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Research Report

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Neoclassical Growth Model with Overlapping Generations

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Let's start with an example for an endowment economy. There is a single good and time is discrete. At each time period, t , a new generation is born, which consists of a single individual. Each generation lives for 2 periods. Each generation has utility $u(c_{1t}, c_{2t})$ where c_{it} is consumption in the i -th period of life for the generation born at time t . Each generation has endowment $(\omega_{1t}, \omega_{2t})$. At $t = 0$, there is an "initial old" individual, who has endowment $w_{2,-1}$ and preferences $\hat{u}(c_{2,-1})$. Assume u is strictly increasing in both arguments and strictly concave. \hat{u} is strictly increasing and strictly concave.

An ADCE for this economy is a list of sequences $\{(c_{1t}^*, c_{2t}^*)\}$, $\{p_t^*\}$, and c_{2t}^* s.t.

1. Consumers Maximize. For each $t = 0, 1, 2, \dots$, taking $\{p_t^*\}$ as given, (c_{1t}^*, c_{2t}^*) solves:

$$\begin{aligned} & \max_{c_{1t}, c_{2t}} u(c_{1t}, c_{2t}) \\ \text{s.t. } & p_t^* c_{1t} + p_{t+1}^* c_{2t} = p_t^* \omega_t + p_{t+1}^* w_{2t} \\ & c_{it} \geq 0 \forall i = 1, 2 \end{aligned}$$

and taking p_0^* as given, $c_{1,-1}^*$ solves:

$$\begin{aligned} & \max_{c_{2,-1}} \hat{u}(c_{2,-1}) \\ \text{s.t. } & p_0^* c_{2,-1} = p_0^* \omega_{2,-1} \\ & c_{2,-1} \geq 0 \end{aligned}$$

2. Market Clearing

$$c_{t-1,2}^* + c_{t,1}^* = \omega_{t-1,2} + \omega_{t,1} \forall t = 0, 1, 2, \dots$$

Proposition 0.1 *If $\{(c_{1t}^*, c_{2t}^*)\}$, $c_{i,2}^*$, $\{p_t^*\}$ is an ADCE, then $c_{1t}^* = \omega_{it} \forall i, t$ and $c_{2,-1}^* = \omega_{2,-1}$.*

Proof Start with the initial old. Since $\hat{u}(\cdot)$ is strictly increasing. It must be that $c_{2,-1}^* = \omega_{2,-1}$. But market clearing for period 0 implies $c_{1,0}^* = \omega_{1,0}$. The budget equation for generation 0 then implies $c_{2,0}^* = \omega_{2,0}$. Market clearing for $t = 1$ then implies $c_{1,1}^* = \omega_{1,1}$ and repeating for all t .

Consider the following equilibrium allocation: $u(c_{1t}, c_{2t}) = \log(c_{1t}) + \log(c_{2t})$ and $(\omega_{1t}, \omega_{2t}) = (.75, .25)$, $\omega_{2,-1} = .25$. Now Consider the following allocation: $c_{1t} = .5\forall i, t$ and $c_{2,-1} = .5$. This allocation is feasible.

Compare this allocation to the equilibrium allocation: The initial old are better off and all generations are better off. Therefore all generations are better off under the alternate allocation. In this model it is not *necessary* that CE is PE.

Consider an endowment economy with I consumers and J goods.

$$u_i : \mathbb{R}_+^J \rightarrow \mathbb{R}$$

$$\omega_i \in \mathbb{R}_+^J$$

Assume u_i is strictly increasing in all good.

Theorem 0.2 If $(c_i^*, i = 1, \dots, I)$, p^* is an ADCE for this economy, then c_i^* , $i = 1, \dots, I$ is Pareto efficient.

Proof 1. Case 1: $I, J < \infty$. Suppose $c_i^* \forall i$ is not PE, then there exists $\hat{c}_i, \forall i$ such that $u_i(\hat{c}_i) \geq u_i(c_i^*) \forall i$, with strict inequality for at least one i . Given strict monotonicity of preferences, it follows $p^* \hat{c}_i \geq p^* c_i^* \forall i$, with strict inequality in at least one i . Adding up across all consumers, we have:

$$\sum_{i=1}^I p^* \hat{c}_i > \sum_{i=1}^I p^* c_i^*$$

$$\Rightarrow p^* \sum_{i=1}^I \omega_i > p^* \sum_{i=1}^{\infty} \omega_i$$

A contradiction.

2. Case 2: $I \rightarrow \infty, J < \infty$. The same logic holds as in Case 1. We must have $\sum_{i=1}^{\infty} p^* \omega_i < \infty$, otherwise there would be nothing to maximize, i.e. aggregate endowment must be finite. So we reach the same contradiction.
3. Case 3: $J \rightarrow \infty, I < \infty$. Since the theorem assumes an ADCE exists, $\sum_{i=1}^I p^* c_i^* < \infty$. Otherwise, there would be no equilibrium. So we reach the same contradiction.

4. *Case 4: $I, J \rightarrow \infty$. The argument breaks down here. But the result can still hold when the equilibrium price will be such that $\sum_{i=1}^{\infty} p^* \omega_i < \infty$.*

Consider the following counter-example from above. Normalize $p_0^* = 1$. Since,

$$\frac{u_1(c_{1t}^*, c_{2t}^*)}{u_2(c_{1t}^*, c_{2t}^*)} = \frac{p_t^*}{p_{t+1}^*} \forall i$$

we can deduce prices since,

$$\frac{c_{2t}^*}{c_{1t}^*} = \frac{p_t^*}{p_{t+1}^*} \Rightarrow p_{t+1}^* = 3p_t^*$$

If $p_0^* = 1$, then $p_t^* = 3^{t-1}$. Aggregate endowment is equal to 1 for all t . Therefore $\sum_{t=0}^{\infty} p_t^* = \sum_{t=0}^{\infty} 3^{t-1} = \infty$. Suppose we consider an arbitrary set of endowments. Let p_t^* be the equilibrium price sequence and let aggregate endowment equal 1 for all t . Our previous argument tells us that if $\sum_{t=0}^{\infty} p_t^*$ is finite, then the aggregate endowment is finite.

Assuming log utility, we know that $p_{t+1}^* = \omega_{1t}/\omega_{2t}p_t^*$. If $\omega_{1t}^*/\omega_{2t} < 1$, then $\sum_t p_t^* < \infty$ and the CE allocation is PE. If the ratio is greater than one, then a CE allocation is not PE. Consider $p_{t+1}^*/p_t^* = 1/(1 + R_t)$. If $p_{t+1}^*/p_t^* < 1$, then $R_t > 0$. Likewise if $p_{t+1}^*/p_t^* > 1$, then $R_t < 0$. From the social planner's perspective, an individual who wants to turn c_{1t} into c_{2t} can do so according to p_{t+1}^*/p_t^* .

A social planner can exchange c_{1t} for c_{2t} at the rate 1-1 by transferring among generations at any point in time. Suppose we consider $(\omega_{1t}, \omega_{2t}) = (.25, .75)$, $\omega_{2t} = .75$. What if we gave everybody $(c_{1t}, c_{2t}) = (.5, .5)$. Here the initial old are made worse off.

1 Overlapping Generations Model with Production

Time is discrete. Each period a new individual is born, who lives for 2 periods.

Preferences. The individual born in period t has preferences defined by $u(c_{1t}) + \beta u(c_{2t})$, where c_{it} is consumption in the i -th period of life for individual born in period t . At $t = 1$, there is also an initial old individual who has utility $u(c_{2,0})$.

Endowments. Individuals are endowed with 1 unit of time when young. Initial old has an endowment of k_1 units of capital.

Technology.

$$\begin{aligned} y_t &= F(k_t, h_t) \\ c_t + i_t &= y_t \\ k_{t+1} &= (1 - \delta)k_t + i_t \end{aligned}$$

An ADCE for this economy is a list of sequences $\{c_{1t}^*, c_{2t}^*\}$, $\{h_t^*\}$, $\{k_t^*\}$, $\{p_t^*\}$, $\{r_t^*\}$, $\{w_t^*\}$ s.t.

1. Consumers Maximize. For all $t = 0, 1, 2, \dots$, taking $\{p_t^*\}$, $\{r_t^*\}$, $\{w_t^*\}$ as given, $\{c_{1t}^*, c_{2t}^*\}$, $\{h_t^*\}$, $\{k_t^*\}$ solves:

$$\begin{aligned} & \max_{\{c_{1t}\}, \{c_{2t}\}, \{h_t\}, \{k_t\}} u(c_{1t}) + \beta u(c_{2t}) \\ \text{s.t. } & p_t^*(c_{1t} + k_{t+1}) + p_{t+1}^* c_{2t} = w_t^* + r_{t+1}^* k_{t+1} + p_{t+1}^*(1 - \delta)k_{t+1} \\ & c_{1t}, c_{2t} \geq 0 \\ & 0 \leq h_t \leq 1 \end{aligned}$$

and $c_{2,0}^*$ solves:

$$\begin{aligned} & \max_{c_{2,0}} u(c_{2,0}) \\ \text{s.t. } & p_1^* c_{2,0} = r_1^* k_1 + p_1^*(1 - \delta)k_1 \\ & c_{2,0} \geq 0 \\ & k_1 \text{ given} \end{aligned}$$

2. Firms Maximize. For all t , taking p_t^* , r_t^* , w_t^* as given, h_t^* , k_t^* , solves:

$$\begin{aligned} & \max_{k_t, h_t} p_t^* F(k_t, h_t) - r_t^* k_t - w_t^* h_t \\ \text{s.t. } & k_t \geq 0, h_t \geq 0 \end{aligned}$$

3. Markets Clear

$$c_{2,t-1}^* + c_{1,t}^* + k_{t+1}^* - (1 - \delta)k_t^* = F(k_t^*, h_t^*) \forall t$$

We can proceed as usual in solving this problem. From the FOCs, we obtain:

$$\frac{u'(c_{1t}^*)}{\beta u'(c_{1t+1}^*)} = (1 - \delta) + F_1(k_{t+1}^*, 1)$$

Compare this to the infinite case. They are not the same despite the similar functional form.

Aggregate consumption here is $c_{1t} + c_{2,t-1}$.

Some observations about equilibrium. Claim: In equilibrium, it must be that:

$$c_{2t}^* = F_1(k_{t+1}^*, 1) + k_{t+1}^* + (1 - \delta)k_{t+1}^* \quad (1)$$

$$c_{1t}^* = F_2(k_t^*, 1) - k_{t+1}^* \quad (2)$$

Proof. Start in period 1. From the budget equation for initial old: $c_{20} = (r_1^*/p_1^*)k_1 + (1 - \delta)k_1$. From firm FOC: $(r_1^*/p_1^*) = F_1(k_1^*, 1)$. Therefore $c_{20}^* = k_1^* F_1(k_1^*, 1) + (1 - \delta)k_1$. To show (2) holds for $t = 1$, look at the market clearing condition for period 1:

$$c_{20}^* + c_{11}^* + k_2^* - (1 - \delta)k_1^* = F(k_1^*, 1) = k_1^* F_1(k_1^*, 1) + F_2(k_1^*, 1)$$

Now use (1) for $t = 1$,

$$c_{11}^* + k_2^* = F_2(k_1^*, 1)$$

Therefore both equations hold for $t = 1$. Now we go to $t = 2$. The budget equation for the generation born in period 1 is:

$$p_1^*(c_{11}^* + k_2^*) + p_2^*c_{21}^* = w_2^* + r_2^*k_2^* + p_2^*(1 - \delta)k_2^*$$

But we just showed that $c_{11}^* = F_2(k_1^*, 1) - k_2^* = w_1^*/p_1^* - k_2^*$ using the firm FOC. This implies

$$\frac{p_2^*}{p_1^*}c_{12}^* = \frac{p_2^*}{p_1^*}(1 - \delta)k_2^*$$

or

$$c_{21}^* = F_1(k_1^*, 1)k_2^* + (1 - \delta)k_2^*$$

Using the market clearing condition for period 2, and these results implies that the equation holds for $t = 2$. These properties tell us how we can express $\{c_{1t}^*\}$, $\{c_{2t}^*\}$ in terms of $\{k_t^*\}$. Using these expressions, it follows that in equilibrium $\{k_t^*\}$ must satisfy:

$$\frac{u'(F_2(k_t^*, 1) - k_{t+1}^*)}{\beta u'(k_{t+1}^* F_1(k_{t+1}^*, 1) + (1 - \delta)k_{t+1}^*)} = F_1(k_{t+1}^* + (1 - \delta)k_t^*)$$

and k_0 given. Note that the budget equation for individuals born at time t is:

$$p_t^*(c_{1t} + k_{t+1}) + p_{t+1}^*c_{2t} = w_t^* + r_t^*k_{t+1} + (1 - \delta)k_{t+1}p_{t+1}^*$$

this implies:

$$c_{1t} + \frac{p_{t+1}^*}{p_t^*}c_{2t} = \frac{w_t}{p_t^*} + k_{t+1}\left[\frac{r_{t+1}^*}{p_t^*} + \frac{p_{t+1}^*(1 - \delta)}{p_t^*} - 1\right]$$

or

$$c_{1t} + \frac{p_{t+1}^*}{p_t^*}c_{2t} = \frac{w_t}{p_t^*} + k_{t+1}\frac{p_{t+1}^*}{p_t^*}\left[\frac{r_{t+1}^*}{p_{t+1}^*} + (1 - \delta) - \frac{p_t^*}{p_{t+1}^*}\right]$$

which in equilibrium reduces to:

$$c_{1t} + \frac{p_{t+1}^*}{p_t^*}c_{2t} = \frac{w_t^*}{p_t^*}$$

Note $p_{t+1}^*/p_t^* = 1/(1 + R)$ and $w_t^*/p_t^* = PV$ income.

Lets start with a simple example. Assume $u(c) = \log(c)$, $F(k, h) = k^\alpha h^{1-\alpha}$. Start in period 1. Given k_1 , this implies we know w_1^* . Consider the maximization problem of the generation born at time 1.

$$\begin{aligned} \max_{c_{11}, c_{21}} \quad & \frac{1}{1 + \beta} [\log(c_{11}) + \beta \log(c_{21})] \\ \text{s.t.} \quad & c_{11} + \frac{p_2^*}{p_1^*}c_{21} = \frac{w_1^*}{p_1^*} \end{aligned}$$

Given p_2^*/p_1^* solves the individuals problem for c_{11}, c_{21} from which we can derive k_2 since $k_2 = w_1^*/p_1^* - c_{11}$. Conversely, if you tell me a value for k_2 , then I can tell you what p_2^*/p_1^* must be, by using the arbitrage equation $p_1^*/p_2^* = r_2^*/p_2^* + (1 - \delta)$ and the FOC from the firm in period 2.

In equilibrium, these two must be consistent, i.e. the k_2 that this individual chooses given p_2^*/p_1^* must be consistent with the value of p_2^*/p_1^* . Let's try and solve for the value. From the consumer's problem, we know:

$$c_{22} = \frac{1}{1 + \beta} \frac{w_1^*}{p_1^*}$$

Therefore, $k_2 = \beta/(1 + \beta)w_1^*/p_1^*$. Note that this value does not depend on p_2^*/p_1^* . Given this k_2 , we know from the arbitrage equation and the firm FOC that $p_1^*/p_2^* = F_1(k_2^*, 1) + (1 - \delta)$. We need to find value of p_1^*/p_2^* and k_2 that solve both of these equations. Iterating we get $k_{t+1} = \beta/(\beta + 1)F_2(k_t^*, 1)$. Using C-D production, $k_{t+1}^* = \beta/(1 + \beta)(1 - \alpha)k_t^\alpha$. Note that in the optimal growth model, we do not get this. Let's solve for k^* to arrive at:

$$k^* = \left(\frac{\beta}{1 + \beta}\right)(1 - \alpha)^{\frac{\alpha}{1 - \alpha}}$$

and $f'(k^*) = F_1(k^*, 1) = \alpha/(1 - \alpha)(1 + \beta)/\beta$.

Recall the golden rule: k^G is the value of k that maximizes steady state consumption. Recall dynamic inefficiency: If $k^* > k^G$, then the allocation is necessarily inefficient. k^G is defined by $f'(k^G) = \delta$. If $f'(k^*) < \delta$, then $k^* > k^G$ and the economy is "dynamically inefficient." Given β , we can find a sufficiently small α so that $f'(k^*) < \delta$, i.e. the steady state *may* be dynamically inefficient. Therefore if marginal product of capital is low, then moving income is inefficient, so we must hold more capital. This is because capital is the only way to move income across periods.

Consider the endowment economy with log-log preferences and $(w_{1t}, w_{2t}) = (.75, .25)$. Note that $(.5, .5)$ is not optimal, but we can construct a pay-as-you-go security system.

Consider the following system: when young, government takes away .25 units to give to current old. When old, government gives you .25 units of consumption taking it from that period's young. Then $(.5, .5)$ is efficient. In the production economy, if the rate of return is low, but people are still investing, government reallocation can do so at a 1-1 rate and get rid of inefficiency.

Problem for generation 1.

$$\begin{aligned} & \max_{c_{1t}, c_{2t}} u(c_{1t}) + \beta u(c_{2t}) \\ \text{s.t. } & c_{11} + \frac{p_2^*}{p_1^*} c_{21} = \frac{w_1^*}{p_1^*} \end{aligned}$$

Given p_2^*/p_1^* , we want to find k_2 . Given k_2 , we know p_2^*/p_1^* in equilibrium. We can solve for c_{1t} , which implies $k_2 = w_1^*/p_1^* - c_{1t}$. Alternate way to write this is $k_2 = s(p_2^*/p_1^*)w_1^*/p_1^*$ where $s(\cdot)$ is the savings rate as a function of p_2^*/p_1^* and taking w_1^*/p_1^* as given. The shape of s is tied to the dynamics that we see. Consider $u(c) = (c^{1-\sigma} - 1)/(1 - \sigma)$, when $\sigma = 1$, s is constant. When $\sigma < 1$, $s(\cdot)$ is increasing, when $\sigma > 1$, $s(\cdot)$ is decreasing. The FOCs are:

$$\begin{aligned} c_{1t}^{-\sigma} &= \lambda \\ \beta c_{2t}^\sigma &= \frac{p_2^*}{p_1^*} = \lambda \end{aligned}$$

or,

$$c_{2t} = \left(\frac{\beta p_1^*}{p_2^*}\right)^{1/\sigma} c_{1t}$$

Combining with the budget equation yields,

$$\begin{aligned} c_{1t} + \left(\frac{p_2^*}{p_1^*}\right)\left(\frac{\beta p_1^*}{p_2^*}\right)^{1/\sigma} c_{1t} &= \frac{w_1^*}{p_1^*} \\ c_{1t}\left[1 + \left(\frac{p_1^*}{p_2^*}\right)^{1/\sigma} \beta^{1/\sigma}\right] &= \frac{w_1^*}{p_1^*} \end{aligned}$$

Solving for c_{1t}^* , we get $(p_1^*/p_2^*)^{\sigma-1}$. This is increasing in p_1^*/p_2^* if $\sigma > 1$ and decreasing in p_1^*/p_2^* if $\sigma < 1$.

1.1 OLG Iterative Algorithm

Consider the young generation at time t . Their maximization problem can be written:

$$\begin{aligned} \max_{c_{1t}, c_{2t}} & u(c_{1t}) + \beta u(c_{2t}) \\ \text{s.t. } & c_{1t} + \frac{c_{2t}}{1 + r_{t+1} - \delta} = \frac{w_t}{p_t} \\ & c_{1t} \geq 0, c_{2t} \geq 0 \end{aligned}$$

or $c_{1t} + p_{t+1}/p_t c_{2t} = \frac{w_t}{p_t}$. We also showed that $k_{t+1} = w_t/p_t - c_{1t}$. And in equilibrium we have that:

$$\frac{r_{t+1}}{p_{t+1}} = F_1(k_{t+1}, 1) = f'(k_{t+1})$$

Simple Example. Assume $u(c) = \log(c)$, $f(k) = k^\alpha$. In this case we showed that the consumer problem implies that savings is a constant fraction of income when young, independent of the value of r_{t+1}/p_{t+1} . For this case, we showed that the dynamics are easily characterized. Note that the OLG can yield outcomes that are well-behaved, even so the welfare properties might be vastly different.

Second Example. Generalized preferences $u(c) = c^{1-\sigma} - 1/1 - \sigma$. We solve the consumer maximization problem for this case and get:

$$c_{1t} = \frac{w_t/p_t}{1 + \beta^{1/\sigma}(1 + r_{t+1}/p_{t+1} - \delta)^{1/\sigma-1}}$$

$$k_{t+1} = [\beta^{-1/\sigma}(1 + r_{t+1}/p_{t+1} - \delta)^{1-1/\sigma} + 1]w_t/p_t$$

As $\sigma \rightarrow 1$, we get that k_{t+1} is a constant fraction of first period income. If $0 < \sigma < 1$, k_{t+1} is increasing in r_{t+1}/p_{t+1} . If $\sigma > 1$, k_{t+1} is decreasing in r_{t+1}/p_{t+1} . Recall that $1 + r_{t+1}/p_{t+1} - \delta$ is the real rate of return to accumulating capital in period t . Rewrite:

$$k_{t+1}[\beta^{-1/\sigma}(1 + f'(k_{t+1} - \delta)^{1-1/\sigma} + 1)] = w_t/p_t = F_2(k_t, 1)$$

The term in brackets can be either increasing or decreasing in k_{t+1} . If LHS is increasing in k_{t+1} , then everything is nicely behaved. If we look for a steady state we require a k^* s.t.

$$k^*[\beta^{-1/\sigma}(1 + f'(k^*) - \delta)^{1-1/\sigma} + 1] = F_2(k^*, 1)$$

If things are not monotone then we get weird looking stuff.