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## Fixed-effect regressions on network data

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# Supplement for "Fixed-effect regressions on network data"

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**cemmap** working paper CWP32/16

# FIXED-EFFECT REGRESSIONS ON NETWORK DATA

## SUPPLEMENT

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## S.1 Proofs

### PROOF OF LEMMA 1 (EXISTENCE)

The estimator is defined by the constraint minimization problem in (2.3). For convenience we express the constraint in quadratic form,  $(\mathbf{a}'\boldsymbol{\iota}_n)^2 = 0$ . By introducing the Lagrange multiplier  $\lambda > 0$  we can write

$$\hat{\boldsymbol{\alpha}} = \arg \min_{\mathbf{a} \in \mathbb{R}^n} (\mathbf{y} - \mathbf{B}\mathbf{a})'(\mathbf{y} - \mathbf{B}\mathbf{a}) + \lambda (\mathbf{a}'\boldsymbol{\iota}_n)^2.$$

Solving the corresponding first-order condition we obtain

$$\hat{\boldsymbol{\alpha}} = (\mathbf{B}'\mathbf{B} + \lambda \boldsymbol{\iota}_n \boldsymbol{\iota}_n')^{-1} \mathbf{B}'\mathbf{y}.$$

Here, the matrix  $\mathbf{B}'\mathbf{B} + \lambda \boldsymbol{\iota}_n \boldsymbol{\iota}_n'$  is invertible, because  $\mathbf{L} = \mathbf{B}'\mathbf{B}$  only has a single zero eigenvalue (because we assume the graph to be connected) with eigenvector  $\boldsymbol{\iota}_n$ , so that adding  $\lambda \boldsymbol{\iota}_n \boldsymbol{\iota}_n'$  gives a non-degenerate matrix. The matrices  $\mathbf{B}'\mathbf{B}$  and  $\boldsymbol{\iota}_n \boldsymbol{\iota}_n'$  commute, and by properties of the Moore-Penrose inverse we thus have

$$(\mathbf{B}'\mathbf{B} + \lambda \boldsymbol{\iota}_n \boldsymbol{\iota}_n')^{-1} = (\mathbf{B}'\mathbf{B})^\dagger + \lambda^{-1} (\boldsymbol{\iota}_n \boldsymbol{\iota}_n')^\dagger.$$

We furthermore have  $(\boldsymbol{\iota}_n \boldsymbol{\iota}_n')^\dagger = n^{-2} \boldsymbol{\iota}_n \boldsymbol{\iota}_n'$  and, because  $\mathbf{B}\boldsymbol{\iota}_n = \mathbf{0}$ , the contribution from  $(\boldsymbol{\iota}_n \boldsymbol{\iota}_n')^\dagger$  drops out of the above formula for  $\hat{\boldsymbol{\alpha}}$ , and we obtain  $\hat{\boldsymbol{\alpha}} = (\mathbf{B}'\mathbf{B})^\dagger \mathbf{B}'\mathbf{y}$ . This concludes the proof. ■

### PROOF OF THEOREM 1 (SAMPLING DISTRIBUTION)

As  $\mathbf{y} = \mathbf{B}\boldsymbol{\alpha} + \mathbf{u}$ , Lemma 1 gives

$$\hat{\boldsymbol{\alpha}} = \boldsymbol{\alpha} + (\mathbf{B}'\mathbf{B})^\dagger \mathbf{B}'\mathbf{u}.$$

Conditional on  $\mathbf{B}$ ,  $\mathbf{u} \sim \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$ , and so

$$\hat{\boldsymbol{\alpha}} \sim \mathcal{N}(\boldsymbol{\alpha}, \sigma^2 (\mathbf{B}'\mathbf{B})^\dagger),$$

where the variance expression follows from properties of the Moore-Penrose pseudoinverse. This concludes the proof. ■

## PROOF OF COROLLARY 1 (INFERENCE)

The result follows from Theorem 1 by standard arguments on the  $F$ -statistic in linear regression models. Here, the degrees-of-freedom correction from  $m - n$  to  $m - (n - 1)$  arises, because the projection matrix

$$\mathbf{I}_m - \mathbf{B}(\mathbf{B}'\mathbf{B})^\dagger \mathbf{B}'$$

has rank  $m - (n - 1)$ . Notice that although  $\mathbf{B}$  has  $n$  columns, we have that  $\text{rank } \mathbf{B} = (n - 1)$ . This concludes the proof.  $\blacksquare$

## PROOF OF THEOREMS 2 AND 5 (ZERO-ORDER BOUNDS)

There are no isolated vertices, because  $\mathcal{G}$  is connected and  $n > 2$ . That is,  $d_i > 0$  for all  $i$ , and so  $\mathbf{D}$  is invertible. From Theorem 1 and the definition of the normalized Laplacian  $\mathbf{S}$  we find

$$\text{var}(\hat{\boldsymbol{\alpha}}) = \sigma^2 \mathbf{D}^{-\frac{1}{2}} \mathbf{S}^\dagger \mathbf{D}^{-\frac{1}{2}}.$$

In the following we write  $\mathbf{M}_1 \leq \mathbf{M}_2$  for symmetric matrices  $\mathbf{M}_1$  and  $\mathbf{M}_2$  to indicate that  $\mathbf{M}_2 - \mathbf{M}_1$  is positive semi-definite. We have  $\mathbf{S}^\dagger \leq \lambda_2^{-1} \mathbf{I}_n$ , because  $\lambda_2$  is the smallest non-zero eigenvalue of  $\mathbf{S}$ . Therefore,

$$\text{var}(\hat{\boldsymbol{\alpha}}) \leq \frac{\sigma^2}{\lambda_2} \mathbf{D}^{-1}.$$

This result implies that, for any vector  $\mathbf{v} \in \mathbb{R}^n$ ,

$$\text{var}(\mathbf{v}'\hat{\boldsymbol{\alpha}}) = \mathbf{v}'\text{var}(\hat{\boldsymbol{\alpha}})\mathbf{v} \leq \frac{\sigma^2}{\lambda_2} \mathbf{v}'\mathbf{D}^{-1}\mathbf{v} = \frac{\sigma^2}{\lambda_2} \mathbf{v}' \text{diag}(d_1^{-1}, d_2^{-1}, \dots, d_n^{-1}) \mathbf{v}.$$

The bound in Theorem 2 follows on setting  $\mathbf{v} = \mathbf{e}_i$ , the  $i$ th unit vector. The corresponding bound for the differences in Theorem 5 follows on setting  $\mathbf{v} = \mathbf{e}_i - \mathbf{e}_j$  for  $i \neq j$ . This concludes the proof.  $\blacksquare$

## PROOF OF THEOREMS 3 AND 6 (FIRST-ORDER BOUNDS)

We first show that, if  $\mathcal{G}$  is connected, then

$$0 \leq \left[ \text{var}(\hat{\boldsymbol{\alpha}}) - \sigma^2 \left( \mathbf{D}^{-1} + \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} - \frac{\boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{D}^{-1}}{n} - \frac{\mathbf{D}^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n'}{n} \right) \right] \leq \frac{\sigma^2}{\lambda_2} \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1}. \quad (\text{S.1})$$

Theorems 3 and 6 will then follow readily. First note that, because  $\mathcal{G}$  is connected, we know that the zero eigenvalue of the Laplacian matrix  $\mathbf{L}$  has multiplicity one, and the corresponding eigenvector is given by  $\boldsymbol{\iota}$ . The Moore-Penrose pseudoinverse of  $\mathbf{L}$  therefore satisfies  $\mathbf{L}^\dagger \mathbf{L} = \mathbf{I}_n - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n'$ , where the right hand side is the idempotent matrix that projects orthogonally to  $\boldsymbol{\iota}_n$ . Using that  $\mathbf{L} = \mathbf{D} - \mathbf{A}$  and solving this equation for  $\mathbf{L}^\dagger$  gives

$$\mathbf{L}^\dagger = \mathbf{D}^{-1} + \mathbf{L}^\dagger \mathbf{A} \mathbf{D}^{-1} - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{D}^{-1}. \quad (\text{S.2})$$

The Laplacian is symmetric, and so transposition gives

$$\mathbf{L}^\dagger = \mathbf{D}^{-1} + \mathbf{D}^{-1} \mathbf{A} \mathbf{L}^\dagger - n^{-1} \mathbf{D}^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n'. \quad (\text{S.3})$$

Replacing  $\mathbf{L}^\dagger$  on the right-hand side of (S.2) by the expression for  $\mathbf{L}^\dagger$  given by (S.3) yields

$$\mathbf{L}^\dagger = \mathbf{D}^{-1} + \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} + \mathbf{D}^{-1} \mathbf{A} \mathbf{L}^\dagger \mathbf{A} \mathbf{D}^{-1} - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{D}^{-1} - n^{-1} \mathbf{D}^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n',$$

where we have also used the fact that  $\mathbf{D}^{-1} \mathbf{A} \boldsymbol{\iota}_n = \boldsymbol{\iota}_n$ . Re-arranging this equation allows us to write

$$\mathbf{L}^\dagger - (\mathbf{D}^{-1} + \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{D}^{-1} - n^{-1} \mathbf{D}^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n') = \mathbf{D}^{-1} \mathbf{A} \mathbf{L}^\dagger \mathbf{A} \mathbf{D}^{-1}.$$

Because  $\mathbf{L} \geq 0$  and by the arguments in the preceding proof we also have the bounds

$$\mathbf{0} \leq \mathbf{L}^\dagger \leq \lambda_2^{-1} \mathbf{D}^{-1}.$$

Put together this yields

$$0 \leq \mathbf{L}^\dagger - (\mathbf{D}^{-1} + \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{D}^{-1} - n^{-1} \mathbf{D}^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n') \leq \lambda_2^{-1} \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1},$$

and multiplication with  $\sigma^2$  gives the bounds stated in (S.1).

To show Theorems 3 and 6 we calculate, for  $i \neq j$ ,

$$\begin{aligned} \mathbf{e}_i' \mathbf{D}^{-1} \mathbf{e}_i &= d_i^{-1}, & \mathbf{e}_i' \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} \mathbf{e}_i &= d_i^{-1} h_i^{-1}, \\ \mathbf{e}_i' \mathbf{D}^{-1} \mathbf{e}_j &= 0, & \mathbf{e}_i' \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} \mathbf{e}_j &= d_i^{-1} d_j^{-1} d_{ij} h_{ij}^{-1}, \\ \mathbf{e}_i' \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} \mathbf{e}_i &= 0, & \mathbf{e}_i' \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{D}^{-1} \mathbf{e}_i &= \boldsymbol{\iota}_n' \mathbf{D}^{-1} \mathbf{e}_i = d_i^{-1}, \\ \mathbf{e}_i' \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1} \mathbf{e}_j &= d_i^{-1} d_j^{-1} (\mathbf{A})_{ij}, & \mathbf{e}_i' \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{D}^{-1} \mathbf{e}_j &= \boldsymbol{\iota}_n' \mathbf{D}^{-1} \mathbf{e}_j = d_j^{-1}. \end{aligned}$$

Combining these results with (S.1) gives the bounds on  $\text{var}(\hat{\alpha}_i) = \mathbf{e}_i' \text{var}(\hat{\boldsymbol{\alpha}}) \mathbf{e}_i$  and  $\text{var}(\hat{\alpha}_i - \hat{\alpha}_j) = (\mathbf{e}_i - \mathbf{e}_j)' \text{var}(\hat{\boldsymbol{\alpha}}) (\mathbf{e}_i - \mathbf{e}_j)$  stated in the theorems and concludes the proof.  $\blacksquare$

## PROOF OF THEOREM 4 (GENERALIZED APPROXIMATION)

From the proof of Lemma 1, the least-squares estimator satisfies the first-order condition

$$\mathbf{L} \hat{\boldsymbol{\alpha}} = \mathbf{B}' \mathbf{y}.$$

Using that  $\mathbf{y} = \mathbf{B}\boldsymbol{\alpha} + \mathbf{u}$  and that  $\mathbf{L} = \mathbf{D} - \mathbf{A}$  this yields  $\mathbf{D}^{1/2}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}) = \mathbf{D}^{-1/2}\mathbf{B}'\mathbf{u} + \boldsymbol{\epsilon}$ , where

$$\boldsymbol{\epsilon} := \mathbf{D}^{-1/2}\mathbf{A}(\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha}).$$

Note that this is the vector version of the expression for  $\sqrt{d_i}(\hat{\alpha}_i - \alpha_i)$  as given in the theorem. From  $\hat{\boldsymbol{\alpha}} - \boldsymbol{\alpha} = (\mathbf{B}'\mathbf{B})^\dagger \mathbf{B}'\mathbf{u}$  it follows that  $\mathbb{E}(\boldsymbol{\epsilon}) = \mathbf{0}$  while from the assumption that  $\mathbb{E}(\mathbf{u}\mathbf{u}') \leq \bar{\sigma}^2 \mathbf{I}_n$  we have that

$$\mathbb{E}(\boldsymbol{\epsilon}\boldsymbol{\epsilon}') = \mathbf{D}^{-1/2}\mathbf{A}(\mathbf{B}'\mathbf{B})^\dagger \mathbf{B}' \mathbb{E}(\mathbf{u}\mathbf{u}') \mathbf{B}(\mathbf{B}'\mathbf{B})^\dagger \mathbf{A}\mathbf{D}^{-1/2} = \bar{\sigma}^2 \mathbf{D}^{-1/2}\mathbf{A}\mathbf{L}^\dagger \mathbf{A}\mathbf{D}^{-1/2}.$$

As in the preceding proofs, we still have that  $\mathbf{L}^\dagger \leq \lambda_2^{-1}\mathbf{D}^{-1}$ , and so

$$\mathbb{E}(\boldsymbol{\epsilon}\boldsymbol{\epsilon}') \leq \bar{\sigma}^2 \lambda_2^{-1} \mathbf{D}^{-1/2} \mathbf{A} \mathbf{D}^{-1} \mathbf{A} \mathbf{D}^{-1/2}.$$

From this we find

$$\mathbb{E}(\epsilon_i^2) \leq \frac{\bar{\sigma}^2}{\lambda_2 h_i}.$$

Thus, if  $\bar{\sigma}^2 \lambda_2^{-1} h_i^{-1} \rightarrow 0$  as  $n \rightarrow \infty$ , then by Markov's inequality we have  $\epsilon_i \rightarrow_p 0$ . By the continuous mapping theorem we therefore have

$$\sqrt{d_i}(\hat{\alpha}_i - \alpha_i) \rightarrow_p \frac{1}{\sqrt{d_i}} \sum_{j \in [i]} u_{ij}.$$

Moreover, if  $\frac{1}{\sqrt{d_i}} \sum_{j \in [i]} u_{ij}$  is asymptotically normal, then so is  $\sqrt{d_i}(\hat{\alpha}_i - \alpha_i)$ . This concludes the proof. ■

## PROOF OF LEMMA 2 (VARIANCE OF COMPONENT ESTIMATORS)

**Additional notation.** Without loss of generality we relabel the elements of  $V$  such that

$$V_1 = \{1, \dots, n_1\}, \quad V_2 = \{n_1 + 1, \dots, n_2\}, \quad \dots \quad V_q = \{n - n_q + 1, \dots, n\}.$$

We decompose  $\boldsymbol{\beta} = (\boldsymbol{\beta}'_1, \boldsymbol{\beta}'_2, \dots, \boldsymbol{\beta}'_q)'$  and  $\widehat{\boldsymbol{\beta}} = (\widehat{\boldsymbol{\beta}}'_1, \widehat{\boldsymbol{\beta}}'_2, \dots, \widehat{\boldsymbol{\beta}}'_q)'$ , where each  $\boldsymbol{\beta}_r$  and  $\widehat{\boldsymbol{\beta}}_r$  are  $n_r$  column vectors. Note that the Laplacian matrix  $\mathbf{L}_W$  is block-diagonal; moreover,

$$\mathbf{L}_W = \text{diag}(\mathbf{L}_1, \mathbf{L}_2, \dots, \mathbf{L}_q),$$

where  $\mathbf{L}_r$  is the Laplacian of the graph  $\mathcal{G}_r$ . We also decompose  $\mathbf{A} = \mathbf{A}_W + \mathbf{A}_B$ , where  $\mathbf{A}_W$  and  $\mathbf{A}_B$  are the adjacency matrix of  $\mathcal{G}_W$  and  $\mathcal{G}_B$ , respectively, and write  $\mathbf{D}_W = \text{diag}(\mathbf{A}_W \boldsymbol{\iota}_n)$  and  $\mathbf{D}_B = \text{diag}(\mathbf{A}_B \boldsymbol{\iota}_n)$  for the corresponding degree matrices. We have  $\mathbf{L}_W = \mathbf{D}_W - \mathbf{A}_W$  and  $\mathbf{L}_B = \mathbf{D}_B - \mathbf{A}_B$ . We also relabel the elements of  $E$  such that

$$E_B = \{1, \dots, m_B\}, \quad E_W = \{m_B + 1, \dots, m\},$$

and correspondingly we decompose  $\mathbf{B} = (\mathbf{B}'_B, \mathbf{B}'_W)'$ , where  $\mathbf{B}_B$  and  $\mathbf{B}_W$  are  $m_B \times n$  and  $m_W \times n$  matrices, respectively, whose rows correspond to edges in  $\mathcal{G}_B$  and  $\mathcal{G}_W$ , respectively. We then have  $\mathbf{L} = \mathbf{B}'\mathbf{B} = \mathbf{B}'_W \mathbf{B}_W + \mathbf{B}'_B \mathbf{B}_B = \mathbf{L}_W + \mathbf{L}_B$ .

**Inverse expressions.** Notice that, under the conventions from above,  $\mathbf{P}$  is simply given by

$$\mathbf{P} = \begin{pmatrix} \boldsymbol{\iota}'_{n_1} & 0 & \dots & 0 \\ 0 & \boldsymbol{\iota}'_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \boldsymbol{\iota}'_{n_q} \end{pmatrix}.$$

We define the block-diagonal  $n \times n$  matrix

$$\mathbf{M} = \mathbf{I}_n - \mathbf{P}'\mathbf{H}^{-1}\mathbf{P} = \begin{pmatrix} \mathbf{I}_{n_1} - n_1^{-1} \boldsymbol{\iota}_{n_1} \boldsymbol{\iota}'_{n_1} & 0 & \dots & 0 \\ 0 & \mathbf{I}_{n_2} - n_2^{-1} \boldsymbol{\iota}_{n_2} \boldsymbol{\iota}'_{n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mathbf{I}_{n_q} - n_q^{-1} \boldsymbol{\iota}_{n_q} \boldsymbol{\iota}'_{n_q} \end{pmatrix}.$$



Some useful relations are  $\mathbf{H} = \mathbf{P}\mathbf{P}'$ ,  $\mathbf{M}^2 = \mathbf{M}$ ,  $\mathbf{P}\mathbf{M} = 0$ ,  $\mathbf{P}'\boldsymbol{\iota}_q = \boldsymbol{\iota}_n$ ,  $\mathbf{P}\boldsymbol{\iota}_n = \mathbf{H}\boldsymbol{\iota}_q$ , and thus also  $\mathbf{H}^{-1}\mathbf{P}\boldsymbol{\iota}_n = \boldsymbol{\iota}_q$ . The various pseudo-inverses that appear in the following satisfy

$$\begin{aligned} \mathbf{L}_W^\dagger \mathbf{L}_W &= \mathbf{M}, \\ (\mathbf{H}^{-1/2} \mathbf{L}_* \mathbf{H}^{-1/2})^\dagger (\mathbf{H}^{-1/2} \mathbf{L}_* \mathbf{H}^{-1/2}) &= \mathbf{I}_q - n^{-1} \mathbf{H}^{1/2} \boldsymbol{\iota}_q \boldsymbol{\iota}_q' \mathbf{H}^{1/2}, \\ (\mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P})^\dagger (\mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P}) &= \mathbf{P}' \mathbf{H}^{-1} \mathbf{P} - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n', \\ (\mathbf{L}_W + \mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P})^\dagger (\mathbf{L}_W + \mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P}) &= \mathbf{I}_n - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n', \end{aligned} \quad (\text{S.4})$$

where on the right-hand side always appears the projector orthogonal to the null-space of the respective matrix, e.g. we have  $\mathbf{L}_W \mathbf{M} = \mathbf{L}_W$ . Using that  $(\mathbf{H}^{-1/2} \mathbf{P})^\dagger = \mathbf{P}' \mathbf{H}^{-1/2}$  and the definition of  $\mathbf{L}_*^{\text{inv}}$  we find that

$$(\mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P})^\dagger = \mathbf{P}' \mathbf{H}^{-1/2} (\mathbf{H}^{-1/2} \mathbf{L}_* \mathbf{H}^{-1/2})^\dagger \mathbf{H}^{-1/2} \mathbf{P} = \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P}.$$

**Proof of Lemma 2.** We derive the result for  $\hat{\boldsymbol{\beta}}$  first. By applying Theorem 1 to each  $\mathcal{G}_r$  separately we obtain

$$\hat{\boldsymbol{\beta}}_r \sim \mathcal{N}(\boldsymbol{\beta}_r, \sigma^2 \mathbf{L}_r^\dagger),$$

for  $r = 1, \dots, q$ . Note that we do not rule out  $n_r = 1$  (i.e.,  $\mathcal{G}_r$  may be a graph with one vertex and no edges), but in this case we simply have  $\hat{\boldsymbol{\beta}}_r = \boldsymbol{\beta}_r = \mathbf{L}_r = \mathbf{L}_r^\dagger = \mathbf{0}$ , so the result for  $\hat{\boldsymbol{\beta}}_r$  holds trivially. Independence of the errors  $u_{ij}$  across observations implies independence of  $\hat{\boldsymbol{\beta}}_r$  and  $\hat{\boldsymbol{\beta}}_s$  for all  $r \neq s$ . We thus find  $\hat{\boldsymbol{\beta}} \sim \mathcal{N}(\boldsymbol{\beta}, \sigma^2 \mathbf{L}_W^\dagger)$ , which is the result in the theorem.

Now turn to  $\hat{\boldsymbol{\gamma}}$ . Analogous to the proof of Lemma 1 we can write the minimization problem for  $\hat{\boldsymbol{\gamma}}$  as

$$\hat{\boldsymbol{\gamma}} = \arg \min_{\mathbf{g} \in \mathbb{R}^q} (\mathbf{y} - \mathbf{B}_W \boldsymbol{\beta} - \mathbf{B}_B \mathbf{P}' \mathbf{g})' (\mathbf{y} - \mathbf{B}_W \boldsymbol{\beta} - \mathbf{B}_B \mathbf{P}' \mathbf{g}) + \lambda (\mathbf{g}' \mathbf{P} \boldsymbol{\iota}_n)^2,$$

where  $\lambda > 0$  is a Lagrange multiplier. Solving the corresponding first-order condition gives

$$\begin{aligned} \hat{\boldsymbol{\gamma}} &= (\mathbf{P} \mathbf{B}_B' \mathbf{B}_B \mathbf{P}' + \lambda \mathbf{P} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{P}')^{-1} \mathbf{P} \mathbf{B}_B' (\mathbf{y} - \mathbf{B}_W \boldsymbol{\beta}) \\ &= (\mathbf{P} \mathbf{B}_B' \mathbf{B}_B \mathbf{P}' + \lambda \mathbf{P} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{P}')^{-1} [(\mathbf{P} \mathbf{B}_B' \mathbf{B}_B \mathbf{P}' + \lambda \mathbf{P} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{P}') \boldsymbol{\gamma} + \mathbf{P} \mathbf{B}_B' \mathbf{u}], \\ &= \boldsymbol{\gamma} + (\mathbf{P} \mathbf{B}_B' \mathbf{B}_B \mathbf{P}' + \lambda \mathbf{P} \boldsymbol{\iota}_n \boldsymbol{\iota}_n' \mathbf{P}')^{-1} \mathbf{P} \mathbf{B}_B' \mathbf{u}, \end{aligned}$$

where in the second step we used the model  $\mathbf{y} = \mathbf{B}_W \boldsymbol{\beta} + \mathbf{B}_B \mathbf{P}' \boldsymbol{\gamma} + \mathbf{u}$ , and we added a term proportional to  $\lambda$  in the square brackets, which is zero due to the normalization of  $\boldsymbol{\gamma}$ , which can be written as  $\boldsymbol{\iota}'_n \mathbf{P}' \boldsymbol{\gamma} = 0$ . Notice that  $\mathbf{P} \mathbf{B}'_B \mathbf{B}_B \mathbf{P}' = \mathbf{P} \mathbf{L}_B \mathbf{P}' = \mathbf{L}_*$ . However, compared to the proof of Lemma 1 the difficulty is that, here, the matrices  $\mathbf{L}_*$  and  $\mathbf{P} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n \mathbf{P}'$  do not commute. To resolve this problem we rewrite the last result as

$$\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma} = \mathbf{H}^{-1/2} \left( \mathbf{H}^{-1/2} \mathbf{L}_* \mathbf{H}^{-1/2} + \lambda \mathbf{H}^{-1/2} \mathbf{P} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n \mathbf{P}' \mathbf{H}^{-1/2} \right)^{-1} \mathbf{H}^{-1/2} \mathbf{P} \mathbf{B}'_B \mathbf{u}.$$

Now, the matrices  $\mathbf{H}^{-1/2} \mathbf{L}_* \mathbf{H}^{-1/2}$  and  $\mathbf{H}^{-1/2} \mathbf{P} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n \mathbf{P}' \mathbf{H}^{-1/2}$  commute, because the zero eigenvalue of  $\mathbf{H}^{-1/2} \mathbf{L}_* \mathbf{H}^{-1/2}$  has multiplicity one (as we assume  $\mathcal{G}_B$  to be connected) with eigenvector given by  $\mathbf{H}^{-1/2} \mathbf{P} \boldsymbol{\iota}_n$ , namely we have  $\mathbf{L}_* \mathbf{H}^{-1} \mathbf{P} \boldsymbol{\iota}_n = \mathbf{L}_* \boldsymbol{\iota}_q = 0$ . Here, we used  $\mathbf{H}^{-1} \mathbf{P} \boldsymbol{\iota}_n = \boldsymbol{\iota}_q$ , which follows from the definition of  $\mathbf{H}$  and  $\mathbf{P}$ . We therefore have

$$\begin{aligned} \left( \mathbf{H}^{-1/2} \mathbf{L}_* \mathbf{H}^{-1/2} + \lambda \mathbf{H}^{-1/2} \mathbf{P} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n \mathbf{P}' \mathbf{H}^{-1/2} \right)^{-1} &= \left( \mathbf{H}^{-1/2} \mathbf{L}_* \mathbf{H}^{-1/2} \right)^\dagger + \frac{1}{\lambda} \left( \mathbf{H}^{-1/2} \mathbf{P} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n \mathbf{P}' \mathbf{H}^{-1/2} \right)^\dagger \\ &= \mathbf{H}^{1/2} \mathbf{L}_*^{\text{inv}} \mathbf{H}^{1/2} + \frac{1}{\lambda n^2} \mathbf{H}^{1/2} \boldsymbol{\iota}_q \boldsymbol{\iota}'_q \mathbf{H}^{1/2}, \end{aligned}$$

and the last term does not contribute to  $\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}$  because we have  $\boldsymbol{\iota}'_q \mathbf{P} \mathbf{B}'_B = \boldsymbol{\iota}'_n \mathbf{B}'_B = 0$ . We therefore have, independent from the choice of  $\lambda$ , that

$$\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma} = \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{B}'_B \mathbf{u}.$$

Using  $\mathbb{E}(\mathbf{u} \mathbf{u}') = \sigma^2 \mathbf{I}_m$  we thus find

$$\text{var}(\hat{\boldsymbol{\gamma}}) = \sigma^2 \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{B}'_B \mathbf{B}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} = \sigma^2 \mathbf{L}_*^{\text{inv}} \mathbf{L}_* \mathbf{L}_*^{\text{inv}} = \sigma^2 \mathbf{L}_*^{\text{inv}}.$$

Because  $\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}$  is a linear combination of the jointly normal errors it is also normally distributed, so we have  $\hat{\boldsymbol{\gamma}} \sim \mathcal{N}(\boldsymbol{\gamma}, \sigma^2 \mathbf{L}_*^{\text{inv}})$ . This concludes the proof.  $\blacksquare$

## PROOF OF THEOREM 7 (GRAPH PARTITIONING)

Throughout the proof we maintain the same notational conventions as for the proof of Lemma 2. Recall that the variance of  $\hat{\boldsymbol{\alpha}}$  is  $\sigma^2 \mathbf{L}^\dagger$ . The variance of the infeasible estimator based on (5.4) equals  $\sigma^2 (\mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P})$ . We show below that

$$-\mathbf{Z}_{\text{low}} - (\mathbf{Q} + \mathbf{Q}') \leq \mathbf{L}^\dagger - \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) \leq \mathbf{Z}_{\text{up}} - (\mathbf{Q} + \mathbf{Q}'), \quad (\text{S.5})$$

for matrices

$$\mathbf{Z}_{\text{low}} := \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{L}_W^\dagger, \quad \mathbf{Z}_{\text{up}} := \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P},$$

and  $\mathbf{Q} := \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P}$ . We also establish that

$$\mathbf{Z}_{\text{low}} \leq \kappa \mathbf{L}_W^\dagger, \quad \mathbf{Z}_{\text{up}} \leq \kappa \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} = \kappa \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P}, \quad (\text{S.6})$$

and that

$$|\mathbf{v}' \mathbf{Q} \mathbf{v}| \leq \kappa^{1/2} \left( \mathbf{v}' \mathbf{L}_W^\dagger \mathbf{v} \right)^{1/2} \left( \mathbf{v}' \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{v} \right)^{1/2}, \quad (\text{S.7})$$

for any  $\mathbf{v} \in \mathbb{R}^n$ . Combining these results yields that, for any  $\mathbf{v} \in \mathbb{R}^n$ ,

$$\begin{aligned} & -\kappa \mathbf{v}' \mathbf{L}_W^\dagger \mathbf{v} - 2\kappa^{1/2} \left[ \left( \mathbf{v}' \mathbf{L}_W^\dagger \mathbf{v} \right) \left( \mathbf{v}' \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{v} \right) \right]^{1/2} \\ & \leq \mathbf{v}' \left( \mathbf{L}^\dagger - \mathbf{L}_W^\dagger - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) \mathbf{v} \leq \\ & \quad \kappa \mathbf{v}' \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{v} + 2\kappa^{1/2} \left[ \left( \mathbf{v}' \mathbf{L}_W^\dagger \mathbf{v} \right) \left( \mathbf{v}' \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{v} \right) \right]^{1/2}. \end{aligned}$$

By Lemma 2 this is the result of Theorem 7. It remains only to show (S.5), (S.6), and (S.7), which we do, in turn, next. ■

**Proof of (S.5).** Start with the upper bound. Because  $\mathbf{L}_B \geq 0$  and  $\mathbf{L}_W \geq 0$ , it holds that

$$\begin{aligned} 0 & \leq \left( \mathbf{L}^\dagger - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) \mathbf{L}_B \left( \mathbf{L}^\dagger - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) \\ & \quad + \left( \mathbf{L}^\dagger - \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger \right) \mathbf{L}_W \left( \mathbf{L}^\dagger - \mathbf{L}_W^\dagger + \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right). \end{aligned}$$

Expanding those terms, and using that  $\mathbf{L}_W + \mathbf{L}_B = \mathbf{L}$ , and  $\mathbf{L}_W^\dagger \mathbf{L}_W = \mathbf{M}$ , and  $\mathbf{L}_W^\dagger \mathbf{L}_W \mathbf{L}_W^\dagger = \mathbf{L}_W^\dagger$ , we obtain

$$\begin{aligned} 0 & \leq \mathbf{L}^\dagger \mathbf{L} \mathbf{L}^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} - \mathbf{L}^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}^\dagger \\ & \quad - \mathbf{M} \mathbf{L}^\dagger - \mathbf{L}^\dagger \mathbf{M} + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{M} \mathbf{L}^\dagger + \mathbf{L}^\dagger \mathbf{M} \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \\ & \quad + \mathbf{L}_W^\dagger - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger - \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P}. \end{aligned}$$

Using that  $\mathbf{L}^\dagger \mathbf{L} \mathbf{L}^\dagger = \mathbf{L}^\dagger$ , and  $\mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} = \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P}$ , and

$$\begin{aligned} -\mathbf{L}^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} + \mathbf{L}^\dagger \mathbf{M} \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} &= -\mathbf{L}^\dagger (\mathbf{I}_n - \mathbf{M}) \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \\ &= -\mathbf{L}^\dagger \mathbf{P}' \mathbf{H}^{-1} \mathbf{P} \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \\ &= -\mathbf{L}^\dagger \mathbf{P}' \mathbf{H}^{-1} \mathbf{P}, \end{aligned}$$

and also the transpose of the last result, we obtain

$$\begin{aligned} 0 \leq \mathbf{L}^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} - (\mathbf{M} + \mathbf{P}' \mathbf{H}^{-1} \mathbf{P}) \mathbf{L}^\dagger - \mathbf{L}^\dagger (\mathbf{M} + \mathbf{P}' \mathbf{H}^{-1} \mathbf{P}) \\ + \mathbf{L}_W^\dagger - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger - \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P}. \end{aligned}$$

Because  $\mathbf{M} + \mathbf{P}' \mathbf{H}^{-1} \mathbf{P} = \mathbf{I}_n$  we thus find

$$\mathbf{L}^\dagger \leq \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger - \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P},$$

which is the upper bound given in the lemma.

Now turn to the lower bound. Introduce

$$\mathbf{\Delta} := \mathbf{M} \mathbf{L}_B \mathbf{M} + \mathbf{P}' \mathbf{H}^{-1} \mathbf{P} \mathbf{L}_B \mathbf{M} + \mathbf{M} \mathbf{L}_B \mathbf{P}' \mathbf{H}^{-1} \mathbf{P}.$$

We then have

$$\mathbf{L} = \mathbf{L}_W + \mathbf{L}_B = \mathbf{L}_W + \mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P} + \mathbf{\Delta}.$$

Plugging this in the equality  $\mathbf{L} \mathbf{L}^\dagger = \mathbf{I}_n - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n'$  we obtain

$$(\mathbf{L}_W + \mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P} + \mathbf{\Delta}) \mathbf{L}^\dagger = \mathbf{I}_n - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n'.$$

Bringing  $\mathbf{\Delta} \mathbf{L}^\dagger$  to the right-hand side, multiplying with  $(\mathbf{L}_W + \mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P})^\dagger$  from the left, and using the last equality in (S.4), we obtain

$$(\mathbf{I}_n - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n') \mathbf{L}^\dagger = (\mathbf{L}_W + \mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P})^\dagger (\mathbf{I}_n - \mathbf{\Delta} \mathbf{L}^\dagger - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}_n'). \quad (\text{S.8})$$

The matrices  $\mathbf{L}_W$  and  $\mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P}$  commute, and we therefore have

$$(\mathbf{L}_W + \mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P})^\dagger = \mathbf{L}_W^\dagger + (\mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P})^\dagger = \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P}.$$

Using this as well as  $\mathbf{L}^\dagger \boldsymbol{\iota}_n = 0$  and  $(\mathbf{L}_W + \mathbf{P}' \mathbf{H}^{-1} \mathbf{L}_* \mathbf{H}^{-1} \mathbf{P})^\dagger \boldsymbol{\iota}_n = 0$  we find that the equation in (S.8) becomes

$$\mathbf{L}^\dagger = \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} - \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) \boldsymbol{\Delta} \mathbf{L}^\dagger. \quad (\text{S.9})$$

Taking the transpose of this last equation gives

$$\mathbf{L}^\dagger = \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} - \mathbf{L}^\dagger \boldsymbol{\Delta} \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right).$$

Replacing  $\mathbf{L}^\dagger$  on the right-hand side of the last equation by the expression for  $\mathbf{L}^\dagger$  in (S.9) we get

$$\mathbf{L}^\dagger = \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} - \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) (\boldsymbol{\Delta} - \boldsymbol{\Delta} \mathbf{L}^\dagger \boldsymbol{\Delta}) \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right).$$

Using the definition of  $\boldsymbol{\Delta}$  we obtain

$$\boldsymbol{\Delta} \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) = \mathbf{M} \mathbf{L}_B \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) + \mathbf{P}' \mathbf{H}^{-1} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger,$$

and the last result on  $\mathbf{L}^\dagger$  can therefore be rewritten as

$$\begin{aligned} \mathbf{L}^\dagger - \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} - \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{L}_W^\dagger - \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger \right) \\ = \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) \boldsymbol{\Delta} \mathbf{L}^\dagger \boldsymbol{\Delta} \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right). \end{aligned}$$

Because  $\mathbf{L}^\dagger \geq 0$  the last expression is positive semi-definite, which gives the lower bound on  $\mathbf{L}^\dagger$  in the lemma. This concludes the proof.  $\blacksquare$

**Proof of (S.6).** Let

$$\boldsymbol{\Lambda} := \text{diag}(\lambda_2^r : i \in \mathbf{V}, \text{ with } r \text{ such that } i \in V_r),$$

where we set  $\lambda_2^r = 0$  if  $n_r = 1$ . Then  $\mathbf{L}_W^\dagger \leq (\boldsymbol{\Lambda} \mathbf{D}_W)^\dagger$ .<sup>1</sup> Also define the symmetrically normalized Laplacian of  $\mathcal{G}_B$ <sup>2</sup>

$$\mathbf{S}_B := \left( \mathbf{D}_B^\dagger \right)^{1/2} \mathbf{L}_B \left( \mathbf{D}_B^\dagger \right)^{1/2}.$$

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<sup>1</sup>The diagonal matrix  $\boldsymbol{\Lambda} \mathbf{D}_W$  has non-negative elements but may be non-invertible as, for  $n_r = 1$ , we have  $\lambda_2^r d_i^W = 0$ , with  $i \in V_r$ . We therefore write  $(\boldsymbol{\Lambda} \mathbf{D}_W)^\dagger$  instead of just  $(\boldsymbol{\Lambda} \mathbf{D}_W)^{-1}$ .

<sup>2</sup>Again we write  $\mathbf{D}_B^\dagger$  because we may have  $d_i^B = 0$  for some  $i \in V$ .

From [Chung \(1997, Lemma 1.7\)](#) we know  $\lambda_n(\mathbf{S}_B) \leq 2$ , which can also be written as  $\mathbf{S}_B \leq 2\mathbf{I}_n$ . We have  $\mathbf{L}_B = \mathbf{D}_B^{1/2} \mathbf{S}_B \mathbf{D}_B^{1/2}$ , and thus find  $\mathbf{L}_B \leq 2\mathbf{D}_B$ .

The diagonal matrix  $\mathbf{D}_B^{1/2} (\mathbf{A} \mathbf{D}_W)^\dagger \mathbf{D}_B^{1/2}$  has  $i$ th diagonal element equal to  $d_i^B / (\lambda_2^r d_i^W)$  for  $n_r > 1$ ,  $i \in V_r$ , and equal to zero otherwise. From the definition of  $\kappa$  in the main text we thus find

$$\mathbf{D}_B^{1/2} (\mathbf{A} \mathbf{D}_W)^\dagger \mathbf{D}_B^{1/2} \leq \frac{\kappa}{2} \mathbf{I}_n,$$

and therefore

$$\mathbf{D}_B^{1/2} \mathbf{L}_W^\dagger \mathbf{D}_B^{1/2} \leq \mathbf{D}_B^{1/2} (\mathbf{A} \mathbf{D}_W)^\dagger \mathbf{D}_B^{1/2} \leq \frac{\kappa}{2} \mathbf{I}_n. \quad (\text{S.10})$$

The matrix  $\mathbf{D}_B^{1/2} \mathbf{L}_W^\dagger \mathbf{D}_B^{1/2}$  is similar to  $(\mathbf{L}_W^\dagger)^{1/2} \mathbf{D}_B (\mathbf{L}_W^\dagger)^{1/2}$ , and so they share the same eigenvalues.<sup>3</sup> We therefore have that

$$(\mathbf{L}_W^\dagger)^{1/2} \mathbf{D}_B (\mathbf{L}_W^\dagger)^{1/2} \leq \frac{\kappa}{2} \mathbf{I}_n \quad (\text{S.11})$$

holds.

Using the inequalities  $\mathbf{S}_B \leq 2\mathbf{I}_n$  and  $\mathbf{L}_B \leq 2\mathbf{D}_B$  along with (S.10) and (S.11) we obtain

$$\begin{aligned} \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{L}_W^\dagger &\leq 2\mathbf{L}_W^\dagger \mathbf{D}_B \mathbf{L}_W^\dagger \\ &= 2 (\mathbf{L}_W^\dagger)^{1/2} (\mathbf{L}_W^\dagger)^{1/2} \mathbf{D}_B (\mathbf{L}_W^\dagger)^{1/2} (\mathbf{L}_W^\dagger)^{1/2} \\ &\leq \kappa (\mathbf{L}_W^\dagger)^{1/2} (\mathbf{L}_W^\dagger)^{1/2} \\ &\leq \kappa \mathbf{L}_W^\dagger, \end{aligned} \quad (\text{S.12})$$

and

$$\begin{aligned} \mathbf{L}_B \mathbf{L}_W^\dagger \mathbf{L}_B &= \mathbf{D}_B^{1/2} \mathbf{S}_B^{1/2} \mathbf{S}_B^{1/2} \mathbf{D}_B^{1/2} \mathbf{L}_W^\dagger \mathbf{D}_B^{1/2} \mathbf{S}_B^{1/2} \mathbf{S}_B^{1/2} \mathbf{D}_B^{1/2} \\ &\leq \frac{\kappa}{2} \mathbf{D}_B^{1/2} \mathbf{S}_B^{1/2} \mathbf{S}_B \mathbf{S}_B^{1/2} \mathbf{D}_B^{1/2} \\ &\leq \kappa \mathbf{D}_B^{1/2} \mathbf{S}_B^{1/2} \mathbf{S}_B^{1/2} \mathbf{D}_B^{1/2} \\ &= \kappa \mathbf{L}_B. \end{aligned} \quad (\text{S.13})$$

These yield the inequalities stated in (S.6). This concludes the proof. ■

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<sup>3</sup>Two square matrices  $\mathbf{M}_1$  and  $\mathbf{M}_2$  are similar if there exists an invertible matrix  $\mathbf{M}_3$  such that  $\mathbf{M}_1 = \mathbf{M}_3^{-1} \mathbf{M}_2 \mathbf{M}_3$ . Two similar matrices have the same eigenvalues.

**Proof of (S.7).** Recall that

$$\mathbf{Q} = \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P}.$$

Applying the Cauchy-Schwarz inequality  $(\mathbf{x}'\mathbf{y})^2 \leq (\mathbf{x}'\mathbf{x})(\mathbf{y}'\mathbf{y})$  with  $\mathbf{x} = \mathbf{L}_B^{1/2} \mathbf{L}_W^\dagger \mathbf{v}$  and  $\mathbf{y} = \mathbf{L}_B^{1/2} \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{v}$ , and using (S.6) we find

$$|\mathbf{v}' \mathbf{Q} \mathbf{v}|^2 \leq \kappa \left( \mathbf{v}' \mathbf{L}_W^\dagger \mathbf{v} \right) \left( \mathbf{v}' (\mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P}) \mathbf{v} \right),$$

which gives (S.7). This concludes the proof. ■

## PROOF OF THEOREM A.1 (SECOND-ORDER BOUND)

**Proof of Theorem A.1.** We start with the lower bound given in the theorem. Let  $V_o := \{i\} \cup [i]$ ; then  $n_o := |V_o| = 1 + d_i$ . Without loss of generality we fix  $i = 1$  and relabel the elements of  $V$  so that  $V_o = \{1, 2, \dots, 1 + d_i\}$ . Let

$$\mathbf{L}_o := \begin{pmatrix} d_i & -\boldsymbol{\nu}'_{d_i} \\ -\boldsymbol{\nu}'_{d_i} & \mathbf{L}_{[i]} \end{pmatrix}, \quad \mathbf{L}_{[i]} := \mathbf{D}_{[i]} - \mathbf{A}_{[i]},$$

using obvious notation for the  $d_i \times d_i$  degree and adjacency matrices in the latter definition.

Now, by the inversion formula for partitioned matrices,

$$\mathbf{L}_o^{-1} = \frac{1}{d_i - \boldsymbol{\nu}'_{d_i} \mathbf{L}_{[i]}^{-1} \boldsymbol{\nu}_{d_i}} \begin{pmatrix} 1 & \boldsymbol{\nu}'_{d_i} \mathbf{L}_{[i]}^{-1} \\ \mathbf{L}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} & \left[ \frac{\mathbf{L}_{[i]} - d_i^{-1} \boldsymbol{\nu}_{d_i} \boldsymbol{\nu}'_{d_i}}{d_i - \boldsymbol{\nu}'_{d_i} \mathbf{L}_{[i]}^{-1} \boldsymbol{\nu}_{d_i}} \right]^{-1} \end{pmatrix}.$$

Below we show that

$$0 \leq \left\{ \text{var}(\widehat{\alpha}_i) - \frac{\sigma^2 \left[ 1 - \frac{2}{n} \left( 1 + \boldsymbol{\nu}'_{d_i} \mathbf{L}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} \right) \right]}{d_i - \boldsymbol{\nu}'_{d_i} \mathbf{L}_{[i]}^{-1} \boldsymbol{\nu}_{d_i}} \right\} \leq \frac{\sigma^2 \boldsymbol{\nu}'_{d_i} \mathbf{L}_{[i]}^{-1} (\mathbf{A}_{o\#}) \mathbf{D}_{\#}^{-1} (\mathbf{A}_{o\#})' \mathbf{L}_{[i]}^{-1} \boldsymbol{\nu}_{d_i}}{\lambda_2 \left( d_i - \boldsymbol{\nu}'_{d_i} \mathbf{L}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} \right)^2}, \quad (\text{S.14})$$

where  $\mathbf{L}_o$  is the upper left  $n_o \times n_o$  block of  $\mathbf{L}$ ,  $\mathbf{A}_{o\#}$  is the upper right  $n_o \times n_{\#}$  block of  $\mathbf{A}$ , and  $\mathbf{D}_{\#}$  is the lower right  $n_{\#} \times n_{\#}$  block of  $\mathbf{D}$ . To make further progress, note that the expansion

$$\mathbf{L}_{[i]}^{-1} = \sum_{q=0}^{\infty} \left( \mathbf{D}_{[i]}^{-1} \mathbf{A}_{[i]} \right)^q \mathbf{D}_{[i]}^{-1}$$

is convergent, because we have  $\|\underline{\mathbf{D}}_{[i]}^{-1} \mathbf{A}_{[i]}\|_\infty < 1$ , where  $\|\cdot\|_\infty$  denotes the maximum absolute row sum matrix norm. We therefore have

$$\begin{aligned} \boldsymbol{\nu}'_{d_i} \mathbf{L}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} &= \boldsymbol{\nu}'_{d_i} \underline{\mathbf{D}}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} + \boldsymbol{\nu}'_{d_i} \sum_{q=1}^{\infty} \left( \underline{\mathbf{D}}_{[i]}^{-1} \mathbf{A}_{[i]} \right)^q \underline{\mathbf{D}}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} \\ &\geq \boldsymbol{\nu}'_{d_i} \underline{\mathbf{D}}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} = \sum_{j \in [i]} d_j^{-1}, \end{aligned} \quad (\text{S.15})$$

where we used that  $\boldsymbol{\nu}'_{d_i} \sum_{q=1}^{\infty} \left( \underline{\mathbf{D}}_{[i]}^{-1} \mathbf{A}_{[i]} \right)^q \underline{\mathbf{D}}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} \geq 0$ , because this is a product and sum of vector and matrices that all have non-negative entries. Define the  $n_o \times n_o$  diagonal matrix  $\underline{\mathbf{D}}_{[i]} = \text{diag}(\underline{d}_{j,i} : j \in [i])$ . We have

$$\mathbf{L}_{[i]} - \underline{\mathbf{D}}_{[i]} = \text{diag}(\mathbf{A}_{[i]} \boldsymbol{\nu}_{d_i}) - \mathbf{A}_{[i]} \geq 0, \quad (\text{S.16})$$

because  $\text{diag}(\mathbf{A}_{[i]} \boldsymbol{\nu}_{d_i}) - \mathbf{A}_{[i]}$  can be expressed as a sum of matrices of the form

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \geq 0,$$

embedded into an  $n_o \times n_o$  matrix. We therefore have  $\mathbf{L}_{[i]}^{-1} \leq \underline{\mathbf{D}}_{[i]}^{-1}$ , implying

$$\boldsymbol{\nu}'_{d_i} \mathbf{L}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} \leq \boldsymbol{\nu}'_{d_i} \underline{\mathbf{D}}_{[i]}^{-1} \boldsymbol{\nu}_{d_i} = \sum_{j \in [i]} \underline{d}_{j,i}^{-1}. \quad (\text{S.17})$$

Combining (S.14), (S.15) and (S.17) gives

$$\text{var}(\hat{\alpha}_i) \geq \frac{\sigma^2 \left[ 1 - \frac{2}{n} \left( 1 + \sum_{j \in [i]} \underline{d}_{j,i}^{-1} \right) \right]}{\sum_{j \in [i]} (1 - d_j^{-1})} = \frac{\sigma^2}{d_i(1 - h_i^{-1})} \left( 1 - \frac{2}{n} - \frac{2}{n} \frac{d_i}{\underline{h}_i} \right),$$

which is the lower bound stated in the theorem.

To show the upper bound, consider the the graph  $\tilde{\mathcal{G}} := (V, \tilde{E})$ , with  $\tilde{E} := E \setminus [i] \times [i]$ . That is, we construct  $\tilde{\mathcal{G}}$  by deleting all edges between neighbors of  $i$  from  $\mathcal{G}$ . Note that  $\tilde{\mathcal{G}}$  is still connected, because all vertices in  $[i]$  are connected through  $i$ . Let  $\tilde{\boldsymbol{\alpha}}$  be the estimator for  $\boldsymbol{\alpha}$  obtained for  $\tilde{\mathcal{G}}$ , in the same way that  $\hat{\boldsymbol{\alpha}}$  was obtained for  $\mathcal{G}$ . Let  $\tilde{\mathbf{L}}$  be the Laplacian matrix of  $\tilde{\mathcal{G}}$ . Analogous to (S.16) we have  $\tilde{\mathbf{L}} \leq \mathbf{L}$ , and therefore  $\tilde{\mathbf{L}}^\dagger \geq \mathbf{L}^\dagger$ . The result (S.14) holds for any connected graph, and so can equally be applied to  $\tilde{\mathcal{G}}$ , we only need to



replace  $\text{var}(\hat{\alpha}_i)$  by  $\text{var}(\tilde{\alpha}_i)$  and  $\mathbf{L}$  by  $\tilde{\mathbf{L}}$ . The matrices  $\mathbf{A}_{\circ\#}$  and  $\mathbf{D}_{\#}^{-1}$  are identical for  $\tilde{\mathcal{G}}$  and  $\mathcal{G}$ . However, for  $\tilde{\mathcal{G}}$  we find  $\tilde{\mathbf{D}}_{[i]} = \underline{\mathbf{D}}_{[i]}$ , because the degree of vertex  $j$  is given by  $\underline{d}_{j,i}$ , and we have  $\tilde{\mathbf{A}}_{[i]} = 0$ , because there are no edges that connect elements in  $[i]$ . We thus have  $\tilde{\mathbf{L}}_{[i]} = \tilde{\mathbf{D}}_{[i]} - \tilde{\mathbf{A}}_{[i]} = \underline{\mathbf{D}}_{[i]}$ . Therefore,

$$\text{var}(\hat{\alpha}_i) \leq \text{var}(\tilde{\alpha}_i) \leq \frac{\sigma^2 \left[ 1 - \frac{2}{n} \left( 1 + \boldsymbol{\iota}'_{d_i} \underline{\mathbf{D}}_{[i]}^{-1} \boldsymbol{\iota}_{d_i} \right) \right]}{d_i - \boldsymbol{\iota}'_{d_i} \underline{\mathbf{D}}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}} + \frac{\sigma^2 \boldsymbol{\iota}'_{d_i} \underline{\mathbf{D}}_{[i]}^{-1} (\mathbf{A}_{\circ\#}) \mathbf{D}_{\#}^{-1} (\mathbf{A}_{\circ\#})' \underline{\mathbf{D}}_{[i]}^{-1} \boldsymbol{\iota}_{d_i}}{\lambda_2 \left( d_i - \boldsymbol{\iota}'_{d_i} \underline{\mathbf{D}}_{[i]}^{-1} \boldsymbol{\iota}_{d_i} \right)^2},$$

and evaluating the right-hand side of the last inequality gives the upper bound on  $\text{var}(\hat{\alpha}_i)$  in the theorem. This concludes the proof.  $\blacksquare$

**Proof of (S.14).** We prove the following more general result. Let  $\mathcal{G}$  be connected. Choose  $V_{\circ} \subset V$  with  $0 < |V_{\circ}| < n$ , and let  $V_{\#} = V \setminus V_{\circ}$ . Let  $n_{\circ} = |V_{\circ}|$  and  $n_{\#} = n - n_{\circ}$ . Relabel the elements in  $V$  such that  $V_{\circ} = \{1, 2, \dots, n_{\circ}\}$ . Let  $\hat{\boldsymbol{\alpha}}_{\circ} = (\hat{\alpha}_1, \dots, \hat{\alpha}_{n_{\circ}})'$ ,  $\mathbf{L}_{\circ}$  be the upper left  $n_{\circ} \times n_{\circ}$  block of  $\mathbf{L}$ ,  $\mathbf{A}_{\circ\#}$  be the upper right  $n_{\circ} \times n_{\#}$  block of  $\mathbf{A}$ , and  $\mathbf{D}_{\#}$  be the lower right  $n_{\#} \times n_{\#}$  block of  $\mathbf{D}$ . Then,

$$0 \leq \left[ \text{var}(\hat{\boldsymbol{\alpha}}_{\circ}) - \sigma^2 \left( \mathbf{L}_{\circ}^{-1} - \frac{\boldsymbol{\iota}_{n_{\circ}} \boldsymbol{\iota}'_{n_{\circ}} \mathbf{L}_{\circ}^{-1} + \mathbf{L}_{\circ}^{-1} \boldsymbol{\iota}_{n_{\circ}} \boldsymbol{\iota}'_{n_{\circ}}}{n} \right) \right] \leq \frac{\sigma^2}{\lambda_2} \mathbf{L}_{\circ}^{-1} (\mathbf{A}_{\circ\#}) \mathbf{D}_{\#}^{-1} (\mathbf{A}_{\circ\#})' \mathbf{L}_{\circ}^{-1}$$

holds.

To show the result, define the  $n \times n$  matrices

$$\mathbf{L}_{\text{b}} := \begin{pmatrix} \mathbf{L}_{\circ} & 0 \\ 0 & \mathbf{L}_{\#} \end{pmatrix}, \quad \mathbf{A}_{\text{b}} := \begin{pmatrix} 0 & \mathbf{A}_{\circ\#} \\ (\mathbf{A}_{\circ\#})' & 0 \end{pmatrix},$$

with obvious definition of  $\mathbf{L}_{\#}$  such that  $\mathbf{L} = \mathbf{L}_{\text{b}} - \mathbf{A}_{\text{b}}$ . Because the graph is connected the pseudo-inverse  $\mathbf{L}^{\dagger}$  satisfies  $\mathbf{L}^{\dagger} \mathbf{L} = \mathbf{I}_n - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n$ . Plugging  $\mathbf{L} = \mathbf{L}_{\text{b}} - \mathbf{A}_{\text{b}}$  into this expression we obtain

$$\mathbf{L}^{\dagger} = \mathbf{L}_{\text{b}}^{-1} (\mathbf{I}_n + \mathbf{A}_{\text{b}} \mathbf{L}^{\dagger} - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n).$$

Using the transposed of this last equation to replace  $\mathbf{L}^{\dagger} = (\mathbf{L}^{\dagger})'$  on the right-hand side of that same equation we obtain

$$\begin{aligned} \mathbf{L}^{\dagger} &= \mathbf{L}_{\text{b}}^{-1} + \mathbf{L}_{\text{b}}^{-1} \mathbf{A}_{\text{b}} \mathbf{L}_{\text{b}}^{-1} + \mathbf{L}_{\text{b}}^{-1} \mathbf{A}_{\text{b}} \mathbf{L}^{\dagger} \mathbf{A}_{\text{b}} \mathbf{L}_{\text{b}}^{-1} - n^{-1} \mathbf{L}_{\text{b}}^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n - n^{-1} \mathbf{L}_{\text{b}}^{-1} \mathbf{A}_{\text{b}} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n \mathbf{L}_{\text{b}}^{-1} \\ &= \mathbf{L}_{\text{b}}^{-1} + \mathbf{L}_{\text{b}}^{-1} \mathbf{A}_{\text{b}} \mathbf{L}_{\text{b}}^{-1} + \mathbf{L}_{\text{b}}^{-1} \mathbf{A}_{\text{b}} \mathbf{L}^{\dagger} \mathbf{A}_{\text{b}} \mathbf{L}_{\text{b}}^{-1} - n^{-1} \mathbf{L}_{\text{b}}^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n - n^{-1} \boldsymbol{\iota}_n \boldsymbol{\iota}'_n \mathbf{L}_{\text{b}}^{-1}, \end{aligned}$$

where in the last step we have used that  $\mathbf{L}_b^{-1} \mathbf{A}_b \boldsymbol{\iota}_n = \boldsymbol{\iota}_n$ , which follows from  $0 = \mathbf{L} \boldsymbol{\iota}_n = (\mathbf{L}_b - \mathbf{A}_b) \boldsymbol{\iota}_n$ . Evaluating the last result for the upper left  $n_o \times n_o$  block gives

$$(\mathbf{L}^\dagger)_o = \mathbf{L}_o^{-1} + \mathbf{L}_o^{-1} (\mathbf{A}_{o\#}) (\mathbf{L}^\dagger)_\# (\mathbf{A}_{o\#})' \mathbf{L}_o^{-1} - n^{-1} \mathbf{L}_o^{-1} \boldsymbol{\iota}_{n_o} \boldsymbol{\iota}_{n_o}' - n^{-1} \boldsymbol{\iota}_{n_o} \boldsymbol{\iota}_{n_o}' \mathbf{L}_o^{-1},$$

with obvious definition of  $(\mathbf{L}^\dagger)_\#$ . We obtain the result searched for for  $\text{var}(\hat{\boldsymbol{\alpha}}_o) = \sigma^2 (\mathbf{L}^\dagger)_o$  by also using  $0 \leq (\mathbf{L}^\dagger)_\# \leq \lambda_2^{-1} \mathbf{D}_\#^{-1}$ . This concludes the proof.  $\blacksquare$

## S.2 Component estimators from graph partitioning

Here we strenghten the result of Theorem 7 by showing that the estimator  $\hat{\boldsymbol{\alpha}}$  is close to the (infeasible) estimator  $\hat{\boldsymbol{\beta}} + \mathbf{P}' \hat{\boldsymbol{\gamma}}$  when  $\kappa$  is small. We also provide a corresponding result for the feasible version  $\hat{\boldsymbol{\beta}} + \mathbf{P}' \tilde{\boldsymbol{\gamma}}$ , where

$$\tilde{\boldsymbol{\gamma}} := \arg \min_{\mathbf{g} \in \mathbb{R}^q} \sum_{(i,j) \in E_B} \left( y_{ij} - (\hat{\beta}_i + g_{r(i)}) + (\hat{\beta}_j + g_{r(j)}) \right)^2 \quad \text{s.t.} \quad \sum_{r=1}^q n_r g_r = 0.$$

Our focus in the main text is on the infeasible estimator. This is so because we use it as a device to analyze the variance of  $\hat{\boldsymbol{\alpha}}$ , and  $\hat{\boldsymbol{\gamma}}$  is independent of  $\hat{\boldsymbol{\beta}}$  while its feasible version is clearly not. If an alternative estimator to  $\hat{\boldsymbol{\alpha}}$  is desired,  $\hat{\boldsymbol{\beta}} + \mathbf{P}' \tilde{\boldsymbol{\gamma}}$  will obviously be of interest. Note, however, that  $\text{var}(\hat{\boldsymbol{\alpha}}_i) \leq \text{var}(\hat{\boldsymbol{\beta}} + \mathbf{P}' \tilde{\boldsymbol{\gamma}})$  by the Gauss-Markov theorem (this, in fact, yields the upper bound given in (S.5)).

The following theorem is the main result of this section.

**Theorem S.1.** *Let  $\mathcal{G}$  and  $\mathcal{G}_1, \dots, \mathcal{G}_q$  be connected. For  $i \in V$  define  $r_i, R_i \in \mathbb{R}$  by*

$$\hat{\alpha}_i = \hat{\beta}_i + \tilde{\gamma}_{r(i)} + r_i, \quad \hat{\alpha}_i = \hat{\beta}_i + \hat{\gamma}_{r(i)} + r_i + R_i.$$

*We then have*

$$\mathbb{E}(r_i^2) \leq \kappa \left[ \text{var}(\hat{\beta}_i) + \text{var}(\tilde{\gamma}_{r(i)}) \right], \quad \mathbb{E}(R_i^2) \leq \kappa \text{var}(\hat{\gamma}_{r(i)}).$$

The theorem shows that, if  $\kappa$  is small, then the differences between  $\hat{\alpha}_i$  and  $\hat{\beta}_i + \tilde{\gamma}_{r(i)}$ , and between  $\hat{\alpha}_i$  and  $\hat{\beta}_i + \hat{\gamma}_{r(i)}$ , are both small compared to the stochastic variability of  $\hat{\beta}_i$  and  $\hat{\gamma}_{r(i)}$

themselves. Thus, the result of Theorem 7 generalizes from the variances to the estimators themselves.

The result (and its proof) also immediately extends to a setting as in Theorem 4, where the errors  $u_{ij}$  can be non-normal, heteroscedastic, or correlated. One only needs to replace  $\text{var}(\hat{\beta}_i)$  by  $\bar{\sigma}^2(\mathbf{L}_W^\dagger)_{ii}$  and  $\text{var}(\hat{\gamma}_{r(i)})$  by  $\bar{\sigma}^2(\mathbf{L}^{\text{inv}})_{rr}$ , where  $\bar{\sigma}^2$  is a bound on the largest eigenvalue of  $\mathbb{E}(\mathbf{u}\mathbf{u}')$ .

**Proof of Theorem S.1.** In vector notation the estimator decompositions reads

$$\hat{\alpha} = \hat{\beta} + \mathbf{P}'\tilde{\gamma} + \mathbf{r}, \quad \hat{\alpha} = \hat{\beta} + \mathbf{P}'\hat{\gamma} + \mathbf{r} + \mathbf{R}.$$

Analogous to the proof of Lemma 1 and Lemma 2 above we can use the first-order conditions of their respective minimization problem to obtain explicit formulas for  $\hat{\beta}$ ,  $\tilde{\gamma}$  and  $\hat{\gamma}$ . We thus find

$$\mathbf{r} = \mathbf{C}_1\mathbf{Y}, \quad \mathbf{C}_1 = \left( \mathbf{L}^\dagger \mathbf{B}' - \mathbf{L}_W^\dagger \mathbf{B}_W' - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{B}_B' + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger \mathbf{B}_W' \right),$$

and

$$\mathbf{r} + \mathbf{R} = \mathbf{C}_2\mathbf{Y} + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \beta, \quad \mathbf{C}_2 = \left( \mathbf{L}^\dagger \mathbf{B}' - \mathbf{L}_W^\dagger \mathbf{B}_W' - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{B}_B' \right).$$

It is easy to verify that  $\mathbf{C}_1\mathbf{B} = 0$  and  $\mathbf{C}_2\mathbf{B}\alpha + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \beta = 0$ , and therefore

$$\mathbf{r} = \mathbf{C}_1\mathbf{U}, \quad \mathbf{r} + \mathbf{R} = \mathbf{C}_2\mathbf{U}.$$

Using this we find

$$\begin{aligned} \sigma^{-2} \mathbb{E}(\mathbf{r}\mathbf{r}') &= \mathbf{C}_1 \mathbf{C}_1' \\ &= -\mathbf{L}^\dagger + \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \\ &\quad - \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger - \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \\ &\leq \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \mathbf{L}_B \mathbf{L}_W^\dagger \mathbf{L}_B \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \\ &\leq \kappa \left( \mathbf{L}_W^\dagger + \mathbf{P}' \mathbf{L}_*^{\text{inv}} \mathbf{P} \right) = \kappa \left[ \text{var}(\hat{\beta}) + \text{var}(\mathbf{P}'\hat{\gamma}) \right], \end{aligned}$$

where in the second to last inequality we used the lower bound for  $\mathbf{L}^\dagger$  in (S.5) above, and in the last inequality we used results from the proof of Theorem 7. We have thus shown the result for  $\mathbb{E}(r_i^2)$  in the theorem. Similarly we find

$$\begin{aligned}\sigma^{-2}\mathbb{E}(\mathbf{R}\mathbf{R}') &= (\mathbf{C}_1 - \mathbf{C}_2)(\mathbf{C}_1 - \mathbf{C}_2)' \\ &= \mathbf{P}'\mathbf{L}_*^{\text{inv}}\mathbf{P}\mathbf{L}_B\mathbf{L}_W^\dagger\mathbf{L}_B\mathbf{P}'\mathbf{L}_*^{\text{inv}}\mathbf{P} \\ &\leq \kappa\mathbf{P}'\mathbf{L}_*^{\text{inv}}\mathbf{P} \leq \kappa \text{var}(\mathbf{P}'\hat{\gamma}^{\text{inf}}),\end{aligned}$$

which implies the result for  $\mathbb{E}(R_i^2)$  in the theorem. This concludes the proof. ■

## References

Chung, F. R. K. (1997). *Spectral Graph Theory*. Volume 92 of CBMS Regional Conference Series in Mathematics, American Mathematical Society.