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LANGE'S 1938 MODEL: DYNAMICS AND THE "OPTIMUM
PROPENSITY TO CONSUME"

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Abstract

Oskar Lange's 1938 article "The Rate of Interest and the Optimum Propensity to Consume", is usually associated with the original IS-LM approach of the late 1930s. However, Lange's article was not only an attempt to illuminate Keynes's main innovations but the first part of a wide project that included the development of a theory of economic evolution. This paper aims at showing that Lange's article can help illuminating critical aspects of this project: in particular, Lange's idea that a synthesis between Kaldor's and Kalecki's theories and that of Schumpeter, might have been possible and that it represented (in intentions) a "modern" and consistent reconstruction of the Marxist theory of the business cycle. Section 1 clarifies Lange's early reflection on dynamics. Section 2 centers on Lange's 1938 static model and indicates the effects of a change of saving on investment. Section 3 suggests a dynamic reconstruction from which are addressed important arguments raised by Lange in a series of papers written between 1934 and 1942.

Keywords: Lange; Kalecki; Marxian theory of the business cycle; marginal propensity to save; non-linearity

JEL: B22, B24, E32, E12.

Lange's 1938 model: dynamics and the "Optimum propensity to consume"

Oskar Lange's 1938 work "The Rate of Interest and the Optimum Propensity to Consume" is widely recognized as one of the earliest mathematical models of Keynes's *General Theory*. In light of its analytical content, it has usually been associated with the original IS-LM approach of Roy Harrod, James Meade and John Hicks (Young, 1987; Darity and Young, 1995). However, Lange's article was not a reaction to Keynes's works but the first part of an ambitious project that included the development of a theory of economic evolution¹ (see Lampa 2013).

Indeed, Lange manifested his interest in dynamics very early in his career (both his doctoral dissertation and his thesis presented for the 'docent' degree – i.e. assistant professor – were devoted to the analysis of the business cycle in Poland), repeatedly emphasizing the close connection between his view and Karl Marx's ideas. Furthermore, he attached great importance also to the works of Joseph Schumpeter and Michal Kalecki²: from 1934 to 1936, he became tightly connected to the former at Harvard, whereas his interest in Kalecki's business cycle seems to have grown more important after the publication of the *General Theory*³.

Although Lange explicitly suggested that his 1938 static model might have been dynamized, he never devised any mathematical demonstration: he just stated, en passant, that this might have been done by means of a time lag à la Kalecki⁴ (1937).

It may be recalled, however, that in the early 1940s, Paul Samuelson devised some "techniques" in order to dynamize what he called the "Keynesian system" (1941: 113). As he explicitly affirmed in his 1941 *Econometrica* paper, "I shall analyze in some detail the simple Keynesian model as outlined in the *General Theory*. Various writers, such as Meade, Hicks, and Lange, have developed explicitly in mathematical form the meaning of the Keynesian system" (1941: 133). He then proceeded to develop two dynamic systems: both a differential and a

difference set of equations and he presented the condition that assured the stability of the equilibrium (1941: 120). Although Samuelson explicitly referred to Lange, his models were only loosely related to his⁵. Firstly, Samuelson did not stress that the level of consumption was the key determinant of the investment function, as Lange repeatedly did. Secondly, and foremost, Samuelson paid no attention to the dynamics of the capital stock (which is the corner stone of Kalecki's theory of fluctuations) to which Lange explicitly referred.

On the other hand, it might also be remarked that Mabel Timlin (1942) made an attempt to dynamize what she had defined as the "Keynes-Lange" system. Her method mainly consisted of developing a "system of shifting equilibrium" to determine how a monetary shock was likely to induce a transformation in Lange's set of structural functions, which were supposed to embed the "psychological-institutional complex" of the economy. Resorting to Lange's diagrammatic representation, Timlin showed how expectations in both the goods and the financial market became critical elements with respect to the dynamics of the economy. Furthermore, by extending Lange's analysis to the "long-run", she was finally able to address the problem of the effects of a change in thriftiness upon the stationary level of the capital stock⁶.

Nevertheless, unlike Timlin and Samuelson, the present article focuses on Lange's (crucial) notion of the "optimum propensity to consume", whose importance is largely ignored in both the aforementioned analyses.

In particular, the aim of this paper is to suggest a consistent reconstruction of Lange's article in order to explore its potential implications in terms of dynamics. We are persuaded that such a reconstruction may be interesting from several perspectives. Firstly, it may help us to better understand how Lange's notion of the "optimum propensity to consume" (on which he based his whole interpretation of the under-consumption theories) may operate in a dynamic context. Secondly and foremost, a similar reconstruction may be useful for making clear how close Lange's view on dynamics – expressed in a series of articles published between 1934 and 1943 –

was to the “Keynesian” dynamic approach of both Kalecki (1939) and Kaldor (1940) (according to Lange, the most prominent contributors of the late 1930s).

Consequently, section 1 discusses Lange’s early reflection on dynamics with the aim of highlighting its most outstanding features. Section 2 focuses on Lange’s 1938 static model and indicates the effects of a change of saving on investment. Furthermore – by means of an unedited correspondence between Lange and Samuelson (dated 1942) recently discovered in the archives of Duke University by one of the authors – we clarify the meaning and the implications of the notion of “optimum propensity to consume”. Section 3, by means of some additional assumptions concerning the introduction of a time lag, outlines the necessary and sufficient conditions for the generation of self-sustaining cycles. Finally, Lange’s model (once dynamized) is compared to Kalecki’s 1939 business cycle theory, and its consistency with Lange’s view (expressed in a series of contemporary papers) is assessed.

1. The foundations of Lange’s endogenous dynamics: Marx and (a touch of) Schumpeter

According to a qualified judgement, the study of business cycles and the evolution of capitalism were Lange’s chief research concerns from his early youth until the end of the Second World War (Kowalik 2008). This notwithstanding (and paradoxically enough), Lange did not publish any work explicitly dealing with these issues in the aforementioned period⁷. However, it is possible to reconstruct the essentials of his reflection on dynamics by means of a careful re-reading of his main articles. In “Marxian Economics and Modern Economic Theory” (1935), Lange advocated for an approach that could explain the “economic evolution” from “within” the economic process. In this field, Modern Economic Theory was most likely to be misleading⁸. Lange’s argument was that by resorting to a static theory of equilibrium, “bourgeois economists”

– that is, all the economists ranging from the Austrian, Marshallian and the Lausanne schools – were unable to depart from a framework in which all data related to preferences, institutions and technology are supposed to be given so that the only possible explanation to fluctuations and crises was an exogenous one. In Lange’s eye, this line of thought was likely to consolidate the unrealistic view that capitalist economies were intrinsically stable, whereas the 1930s contingency showed their destructive instability both in the United States and in Europe. Sarcastically enough, Lange wrote:

“It was very generally held among “bourgeois” economists both at the beginning of the twentieth century and in the years preceding 1929, that the economic stability of Capitalism was increasing and that business fluctuations were becoming less and less intense. Thus the Marxian claim that “bourgeois” economists failed to grasp the fundamental tendencies of the evolution of the Capitalist system proves to be true.”

(Lange, 1935: 190)

However, it must be remarked that the “real” superiority of Marxian economics was not supposed to stem from any specific analytical tool originally used by Marx. Firstly, Lange considered that the labour theory of value, at best, can explain equilibrium’s price and production, once a given amount of labour necessary to produce a commodity is known. On the other hand, it is of no use to highlight how changes (particularly, technological changes) occur. (Lange 1935: 194) Secondly, he thought that also the original version of Marx’s schemes of reproduction were of little help in the field of business cycle, because of its analytical backwardness⁹:

“The inability of Marxian economics to solve the problem of the business cycle is demonstrated by the considerable Marxist literature concerned with the famous reproduction schemes of the second volume of *Das Kapital*. This whole literature tries to solve the fundamental problems of economic equilibrium and disequilibrium without

even attempting to make use of the mathematical concept of functional relationship.”

(Lange 1935: 196)

The alleged superiority of Marxian economics laid instead on the exact specification of the institutional datum within which the economic process was studied. Its merit, in particular, was to study the functioning of an economy made of two main social classes:

“[...] the consequences of the additional institutional' datum which distinguishes Capitalism from other forms of exchange economy, i.e. the existence of a class of people who do not possess any means of production, is scarcely examined. Now, Marxian economics is distinguished by making the specification of this additional institutional datum *the very corner-stone of its analysis*, thus discovering the clue to the peculiarity of the Capitalist system by which it differs from other forms of exchange-economy. (Lange, 1935: 192, emphasis added)

In Lange's eyes, it is thanks to this institutional datum that Marx could establish a theory of economic evolution which was conceptually consistent, despite its analytical faults: above all, it was certainly the only theory able to explain the origin of the changes in the economy and also in the extra-economic factors¹⁰. The specificity of this theory was supposed to come from the analysis of the interactions between the dynamics of income distributive shares and the dynamics of investment, in presence of technological progress¹¹. In the first place, Lange recalled the essentials of Marx's analysis of the general law of capitalist accumulation of Volume I of 'Das Kapital' (Lange, 1935, Section 8)¹² emphasizing how a high rate of capital accumulation was likely to trigger an increase in employment and in real wages and eventually to drive firms to introduce "labour-saving technical innovation". Once this tendency spreads throughout the economy, this process is accompanied by a fall in the profit rate that ceases only once new technological innovations are introduced:

“For Capitalism creates, according to Marx, its own surplus population (industrial reserve army) through technical progress, replacing workers by machines. The existence of the surplus population created by technical progress prevents wages from rising so as to swallow profits. Thus technical progress is necessary to maintain the capitalist system and the dynamic nature of the capitalist system, which explains the constant increase of the organic composition of capital, is established.” (Lange, 1935, p. 199)

By connecting the dynamics of investment to the dynamics of income distribution in the presence of two antagonistic social classes, Marx would hence have succeeded in developing a consistent theory of the causes of the intrinsic instability of capitalism. Of course, in Lange’s eye, to emphasize the instability of capitalism implies that economic growth cannot be taken for granted:

“[...] the necessity of the fact that labour-saving technical innovations are always available at the right moment cannot be deduced by economic theory and in this sense the “necessity” of economic evolution cannot be proved. But *Marxian economics* does not attempt to prove this. All it *establishes is that the capitalist system cannot maintain itself without such innovations. And this proof is given by an economic theory which shows that profit and interest on capital can exist only on account of the instability of a certain datum i.e. the technique of production*, and that it would necessarily disappear the moment further technical progress proved impossible.” (Lange, 1935, pp. 199-200, emphasis added)

In other words, innovations and technological change represent, on the one hand, the endogenous driving forces of the economic “evolution”. On the other hand, far from implying any enduring growth, they become a condition of instability (and eventually crisis), since changes in the (social) sphere of production induced by the innovations are uncoordinated, unplanned and merely driven by the (individual) profit motive. Not coincidentally, in a subsequent work,

Lange explicitly praised Schumpeter's 1939 *Business Cycles* precisely because of a certain proximity to Karl Marx's previous analysis:

In intention and horizon Professor Schumpeter's book can be compared with *Das Kapital* of Karl Marx which set out to investigate the "law of motion" of capitalism (...) and found that "*crises*" *play the pivotal role*. This comparison is intended by the reviewer as highest praise. (Lange, 1941b, p. 190. Emphasis added)

However, Lange firmly rejected Schumpeter's definition of "innovation":

Professor Schumpeter says, "We will simply define an innovation as the setting up of a new production function". (...) This definition, however, is too wide. A large (possibly even infinite) number of ways always exists in which production functions can be changed. But an innovation appears only when there is a possibility of such a change, which increases the (discounted) maximum effective profit the firm is able to make. *All other possible changes are disregarded by the firms*. (Lange, 1943, p. 21, n.8, emphasis added)

Stated succinctly, Lange is therefore persuaded (following Marx) that innovations are related to the maximization of the rate of profit by means of '*a most indissoluble tie*', which in turn refers to the distributive conflict, typical of any capitalist economy, and eventually to its unequal and anarchic *laws of development*.

In the second place, in a series of parallel works written between 1934 and 1942, Lange also emphasized the close connection between the dynamics of the saving rate and the dynamics of income distributives shares. Since the workers' saving must be considered negligible, the higher the profit share, the higher the saving rate. As a result, any innovation that comes with an increase in the profit share is accompanied by a rise in the saving rate. And since technical

progress is a mere historical datum, the saving rate is most likely to be independent from “the requirements of the maximization of social welfare”. (Lange, 1937: 123)

“ [...] saving is [...] in the present economic order determined only partly by pure utility considerations, and *the rate of saving is affected much more by the distribution of incomes, which is irrational from the economist's point of view.* (Lange, 1937, p.127, emphasis in original)

Along these lines, Lange firmly rejected the traditional idea (put forward by Schumpeter and Robertson) that the adjustment of the rate of interest may significantly affect saving and bring it into equilibrium with investment (see Toporowski 2012). In a series of seminars held in 1942 (and published posthumously in 1987), he emphasized, on the contrary, that in a capitalist economy saving and investment decisions were definitely uncoordinated. While saving depends on income distribution, investment, instead, depends partly on the rate of interest (which, in turn, depends on banking policy, i.e. on the banks' expectations about the safety of the investment) and partly on the profit expected by the entrepreneur who makes the investment, based on poor and volatile expectations, as all that he is able to know is the current state of the market, so that any anticipation of the future inescapably becomes “*a purely haphazard type or even ... subject to the quite erratic influences of mass psychology*” (Lange, 1942, p. 15).

In short, Lange was persuaded that *Capital* (meant as a social relation of production) was the key-concept for treating satisfactorily the problem of capitalist dynamics. However, he was aware that this social relation had a dual dimension in dealing with both the struggle between capitalists and workers, and the competition among the capitalists themselves. Therefore, both the distributive conflict and the (un)coordination between saving and investment (which in turn affects technical progress and accumulation) represent the critical elements of capitalist dynamics. Thus, Lange does not share a widespread assumption among Marxists (e.g. Rosa Luxemburg) that entrepreneurs operate like an individual ‘collective capitalist’ whenever they have to set out

the level of investment. On the contrary, he is persuaded that crisis originates from the *individual decisions* of saving and investment. Along this line, capitalism's irremediable instability becomes strictly related to the separation between *social* and *individual*, in the sphere of both production and consumption.

Finally, it has to be remarked that Lange's endorsement of Marx's theory of economic evolution had at least another, crucial, implication. While analysing the role of time lags, Lange repeatedly insisted that they should in any case be considered the ultimate determinants of fluctuations and instability. At best, they could be considered the *phenomenal form* of deeper changes in the economic and institutional data. Lange denied neither their existence nor their theoretical relevance on certain occasions. He claimed simply that they could not, alone, suffice to justify a dynamic theory. For instance, it is for this reason that Lange rejected the analysis based on the "Cobweb Theorem" which, according to him, rested mainly on the existence of time lags:

These theories deduce the impossibility of an equilibrium ... from the very nature of the adjustment mechanism, but they cannot deduce theoretically the changes of data responsible for the trend on which the fluctuations due to the process of adjustment are superimposed. (Lange, 1935, n.2 pp. 192-193)

Given the aforementioned features of his reflection on dynamics, it is worth considering to what extent Lange believed that also the analytical apparatus created by Keynes might be a useful tool to cope with the problems of capitalist dynamics originally raised by Marx. In particular, it is worth investigating if, in Lange's eyes, Keynes's theory of employment could serve as a basis for the theory of the business cycle. In our view, Lange's 1938 paper "The Rate of Interest and the Optimum Propensity to Consume" can help address this crucial issue.

2. Lange's 1938 static model

Lange developed in explicit mathematical form the meaning of the Keynesian system. He stated three fundamental relationships: (1) the consumption function relating consumption to income, and for generality, to the interest rate as well; (2) the marginal efficiency of capital relating net investment to the interest rate and to the level of consumption (as for a level of capital equipment fixed for the short period under investigation); (3) the schedule of liquidity preference relating the existing amount of money to the interest rate and the level of income¹³.

At a first sight, Lange makes some assumptions similar to those previously made by Hicks (1937). Firstly, Lange's analysis is static. Since net investment varies, so should total capital, eventually influencing output, investment, and savings. But these effects are ignored and the stocks of production factors and technology are treated as constant¹⁴. Secondly, like Hicks, Lange extended his model's assumption by allowing investment to depend on real income as well as the rate of interest¹⁵. However, he justifies such a choice recalling K. Marx's and T.R. Malthus's idea that investment must depend on the level of consumption itself related (also) to distributive shares, given the necessity to maximise profits:

Mr. Keynes treats investment and expenditure on consumption as two independent quantities and thinks that total income can be increased indiscriminately by expanding either of them. But it is a common place which can be read in any textbook of economics that the demand for investment goods is derived from the demand for consumption goods. The real argument of the under consumption theories is that investment depends on the expenditure on consumption and, therefore, cannot be increased without an adequate increase of the later, *at least in a capitalist economy where investment is done for profit*. (Lange, 1938, p. 23, emphasis added).

By introducing the level of consumption as an argument in the investment function, Lange aimed at emphasizing that, in a market economy led by profit maximization, any increase in the

demand of consumer goods (as well as any favorable change in the expectations) was likely to affect the prospective yield of the investment projects:

Investment per unit of time depends, however, (...) also on the expenditure on consumption. For the demand for investment goods is *derived* from the demand for consumers' goods. The smaller the expenditure on consumption the smaller is the demand for consumers' goods and, consequently, the lower is the rate of net return on investment. Thus, the rate of interest being constant, investment per unit of time is the larger, the larger the total expenditure on consumption. (Lange, 1938: 14 – emphasis in original)

Keynes, in Chapter 12 of the *General Theory*, considered that long-term expectations were merely exogenous, which amounts to assuming that investment was insensitive to current levels of output, consumption or national income. In this respect, Lange's model evidently departs from Keynes's theory¹⁶.

In light of this premise, Lange's model can be summarized by the following four equations:

$$\frac{M}{P} = L(i, y) \quad L_i \leq 0, L_y \geq 0 \quad (1)$$

$$c = \phi(y, i) \quad 0 < \phi_y < 1 \quad \phi_i \geq 0 \quad (2)$$

$$I = F(i, c) \quad F_i < 0, F_c > 0 \quad (3)$$

$$y \equiv c + I \quad (4)$$

where M is the amount of money held by individuals or the real value of cash balances, y is total real income, i is the interest rate, c is total expenditure on consumption per unit of time, and I is investment per unit of time. According to Lange, M, y, c and I are measured in wage units. Once the amount of money M (in wage units) is given, these four equations determine the

four unknowns c , I , y and i . Alternatively, i can be assumed to be given (i.e. exogenously set by the banking system) and M can be assumed to be endogenous.

In this case, the process of determination of the rate of interest is depicted by Lange in three diagrams. The first represents the relation between the demand for cash balances and the rate of interest. The quantity of money (in wage-units) is measured on the axis OM and the rate of interest on the axis Oi , yielding a family of liquidity preference curves, one for each level of total income (measured in wage-units). The greater the total income the higher positioned is the corresponding curve. We have a second family of curves (for each rate of interest) representing the relation between income and expenditure on consumption. Income is measured along Oy and expenditure on consumption along Oc . The relation between investment and the rate of interest is represented by the third graph. Measuring investment per unit of time along the OI axis and the rate of interest along the Oi axis we have a family of curves indicating investment corresponding to each value of the interest rate. These curves represent the marginal net return (marginal efficiency) of each amount of investment per unit of time. It is important to note that there is a separate curve for each level of expenditure on consumption. The greater the expenditure on consumption, the higher the position of the corresponding curve.

Having constructed this tool, Lange then determines interest rate, level of consumption and investment in the economy. With a given amount of money, M_0 and a given initial level of income, say y , equation 1 gives us a rate of interest of i_0 . With y and i_0 given, equation (2) determines total consumption, C_0 , and equation (3) provides the level of investment, i_0 .

If we find that the sum of total consumption and investment precisely equals total income, then equation (4) is confirmed. If not, we must start on a process of adjustment until an equilibrium position in the economy is established:

This process of *mutual adjustment* goes on until the curves in our three diagrams have reached a position compatible with each other and with the quantity of money given, i.e., until equilibrium is attained. (Lange, 1938: 17 – emphasis in original).

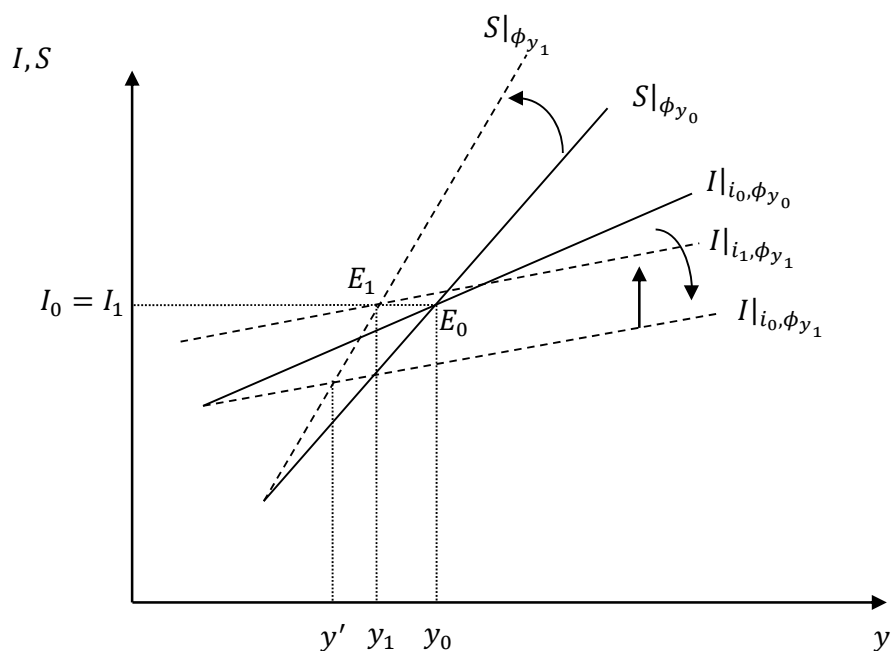
After having examined the *General Theory* of interest, Lange introduces the problem of under-consumption theorists (from Malthus onward) who believed that up to a certain point an increase in saving promotes investment, but beyond this point it would be harmful. To determine an optimal level of saving, Lange emphasized two effects working in opposite directions (Lampa, 2013). First, an increase in the propensity to save induces a decrease in output which causes a negative “accelerator” effect on investment. Second, because of the fall in total income, as long as the money supply remains unchanged, the rate of interest falls, stimulating the level of investment. The optimality of the saving rate can be characterized by the condition that the decrease in investment brought about by decreasing income is just balanced by the increase in investment brought about by the pressure of lower income transaction monetary needs upon the rate of interest. Here is how Lange discusses both effects:

“Since investment per unit of time is a function of both the rate of interest and expenditure on consumption a decrease of the propensity to consume (increase in the propensity to save) has a twofold effect. On the one hand the decrease of expenditure on consumption discourages investment, but the decrease in the propensity to consume also causes... a fall of the rate of interest which encourages investment on the other hand. The optimum propensity to consume is that at which the encouraging and the discouraging effect of a change are in balance. (1938 : 34).

In an unpublished manuscript dated 1942, Paul Samuelson attempted to determine whether Lange’s model was conveying the essence of Keynes’s ideas concerning the effects of an increase in saving or not. According to him, two critical propositions could be found in the *General Theory*: “(a) an increased desire to save serves merely to depress income” and “(b) the net result is

actually less attained saving and investment”. As a result, he stresses that the critical difference between Keynes and Lange appears to be that, notwithstanding the fact that Lange would surely agree with proposition (a), he would agree with (b) only up to the point of the optimum propensity to consume. If this were not to occur, emphasizes Samuelson, “we would have a contradiction: income would then not have decreased, interest would not have fallen, and investment would not have risen. But then income must have fallen and not fallen simultaneously, which is an impossibility. Whether or not (b) is valid, it appears that Keynes contention (a) must hold”. (Samuelson Papers, 1942: 23).

A simple diagram of our own representing the saving and investment schedules may help illuminate this point (see also Lampa, 2013). The marginal propensity to save $1 - \phi_y$ is assumed to be lower than the marginal propensity to invest $F_c \phi_y$.



Graphically, the negative “accelerator” effect of a fall in the marginal propensity to consume ϕ_y materializes in the rotation in the opposite direction of the saving and investment schedules whose slopes have respectively increased and decreased while the positive effect of the fall in the rate of interest comes with the upward shift of the investment schedule¹⁷. In the diagram, we have represented the situation in which the second effect balances the first one and the economy reaches a new equilibrium with a lower level of income but the same level of investment.

In his reply to Samuelson, Lange recognized the validity of Samuelson’s interpretation and he clarified that the notion of “the optimum propensity to consume” is not the propensity which, for a given investment schedule, would guarantee a full employment level of national income:

“My optimum propensity to consume should in no way be interpreted as involving an “optimum” for policy. All it meant was maximizing the rate of investment. The policy optimum is in terms of maximum income and employment (plus stability considerations). In fact, when I wrote the article, I thought of writing another one on the propensity to

consume which maximizes income, which I wanted to recommend as the optimum for policy” (Lange in Samuelson Papers 1942)¹⁸.

Thus, he explicitly recognizes that his theory of interest can be consistent with an analysis of both disequilibrium and underemployment equilibrium (Lampa, 2013)

Although his article exclusively targets static analysis of the theory of interest and output with the aim of examining the effect of a change in thriftiness, Lange seems to recommend that an exhaustive approach to this topic should include dynamics as well. First, he explicitly cites Kalecki’s 1937 business cycle model, thus suggesting a complementary as well as a consistent appendix to his own contribution. In a cryptic footnote he explicitly states that in presence of time lags, the system might give rise to cyclical fluctuations:

If this process of adjustment involves a time lag of a certain kind, a cyclical fluctuation, instead of equilibrium, is the result. Cf. Kalecki ‘A theory of business cycle’.... (Lange, 1938, n1, p. 17)

Second, by way of conclusion, Lange pointed out that static investigation of how the optimum propensity to consume is attained was only part of the question, for:

In a society where the propensity to save is determined by the individuals, there are no forces at work which keep it automatically at its optimum and it is well possible, as the underconsumption theorists maintain, that there is a tendency to exceed it. (Lange, 1938, p. 32)

Unfortunately, Lange’s 1938 article does not provide any further information on the role and characteristics of dynamics with respect to the change in the propensity to consume nor any elements illuminating how dependent the dynamics was to time lags.

In the following development, we try to shed some light on this issue by attempting a formal dynamic reconstruction along Kaleckian lines, as explicitly evoked by Lange in 1938.

3. The dynamization of Lange's model: an interpretation

In order both to render effective and to simplify our exposition, our starting point will be a general IS-LM model. Firstly, it should be noted that such a general IS-LM model can be deduced easily from Lange's equations (1) to (4) only on condition only that the equality (4) is assumed to be an equilibrium equation (whereas Lange emphasizes that it is an identity)^{19 20}.

Secondly, the present analysis will adopt Kalecki's lag structure, as explicitly suggested by Lange himself in a footnote (1938). In a revised version of his 1937 business cycle model, Kalecki (1939) criticized Lange for not exploring the "fundamental importance" of this lag for analysing the dynamics of the economy. His point was that when a relatively long time is needed to complete investment projects, current investments are a result of former investment decisions and become a "datum inherited from the past like the capital equipment" (Kalecki 1937: 81). It follows that the "dynamic process" is similar to "a chain of short period equilibria" each of them prevailing as long as the actual level of investment remains unchanged. Naturally, in order to determine how "the links of the chains are connected" it becomes necessary to explain how investment decisions are related to the current economic situation.

With respect to this point, Lange's model did not suffer from the flaw identified by Kalecki (1936) in Keynes's analysis of investment. Kalecki was convinced that the development of a proper dynamic theory of employment implies the idea that the current rate of investment may itself, by means of current changes in profits, affect current investment decisions. In Lange's model, the influence of current profits (resulting from actual investments) on investment

decisions is taken into account through the introduction of the level of real consumption (which is considered as a proxy of expected profitability in the investment function by Lange, as we have shown). It is true that the assumption of real variable (Lange introduced the level of real consumption as an argument of the investment function) may obscure some important changes in prices that might influence investment decisions: any increase in investment which might improve profit expectations implies, at the same time, an increase in prices of investment goods that might reduce it. Although Lange's model does not allow disentangling these two effects, it helps highlighting, as we will see, how feed-back effects of current investment on investment decisions, depending on the shape of the $F(\cdot)$ and $\phi(\cdot)$ functions, are likely to set off fluctuations.

A possible way to formalize Kalecki's time lag notion consists in assuming that actual investment changes according to the following equation

$$dI/dt = \theta [I^d - I] \quad (5)$$

where I^d refers to the desired investment, I to actual investment and θ to the speed of adjustment of the actual level of investment to the level of investment decisions²¹.

In contrast, it is assumed that the money market adjusts instantaneously through fluctuations in the interest rate. Solving the money market equation for i , we obtain:

$$i = i(y, P, M)$$

To examine the dynamics, it is useful to introduce the following notation: F_x stands for the partial derivative of function F with respect to x : $i_y = -L_y/L_i > 0$, $i_P = -M/(P^2 L_i) \geq 0$ and $i_M = 1/L_i P \leq 0$ ²².

In the long run, the capital accumulation is defined as

$$dK/dt = I - \delta K \quad (6)$$

where $\delta < 1$, the rate of depreciation, is assumed to be proportional to the stock of capital.

The essential dynamic feature that enables the model to display cyclical behaviour is introduced by a Kaleckian assumption about the inverse relation between the capital stock and investment decisions: for any level of consumption and income, an increasing capital stock lowers the marginal efficiency of capital, implying that investment decision is lower.²³ The investment curve is therefore, in the diagram drawn in section 2, shifting downwards for high capital stock levels. Similar reasoning holds for a low level of the capital stock. The desired investment function is thus now:

$$I^d = F(c, i, K)$$

In the short run, the actual level of investment, I , is given. So, as long as the consumption does not depend on the rate of interest, equilibrium in the goods market requires that $Y = (I + \bar{c})/(1 - \phi_y)$ where \bar{c} is the autonomous component of consumption²⁴. Replacing this expression in the equation and interest rate equations, we get:

$$I^d = G(I, K)$$

Where $G_I = (F_c \phi_y + F_i i_y)/(1 - \phi_y)$ and $G_K = F_K$.

Substituting the investment demand function in equation (5), we get

$$\frac{dI}{dt} = \theta(G(I, K) - I) \quad (7)$$

The dynamics for this model therefore are given by equations (6) and (7).

The long-run equilibrium values for I and K , obtained by setting $dI/dt=0$ and $dK/dt=0$ in equations (6) and (7), are given by:

$$I^* = \frac{\left(\frac{F_c\phi_y + F_i i_y}{1 - \phi_y}\right)\bar{c}}{1 - \frac{F_c\phi_y + F_i i_y}{1 - \phi_y} - F_K\delta}$$

and:

$$K^* = \frac{1}{\delta} \frac{\left(\frac{F_c\phi_y + F_i i_y}{1 - \phi_y}\right)\bar{c}}{1 - \frac{F_c\phi_y + F_i i_y}{1 - \phi_y} - F_K\delta}$$

An increase in F_c and in ϕ_y increases the long-run equilibrium values of I and K by increasing effective demand due to an increase in consumption and investment, while an increase in F_K and δ reduces it by increasing the depressive effect of capital stock on investment, and by reducing the steady state level of investment.

To examine the local dynamics, we calculate the Jacobian matrix for this dynamic system, which is given by:

$$J = \begin{bmatrix} \theta(G_I - 1) & \theta F_K \\ 1 & -\delta \end{bmatrix}$$

The trace and determinant of this matrix are:

$$T = -[\delta + \theta(1 - G_I)]D = \theta[-\delta(G_I - 1) - F_K]$$

Since the sufficient conditions for stability of the system are $Tr(\mathbf{J}) < 0$ and $Det(\mathbf{J}) > 0$, the condition $G_I < 1$ and so $F_c\phi_y + F_i i_y < 1 - \phi_y$ is sufficient to ensure stability and dampened oscillations. However, that it is not necessary is evidenced by the fact that $F_c\phi_y + F_i i_y = 1 - \phi_y$ still satisfy the stability condition. Explosive oscillations will occur if $Tr(\mathbf{J}) > 0$ while $Det(\mathbf{J}) > 0$. If $G_I > 1$ and so $F_c\phi_y + F_i i_y > 1 - \phi_y$, and θ is sufficiently large, so that $\theta \geq \delta [1 - \phi_y / (F_c\phi_y + F_i i_y - (1 - \phi_y))]$, the trace condition will be violated and persistent or explosive

oscillations become possible. Thus the stable oscillations depend on the reaction coefficients F_c, ϕ_y, F_i, l_y and the time lag represented by θ .

To further analyse the dynamic behaviour of this model we use the phase diagram in Figure 2 which measures the levels of the two state variables, I and K , on the two axes. The isocline for $dK/dt = 0$ is seen, from equation (6), to be given by the equation:

$$I = \delta K,$$

whose slope is equal to:

$$\left. \frac{dK}{dI} \right|_{K=0} = \frac{1}{\delta}$$

which is a positively-sloped straight line. K rises below this line and falls above it, explaining the direction of the vertical arrows. The isocline for $dI/dt = 0$ is given by:

$$I = \frac{F_K K}{1 - G_I}$$

whose slope is equal to:

$$\left. \frac{dK}{dI} \right|_{I=0} = \frac{1 - G_I}{F_K}$$

This shows that to the left of this isocline I is rising and to its right it is falling, explaining the direction of the horizontal arrows in the figure. If $F_c \phi_y + F_i l_y$ is lower than $1 - \phi_y$, this yields a downward-sloping straight. As depicted in Figure 1, the cyclical behavior is necessarily dampened.

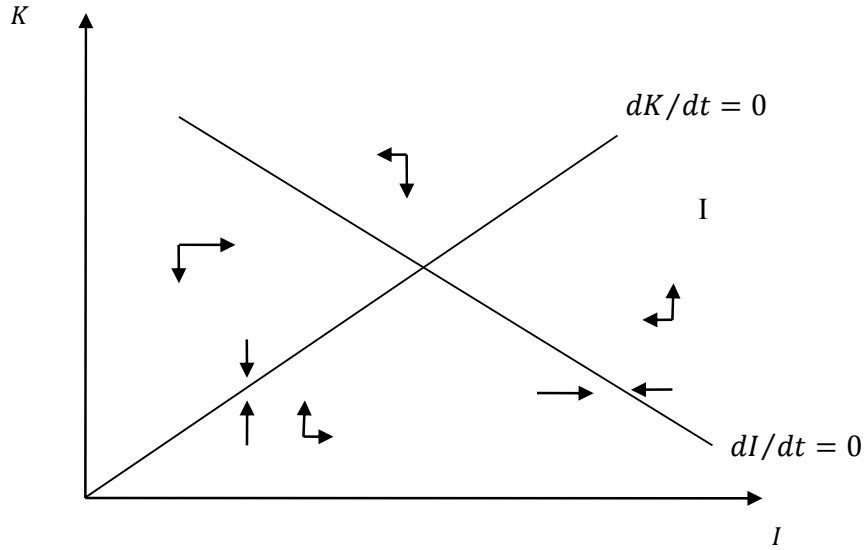


Figure 2. Dampened cycles

If we have $F_c \phi_y + F_i i_Y > 1 - \phi_y$, the $dI/dt=0$ isocline becomes positively sloped. In this case, we can distinguish between two possibilities. In one, in which the $dI/dt=0$ isocline is flatter than the $dK/dt=0$ isocline, we have $(G_I - 1) > \frac{F_K}{\delta}$, which implies that the determinant condition for stability is violated. This means that the dynamics of the system are saddlepoint-unstable, and there are no cycles. In the other case in which the $dI/dt=0$ isocline is steeper than the $dK/dt=0$ isocline, the determinant condition is satisfied, and the dynamics of I are unstable (leading I away from its null-cline) while those for K are stable. The result is cycles: explosive cycles if the trace becomes positive, and dampened ones if it is negative. Since the trace condition can be written as $\theta \left(\frac{F_c \phi_y + F_i i_Y}{1 - \phi_y} - 1 \right) < \delta$, it is more likely to be satisfied the longer the investment lag or the smaller is θ .

When cycles do occur, their occurrence is related to both the slow adjustment of desired investment to actual investment and to the negative effect of capital stock on desired investment. If θ is infinitely large so that adjustment is very rapid, the economy will always be on the $dI/dt =$

0, so that adjustment to the long-run equilibrium will be smooth as long as the determinant condition is satisfied. If the time-lag between the investment decisions and the corresponding income is large relative to the rate at which the amount of equipment is increasing, the rate of investment decisions can continue to fall even below what corresponds to replacement, simply because the fall in income lags behind. Thus introducing a time-lag between the investment decision and the corresponding income may explain a cyclical movement even if the underlying situation is stable; although, in order that the cycle is not highly dampened (i.e., that it does not peter out too quickly in the absence of new disturbing factors), we need to assume that the effect of current investment on total equipment is relatively large, such that the equipment added during the period of the time lag has a considerable influence on the investment decision. It should be noted, however, that Lange's 1938 model, once dynamized along these lines, departs significantly from Kalecki's 1939 business cycle model. By resorting to his theory of profit, Kalecki aims at showing that the dynamics of the profit rate were completely independent from the dynamics of the functional distribution. This is because, in the presence of an investment time lag, the profit level remains uniquely determined by the actual level of investment and capitalists' expenditure. It follows that any change in profit share which may happen during the business cycle will have no effect on total *current* profits, since they continue to be determined by actual investment and are themselves determined by past investment decisions. Hence, as long as investment remains constant there will be the same total amount of profits (i.e. of saving). Naturally, while profits remain unchanged, the real wages and real national product will vary following the variations in aggregate demand, with a consequent fall or rise in output which will adjust as much as the new percentage share of profits in output renders an unchanged absolute amount of profits. We hence find the very striking Kaleckian proposition that capitalists (considered as a whole) cannot impinge the course of the cycle by raising their share in national income²⁵. This does not mean income distribution does not affect the time path of national income, as a higher value of the profit share reduces the amplitude of output over the cycle. It

means that the determinants of the profits share will have no effect on the current level of profit and thus on actual profitability. As a result, Kalecki's model breaks the connection between the dynamics of functional distribution and the dynamics of investment. On the contrary, in Lange's model, one can easily see that any change in the propensity to save that might result from a change in income distribution will directly change investment decisions and the dynamics of the economy. With this respect, Lange's model is closer to Marx's description of the business cycle than to Kalecki's.

Let us now examine how the model, when the coefficients of the differential equations are assumed to change, is likely to produce a self-sustained cycle in absence of any delay between actual investment and investment decisions. This is the case when, for extreme values of the level of investment, $F_c\phi_y + F_i i_y$ is lower than $1 - \phi_y$ while for normal values of investment $F_c\phi_y + F_i i_y$ is higher than $1 - \phi_y$. Under this condition, the trace will be positive for very high and for very low levels of investment, and negative for normal levels of investment. Mathematically, these changes in the sign of the trace will allow the generation of self-sustaining cycles²⁶. This is illustrated in Figure 2 by the fact that the $dl/dt = 0$ curve is decreasing for low and high levels of investment, and increasing for normal values of capital stock. In that case, the economy never reaches a stationary equilibrium. We saw earlier that the shape of the isocline for $dl/dt = 0$ because F_K is negative (negative capacity effect), depends upon the value of G_I

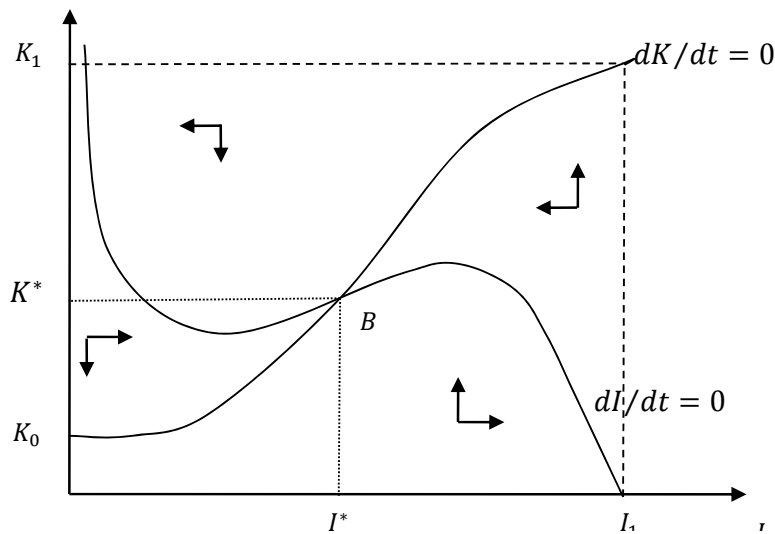


Figure 3. Self-sustained cycles

Figure 3 shows that at very low and very high values of I , the isocline is negatively-sloped; this corresponds to the areas where the savings function is steeper than the investment function. However, for normal values of I , the isocline is positively-sloped, which corresponds to the region where the investment curve is steeper than the savings curve. Above the isocline, $dI/dt < 0$, and investment falls; below the isocline, $dI/dt > 0$, and investment rises. The directional arrows indicate these tendencies. Both curves are drawn such that the locus $dK/dt = 0$ intersects the ordinate at $K_0 > 0$ and that the curve $dI/dt = 0$ intersects the abscissa at $I_1 > I^*$. The curve $dI/dt = 0$ approaches the K -axis for $K \rightarrow \infty$.

When the $dI/dt=0$ isocline is flatter than the $dK/dt=0$ line at its intersection, the determinant of the Jacobian matrix is positive and limit cycles can exist. Indeed, if the trace condition is violated, that is, if $F_c \phi_y + F_i i_y > 1 - \phi_y$ is sufficiently large, the equilibrium at B is unstable, and trajectories close to it will push the economy towards the limit cycle. The combination of

the non-linear investment and saving curves, in absence of any delay between actual investment and investment decisions, is thus sufficient to produce limit cycles.

When there is no investment lag (θ is infinitely large) so that the economy is always on the $dI/dt=0$ isocline, the economic dynamics are particularly interesting. The underlying dynamics, with economy always on the $dI/dt=0$ line, implies that there will be a catastrophic drop from the high to the lower equilibrium. Notice that during this catastrophic fall, output is driven solely by the fast multiplier dynamic, the slower-moving capital dynamic is inoperative since, in moving from a high to a low equilibrium, capital is constant. Therefore, we show that cycles can exist without investment lags, which is perfectly consistent with Lange's idea that time lags represent the *phenomenal form* of capitalist dynamics and not its primary determinant.

With this respect, Lange's model offers an interesting framework for investigating the effects of structural changes that he wished for in his 1935 plea for Marxian economics. It is clear, however, that this model in itself, sheds no light on the origin of the "data" responsible for the trend [just like in Kalecki's (1939) and Kaldor's (1940) models] and, consequently, does not contain the solution to Lange's pursuit of a satisfactory "theory of economic evolution". The reason why is that Lange thought that this problem could be overcome only by developing a theory combining Schumpeter's analysis of innovation (and productivity) with either Kalecki's or Kaldor's trade cycle models, as evidenced by his 1941 review to Schumpeter's *Business Cycles*. On the one hand, he criticizes Schumpeter's theory because it is not connected with the fluctuation of the level of employment (as well as the degree of utilization of resources) and he praises both Kalecki's and Kaldor's analyses, based on what he defines "adaptive fluctuations of the rate of investment" (p.193). On the other hand, he stresses the limits of these latter contributions, as they both assume the unrealistic hypothesis that net disinvestment of capital during the recessions permits to turn the downswing to upswing, whereas Lange (following Schumpeter) is

persuaded that the key factor is the higher productivity due to innovations. Therefore, he concluded:

This raises the question of a possible synthesis between the ‘adaptive fluctuations of investment’ theories and that of Professor Schumpeter. The cycle in investment activity may prove to be a consequence of both adaptive fluctuations and fluctuations in the rate of innovation resulting from changes in the risk of failure.

Since:

Professor Schumpeter’s theory (...) provides us with a decisive element of any realistic explanation of the phenomenon. (Lange, 1941, p. 193)

Unfortunately, Lange did not highlight how waves of (profit-led) innovations were likely to cause changes in the parameters of the investment and saving function nor did he clarify how these changes would come through change in income distribution. Therefore, his 1938 analysis remains necessarily incomplete.

Concluding Remarks

The analysis developed in this article can be interpreted as both a reconstruction of Oskar Lange’s early beliefs on capitalist dynamics and (on such bases) a further development in terms of a dynamic analysis of his 1938 theory of interest, contained in “The Rate of Interest and the Optimum Propensity to Consume”.

With respect to the first issue, we have shown that Karl Marx’s (together with Joseph A. Schumpeter’s) analysis of capitalist dynamics represented a crucial source of inspiration for Lange, because of the pivotal (as well as destabilizing) role played by distributive conflict, technical progress (that is, innovation) and accumulation of capital (that is, un-coordination

between investment and saving). Furthermore, we have also highlighted Lange's interest in Kalecki's (1939) and Kaldor's (1940) theories of the business cycle, because of their strict connection with the fluctuation of both employment and investment.

On the other hand – starting from an unedited correspondence between Lange and Paul Samuelson and from a sibylline footnote of Lange himself (1938) – we have clarified that his 1938 static model can easily be translated into a dynamic model in which the marginal propensity to consume (i.e. to save) plays a key role, as this latter may, by means of the investment, determine the stability property of the economy in a particular way. If there is a unique stationary equilibrium and the standard macro condition is globally satisfied, the unique equilibrium will be stable. If there are multiple equilibriums due to varying marginal propensities to save, some will be stable and others will be unstable so that self-sustained cycles become possible.

In turn, our results are able to capture some relevant aspects of Lange's (previous) reflection on dynamics. Firstly, the functional distribution of income depends on the dynamics of the profit rate and it is related to saving, so that some of Marx's main ideas about the intrinsic instability of capitalism (that is, about cycles and growth) are encapsulated.

Secondly, our results show that cycles can exist even without an investment lag, which is perfectly consistent with Lange's broader idea about the meaning and the role of time lags, as exposed in the first section of the article.

Finally – although our model could not capture the crucial interaction between technological progress and income distribution, since Lange (1938) largely ignores such implications – we have shown that Lange was persuaded that a synthesis between Kaldor's and Kalecki's theories and that of Schumpeter, in which the cycle in investment activity may prove to be a consequence of both adaptive fluctuations of investment and fluctuations in the rate of innovation resulting from changes in the risk of failure, might have been possible. In Lange's eye, such a synthesis precisely

represented a “modern” and consistent reconstruction of the Marxist theory of the business cycle: unfortunately, probably due to his wide and ambitious scientific project, as well as to his intense political and diplomatic career in the 1940s, he could not elaborate it.

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¹ Stated succinctly, Lange's project consisted of a theoretical generalization (capable of guaranteeing *universality*) and an analysis of institutional data (in order to have *realism*), intended to separate economic theory from the tacit assumptions of a capitalist economy, as well as to generalize the economic theory for a 'world to come', whose features were portrayed in a series of contemporary works about socialist theory (see Lampa 2011 and 2013).

² According to Kowalik, Lange's decision to study with Schumpeter stemmed from "Lange's wish to learn as much as possible from the universally recognized specialist in business cycles" (1994: xiii)

³ Lange spent seven months at Cambridge in 1936 where he met Kalecki who, at that time, was working on a new version of his business cycle model, attempting to illuminate how his approach may be related to Keynes's theory. Given their proximity to the Polish socialist circle and their common friendship with their fellow countryman and economist Marek Breit, Lange might have certainly read Kalecki's 1933 *Essay*, published in Polish. It is, however, only after reading the *General Theory* that he really seemed to have come to grips with it.

⁴ According to Kalecki, there is, however, no doubt that Lange's 1938 paper was ultimately about dynamics. While discussing the scope of Lange's works, he claims: "Already in the early stage of Lange's work the versatility of his interests is apparent. I have in mind his three papers published in the years 1936-38: "On the Economic Theory of Socialism," "The Rate of Interest and the Optimum Propensity to Consume" and "Ludwik Krzywicki as a Theoretician of Historical Materialism" (in Polish). The first of these papers was concerned with the role of the quasi-market mechanism in a socialist system, the second with the dynamics of a capitalist economy (our emphasis), the third with historical materialism. These were the three directions in which Lange's work was to develop in its later stages." (Kalecki 1966: 431)

⁵ This is true also concerning Meade's 1937 model (see Rappoport 1992)

⁶ See Young (2008) for a more extensive study of the attempts to make the IS-LM a dynamic system.

⁷ A possible explanation is the existence of an ambitious scientific project, to which Lange gave priority (see note 1)

⁸ Lange, however, recognized that the economic equilibrium approach, insofar as it precludes institutional data, has the merit of being abstract and, therefore, universal (since its basic notions hold true in any kind of economic system, included a socialist one). Consequently, it provides "*a scientific basis*" for current administrations of the economy in many respects, such as prices, market-structure, or the allocation of resources.

⁹ Lange's critique of Marx's reproduction schemes was also influenced by the Polish Marxist *milieu*, which on several occasions, since the well-known attempt of Rosa Luxemburg, had dealt with this issue trying to correct "Marx's error" (see Kowalik, 2009)

¹⁰ Lange was especially aware that economic factors were likely to interact with extra-economic factors while the study of such interactions became the field of his application of historical materialism:

"[...] the full evolution of Capitalism in all its concreteness cannot be explained by a theory of economic evolution alone. It can be explained only by a joint use of both economic theory and the theory of historical materialism. The latter is an inseparable part of the Marxian analysis of Capitalism. (Lange, 1935, p. 201)

¹¹ On the other hand, Lange notes that a purely dynamic analysis would be "*a poor basis*" for solving more "ordinary" problems, such as monopoly prices, distribution of productive resources, etc. Therefore Lange concludes that a correct method of investigation presupposes both statics and dynamics, as clearly shown by business cycle theories.

¹² The analytical tools used by Lange were quite different from the traditional Marxist approach because of the former's firm refusal of the labour theory of value: "In the Marxian system the labour theory of values serves also to demonstrate the exploitation of the working class under Capitalism, i.e. the difference between the personal distribution of income in a capitalist economy and in an 'einfache Warenproduktion'. It is this deduction from the labour theory of value which makes the orthodox Marxist stick to it. But the same fact of exploitation can also be deduced without the help of the labour theory of value" (Lange, 1935, n3 p.195). Lange's rejection of the Marxian labor theory of value is expressed, even clearer, in the Appendix to "On the Economic Theory of Socialism" (1937).

¹³ For a detailed analysis see Kowalik (1964, 1994, 2008) and Lampa (2013)

¹⁴ "It also ought to be observed that the investment function holds only for a given capital equipment and for a given distribution of the expenditure for consumption between the different industries" (Lange, 1938, p. 13).

¹⁵ The argument is close to that raised by Hicks who wrote: "Surely there is every reason to suppose that an increase in the demand for consumers' goods, arising from an increase in employment, will often directly stimulate an increase in investment, at least as soon as an expectation develops that the increased demand will continue. If this is so, we ought to include I [national income] in the second equation [investment function], though it must be confessed that the effect of I on the marginal efficiency of capital will be fitful and irregular" (Hicks, 1937, p. 156). One may also notice that it is assonant with J.A. Hobson's analysis, contained in his well-known 1910 book.

¹⁶ Kregel (1976, pp. 215-17) notes, however, that when dealing with money wage dynamics, and general policy, Keynes admitted that long-term expectations could be affected by current events.

¹⁷ For sake of simplicity, we have left out the influence of the rate of interest on consumption.

¹⁸ In a footnote added to the 1943 edition of his 1938 paper, he returned to this issue: "Optimum means here merely "maximizing investment". This need not be the most desirable propensity to save from the point of view of social policy. From the latter point of view a propensity to save which maximizes real income may be more desirable." (Kowalik 1994: 211)

¹⁹ One could add that also the shape of the LM curve is the result of a theoretical assumption, whereas in Lange it was "empirically determined". See: Boianovski, M., 2004: 106-107.

²⁰ The IS curve is therefore given by the equation:

$$Y = \phi(y, i) + F(\phi(y, i), i)$$

whose slope is

$$\left. \frac{di}{dy} \right|_{IS} = \frac{1 - \phi_y - F_c \phi_y}{\phi_i + F_c \phi_i + F_i}$$

²¹ The investment dynamics formalized by Kalecki was more complex since his analysis, by distinguishing three stages in the investment-production nexus: decisions to invest, the production period and the time of delivery lead up to a mixed difference and differential equation.

²² Kalecki recognized in his 1937 paper that the coefficient i_y was likely to change during the business cycle: "We see that the rise of investment I increases the demand for cash and has in that way the tendency to raise the rate of interest. It is, however, not the only way in which the rate of interest is affected by a change in investment I . The investment I as we know determines (with a given capital equipment) the short-period equilibrium and thus the "general state of affairs." But the better this state of affairs the greater is the "lender's confidence and, therefore, through this channel the rise of investment has a tendency to lower the rate of interest." (Kalecki 1937: 87)

²³ Lange (1941) attached great importance to this assumption.

²⁴ Equilibrium between aggregate demand and aggregate output requires that $Y = I + C$ and thus $Y = (I + \bar{c}) / (1 - \phi_y)$

²⁵ See Lopez and Assous 2010 for further details.

²⁶ The proof is a straightforward application of the Poincaré-Bendixson theorem.