extend the static framework to incorporate firms and production.

[END OF LECTURE 1]

1.1.2 Static Production Economy

Economy E is described by the list $\{(x_i, u_i, w_i), i = 1, \dots, I, Y_j, j = 1, \dots, J\}$, where:

$$\begin{aligned} X_i &\subset \mathbb{R}_+^K \\ u_i : X_i &\rightarrow \mathbb{R} \\ w_i &\in X_i. \\ Y_j &\subset \mathbb{R}^K, j = 1, \dots, J \end{aligned}$$

The production economy is described by J firms, and each Y_j denote the technology of a firm. Y_j describes feasible choice of inputs and outputs, where a negative number is an input and a positive number an output. $y_j \in Y_j$ means y_j is a feasible choice of inputs and outputs for firm j .

Note we are denoting production using sets and not production functions. Later we will use production functions, but in essence they are equivalent. For example, we if we specify a production function $y = f(h)$ where h is labor input and y is the output good, then we can also specify a set $Y = \{(z_1, z_2) \in \mathbb{R}^2 : z_2 = f(-z_1), z_1 \leq 0\}$

Definition An *allocation* for economy E is a list $\{x_1, x_2, \dots, x_I, y_1, y_2, \dots, y_J\}$ with $x_i \in X_i \forall i$ and $y_j \in Y_j \forall j$.

Definition A *feasible allocation* for E is a list $\{x_1, x_2, \dots, x_I, y_1, y_2, \dots, y_J\}$ with $x_i \in X_i \forall i$ and $y_j \in Y_j, \forall j$ and,

$$\sum_{i=1}^I x_i = \sum_{i=1}^I w_i + \sum_{j=1}^J y_j$$

Definition An allocation is *Pareto efficient* if,

1. $\{x_1, x_2, \dots, x_I, y_1, y_2, \dots, y_J\}$ is feasible,
2. there does not exist another feasible allocation $\{x'_1, x'_2, \dots, x'_I, y'_1, y'_2, \dots, y'_J\}$ with $u_i(x'_i) \geq u_i(x_i), i = 1, \dots, I$, with one strict inequality for at least one i .

Note that here individuals own firms. Therefore, let θ_{ij} represent the fraction of firm j that is owned by individual i . We require $0 \leq \theta_{ij} \leq 1$ for all ij and $\sum_{i=1}^I \theta_{ij} = 1, \forall j$.

Definition A *competitive equilibrium* for the economy E is a price vector $p^* \in \mathbb{R}^K$ (we can allow for negative prices if there are "bads" such as radioactive waste) and an allocation $\{x_1^*, x_2^*, \dots, x_I^*, y_1^*, y_2^*, \dots, y_J^*\}$, and profits $\{\pi_1^*, \pi_2^*, \dots, \pi_J^*\}$ subject to:

1. (Consumer Maximization)

For each $i = 1, \dots, I$, taking p^* and $\{\pi_1^*, \pi_2^*, \dots, \pi_J^*\}$ as given, x_i^* is a solution to:

$$\begin{aligned} \max_{x_i} \quad & u_i(x_i) \\ \text{s.t.} \quad & p^* \cdot x_i \leq p^* \cdot w_i + \sum_{j=1}^J \theta_{ij} \pi_j^* \\ & x_i \in X_i \end{aligned}$$

2. (Firm Maximization)

For each $j = 1, \dots, J$, taking p^* as given, y_j^* is a solution to:

$$\begin{aligned} \max_{y_j} \quad & p^* \cdot y_j \\ \text{s.t.} \quad & y_j \in Y_j \end{aligned}$$

3. (Market Clearing) $\sum_{i=1}^I x_i^* = \sum_{i=1}^I w_i + \sum_{j=1}^J y_j^*$

Note: we do not know whether the equilibrium is unique. If the technology features Constant Returns to Scale (CRS), in equilibrium profits will be zero. If profits of a firm are not zero, it is not a competitive equilibrium. Generically, the solution of a general equilibrium problem is not unique.

Note: The budget constraint may hold with a weak inequality - you may prefer not to spend all.

Note: Primitives of a model are preferences, technology, endowments. Those have to be specified for every economy.

General remarks about models: What is the objective of a certain model? - to be used to analyze certain issues. Obviously, the ultimate model is too complicated, and unmanageable for a single individual to handle. A good model is thus a parsimonious one. What issues need to be addressed? The relevance of the details in the model should be evaluated

on a case by case basis.

DSGE Modeler's Motto: "All models are wrong, but some are useful" (N. Kocherlakota)

For all models we can find an assumption that somewhere in the real world does not hold. We should accept that deficiency and move forward. We want to capture the important forces at play and abstract away from the unimportant ones. To isolate the quantitatively important issue, to keep it clean. Research is a learning process, literal descriptions are not very important. We do not evaluate the models only based on assumptions.

We now expand our simple model to include dynamics.

2 Neoclassical Growth Model with Infinitely Lived Agents

The natural starting point for this is the neoclassical growth model. This model is also useful to analyze business cycles, questions related to the standard of living, fiscal and monetary policy. The model is the first word on those (but not the last). The growth model captures non-neutral (non-degenerate) trade-offs. In general, a resource allocation problem implicitly defines trade-offs. Two fundamental trade-offs relevant to many aggregate issues are:

1. leisure \longleftrightarrow consumption: One way to get more consumption is to give up leisure and work more.
2. consumption \longleftrightarrow investment: This is equivalent to a trade-off between consumption today and consumption in the future. Investment goods cannot be consumed immediately, but can be used to produce stuff to be consumed in the future. Investment influences how much capital the household owns, and how much it works. Production is split between consumption and investment.

The neoclassical growth model is the simplest framework to address those issues and the interesting trade-offs. But what makes the growth model simple? The common simplifying assumptions of the growth model are (1) there is only one good. Obviously this setup cannot be used for countries that invest a lot in education or health services;

(2) all individuals are identical. In many contexts that is identical to having one agent. What happens in the economy if everyone is the same? This allows us to compare averages, and to abstract away from distributional aspects of the analysis;

(3) the same good is used for consumption and investment. We will not distinguish between consumption and investment goods for now.

2.1 Dynamic General Equilibrium Model with Production, Part I

When writing down a model, we must always begin by specifying *preferences*, *technology*, and *endowment*. In this case time is discrete, $t = 0, 1, 2, \dots$, and infinite.

Preferences: There is a single consumer with preferences defined over consumption, $\{c_t\}_{t=0}^{\infty}$ and leisure, $\{1 - h_t\}_{t=0}^{\infty}$, given by,

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) = u(c_0, 1 - h_0) + \beta u(c_1, 1 - h_1) + \dots$$

where $u : \mathbb{R}_+ \times [0, 1] \rightarrow \mathbb{R}$ and the discount factor $\beta \in (0, 1)$. Assume $u_i(c_t, 1 - h_t) > 0$ and $u_{ii}(c_t, 1 - h_t) < 0$ for $i = 1, 2$, and $u \in C^2$. That is, u is strictly increasing in both arguments, strictly concave and is twice differentiable. Under this assumptions, the utility function describes the preferences of the households (but may not be unique), how much they consume and how much utility they obtain out of it. h_t is the time spent working. The discount factor β reflects the fact that the household obtains less utility from stuff in the future. $u(., .)$ is called the momentary (period) utility function, and we will assume it is additively separable over time.

Technology: Aggregate production function:

$$y_t = F(k_t, h_t)$$

where y_t is output, k_t is capital input and h_t is labor input at t and $F : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}$ (not defined on negative values). Output can be used for either consumption or investment,

$$c_t + i_t = y_t$$

and $c_t \geq 0$. Investment leads to increased capital in the next period along with capital depreciation,

$$k_{t+1} = (1 - \delta)k_t + i_t$$

where $\delta \in (0, 1)$ and $k_t \geq 0$. The assumption that i_t is allowed to be negative corresponds to having a "putty-putty" technology: output can be invested and turned back into consumption. Compare this to "putty-clay" technology, where once output is turned into investment, it cannot be turned back into consumption.

Assumptions on the production function:

1. F satisfies constant returns to scale ($\forall a \geq 0, k, h \geq 0, F(ak, ah) = aF(k, h)$). Replication should be the worst case, not DRS (decreasing returns to scale), which is not interesting possibility on aggregate level.

Example: oil is a finite resource, hence we will not be able to always upscale the use of a resource. However, the returns to scale is only a description of the technology. It says nothing about the availability of a resource.

2. $F(0, h) = 0 \forall h$. This is also called "no free lunch" condition.
3. $F_i(k_t, h_t) > 0$ for $i = 1, 2$ and $k_t, h_t > 0$,
4. $F_{ii}(k_t, h_t) < 0$ for $i = 1, 2$ and $F_{ij}(k_t, h_t) \leq 0$ when $i \neq j$. In other words, the function is strictly concave in each argument individually, and weakly concave jointly (CRS).
5. $F \in C^2$,
6. (Inada Conditions)

$$\begin{aligned}\lim_{k \rightarrow 0} F_1(k, h) &= +\infty \\ \lim_{k \rightarrow \infty} F_1(k, h) &= 0\end{aligned}$$

where $h > 0$.

Endowment The consumer is endowed with one unit of time in each period. Capital stock for $t = 0$ is defined as $k_0 > 0$.

Given that we defined preferences, technology, and endowment, the economy now defines a resource allocation problem.

Definition An *allocation* for this economy is a list of sequences $\{c_t^c\}_{t=0}^\infty$, $\{h_t^c\}_{t=0}^\infty$, $\{y_t^f\}_{t=0}^\infty$, $\{h_t^f\}_{t=0}^\infty$, $\{k_t^f\}_{t=0}^\infty$ subject to,

- (a) $c_t^c \geq 0 \forall t$
- (b) $h_t^c \in [0, 1] \forall t$
- (c) $h_t^f \geq 0, k_t^f \geq 0 \forall t$
- (d) $y_t^f = F(k_t^f, h_t^f) \forall t$

We can express the following definition more succinctly by doing away with superscripts found above.

Definition A *feasible allocation* for this economy is an allocation $\{c_t\}_{t=0}^{\infty}$, $\{k_t\}_{t=0}^{\infty}$, $\{y_t\}_{t=0}^{\infty}$, $\{h_t\}_{t=0}^{\infty}$ subject to,

$$\begin{aligned} k_{t+1} &= (1 - \delta)k_t + y_t - c_t \forall t \\ y_t &= F(k_t, h_t) \forall t \\ h_t &\in [0, 1], c_t \geq 0, k_t \geq 0 \forall t \text{ and } k_0 \text{ is given.} \end{aligned}$$

Before we go any further, in the next section we present a special case of the growth model.

2.2 Solow Growth Model

Given the above description of the environment, we impose a few ad-hoc restrictions that yield the Solow-Swan growth model (what follows is a discrete time version of Chapter 1 in Barro and Sala-i-Martin). Assume that for each period:

- (a) $h_t = 1$ (consumer spends all time working)
- (b) $i_t = sy_t$, $s \in [0, 1]$ (savings rate is constant)

[END LECTURE 2]

Now define $f(k_t) \equiv F(k_t, 1)$. Our first task, given these decision rules, is to find what allocation will emerge for this economy. We can completely characterize the economy by *iterating*. To start, we are given an initial k_0 , and $h_0 = 1$. Therefore, at $t = 0$ we have:

$$\begin{aligned} y_0 &= f(k_0) \\ i_0 &= sy_0 \\ c_0 &= (1 - s)y_0 \end{aligned}$$

Substituting in these values yields the equivalent for $t = 1$:

$$\begin{aligned} k_1 &= (1 - \delta)k_0 + sy_0 \\ h_1 &= 1 \\ y_1 &= f(k_1) \\ c_1 &= (1 - s)y_1 \end{aligned}$$

Repeating this process infinitely more times, we obtain the required sequences. In general, $k_{t+1} = (1 - \delta)k_t + sf(k_t)$, for every t with k_0 given, is a first-order difference equation. We can also understand some of the properties the solution will have. Define:

$$g(k) = (1 - \delta)k + sf(k)$$

It follows that the sequence $\{k_t\}_{t=0}^{\infty}$ is defined by $k_{t+1} = g(k_t) \forall t$ and k_0 is given.

Next we examine $g(\cdot)$ for the case $s > 0$:

- (a) $g(0) = 0$ (because $f(0) = 0$)
- (b) g is strictly increasing (because f is strictly increasing)
- (c) g is strictly concave (because f is strictly concave)
- (d) $\lim_{k \rightarrow 0} g'(k) = +\infty$ (Inada conditions on f)
- (e) $\lim_{k \rightarrow \infty} g'(k) = (1 - \delta) < 1$ (Inada conditions on f). At some point the function crosses the 45 degree line from above. The slope is less than unity at the point of crossing and ever after.

From a graph of $g(k)$ we can reach the following conclusions:

- (a) There exists a unique $k^* > 0$ such that $g(k^*) = k^*$,
- (b) if $0 < k < k^*$, then $g(k) > k$,
- (c) if $k > k^*$ then $g(k) < k$.

In addition we can something stronger regarding observations 2 and 3. If $0 < k < k^*$, then $k^* > g(k) > k$ by strict monotonicity of g . Also if $k > k^*$, $k^* < g(k) < k$.

Remember $k_{t+1} = g(k_t)$. We can analytically derive the following propositions.

Proposition 2.1 *Assume $s > 0$ and $0 < k_0 < k^*$. Then $\{k_t\}$ has the following properties:*

- (a) $\{k_t\}$ is strictly increasing,
- (b) $\lim_{t \rightarrow \infty} k_t = k^*$.

Proof (1) Since g is monotonically increasing, $g(k_t) > k_t \forall t$. Since $k_0 \in (0, k^*)$ then we know $k^* > k_1 > k_0$ (since $k_1 > k_0$ and $0 < k_1 < k^*$). Then by repeated iteration we have $k^* > k_{t+1} > k_t \forall t$.

(2) Since $\{k_t\}$ is strictly monotonic and bounded, then it converges. Let k^l be the limit of the sequence. It remains to be shown that $k^l = k^*$. We know $k_{t+1} = g(k_t)$ and g is continuous function, then $\lim_{t \rightarrow \infty} k_{t+1} = \lim_{t \rightarrow \infty} g(k_t) = g(k^l)$. Therefore, $k^l = g(k^l)$. Since k^* is a unique positive solution to the difference equation, then it must be that $k^l = k^*$.

Proposition 2.2 Assume $s > 0$, $k_0 > k^*$. Then $\{k_t\}$ has the following properties:

(a) $\{k_t\}$ is strictly decreasing,

(b) $\lim_{t \rightarrow \infty} k_t = k^*$.

Proof (1) Since $k_0 > k^*$, then $k_0 > k_1 > k^*$ because of strict monotonicity of g . By repeatedly iterating we have $k_t > k_{t+1} > k^* \forall t$.

(2) Once again since $\{k_t\}$ is monotonically decreasing and bounded from below by k^* , the sequence converges to some value k^l . Since $k_{t+1} = g(k_t)$ and g is continuous, $\lim_{t \rightarrow \infty} k_{t+1} = \lim_{t \rightarrow \infty} g(k_t) = g(k^l)$. Therefore, $k^l = g(k^l) = k^*$ since $k^* > 0$ is a unique solution.

Proposition 2.3 Assume $s > 0$, $k_0 = k^*$. Then $\{k_t\} = k^* \forall t$.

Proof Since $k_0 = k^*$, it follows that $k_1 = k_0$ since $g(k_t) = k_t \forall t$. Then $k_t = k_{t+1} = k^* \forall t$.

Proposition 2.4 Assume $s > 0$, $k_0 = 0$. Then $\{k_t\} = 0 \forall t$.

Proof Since $k_0 = k^*$, by definition of g , $g(k_0) = g(0) = 0 = k_1$. Then clearly $k_{t+1} = g(k_t) = 0 \forall t$.

Assume $k^* > 0$ satisfies $g(k^*) = k^*$, $k^* > 0$.

Definition A steady state value of k is k^* such that $k_t = k^*$ implies $k_{t+1} = k^*$.

Remark: In our model, there are two steady states: 0 and some positive value. Also note that if $g(k^*) = k^* = (1 - \delta)k^* + sf(k^*)$ implies $sf(k^*) = \delta k^*$.

More generally, $k_{t+1} = g(k_t)$, k_0 given is an example of a dynamical system. Classic questions that arise in the context of dynamical systems are:

- (a) Does the system have a steady state? If so, is it unique?
- (b) Do you get convergence to a steady state from arbitrary initial values?
- (c) If you get convergence, what is the nature of convergence, i.e. is it monotone/oscillatory or something else?

The equation that generally characterizes $\{k_t\}$ is $k_{t+1} = (1 - \delta)k_t + sf(k_t)$. Assuming $k_0 > 0$, we can rearrange to get:

$$\gamma_t^k = \frac{k_{t+1} - k_t}{k_t} = \frac{sf(k_t)}{k_t} - \delta$$

This is the growth rate of k_t , which we denote γ_t^k . The relationship describes the connection between the value of today's capital and the growth rate of capital tomorrow. Properties of $sf(k)/k$:

- (a) $\lim_{k \rightarrow 0} sf(k)/k = sf'(k)/1 = +\infty$
- (b) $\lim_{k \rightarrow \infty} sf(k)/k = sf'(k)/1 = 0$

Therefore,

$$\frac{\partial \gamma}{\partial k} = \frac{sf'(k)k - sf(k)}{k^2} = \frac{s[kf'(k) - f(k)]}{k^2}$$

Recall Euler's Theorem states that $F(k, h) = F_1k + F_2h = 0$, so $kf'(k) - f(k) < 0$ and it must be that $\partial \gamma / \partial k < 0$. By the properties of $sf(k)/k$ established earlier, we know growth approaches ∞ as $k \rightarrow 0$ and 0 as $k \rightarrow \infty$. Also since growth is decreasing in k implies that if $0 < k_0 < k^*$, then $\gamma_t^k > 0$ and is decreasing as $k_0 \rightarrow k^*$. Likewise, if $k_0 > k^*$, then $\gamma_t^k < 0$ as $k_0 \rightarrow k^*$.

In this case, monotonicity in k , implies that we have monotonicity in y , c , i as well. Define:

$$\gamma_t^y = \frac{y_{t+1} - y_t}{y_t} = \frac{f(k_{t+1}) - f(k_t)}{f(k_t)}$$

We perform Taylor expansion around $f(k_{t+1})$ where $f(k_{t+1}) = f(k_t) + (k_{t+1} - k_t)f'(k_t)$ to get:

$$\gamma_t^y = \frac{(k_{t+1} - k_t)f'(k_t)}{f(k_t)}$$

Which we can expand to:

$$\gamma_t^y = \frac{(k_{t+1} - k_t)}{k_t} \frac{f'(k_t)}{f(k_t)} k_t = \gamma_t^k \frac{k_t f'(k_t)}{f(k_t)}$$

Therefore, for γ_t^y to be monotonically decreasing, we must have $k_t f'(k_t)/f(k_t)$ be monotonically increasing in k as well. This might not always be the case.

Monotonicity properties of $\frac{sf(k)}{k}$

$$\frac{d}{dk} \left[\frac{sf(k)}{k} \right] = \frac{sf'(k)k - sf(k)}{k^2} = \frac{s[f'(k)k - f(k)]}{k^2} < 0$$

Why? because $f'(k)k$ is payment to capital, and $f'(k)k < f(k)$. The latter follows from the Euler theorem

$$F(k, h) = F_1(k, h)k + F_2(k, h)h$$

As k_t increases, γ_t^k decreases, the curvature of the capital stock path decreases.

γ_t^k is also monotone. If $0 < k_0 < k^*$, then γ_t^k is decreasing. If $k_0 > k^*$, then γ_t^k is increasing (less negative). In other words: the further away the economy is from the steady-state, the higher the rate of your capital accumulation.

Next, how does the monotonicity of k translates to the other variables?

$y = f(k)$, $c = (1 - s)f(k)$, $i = sf(k)$, there is a 1:1 mapping.

What about the growth rates? If γ_t^k is monotone, what can we say about γ_t^y , γ_t^c , γ_t^i ? In terms of growth rates, $\gamma_t^y = \gamma_t^c = \gamma_t^i$ (because in this setup consumption and investment are scalars of output).

$$\gamma_t^y = \frac{y_{t+1} - y_t}{y_t} = \frac{f(k_{t+1}) - f(k_t)}{f(k_t)} = \frac{f(k_t) + (k_{t+1} - k_t)f'(k_t) - f(k_t)}{f(k_t)} = (k_{t+1} - k_t) \frac{f'(k_t)}{f(k_t)} = \gamma_t^k \frac{f'(k_t)}{f(k_t)}$$

where the first line is obtained by using a first-order Taylor approximation of $f(k_{t+1})$ around k_t , and the second - by multiplying and dividing by k_t .

As k_t goes down, γ_t^k increases. In order for the growth rate to be monotone, we require that $\frac{k f'(k)}{f(k)}$ be strictly decreasing in k . Note that generally the growth rate of variables is non-constant, except for the case of Cobb-Douglas production function when the fraction is α , the constant capital share.

We can do comparative statics exercises - by taking two different economies, or comparative dynamics and trace some dynamics. This is described in the next section.

2.2.1 Comparative Dynamics

How does s^* affect the (positive) steady state in our model? What the new steady-state would be? Recall, for a given value of s , k^* satisfies,

$$k^* = (1 - \delta)k^* + sf(k^*)$$

or,

$$\delta k^* = sf(k^*)$$

Let $k^*(s)$ denote the value of capital in the positive steady state as a function of savings. k_0 can be zero. Still, there are two steady states. Important note: how many steady states an economy has is independent of the initial conditions. Initial conditions are needed for the time path of the variables. If $k_0 > 0$, $k = 0$ will never be achieved but that is another issue. We should make a clear distinction between the two statements.

Similarly let $y^*(s)$ and $c^*(s)$ denote output and consumption in the positive steady state.

What can we say about $k^*(s)$? Since $k = \phi(s)$, we can use the equation $k^* = (1 - \delta)k^* + sf(k^*)$ and the implicit function theorem to differentiate with respect to s :

$$k^{*'}(s) = \frac{f(k^*(s))}{\delta - sf'(k^*(s))} > 0$$

Note that

$$\delta = s(f(k^*(s)))/k^* > f'(k^*(s))] > 0.$$

The inequality follows from the fact that the *average* product of capital less the *marginal* product of capital. By concavity of f we know that the marginal product decreases faster than the average product.

Remark: $s \in [0, 1]$. By our last result $k^*(s)$ is the largest capital stock. Even if $s = 1$, the capital stock is bounded. Why? (Inada conditions of the production function, $k^*(1) < \infty$. At that point people do not consume anything.)

Note that

$$y^*(s) = f(k^*(s)).$$

* It follows that y inherits any monotonicity properties of $k^*(s)$, so $y^*(s)$ is monotonically increasing.

What about $c^*(s)$? Simple observation tells us $c^*(0) = k^*(0) = 0$, i.e. if everything will

be consumed, nothing will be invested, thus nothing will be produced in equilibrium, and in the steady-state. Also, $c^{*(1)=0}$ when everything is invested, a lot of output is produced, but nothing is consumed again. $c^*(s)$ is thus not monotonic function. Since $c^*(s) = y^*(s) - i^*(s)$ and $i = \delta k$, then $c^*(s) = f(k^*(s)) - \delta k^*(s)$. Differentiating $c^*(s)$, we get:

$$c^{*'} = [f'(k^*(s)) - \delta]k^{*'}(s).$$

If we show

$$f'(k^*(1)) - \delta < 0$$

and

$$f'(k^*(0)) - \delta = \infty$$

then

$$f'(k^*(s)) - \delta = 0$$

for some s . Also,

$$f'(k^*(0)) - \delta = \infty$$

by the Inada conditions on f . Recall that $\delta k^*(1) = f(k^*(1))$ implies

$$\delta = f(k^*(1))/k^*(1) > f'(k^*(1))$$

by the fact that average product of capital is greater than marginal product of capital. Therefore,

$$f'(k^*(1)) - \delta < 0$$

, i.e. $c^*(s)$ is increasing and then decreasing. At the value of s where $f'(k^*(s^G)) = \delta$ we define s^G as the *Golden Rule* savings rate. At this intermediate point consumption is maximized. Note that associated with s^G , we have the values of c^G , k^G , y^G .

2.2.2 Transition

Consider an economy with a particular savings rate s , that is at the corresponding steady state. What happens if we change s ? Countries with s for a long time will be assumed to be at the steady state (or at least very close to it).

- (a) Switch from $s < s^G$ to s^G . At time 0, k_0 corresponds to $k^*(s)$. k is increasing at a decreasing rate in s , growth is decreasing. Output follows the same dynamics. Consumption initially drops below $c^*(s)$, then follows the dynamics of $y(s_G)$.

Question: Is it good for the economy to increase s to s_G ? Unclear, depends on

discounting. In particular, depends on the utility function, which describes the trade-off between current and future consumption. The utility function takes into consideration all the periods until the economy arrives at $c(s_G)$.

- (b) Switch from $s > s^G$ to s^G . Capital and output monotone decreasing, consumption initially jumps above $c^*(s_G)$, then decreases monotonically to that level. At the end, we have a higher level of steady-state consumption, that is why initially consumption has to be higher (above the $c^*(s_G)$ level, and then decrease to that level.

The graphs for these two scenarios to follow.

In Case 1, $k^*(s^G) > k^*(s)$ (because k is increasing in s). Likewise, $y^*(s^G) > y^*(s)$. Because s^G maximizes consumption, $c^*(s^G) > c^*(s)$. Note however, that in $t = 0$ (the t at which s is changed, $c_0(s^G) < c_0(s)$). Therefore, the consumer suffers a decrease in consumption before the benefits of s are realized. Consequently, it is not clear whether an increase in the savings rate to s^G is a good idea.

In Case 2, $k^*(s^G) < k^*(s)$ for the same reason. Likewise, $y^*(s^G) < y^*(s)$. Because s^G maximizes consumption, $c^*(s^G) > c^*(s)$. Note that in this scenario $c_0(s^G) > c_0(s)$, so the benefits of s are immediate. This economy is said to be *dynamically inefficient*. Therefore, it is a good idea to reduce s because consumption is always higher. This is despite the fact that we are *reducing* both capital stock and output.

Pareto efficiency is not a useful concept in dynamic setting. We can only say the economy is efficient or not.

: A good homework question - the speed of which the economy will converge to the steady state. Assume $h_t = 1$, hence $y_t = k_t^\theta$, with $\theta = 0.3$, $\delta = 0.1$, $s = 0.2$. It takes infinite amount of time to get there. So we will compute how long it takes to get half-there. Starting from $k_0 = 0.5 * k^*$, iterate in MATLAB on the difference equation $k_{t+1} = (1 - \delta)k_t + sk_t^\theta$. When will the concave function cross the 45 degree line. Application of FOR loop, or WHILE comment, do while $k_t < 0.75 * k^*$. Plot. Show

the number of iterations.

2.3 Solow Growth Model with Exogenous Labor-Augmenting Technical Change

Independently of s , both $\{k_t\}$ and $\{y_t\}$ are bounded. Asymptotically growth rate goes to 0. Yet a simple fact from the real world: even over long periods, 100-200 years, we do not see any tendency of growth of output/capital to tend to zero. On the contrary, it is fairly constant (2 percent). Also, Solow model predicts no matter where you start, in 50 years the economy is pretty close to its steady state. Such a fast convergence is not what we see in reality.

This is a problem. What is missing? The answer is technological progress. Recall that in the above production function, the relationship between inputs does not change over time, i.e. the share in production remains the same. Incorporating technological progress into the model is a easy way to allow continued growth.

How do we model technological progress? In other words, (1) how can we express being able to product more output given fixed input, (2) organize inputs in a more efficient way (Henry Ford's assembly line), (3) distinguish between methods of production (a machine doing something quicker than people in sectors like textile, agriculture), or (4) distinguish between quality versus quantity in production (computers).

Let A_t represent an index of technology at time t (state of the technology). If technology is improving over time, then A_t is increasing. From the production function $y_t = F(k_t, h_t)$ we incorporate technology in three ways

- (a) (Neutral Technology) $y_t = A_t F(k_t, h_t)$. Distributed equally between inputs.
- (b) (Labor-Augmenting Technological Change) $y_t = F(k_t, A_t h_t)$. Like having more workers.
- (c) (Capital-Augmenting Technological Change) $y_t = F(A_t k_t, h_t)$

Note since $F(\cdot)$ exhibits CRS, it can be expressed as: $y_t = F(A_t k_t, A_t h_t)$. Only (2) will deliver constant growth rates over long periods of time (more on this later). [Of course, not all technological progress in reality is labor-augmenting.] We are now ready

to present the model.

How about technological regress? Where is the frontier? Are they producing efficiently? Are there distortions? Thus we measure regress not in terms of frontier but in terms of efficiency, the nature of the economic environment. Poor economies are a lot less efficient, not that the technology frontier is lower for them.

Exogenous - take as given. Usually a bunch of people push the frontiers in research labs.

Preferences. Single agent with preferences,

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)$$

Technology.

$$\begin{aligned} y_t &= F(k_t, A_t h_t) \\ c_t + i_t &= y_t \\ k_{t+1} &= (1 - \delta)k_t + i_t \\ A_{t+1} &= (1 + \gamma^A)A_t \end{aligned}$$

where $\gamma^A > 0$. Usually technological progress arrives over time, and there are fluctuations, but here the rate is constant.

Endowments The agent is endowed with one unit of time each period, k_0 units of capital at time 0, and $A_0 = 1$ (just an index, a normalization).

Note that this setup is very similar to the plain Solow growth model. We proceed exactly as before. Consider the decision rules:

$$\begin{aligned} i_t &= sy_t, \forall t \\ h_t &= 1, \forall t \end{aligned}$$

Now define,

$$k_{t+1} = (1 - \delta)k_t + sF(k_t, A_t)$$

where $A_t = (1 + \gamma^A)^t$. Does this system possess a steady state with positive capital? If k^* is a steady state, then k^* satisfies:

$$k^* = (1 - \delta)k^* + sF(k^*, (1 + \gamma^A)^t), \forall t$$

and $\delta k^* = sF(k^*, (1 + \gamma^A)^t)$ for all t . Note the right hand side is increasing over time if $k^* > 0$, since $(1 + \gamma^A)^t$ is increasing. The left hand side is constant for all t .

It remains to be shown that the system has a solution in which $\{k_t\}$ grows at a constant rate.

To do this, define the following transformation: $\tilde{k} = k_t/(1 + \gamma^A)^t$. Now divide $k_{t+1} = (1 - \delta)k_t + sF(k_t, A_t)$ by $(1 + \gamma^A)^t$,

$$\frac{k_{t+1}}{(1 + \gamma^A)^t} = \frac{(1 - \delta)k_t}{(1 + \gamma^A)^t} + \frac{sF(k_t, A_t)}{(1 + \gamma^A)^t}$$

to get:

$$\tilde{k}_{t+1}(1 + \gamma^A) = (1 - \delta)\tilde{k}_t + sF(\tilde{k}_t, 1)$$

Now divide by $(1 + \gamma^A)$ and define $(1 - \tilde{\delta}) = (1 - \delta)/(1 + \gamma^A)$ and $\tilde{s} = s/(1 + \gamma^A)$. Note that $\tilde{\delta}, \tilde{s} \in [0, 1]$. Thus we arrive at:

$$\tilde{k}_{t+1} = (1 - \tilde{\delta})\tilde{k}_t + \tilde{s}f(\tilde{k}_t)$$

This equation has the same properties as the untransformed variant. If $\tilde{s} > 0$, the system has a unique steady state \tilde{k}^* and we get monotone convergence to \tilde{k}^* from any $\tilde{k}_0 > 0$. Note also that if $\tilde{k}_0 = \tilde{k}^*$, then $\tilde{k}_t = \tilde{k}^* \forall t$.

By our transformation, we know:

$$k_t = (1 + \gamma^A)^t \tilde{k}_t$$

If $\tilde{k}_t = \tilde{k}^* \forall t$, then k_t grows at a constant rate $(1 + \gamma^A)$.

Remark: Any path $\{k_t\}$ which exhibits constant growth at rate $(1 + \gamma^A)$ necessarily corresponds to a steady state \tilde{k}^* . Note that if $\tilde{k}_0 > \tilde{k}^*$, then \tilde{k}_t is decreasing, but $(1 + \gamma^A)$ is increasing, so $\{k_t\}$ does not have to be monotone.

Definition A solution $\{k_t^{BG}\}$ to $k_{t+1} = (1 - \delta)k_t + sF(k_t, (1 - \gamma^a)^t)$, $k_t \geq 0 \forall t$ is called a *balanced growth path solution* if $k_{t+1}^{BG} = (1 + \gamma^k)k_t^{BG}$ for some γ^k .

In other words a balanced growth path solution is a solution with a constant growth rate (note there is no mention of initial conditions).

How many balanced growth path solutions do we have?

- (a) $k_t = 0 \forall t$

(b) $k_0 = (1 + \gamma^k)\tilde{k}^*, \gamma^k = \gamma^A$

Can we have other balanced growth path solutions in which $k_t > 0 \forall t$ and $\gamma^k \neq \gamma^A$?

One way to check is to divide $k_{t+1} = (1 - \delta)k_t + sF(k_t, A_t)$ by k_t . On the balanced growth path k_{t+1}/k_t is constant.

$$\frac{k_{t+1}}{k_t} = (1 - \delta) + sF\left(1, \frac{(1 + \gamma^A)^t}{k_t}\right)$$

Since the left-hand side is constant, we know $k_t = (1 + \gamma^A)^t/c$, where c is some constant. Therefore k_t grows at γ^A . We conclude there is exactly one balanced growth path solution with positive k_t . Note again that convergence is monotone from below, but not from above.

If $\{k_t^{BG}\}$ is a balanced growth path solution with $\gamma^k = \gamma^A$, then what properties do $\{y_t\}, \{c_t\}, \{i_t\}$ possess?

$$y_t = F(k_t^{BG}, (1 + \gamma^A)^t)$$

Since $k_{t+1} = (1 + \gamma^A)k_t^{BG}$, we can write:

$$y_t = F((1 + \gamma^A)^t k_0^{BG}, (1 + \gamma^A)^t) = (1 + \gamma^A)^t F(k_0^{BG}, 1) = (1 + \gamma^A)^t f(k_0^{BG})$$

$\{y_t\}$ grows at a constant rate, γ^A . It follows that c_t and i_t also grow at constant rate γ^A . All variables grow at the same rate. Technological progress is critical to keeping growth bounded from zero.

Let's go back to the beginning. We mentioned there are three types of technological change. If technological change is *not* labor-augmenting, then there does *not* exist a balanced growth path with $k > 0$ (Proof in Barro and Sala-i-Martin).

Now suppose that F is Cobb-Douglas: $y_t = A_t k_t^\theta h_t^{1-\theta}$. Then,

$$y_t = k_t (A_t^{\frac{1}{1-\theta}} h_t)^{1-\theta} = (A_t^{1/\theta} k_t) h_t^{1-\theta}$$

With Cobb-Douglas production there is *no distinction* between various types of technological change.

There is no point to look for a growth path where $\gamma^k \neq \gamma^A$.

Different BGPs correspond to different initial conditions. Thus, we have a map of parallel lines with the same slope (growth rate).

How to describe a sequence: by (1) initial conditions and (2) growth rate.

3 Optimal Growth with Endogenous Savings

Different countries feature different fractions of time devoted to market work. Also, there are important differences in decision rules - saving rate, population growth, etc. Thus, the main limitation of the Solow model are the ad hoc decision rules. Such a model does not explain why saving rates are lower in some countries.

Preferences. Single consumer with preferences defined:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

Technology.

$$\begin{aligned} y_t &= F(k_t, h_t) \\ c_t + i_t &= y_t \\ k_{t+1} &= (1 - \delta)k_t + i_t \end{aligned}$$

Endowments. The consumer receives k_0 units of capital at $t = 0$ and 1 unit of time per period.

Characterizing the Pareto efficient allocation when given a single agent, is equivalent to maximizing the utility of the single agent subject to feasibility. We can write the consumer's problem using the following compact notation:

$$\begin{aligned} \max_{\{c_t\}, \{k_t\}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t.} \quad & c_t = F(k_t, 1) + (1 - \delta)k_t - k_{t+1} \\ & c_t \geq 0, k_t \geq 0, k_0 \text{ given} \end{aligned}$$

We are guaranteed a solution by the (generalized) Weierstrass Theorem: A continuous function on a compact set attains a solution. Does this problem have a unique solution? If our objective function is strictly concave on a convex set, then we can have at most one solution.

Given that u, f are strictly concave, these conditions are satisfied.

Note it is sufficient to rule of consumption being equal to zero. Given that $\lim_{c_t \rightarrow 0} u'(c) = +\infty$, it follows that $u(0)$ is not optimal if $c_t > 0$ is feasible. If $k_0 > 0$, then trivially $c_t > 0$. Therefore, given $k_0 > 0$ we can focus on an interior solution.

The first order condition for this problem can be written as:

$$-\beta^{t-1}u'(f(k_{t-1}) + (1 - \delta)k_{t-1} - k_t) + \beta^t u'(f(k_t) + (1 - \delta)k_t - k_{t+1})(f'(k_t) + (1 - \delta)) = 0$$

or

$$u'(f(k_{t-1}) + (1 - \delta)k_{t-1} - k_t) = \beta u'(f(k_t) + (1 - \delta)k_t - k_{t+1})(f'(k_t) + (1 - \delta))$$

In addition, we also have a transversality condition for this problem:

$$\lim_{t \rightarrow \infty} \beta^t u'(f(k_t) + (1 - \delta)k_t - k_{t+1})(f'(k_t) + (1 - \delta))k_{t+1} = 0$$

and k_0 given.

Given that our problem is a concave programming problem, these conditions are sufficient. We can rearrange and move forward one period the FOC:

$$\frac{u'(f(k_t) + (1 - \delta)k_t - k_{t+1})}{\beta u'(f(k_{t+1}) + (1 - \delta)k_{t+1} - k_{t+2})} = f'(k_{t+1}) + (1 - \delta)$$

or

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1}) + (1 - \delta)$$

Suppose we decrease c_t by one unit at the margin. How much additional consumption could we get in period $t + 1$ subject to keeping everything constant beyond period $t + 1$. In period t , if we decrease c_t by one unit, then we increase investment and have k_{t+1} increase by one unit. If k_{t+1} increase by one unit, how much does y increase? It increases by $f'(k_{t+1})$. Because we want to leave everything unchanged beyond period $t + 1$, the individual can also increase consumption in period $t + 1$ by $(1 - \delta)$ units or equivalently decrease investment by that amount. Therefore c_{t+1} increases by $f'(k_{t+1}) + (1 - \delta)$. This problem is equivalent to the following state allocation problem: There are 2 goods, one individual, and 2 firms.

Preferences. Single consumer with preferences defined:

$$u(c_1, c_2)$$

Technology.

$$c_1 = f_1(h_1)$$

$$c_2 = f_2(h_2)$$

Endowments. 1 unit of time.

We want to find a Pareto Efficient allocation to the following problem:

$$\begin{aligned} & \max_{c_1, c_2, h_1, h_2} u(c_1, c_2) \\ \text{s.t. } & c_1 = f_1(h_1), c_2 = f_2(h_2) \\ & 0 \leq h_1 + h_2 = 1 \\ & h_1 \geq 0, h_2 \geq 0 \\ & c_1 \geq 0, c_2 \geq 0 \end{aligned}$$

We can state the problem in a more compact form:

$$\begin{aligned} & \max_{h_1, h_2} u(f_1(h_1), f_2(1 - h_1)) \\ \text{s.t. } & h_1 \geq 0, 1 - h_1 \geq 0 \end{aligned}$$

The first order condition is:

$$\frac{u_1(f_1(h_1), f_2(1 - h_1))}{u_2(f_1(h_1), f_2(1 - h_1))} = \frac{f_2' h_1}{f_1'(1 - h_1)}$$

This first-order condition represents the trade-off between two goods. This is equivalent to the dynamic first-order equation above, i.e. in a dynamic program we differentiate one good over two periods of time and here we differentiate goods over one period of time.

Given the first-order conditions to the optimal growth model, can we find a steady state solution, i.e. a value of k^* s.t. $k_t = k^* \forall t$, that satisfied all of the necessary conditions? Given,

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1} + (1 - \delta))$$

and the transversality condition, we set $k_t = k^* \forall t$:

$$f'(k^*) = \frac{1}{\beta} - (1 - \delta)$$

We have only one solution that relates to a positive steady state.

4 Digression: Generalization of Kuhn-Tucker

We can rewrite the social planner's problem above in more general notation:

$$\begin{aligned}
W(k_0) &= \max_{\{k_t\}} \sum_{t=0}^{\infty} \beta^t v(k_t, k_{t+1}) \\
&\text{s.t. } k_0 \text{ given} \\
&\{k_t\} \in \Omega^\infty(k_0) \\
&\Omega^\infty(k_0) \subseteq \mathfrak{R}_+^\infty
\end{aligned}$$

Proposition 4.1 *Each solution $\{k_t\}$ to the SP problem satisfies the Euler equations and transversality conditions:*

$$\begin{aligned}
v_2 - \beta v_1 &= 0 \\
\lim_{t \rightarrow \infty} \beta^t v_1(k_t, k_{t+1}) k_t &\leq 0
\end{aligned}$$

Proof *The first condition is simply the FOC for an interior solution. Since $W(k_0)$ is strictly concave (since each $u(\cdot)$ is concave), we can use Taylor expansion to write:*

$$\begin{aligned}
W(k_t/2) &\leq W(k_t) + W'(k_t)(k_t/2 - k_t) \\
W(k_t/2) - W(k_t) &\leq -W'(k_t)k_t/2 \\
\beta^t [W(k_t/2) - W(k_t)] &\leq -\beta^t W'(k_t)k_t/2
\end{aligned}$$

Taking limits of both sides, notice that since $W(\cdot)$ is bounded, the LHS goes to zero, therefore:

$$0 \leq -\lim_{t \rightarrow \infty} \beta^t W'(k_t)k_t/2$$

Simplifying and using the envelope theorem yields the TVC.

Going in the other direction, we get:

Proposition 4.2 *If $\{k_t\}$ satisfies the Euler equations and TVC, then it is a solution to the maximization SP.*

Proof *By definition of maximum,*

$$\sum_{t=0}^{\infty} \beta^t [v(k_t^*, k_{t+1}^*) - v(k_t, k_{t+1})] \geq 0$$

Use Taylor expansion:

$$v(k_t, k_{t+1}) \leq v(k_t^*, k_{t+1}^*) + v_1(k_t^*, k_{t+1}^*)(k_t - k_t^*) + v_2(k_t^*, k_{t+1}^*)(k_{t+1} - k_{t+1}^*)$$

Substitution yields,

$$\sum_{t=0}^{\infty} \beta^t [v_1(k_t^*, k_{t+1}^*)(k_t^* - k_t) + v_2(k_t^*, k_{t+1}^*)(k_{t+1}^* - k_{t+1})]$$

Since $v_2(k_t^*, k_{t+1}^*) = \beta v_1(k_{t+1}^*, k_{t+2}^*)$, then,

$$\begin{aligned} & \sum_{t=0}^{\infty} \beta^t [v_1(k_t^*, k_{t+1}^*)(k_t^* - k_t) - \beta v_1(k_{t+1}^*, k_{t+2}^*)(k_{t+1}^* - k_{t+1})] \\ & \geq v_1(k_0^*, k_1^*)(k_0^* - k_0) - \lim_{T \rightarrow \infty} \beta^{T+1} v_1(k_{T+1}^*, k_{T+2}^*)(k_{T+1}^* - k_{T+1}) \\ & \geq - \lim_{T \rightarrow \infty} \beta^T v_1(k_T^*, k_{T+1}^*) k_T^* \geq 0 \end{aligned}$$

5 Social Planner's Problem

$$\begin{aligned} & \max_{\{k_t\}} \sum_{t=0}^{\infty} \beta^t u(f(k_t) + (1 - \delta)k_t - k_{t+1}) \\ & \text{s.t. } k_t \geq 0 \\ & f(k_t) + (1 - \delta)k_t - k_{t+1} \geq 0 \\ & k_0 \text{ given} \end{aligned}$$

The first-order conditions are:

$$-\beta^{t-1} u'(f(k_{t-1}) + (1 - \delta)k_{t-1} - k_t) + \beta^t u'(f(k_t) + (1 - \delta)k_t - k_{t+1})(f'(k_t) + (1 - \delta)) = 0$$

and,

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1} + (1 - \delta))$$

and k_0 given.

Our steady state solution is as follows. If $k_t = k^*$ satisfies FOC $\forall t$, then we derived:

$$f'(k^*) = \frac{1}{\beta} - (1 - \delta)$$

This equation has a unique solution and the solution is strictly positive.

Recall the Golden Rule of capital accumulation. k^G was characterized by $f'(k^*) = \delta$. Contrast this with $f'(k^*) = \frac{1}{\beta} - (1 - \delta)$. Therefore $k^G > k^*$ implies $c^G > c^*$. Why? In order to get to k^G you must forgo consumption earlier. So maybe this is not the best

place to go. Note also that if you started at k^G , then you still would go down to k^* . From,

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = f'(k_{t+1} + (1 - \delta))$$

if we set $c_t = c^G \forall t$, $k_t = k^G$, then the above equality would reduce to:

$$\frac{1}{\beta} > 1$$

This tells us that at $c_t = c^G \forall t$, the agent prefers to increase c_0 and decrease c_1 .

Lets consider the following situation. Suppose we imposed a steady state before taking FOCs. Would this make a difference? Imposing the steady state, the problem becomes:

$$\begin{aligned} \max_k \quad & \sum_{t=0}^{\infty} \beta^t u(f(k) + (1 - \delta)k - k) \\ \text{s.t.} \quad & k \geq 0 \\ & f(k) + (1 - \delta)k - k \geq 0 \end{aligned}$$

The FOC reduce to:

$$f'(k) = \delta$$

i.e. the golden rule allocation. Here we do not allow the consumer the opportunity to differentiate allocations over time. The dyanamic aspect of the problem is effectively ignored. This is incorrect. Never ever do this!

5.0.1 Phase Diagrams

Shooting Algoritm and Fsolve.

5.1 Equilibrium in the Dynamic General Equilibrium Model

So far we have examined efficient resource allocation via the social planner's problem. This section deals with decentralized equilibrium for our economy. Specifically we define the *Arrow-Debreu Competitive Equilibrium*.

One issue to consider is ownership of capital. We have two choices:

- (a) Consumers own capital and undertake the decision of capital accumulation and rent capital to firms, or

- (b) firm own capital, and undertake decisions about accumulating capital. Since consumers own the firms, consumers indirectly get the proceeds generated by this accumulation.

We assume a single firm.

Definition An *Arrow-Debreu Competitive Equilibrium* for our economy is a list of sequences $\{c_t^{c*}\}$, $\{i_t^{c*}\}$, $\{h_t^{c*}\}$, $\{k_t^{c*}\}$, $\{y_t^{f*}\}$, $\{k_t^{f*}\}$, $\{h_t^{f*}\}$, and $\{p_t^*\}$, $\{w_t^*\}$, $\{r_t^*\}$, subject to:

- (a) (Consumer Maximization)

Taking $\{p_t^*\}$, $\{w_t^*\}$, $\{r_t^*\}$ as given, $\{c_t^{c*}\}$, $\{i_t^{c*}\}$, $\{h_t^{c*}\}$, $\{k_t^{c*}\}$ solves:

$$\begin{aligned} \max_{\{c_t^c\}, \{i_t^c\}, \{k_t^c\}, \{h_t^c\}} & \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } \sum_{t=0}^{\infty} p_t^* (c_t + i_t) &= \sum_{t=0}^{\infty} w_t^* h_t + r_t^* k_t \\ k_{t+1} &= (1 - \delta) k_t + i_t \\ c_t &\geq 0, 0 \leq h_t \leq 1, k_0 \text{ given} \end{aligned}$$

- (b) (Firm Maximization)

Taking $\{p_t^*\}$, $\{w_t^*\}$, $\{r_t^*\}$ as given, $\{y_t^{f*}\}$, $\{k_t^{f*}\}$, $\{h_t^{f*}\}$ solves:

$$\begin{aligned} \max_{\{y_t^f\}, \{k_t^f\}, \{h_t^f\}} & \sum_{t=0}^{\infty} (p_t^* y_t - r_t^* k_t - w_t^* h_t) \\ \text{s.t. } & y_t = F(k_t, h_t) \forall t \\ & k_t, h_t \geq 0 \forall t \end{aligned}$$

- (c) (Markets Clear)

$$\begin{aligned} c_t^{c*} + i_t^{c*} &= y_t^{f*} \forall t \\ h_t^{c*} &= h_t^{f*} \forall t \\ k_t^{c*} &= k_t^{f*} \forall t \end{aligned}$$

6 Characterizing the Arrow-Debreu Competitive Equilibrium

Preferences. Single consumer with preferences defined:

$$\sum_{t=0}^{\infty} \beta^t u(c_t)$$

Technology.

$$\begin{aligned} y_t &= F(k_t, h_t) \\ c_t + i_t &= y_t \\ k_{t+1} &= (1 - \delta)k_t + i_t \end{aligned}$$

Endowments. The consumer receives k_0 units of capital at $t = 0$ and 1 unit of time per period.

We can redefine the Arrow-Debreu Competitive Equilibrium in a more compact way. Here the firm's problem reduces to a series of static problems.

Definition An Arrow-Debreu Competitive Equilibrium for this economy is a list of sequences, $\{c_t^*\}$, $\{h_t^*\}$, $\{k_t^*\}$, $\{w_t^*\}$, $\{r_t^*\}$, $\{p_t^*\}$ such that:

(a) (Consumers Maximize) Taking $\{w_t^*\}$, $\{r_t^*\}$, $\{p_t^*\}$ as given, $\{c_t^*\}$, $\{h_t^*\}$, $\{k_t^*\}$ solves:

$$\begin{aligned} &\max_{\{c_t\}, \{k_t\}, \{h_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } &\sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) = \sum_{t=0}^{\infty} (r_t^* k_t + w_t^* h_t) \\ &c_t \geq 0, 0 \leq h_t \leq 1 \quad \forall t, k_0 \text{ given} \end{aligned}$$

(b) (Firms Maximize) Taking $\{w_t^*\}$, $\{r_t^*\}$, $\{p_t^*\}$ as given, $\{h_t^*\}$, $\{k_t^*\}$ solves:

$$\begin{aligned} &\max_{\{k_t\}, \{h_t\}} p_t^* F(k_t, h_t) - w_t^* h_t - r_t^* k_t \\ &\text{s.t. } k_t \geq 0, h_t \geq 0 \end{aligned}$$

(c) (Markets Clear)

$$c_t^* + k_{t+1}^* - (1 - \delta)k_t^* = F(k_t^*, h_t^*) \forall t.$$

6.1 Consumer's Problem

For the consumer's problem, note that leisure is not valued, so in equilibrium $h_t^* = 1 \forall t$. The first-order conditions are as follows.

$$\begin{aligned}
\beta^t u'(c_t) &= \lambda p_t^* \forall t \\
\lambda p_{t-1}^* - (1 - \delta)p_t^* &= \lambda r_t^* \\
\sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) &= \sum_{t=0}^{\infty} (r_t^* k_t + w_t^* h_t) \\
k_0 &\text{ given} \\
\lim_{t \rightarrow \infty} \beta^{t-1} u'(c_{t-1}) k_t &= 0
\end{aligned}$$

6.2 Firm's Problem

$$\begin{aligned}
p_t^* F_1(k_t, h_t) &= r_t^* \\
p_t^* F_2(k_t, h_t) &= w_t^*
\end{aligned}$$

6.3 Conditions that Characterize ADCE

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = \frac{p_t^*}{p_{t+1}^*} \quad (1)$$

$$h_t = 1 \quad (2)$$

$$p_t^* = (1 - \delta)p_{t+1}^* + r_{t+1}^* \quad (3)$$

$$\sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) = \sum_{t=0}^{\infty} (r_t^* k_t + w_t^* h_t) \quad (4)$$

$$\lim_{t \rightarrow \infty} \beta^{t-1} u'(c_{t-1}^*) k_t = 0 \quad (5)$$

$$k_0 \text{ given} \quad (6)$$

$$p_t^* F_1(k_t^*, h_t^*) = r_t^* \quad (7)$$

$$p_t^* F_2(k_t^*, h_t^*) = w_t^* \quad (8)$$

$$c_t^* + k_{t+1}^* - (1 - \delta)k_t^* = F(k_t^*, h_t^*) \quad (9)$$

From (3) we have:

$$\frac{p_t^*}{p_{t+1}^*} = (1 - \delta) + \frac{r_{t+1}^*}{p_{t+1}^*} \quad (10)$$

Combining (10) and (1):

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = (1 - \delta) + \frac{r_{t+1}^*}{p_{t+1}^*} \quad (11)$$

Combining this with (7):

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = (1 - \delta) + F_1(k_{t+1}^*, h_{t+1}^*) \quad (12)$$

Using (2):

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = (1 - \delta) + f'(k_{t+1}^*) \quad (13)$$

Note that (13), (5), (6), (9) are the same conditions that we derived for the solution to the social planner's problem. In particular, these conditions completely characterize the solution to the SP problem, i.e. there is a unique solution to the SP and the solution is the only solution to these equations. Therefore $\{k_t^*\}$, $\{h_t^*\}$, $\{c_t^*\}$ are identical to their counterparts in the SP problem.

From this we can solve for prices. Normalize $p_0^* = 1$. Given that we know p_0^* , then (7) and (8) allows us to determine r_0^* , w_0^* as functions of k_0^* , and h_0^* . Now look at equation (1). Given p_0^* , c_0^* , c_1^* , the equation tells us p_1^* . Then by iterating we obtain the entire sequence.

Some notes on equation (3): $p_t^* = (1 - \delta)p_{t+1}^* + r_{t+1}^*$. Here the only benefit to the consumer from k is to rent it out and generate income. This is a no arbitrage condition. Otherwise, suppose that consumer buys 1 unit of k in period t and sells in $t + 1$. The cost is p_t^* and the income generated is $r_{t+1}^* + (1 - \delta)p_{t+1}^*$. If cost exceeds income or income exceeds cost the consumer can buy capital and sell tomorrow to make profit.

A few other remarks. Given all equations that we derived which hold in an ADCE, it is easy to show that the budget equation is implied by all the other equations. From the firm's maximization problem we have:

$$\begin{aligned} p_t^* F_1(k_t^*, h_t^*) &= r_t^* \\ p_t^* F_2(k_t^*, h_t^*) &= w_t^* \end{aligned}$$

Along with market clearing,

$$c_t^* + k_{t+1}^* - (1 - \delta)k_t^* = F(k_t^*, h_t^*)$$

we note that since F exhibits CRS, and since Euler's Theorem implies $F(k_t^*, h_t^*) = k_t^* F_1(\cdot) + h_t^* F_2(\cdot)$, we get:

$$c_t^* + k_{t+1}^* - (1 - \delta)k_t^* = k_t^* F_1 + h_t^* F_2$$

or

$$p_t^*[c_t^* + k_{t+1}^* - (1 - \delta)k_t^*] = k_t r_t^* + h_t w_t^*$$

Summing across all t , it follows that the budget constraint is necessarily satisfied. This is true because we only have one consumer. There doesn't exist a market for borrowing or lending. You might think, that if this is the case, then why did we not make this simplification when we wrote down the consumer's problem? If we do not use a summation, and instead write that there exists a budget set for every t , then our first order condition would look like this:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{\lambda_t}{\lambda_{t+1}} \frac{p_t^*}{p_{t+1}^*}$$

Which is certainly not the same first-order condition we derived earlier.

In general, what is the implied rate of interest on borrowing and lending between period t and $t + 1$? We make the distinction between nominal interest (prices today vs. prices tomorrow) and real interest (consumption today vs. consumption tomorrow). If an individual gives up one unit of consumption in time t , how much extra consumption can he get in period $t + 1$? He gets p_t^* income saved by buying one less unit of c_t , and spends p_{t+1}^* on c_{t+1} tomorrow. Therefore the real rate of borrowing/lending is p_t^*/p_{t+1}^* . What is the implied rate of return to capital accumulation? Suppose you give up one unit of consumption today and use the proceeds to buy capital, then sell capital in the next period, and use all revenue to purchase consumption in $t + 1$. Giving up one unit of c_t is identical to buying one unit of k_t . This k_t generates $r_{t+1}^*(1 - \delta)p_{t+1}^*$ additional income. We then use this income to purchase consumption in $t + 1$ at price p_{t+1}^* . How much can we purchase? We can purchase exactly $(r_{t+1}^* + (1 - \delta)p_{t+1}^*)/p_{t+1}^*$ units of consumption. Note that in equilibrium, $r_{t+1}^* + (1 - \delta)p_{t+1}^* = p_t^*$. Therefore 1+(real rate of return on capital) is p_t^*/p_{t+1}^* .

7 Steady State Arrow-Debreu Competitive Equilibrium

Definition A Steady State ADCE for our economy is a value of $k_0 = k_0^*$, and an ADCE for the economy with $k_0 = k_0^*$ in which $k_t^* = k_0^* \forall t$.

Note that we did *not* say an ADCE in which $\{k_t^*\}$, $\{h_t^*\}$, $\{c_t^*\}$, $\{p_t^*\}$, $\{w_t^*\}$, $\{r_t^*\}$ are all constant.

For any given $k_0 > 0$, if $\{k_t^*\}$ is part of an ADCE then $\{k_t^*\}$ must satisfy:

$$u'(f(k_t^*) + (1 - \delta)k_t^* - k_{t+1}^*) = \beta u'(f(k_{t+1}^*) + (1 - \delta)k_{t+1}^* - k_{t+2}^*)(f'(k_{t+1}^* + (1 - \delta))\forall t$$

and,

$$\lim_{t \rightarrow \infty} \beta^{t-1} u'(f(k_{t-1}^*) + (1 - \delta)k_{t-1}^* - k_t^*)k_t^* = 0$$

Therefore, finding an SS-ADCE requires that k_0 be such that we can find a solution $k_0^* = k_0 \forall t$ that satisfy these equations.

This problem is equivalent to finding a steady state to the social planner's problem. In particular, if $k_t^* = k^* \forall t$ is a solution, then k^* must solve:

$$f'(k^*) = \frac{1}{\beta} - (1 - \delta)$$

Now let k^* be the solution to this equation. We can now construct the rest of ADCE:

$$c_t^* = f(k_t^*) + (1 - \delta)k_t^* - k_{t+1}^* \forall t$$

Since $k_t^* = k^* \forall t$,

$$c_t^* = f(k^*) + (1 - \delta)k^* - k^* = f(k^*) - \delta k^*$$

i.e. c_t^* is constant. Also $h_t = 1 \forall t$, since this holds for any equilibrium with k_0 .

What about $\{p_t^*\}$, $\{w_t^*\}$, $\{r_t^*\}$? Let's normalize $p_0^* = 1$. The relevant equations are:

$$\begin{aligned} \frac{p_t^*}{p_{t+1}^*} &= (1 - \delta) + \frac{r_{t+1}^*}{p_{t+1}^*} \\ F_1(k_t^*, h_t^*) &= \frac{r_t^*}{p_t^*} \\ F_2(k_t^*, h_t^*) &= \frac{w_t^*}{p_t^*} \end{aligned}$$

Since $c_t^* = c \forall t$

$$\frac{p_t^*}{p_{t+1}^*} = \frac{1}{\beta}$$

Also:

$$\begin{aligned} F_1(k^*, 1) &= \frac{r_t^*}{p_t^*} \\ F_2(k^*, 1) &= \frac{w_t^*}{p_t^*} \end{aligned}$$

Since $p_0^* = 1$,

$$\begin{aligned} r_0^* &= F_1(k^*, 1) \\ w_0^* &= F_2(k^*, 1) \\ p_1^* &= \beta \\ r_1^* &= \beta F_1(k^*, 1) \\ w_1^* &= \beta F_2(k^*, 1) \end{aligned}$$

and so forth. In general:

$$\begin{aligned} p_t^* &= \beta^t \\ r_t^* &= \beta^t F_1(k^*, 1) \\ w_t^* &= \beta^t F_2(k^*, 1) \end{aligned}$$

Here is an alternative method of normalization.

Let w_t^* denote the price of a unit of labor in period t relative to price of output in period t (i.e. p_t^*). Let r_t^* denote the price of renting a unit of capital in period t relative to the price of period t output. Our new budget equation becomes:

$$\sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) = \sum_{t=0}^{\infty} (p_t^* r_t^* k_t + p_t^* w_t^* h_t)$$

In other words, we defined $\tilde{w}_t^* = w_t^*/p_t^*$ and $\tilde{r}_t^* = r_t^*/p_t^*$. The firm objective function becomes:

$$p_t^* (F(k_t, h_t) - r_t^* k_t - w_t^* h_t)$$

If one now characterizes the ADCE the conditions that characterize allocations are unaffected, but conditions that relate allocations to prices take a different form:

$$\begin{aligned} \frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} &= \frac{p_t^*}{p_{t+1}^*} \\ F_1(k_t^*, h_t^*) &= r_t^* \\ F_2(k_t^*, h_t^*) &= w_t^* \end{aligned}$$

If we repeat the steady-state ADCE calculations, we get,

$$\begin{aligned} \frac{1}{\beta} &= \frac{p_t^*}{p_{t+1}^*} \\ F_1(k_t^*, 1) &= r_t^* \\ F_2(k_t^*, 1) &= w_t^* \end{aligned}$$

which implies $r_t^* = r^* \forall t$ and $w_t^* = w^* \forall t$.

We can use the model to answer policy questions. What are the effects of policy x on allocations and welfare? We can assess policy that effects the consumption/savings decision.

8 Policy: Tax Income from Capital

Objective: Use the model to access the consequences of different capital income taxation policies on equilibrium allocations and welfare.

When assessing the consequences of policies in GE models, we must make sure that policies are well-specified. We begin by considering a very simple policy. Suppose there is a constant proportional tax levied on income from capital, denoted τ_k . We consider two different cases to reflect different use of revenue.

Case 1: Government uses revenue to purchase consumption, but then throws it away.

Case 2: Government uses the revenue to finance lump-sum transfer to all individuals, and does this with a period-by-period balanced budget.

For Case 1, an ADCE is a list of sequences $\{c_t^*\}$, $\{h_t^*\}$, $\{k_t^*\}$, $\{g_t^*\}$, $\{w_t^*\}$, $\{r_t^*\}$, $\{p_t^*\}$ such that:

(a) (*Consumers Maximize*) Taking $\{w_t^*\}$, $\{r_t^*\}$, $\{p_t^*\}$ as given, $\{c_t^*\}$, $\{h_t^*\}$, $\{k_t^*\}$ solves:

$$\begin{aligned} & \max_{\{c_t\}, \{k_t\}, \{h_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & \sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) = \sum_{t=0}^{\infty} p_t^* ((1 - \tau_k)r_t^* k_t + w_t^* h_t) \\ & \tau_k \geq 0, c_t \geq 0, 0 \leq h_t \leq 1 \quad \forall t, k_0 \text{ given} \end{aligned}$$

(b) (*Firms Maximize*) Taking $\{w_t^*\}$, $\{r_t^*\}$, $\{p_t^*\}$ as given, h_t^* , k_t^* for all t , solves:

$$\begin{aligned} & \max_{\{k_t\}, \{h_t\}} p_t^* (F(k_t, h_t) - w_t^* h_t - r_t^* k_t) \\ \text{s.t. } & k_t \geq 0, h_t \geq 0 \end{aligned}$$

(c) (*Government*)

$$g_t^* = \tau_k r_t^* k_t^* \forall t$$

(d) (*Market Clearing*)

$$c_t^* + g_t^* + k_{t+1}^* - (1 - \delta)k_t^* = F(k_t^*, h_t^*) \forall t.$$

Note that the first welfare theorem fails with taxes, i.e. we cannot solve the social planner's problem. Writing the ADCE and taking first-order conditions, we get:

$$\begin{aligned} \frac{u'(c_t^*)}{\beta' u(c_{t+1}^*)} &= (1 - \delta) + (1 - \tau_k) f'(k_{t+1}^*) \\ c_t^* + k_{t+1}^* - (1 - \delta)k_t^* + \tau f'(k_t^*) k_t^* &= f(k_t^*) \\ k_0 &\text{ given} \\ \lim_{t \rightarrow \infty} \beta^{t-1} u'(c_{t-1}^*) k_t &= 0 \end{aligned}$$

These conditions characterize the ADCE. The value of k^* in the SS-ADCE is:

$$\frac{1}{\beta} = (1 - \delta) + (1 - \tau_k) f'(k^*)$$

For Case 2, consider taxing constant proportion of income from capital, τ_k . Government uses income on lump-sum transfer to the household, subject to period-by-period budget, T_t . In the ADCE we add $\{T_t^*\}$ instead of $\{g_t\}$. In the consumer's problem, we add T_t^* which show up in the budget constraint. The government constraint is now $T_t^* = \tau_k r_t^* k_t^* \forall t$. The market clearing condition is the same. The SS-ADCE is:

$$\frac{1}{\beta} = (1 - \delta) + (1 - \tau_k) f'(k^*)$$

Remarks: SS-ADCE values of k^* are the same for Case 1 and Case 2. Consumption is different, as seen in market clearing. For k_0 , ADCE sequences for $\{k_t\}$ are different (but k^* is the same). Note this means that the steady state is not the complete solution to the problem. In other words, k^* is not a sufficient statistic for welfare in the economy. Also c^* is higher in Case 2.

Consider an economy which is considering between 2 tax policies, both corresponding to Case 2, but with different tax rates: $\hat{\tau}_k$, τ_k^* , and $\tau_k^* > \hat{\tau}_k$. Let $\{c_t^*\}$, $\{h_t^*\}$, $\{k_t^*\}$

correspond to τ_k^* and $\{\hat{c}_t\}$, $\{\hat{h}_t\}$, $\{\hat{k}_t\}$ correspond to $\hat{\tau}_k$. How do we evaluate welfare consequences of choosing τ_k^* over $\hat{\tau}_k$? Define,

$$U^* = \sum_{t=0}^{\infty} \beta^t u(c_t^*) > \hat{U} = \sum_{t=0}^{\infty} \beta(\hat{c}_t)$$

How does we say something about the magnitude of the welfare gain/loss?

We consider a measure of welfare based on *compensating variation*. Define λ , such that:

$$\sum_{t=0}^{\infty} \beta^t u(c_t^*) = \sum_{t=0}^{\infty} \beta^t u((1 + \lambda)\hat{c}_t)$$

This value of λ is independent of montone transformations. Also, it has a simple interpretation. There are two different types of welfare comparisons: 1. steady-state to steady-state, and 2. including the transition path.

consider two economies that correspond to the model we have been studying with same primitives, but different tax policies.

Assume that initial conditions, capital stock in each economy corresponds to the steady-state value associated with each tax rate, i.e. k_0^i solves:

$$\frac{1}{\beta} = (1 - \delta) + (1\tau_k^i)f'(k^{*i})$$

How much better off is the individual in economy (1) than the individual in economy (2), assuming both economies are in their steady states. To consider this we must compute,

$$\sum_{t=0}^{\infty} u(c^{*1}) = \sum_{t=0}^{\infty} u((1 + \lambda)c^{*2})$$

which implies $1 + \lambda = c^{*1}/c^{*2}$. If individuals could trade places, how much is that worth to them? What is the welfare gain/loss from having country 2 adopt country 1's policy from period 0 on, as opposed to continuing their policy?

Let $\{\hat{c}_t\}$, $\{\hat{h}_t\}$, $\{\hat{k}_t\}$ be the ADCE allocation that corresponds to the outcome with the change in policy:

$$\sum_{t=0}^{\infty} u(c^{*2}) = \sum_{t=0}^{\infty} u((1 + \lambda)\hat{c}_t)$$

This λ is not the same as before.

9 Simple Procedure to Calibrate the Growth Model

Our model is designed to capture certain economic forces/processes and not others. Our strategy is to use the data that we think our model could be capturing to inform us about reasonable model parameters. If convergence to the steady state is relatively fast, the observations for an economy like the US in the post-WW2 period should basically correspond to near steady-state outcomes.

Basic observations from the data, known as the Kaldor facts:

- (a) Y, L, I , should all grow at the same rate and this rate is constant over time: i.e. $K/Y, I/Y, L/Y$ are roughly constant.
- (b) labor share of output is constant over time.
- (c) real rate of return to capital is roughly constant over time.

The basic calibration strategy is as follows. Use data on the average values of $K/Y, I/Y$ and real rate of return to capital to determine the “reasonable” parameters for our model, based on looking at the mapping from model parameters to the steady state CE. The first step is to decide the functional forms subject to minimizing the number of parameters and to having an economics interpretation. The following are parsimonious or good starting points: $u(c_t) = (c_t^{1-\sigma} - 1)/(1 - \sigma)$ and $F(k_t, h_t) = Ak_t^\theta h_t^{1-\theta}$. Therefore we have parameters: $\sigma, A, \theta, \beta, \delta$. We cant to take the values of these parameters and solve for SS-ADCE as a function of these parameters, and then find values of these parameters for $K/Y, I/Y$, and the real rate of return on capital from the data.

Notes: A is useless. In essence, it is equivalent to a chance of units. σ does not appear in steady-state: $1/\beta = f'(k^*) + (1 - \delta)$. Is equilibrium independent of σ . No! In particular σ does not influence the steady state values, and hence, data relating the to the steady-state values cannot help us determine σ . This leaves us with 3 parameters and 3 targets. Benchmark: $\sigma = 1$, i.e. $u(c_t) = \log(c_t)$.

Lets calibrate these targets for US data. Need to divide the length of a period. We set one period to one year. Set $K/Y = 2.5, I/Y = .2$, real rate of return to capital 4% per year. In SS-ADCE $1 + (\text{real rate of return to capital}) = 1/\beta$. Therefore $1.04 = 1/\beta \Rightarrow \beta = 1/1.04 = 0.96$. $I/K = \delta = 10/.2 = (I/Y)/(K/Y) = .2/2.5 = 0.08 = \delta$. To determine the value of θ we start with the following equation which must hold in SS: $1/\beta - (1 - \delta) = f'(k^*)$, where $f(k^*) = \theta - Y/K$, and rearranging yields $\theta = 0.3$.

10 Sequence of Markets Competitive Equilibrium

In the ADCE the decision to exchange labor services, capital services, and output took place for the entire sequence $t = 0, \dots, \infty$ before any action took place. That is consumers could make claims to consumption and firms to rent capital in any distant time period. All allocations for all t were decided before $t = 0$. This might strike you as absurd. In the *sequence of markets competitive equilibrium* (SOMCE) we take a different approach. That is, we decide allocations for each market in each period separately. We start with the following example. Consider the economy that lasts for 2 periods, with no production, a single agent, and a single good. Define utility as $u(c_1, c_2)$ and endowment (ω_1, ω_2) .

The ADCE for this economy is a list $(c_1^*, c_2^*), (p_1^*, p_2^*)$ such that,

- (a) (Consumers Maximize) Taking (p_1^*, p_2^*) as given, (c_1^*, c_2^*) solves:

$$\begin{aligned} & \max_{c_1, c_2} u(c_1, c_2) \\ \text{s.t. } & p_1^* c_1 + p_2^* c_2 = p_1^* \omega_1 + p_2^* \omega_2 \\ & c_1, c_2 \geq 0 \end{aligned}$$

- (b) (Market Clearing)

$$\begin{aligned} c_1^* &= w_1^* \\ c_2^* &= w_2^* \end{aligned}$$

Now we have a market for borrowing and lending in period 1. We will demonstrate this in terms of bonds, b , with the interpretation that a bond gives the individual claims to 1 unit of good in period 2. We will denominate price of a bond by q and measure it relative to period 1 consumption.

We define an SOMCE as a list $(c_1^*, c_2^*), b^*, (p_1^*, p_2^*, q^*)$ such that,

- (a) (Consumers Maximize) Taking (p_1^*, p_2^*, q) as given, (c_1^*, c_2^*, q^*) solves:

$$\begin{aligned} & \max_{c_1, c_2, b} u(c_1, c_2) \\ \text{s.t. } & p_1^* c_1 + p_1^* b q^* = p_1^* \omega_1 \\ & p_2^* c_2 = p_2^* (w_2 + b) c_1, c_2 \geq 0 \end{aligned}$$

(b) (Market Clearing)

$$c_1^* = w_1^*$$

$$c_2^* = w_2^*$$

$$b^* = 0$$

We can set $p_1 = 1$ and define $p_2^* = p^*$. Rewriting the SOMCE above, we define SOMCE as a list (c_1^*, c_2^*) , b^* , (p^*, q^*) such that,

(a) (Consumers Maximize) Taking (p^*, q) as given, (c_1^*, c_2^*, q^*) solves:

$$\begin{aligned} & \max_{c_1, c_2, b} u(c_1, c_2) \\ & \text{s.t. } c_1 + bq^* = \omega_1 \\ & c_2 = w_2 + bc_1, c_2 \geq 0 \end{aligned}$$

(b) (Market Clearing)

$$c_1^* = w_1^*$$

$$c_2^* = w_2^*$$

$$b^* = 0$$

Define $B^{AD}(p) = \{(c_1, c_2) \in \mathbb{R}_+^2 : c_1 + pc_2 = \omega_1 + p\omega_2\}$. From the SOMCE, the consumers problem implies $c_1 + qc_2 = w_1 + q\omega_2$. Then, $B^{SOM}(q) = \{(c_1, c_2) \in \mathbb{R}_+^2 : c_1 + qc_2 = \omega_1 + q\omega_2\}$. If $p = q$, then $B^{AD}(p) = B^{SOM}(q)$.

Proposition 10.1 *The allocation (c_1^*, c_2^*, p^*) is an ADCE iff (c_1^*, c_2^*, b^*, q^*) is an SOMCE where $q^* = p^*$ and $b = (\omega_1 - c_1^*)/q^*$.*

Note that ADCE and SOMCE are not the same, they are different objects, but there is a 1-1 mapping between them.

A SOMCE for the growth model is a list of sequences $\{c_t^*\}$, $\{k_t^*\}$, $\{h_t^*\}$, $\{b_t^*\}$, $\{r_t^*\}$, $\{w_t^*\}$, $\{q_t^*\}$ subject to:

- (a) (Consumer Maximization) Taking $\{r_t^*\}, \{w_t^*\}, \{q_t^*\}$ as given, $\{c_t^*\}, \{k_t^*\}, \{h_t^*\}, \{b_t^*\}$ solves:

$$\begin{aligned} & \max_{\{b_t\}, \{c_t\}, \{h_t\}, \{k_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\ \text{s.t. } & c_t + k_{t+1} - (1 - \delta)k_t + q_t^* b_t = w_t^* h_t + r_t^* k_t + b_{t-1} \forall t \\ & c_t \geq 0, 0 \leq h_t \leq 1 \forall t \\ & k_0 \text{ given} \\ & b_{-1} = 0 \\ & (PONZIHERE) \end{aligned}$$

- (b) (Firm Maximization) Taking $\{r_t^*\}, \{w_t^*\}, \{q_t^*\}$ as given, $\forall t, h_t^*, k_t^*$ solves:

$$\begin{aligned} & \max_{k_t, h_t} F(k_t, h_t) - r_t^* k_t - w_t^* h_t \\ \text{s.t. } & k_t, h_t \geq 0 \end{aligned}$$

- (c) (Market Clearing)

$$\begin{aligned} c_t^* + k_{t+1}^* - (1 - \delta)k_t^* &= F(k_t^*, h_t^*) \forall t \\ k_t^* &= 0 \forall t \end{aligned}$$

Note that if we let $b_t = 0$, we cannot solve for q_t , so we must have q_t in the problem. Lets characterize the equilibrium. From the consumer's problem,

$$\begin{aligned} \beta^t u'(c_t) &= \lambda_t \\ \lambda_{t-1} &= \lambda_t(1 - \delta) + r_t^* \lambda_t \\ \lambda_t q_t^* &= \lambda_{t+1} \end{aligned}$$

and the transversality condition. From the firm's problem,

$$\begin{aligned} F_1(k_t^*, 1) &= r_t^* \\ F_2(k_t^*, 1) &= w_t^* \end{aligned}$$

since $h_t^* = 1$. Rearranging and combining yields,

$$\frac{1}{q_{t-1}^*} = r_t^* + (1 - \delta)$$

and upon combining some more,

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = r_{t+1}^* + (1 - \delta)$$

or,

$$\frac{u'(c_t^*)}{\beta u'(c_{t+1}^*)} = f'(k_{t+1}^*) + (1 - \delta)$$

If you want one unit of consumption at $t + 1$ and want to use it to get consumption at t via borrowing/lending, how much consumption do you get today? In equilibrium, this is equal to q_t^* .

The No Ponzi Scheme says that debt in period t is less than the present value of future income:

$$-b_t + [w_t^* h_t + r_t^* k_t + \sum_{s=1}^{\infty} (\prod_{j=0}^{s-1} q_{t+j}^*) [w_{t+s}^* h_{t+s} + r_{t+s}^* k_{t+s}]] \geq 0$$

If we get rid of the bond market, the FOCs change: we no longer have a no arbitrage condition for borrowing and lending. We can do this if we have only one individual (, check this). In general, we can write the firm's problem dynamically in SOMCE:

$$\sum_{t=0}^{\infty} (\prod_{s=0}^{t-1} q_s^*) (F(k_t, h_t) - k_t r_t^* - h_t w_t^*)$$

Where $q_s^* = 1/(r_{s+1}^* + (1 - \delta))$. So instead of using prices, we use q_t for discounting.

11 Optimal Growth Model with Endogenous Labor Choice

We start with a static allocation problem to illustrate the framework. We have a single consumer with the following environment.

- (a) Preferences: $u(c, 1 - h), c \geq 0, 0 \leq h \leq 1$
- (b) Technology: $c = Ah, h \geq 0$
- (c) Endowment: individual has one unit of time, $A > 0$.

The social planner's problem is:

$$\begin{aligned} \max_{c, h} & u(c, 1 - h) \\ \text{s.t.} & c = Ah \\ & c \geq 0, 0 \leq h \leq 1 \end{aligned}$$

To use the standard method to solve this problem we assume u is weakly concave in $(c, 1 - h)$ jointly, strictly concave in $c, 1 - h$ individually, u is twice continuously differentiable, and increasing in both arguments. Also $\lim_{c \rightarrow 0} u_1(c, 1 - h) = \infty$, $\lim_{h \rightarrow 0} u_2(c, 1 - h) = 0, \lim_{h \rightarrow 1} u_2(c, 1 - h) = \infty$. The first order condition is,

$$\frac{u_2(Ah, 1 - h)}{u_1(Ah, 1 - h)} = A$$

i.e. marginal rate of substitution equals the marginal rate of transformation. What happens to the optimal choice of h if A increases? As A increases, the return to working increases at the margin, this suggests h increases. Therefore c and h increase. On the other hand, as A increases, you have higher c for a given level of h , which implies you have a positive income effect, which suggests that you want more leisure and more consumption if both are normal. Here, c increases and h decreases. Therefore change in h is ambiguous.

Consider the following utility function,

$$u(c, 1 - h) = \log(c) + r(1 - h)$$

where r is any strictly concave, strictly increasing, C^2 function satisfying the Inada condition. The social planner's problem becomes:

$$\begin{aligned} \max_h \log(Ah) + r(1 - h) \\ \text{s.t. } 0 \leq h \leq 1 \end{aligned}$$

The FOC for this problem is $1/h = r'(1 - h)$, i.e. h is independent of A . Then the income effect and substitution effect directly offset each other. You can generalize this to a class of utility functions where income and substitutions effects are perfectly offsetting:

$$u(c, 1 - h) = \frac{[cr(1 - h)]^{1-\sigma} - 1}{1 - \sigma}$$

Consider the following problem:

$$\begin{aligned} \max_{c, h} u(c, 1 - h) \\ \text{s.t. } c = wh + y \\ c \geq 0, 0 \leq h \leq 1 \end{aligned}$$

let $h(w, y)$ and $c(w, y)$ be the solutions as functions of w, y . $c, 1 - h$ are normal if the functions c, h satisfy: c is increasing in y and h is decreasing in y .

Let's now turn to the growth model. As before, we define the economy with a single individual,

(a) *Preference*: $\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t)$

(b) *Technology*:

$$y_t = F(k_t, h_t)$$

$$c_t + i_t = y_t$$

$$k_{t+1} = (1 - \delta)k_t + i_t$$

(c) *Endowment*: k_0 given, one unit of time $\forall t$

plus consumption and leisure are normal goods.

The social planner's problem for this economy is:

$$\begin{aligned} \max_{\{k_t\}, \{h_t\}} \quad & \sum_{t=0}^{\infty} \beta^t u(F(k_t, h_t) + (1 - \delta)k_t - k_{t+1}, 1 - h_t) \\ \text{s.t.} \quad & F(k_t, h_t) + (1 - \delta)k_t - k_{t+1} \geq 0 \forall t \\ & k_t \geq 0, 0 \leq h_t \leq 1 \forall t, k_0 \text{ given} \end{aligned}$$

The first-order conditions for an interior solution are as follows. Recall that by the Weierstrauss Theorem a optimal solution exists and it is unique because of concavity. The interior solution is guaranteed since $k_0 > 0$.

$$\begin{aligned} k_t : \beta^{t-1} u_1(F(k_{t-1}, h_{t-1}) + (1 - \delta)k_{t-1} - k_t, 1 - h_{t-1}) = \\ \beta^t (F(k_t, h_t) + (1 - \delta)k_t - k_{t+1}, 1 - h_t) (F_1(k_t, h_t) + (1 - \delta)) \forall t \\ h_t : \beta^t (c_t, 1 - h_t) (F_2(k_t, h_t)) = \beta^t u_2(c_t, 1 - h_t) \forall t \end{aligned}$$

Rearranging, we get:

$$\begin{aligned} \frac{u_1(c_t, 1 - h_t)}{\beta u_1(c_{t+1}, h_{t+1})} &= F_1(k_{t+1}, h_{t+1}) + (1 - \delta) \forall t \\ \frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} &= F_2(k_t, h_t) \forall t \end{aligned}$$

Note that the first condition is dynamic and the second is static. Both conditions have to hold for optimality. The transversality condition for this problem is: $\beta^t u_1(c_{t-1}, 1 - h_{t-1}) k_t = 0$. Along with k_0 given and $c_t = F(k_t, h_t) + (1 - \delta)k_t - k_{t+1}$ we completely characterize the solution to the SP problem. What about the steady state solution?

Lets ignore k_0 given constraint. Can we find a solution to the remaining conditions, such that k_t, h_t are constant? Let k^*, h^* be a positive steady state. Then k^*, h^* must

satisfy:

$$\begin{aligned}\frac{1}{\beta} &= F_1(k^*, h^*) + (1 - \delta) \\ \frac{u_2(c^*, 1 - h^*)}{u_1(c^*, 1 - h^*)} &= F_2(k^*, h^*)\end{aligned}$$

where $c^* = F(k^*, h^*) = (1 - \delta)k^* - k^*$. Recall that in the case with leisure not valued, $1/\beta = f'(k^*) + (1 - \delta)$, we agreed that this equation had a unique positive solution for k^* . Let's show that there exists a unique solution for the model with leisure valued. Recall that by CRS, marginal product only depends on the ratio k^*/h^* . Moreover, given our assumptions, $F_1 \rightarrow \infty$ as $k^*/h^* \rightarrow \infty$ and $F_1 \rightarrow 0$ as $k^*, h^* \rightarrow \infty$ and F_1 is strictly decreasing in k^*, h^* . From $F(k^*, h^*) = 1/\beta - (1 - \delta)$ we can determine the value of k^*/h^* . Therefore, we can calculate $F_1(k^*, h^*)$ and $F_2(k^*, h^*)$. From the equation governing c^* in the steady state we get:

$$c^* = h[F(k^*/h^*, 1) - \delta k^*/h^*]$$

Given k^*, h^* , we can write that $c^* = Ah^*$, where $A = F(k^*/h^*, 1) - \delta k^*/h^*$. Let $B = F_2(k^*, h^*)$ and then,

$$\frac{u_2(Ah^*, 1 - h^*)}{\beta u_1(Ah^*, 1 - h^*)} = B$$

for some positive constant A, B . This is the equation for the single variable h^* . If this equation has a unique solution for h^* , then we know the SP problem has a single positive steady-state solution. It turns out that of c, h are normal, then k_0 given has a unique solution.

What is the competitive equilibrium for this economy? We define an ADCE for this economy as a list of sequences $\{c_t^*\}, \{h_t^*\}, \{k_t^*\}, \{w_t^*\}, \{r_t^*\}, \{p_t^*\}$ such that:

(a) (Consumers Maximize) Taking $\{w_t^*\}, \{r_t^*\}, \{p_t^*\}$ as given, $\{c_t^*\}, \{h_t^*\}, \{k_t^*\}$ solves:

$$\begin{aligned}\max_{\{k_t\}, \{h_t\}, \{c_t\}} & \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) \\ \text{s.t.} & \sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) = \sum_{t=0}^{\infty} p_t^* (r_t^* k_t^* + w_t^* h_t^*) \\ & 0 \leq h_t \leq 1 \forall t \\ & k_0 \text{ given}\end{aligned}$$

(b) (Firms Maximize) Taking $\{w_t^*\}, \{r_t^*\}, \{p_t^*\}$ as given, $\forall t$ h_t, k_t solves:

$$\begin{aligned}\max_{k_t, h_t} & p_t^* (F(k_t, h_t) - r^* k_t - w_t^* h_t) \forall t \\ & k_t, h_t \geq 0\end{aligned}$$

(c) (Market Clearing)

$$c_t^* + k_{t+1}^* - (1 - \delta)k_t^* = F(k_t^*, h_t^*) \forall t$$

We now characterize the ADCE. For the consumer's problem:

$$\begin{aligned} c_t : \beta^t u_1(c_t^*, 1 - h_t^*) &= \lambda p_t^* \\ h_t : \beta^t u_2(c_t^*, 1 - h_t^*) &= \lambda p_t^* w_t^* \\ k_t : \lambda p_{t-1}^* &= \lambda [p_t^* r_t^* + p_t^* (1 - \delta)] \end{aligned}$$

For the firm's problem:

$$\begin{aligned} F_1(k_t^*, h_t^*) &= r_t^* \\ F_2(k_t^*, h_t^*) &= w_t^* \end{aligned}$$

Combining some terms as usual, we get:

$$\begin{aligned} \frac{u_1(c_t^*, 1 - h_t^*)}{\beta u_1(c_t^*, 1 - h_t^*)} &= F_1(k_{t+1}^*, h_{t+1}^*) + (1 - \delta) \\ \frac{u_2(c_t^*, 1 - h_t^*)}{u_1(c_t^*, 1 - h_t^*)} &= F_2(k_t^*, h_t^*) \end{aligned}$$

The choice of h_t is not a static decision. Divide the first equation at t by the second at $t + 1$:

$$\frac{u_2(c_{t+1}^*, 1 - h_{t+1}^*)}{u_1(c_t^*, 1 - h_t^*)} \frac{u_1(c_{t+1}^*, 1 - h_{t+1}^*)}{u_1(c_{t+1}^*, 1 - h_{t+1}^*)} = \frac{F_2(k_{t+1}^*, h_{t+1}^*)}{F_2(k_t^*, h_t^*)}$$

or,

$$\frac{\beta u_2(c_{t+1}^*, 1 - h_{t+1}^*)}{u_2(c_t^*, 1 - h_t^*)} = \frac{F_2(k_{t+1}^*, h_{t+1}^*)}{F_2(k_t^*, h_t^*)} \frac{1}{[F_1(k_{t+1}^*, h_{t+1}^*) + (1 - \delta)]}$$

From the consumers problem in ADCE we would similarly get:

$$\frac{\beta u_2(c_{t+1}^*, 1 - h_{t+1}^*)}{u_2(c_t^*, 1 - h_t^*)} = \frac{w_{t+1}^*}{w_t^*} \frac{p_{t+1}^*}{p_t^*}$$

11.1 Leisure-Labor Choice with Labor Augmenting Technological Progress

Recall that we can only attain a balanced growth path using $y_t = F(k_t, Ah_t)$. If a solution can be found with technological progress, we can find a solution without

technological progress. Consider $A_t = (1 + \gamma_A)^t$ as before. The Social Planner's problem can be expressed as:

$$\begin{aligned} \max_{k_t} \sum_{t=0}^{\infty} \beta^t (F(k_t, (1 + \gamma_A)^t) + (1 - \delta)k_t - k_{t+1}) \\ \text{s.t. } k_t, c_t \geq 0 \\ k_0 \text{ given} \end{aligned}$$

The first order conditions can be written as:

$$\frac{u_1(c_{t-1})}{\beta u_1(c_t)} = F_1(k_t, (1 + \gamma_A)^t) + (1 - \delta)$$

Can we find a steady state? No! Since in the equation,

$$\frac{1}{\beta} = F(k^*, (1 + \gamma_A)^t) + (1 - \delta)$$

the LHS is constant, but RHS changes over time. This implies the steady-state is changing as well. Instead, we find a balanced growth path solution, i.e. look for a solution where k_t is growing at a constant rate.

We can show that the only possibility is that k grows at the same rate as A (SHOW THIS!).

Consider $c_t = F(k_t, (1 + \gamma_A)^t) + (1 - \delta)k_t - k_{t+1}$. If $k_t^* = (1 + \gamma_A)^t \hat{k}_0$, substitution yields,

$$\begin{aligned} c_t &= F((1 + \gamma_A)^t \hat{k}_0, (1 + \gamma_A)^t) + (1 - \delta)(1 + \gamma_A)^t \hat{k}_0 - (1 + \gamma_A)^{t+1} \hat{k}_0 \\ &= (1 + \gamma_A)^t [F(\hat{k}_0, 1) + (1 - \delta) - (1 + \gamma_A) \hat{k}_0] = (1 + \gamma_A) c_{t-1} \end{aligned}$$

therefore c grows at a constant rate. Using \hat{k}_0 and \hat{c}_0 ,

$$\frac{u'((1 + \gamma_A)^{t-1} \hat{c}_0)}{\beta u'((1 + \gamma_A)^t \hat{c}_0)} = F_1((1 + \gamma_A)^t \hat{k}_0, (1 + \gamma_A)^t) + (1 - \delta) = F_1(\hat{k}_0, 1) + (1 - \delta)$$

The RHS is constant since growth is constant over time. This does not imply that MRS are constant, so we must find a u such that MRS is constant over time. This does not happen in the Solow model, since consumption is exogenously determined. If we want to find a BGP solution, it must be that $u'(c_{t-1})/\beta u'(c_t)$ is constant along a path with $c_t = (1 + \gamma_A) c_{t-1}$.

What would u have to look like for us to achieve this?

$$u(c) = \frac{c^{1-\sigma} - 1}{1 - \sigma} \forall \sigma > 0$$

Check that this utility function has the desired property,

$$\frac{u'(c_{t-1})}{\beta u'(c_t)} = \frac{1}{\beta} \left(\frac{c_t}{c_{t-1}} \right)^\sigma$$

which is only dependent on c_t/c_{t-1} . Along a balanced growth path $c_{t+1} = (1 + \gamma_A)c_t$ and therefore $u'(c)/\beta u'((1 + \gamma_A)c)$ is constant. If this holds for every c , we can differentiate both sides and using algebra this yields:

$$\frac{u''(c)c}{u'(c)} = \frac{u''(c)c}{u'(c)}(1 + \gamma_A)$$

i.e. $CRRA(c) = CRRA((1 + \gamma_A)c)$. If this holds for all c , then this implies c must be CRRA.

Consider the Social Planner's problem with a change of variables.

$$\begin{aligned} & \max_{c_t, k_t} \sum_{t=0}^{\infty} \beta^t \left(\frac{c_t^{1-\sigma} - 1}{1 - \sigma} \right) \\ \text{s.t. } & c_t = F(k_t, (1 + \gamma_A)^t) + (1 - \delta)k_t - k_{t+1} \end{aligned}$$

Define $\tilde{c}_t = c_t/(1 + \gamma_A)^t$ and $\tilde{k}_t = k_t/(1 + \gamma_A)^t$. Our social planner's problem becomes:

$$\begin{aligned} & \max_{\tilde{c}_t, \tilde{k}_t} \sum_{t=0}^{\infty} \beta^t ((\tilde{c}_t(1 + \gamma_A)^t)^{1-\sigma} + \text{constant}) \\ \text{s.t. } & \tilde{c}_t(1 + \gamma_A)^t = F(\tilde{k}_t(1 + \gamma_A)^t, (1 + \gamma_A)^t) + (1 - \delta)(1 + \gamma_A)^t \tilde{k}_t \\ & - (1 + \gamma_A)^{t+1} \tilde{k}_{t+1} \end{aligned}$$

This implies $[\beta(1 + \gamma_A)^{1-\sigma}]^t \tilde{c}_t^{1-\sigma}$. Define $\tilde{\beta} = \beta(1 + \gamma_A)^{1-\sigma}$. With this change of variables, the only change we see is a $(1 + \gamma_A)$ floating around. Note that if $\tilde{\beta} > 1$ the model will fall apart.

11.2 Model with Labor-Leisure Choice and Taxes

In this case we allow for two changes. Namely, we allow for different responses to tax policies and expand the realm of possible tax policies. Consider a government that levies a constant proportional tax on income from labor, τ_h , and uses proceeds to fund a lump-sum transfer with a period-by period balanced budget.

An ADCE is a list of sequences, $\{c_t^*\}$, $\{h_t^*\}$, $\{k_t^*\}$, $\{w_t^*\}$, $\{r_t^*\}$, $\{p_t^*\}$, $\{T_t^*\}$ such that:

(a) (Consumers Maximize) Taking $\{w_t^*\}, \{r_t^*\}, \{p_t^*\}$ as given, $\{c_t^*\}, \{h_t^*\}, \{k_t^*\}$ solves:

$$\begin{aligned} & \max_{\{k_t\}, \{h_t\}, \{c_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) \\ \text{s.t. } & \sum_{t=0}^{\infty} p_t^* (c_t + k_{t+1} - (1 - \delta)k_t) = \sum_{t=0}^{\infty} p_t^* (r_t^* k_t^* + (1 - \tau_h)w_t^* h_t^* + T_t^*) \\ & 0 \leq h_t \leq 1 \forall t \\ & k_0 \text{ given} \end{aligned}$$

(b) (Firms Maximize) Taking $\{w_t^*\}, \{r_t^*\}, \{p_t^*\}$ as given, $\forall t$ h_t, k_t solves:

$$\begin{aligned} & \max_{k_t, h_t} p_t^* (F(k_t, h_t) - r^* k_t - w_t^* h_t) \forall t \\ & k_t, h_t \geq 0 \end{aligned}$$

(c) (Government) $T_t^* = \tau_h w_t^* h_t^*$

(d) (Market Clearing)

$$c_t^* + k_{t+1}^* - (1 - \delta)k_t^* = F(k_t^*, h_t^*) \forall t$$

The equilibrium conditions for this economy are:

$$\frac{u_1(c_t^*, 1 - h_t^*)}{\beta u_1(c_{t+1}^*, 1 - h_{t+1}^*)} = F_1(k_{t+1}^*, h_{t+1}^*) + (1 - \delta)$$

and,

$$\frac{u_2(c_t^*, 1 - h_t^*)}{u_1(c_t^*, 1 - h_t^*)} = (1 - \tau_h) F_2(k_t^*, h_t^*)$$

We can also extend our calibration procedure to the economy with labor-leisure choice.

Let $F(k, h) = k^\theta h^{1-\theta}$ and,

$$c = \frac{c^{1-\sigma} - 1}{1 - \sigma} + \frac{\alpha(1 - h)^{1-\gamma} - 1}{1 - \gamma}$$

We need to find the following parameters: $\beta, \delta, \sigma, \gamma, \theta, \alpha$. In addition to the usual targets, we add that the fraction of time dedicated to market work is 1/3. Using this new information, θ , and α are straightforward to find.

11.3 Balanced Growth in the Model with Labor-Leisure Choice

$$\begin{aligned} \max_{\{h_t\}, \{k_t\}} \quad & \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - h_t) \\ F(k_t, A_t h_t) + (1 - \delta)k_t - k_{t+1} & \geq 0 \\ k_t \geq 0, 0 \leq h_t & \leq 1 \\ k_0 & \text{ given} \end{aligned}$$

where $A_t = (1 + \gamma_A)^t$. Deriving the FOCs we attain the following equilibrium conditions:

$$\begin{aligned} \frac{u_1(c_t, 1 - h_t)}{\beta u_1(c_{t+1}, 1 - h_{t+1})} &= F_1(k_{t+1}, A_{t+1} h_{t+1}) + (1 - \delta) \\ \frac{u_2(c_t, 1 - h_t)}{u_1(c_t, 1 - h_t)} &= A_t F_2(k_t, A_t h_t) \end{aligned}$$

We look for a solution in which k_t, c_t grow at the same rate as A_t and h_t is constant. Recall static problem,

$$\begin{aligned} \max_{h, c} \quad & u(c, 1 - h) \\ \text{s.t.} \quad & c = Ah \end{aligned}$$

Solving this problem we get $u_1/u_2 = A$. If A gets bigger, solution for h stays the same, then c is growing at rate A . Note when A increases, h is more expensive. This is the income and substitution effect. Therefore A, h have to offset each other. We have this with preferences that perfectly offset income and substitution effects. Namely,

$$\begin{aligned} u(c, 1 - h) &= \log(c) + av(1 - h) \\ u(c, 1 - h) &= \frac{[cv(1 - h)]^{1-\sigma} - 1}{(1 - \sigma)} \end{aligned}$$

the latter when $\sigma \neq 1$ and is strictly positive. These are the only preferences consistent with balanced growth. This is because MRS grows at a constant rate.