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Abstract

We analyze a divisible good uniform-price auction that features two groups each with a finite number of identical bidders. Equilibrium is unique, and the relative market power of a group increases with the precision of its private information but declines with its transaction costs. In line with empirical evidence, we find that an increase in transaction costs and/or a decrease in the precision of a bidding group's information induces a strategic response from the other group, which thereafter attenuates its response to both private information and prices. A “stronger” bidding group -which has more precise private information, faces lower transaction costs, and is more oligopsonistic- has more market power and so will behave competitively only if it receives a higher per capita subsidy rate. When the strong group values the asset no less than the weak group, the expected deadweight loss increases with the quantity auctioned and also with the degree of payoff asymmetries. Market power and the deadweight loss may be negatively associated.

JEL-Codes: D440, D820, G140, E580.

Keywords: demand/supply schedule competition, private information, liquidity auctions, treasury auctions, electricity auctions.

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1 Introduction

Divisible good auctions are common in many markets, including government bonds, liquidity (refinancing operations), electricity, and emission markets.¹ In those auctions, both market power and asymmetries among the participants are important; asymmetries can make market power relevant even in large markets. However, theoretical work in this area has been hampered by the difficulties of dealing with bidders that are asymmetric, have market power, and are competing in terms of demand or supply schedules in the presence of private information. This paper helps to fill that research gap by analyzing asymmetric uniform-price auctions in which there are two groups of bidders. Our aims are to characterize the equilibrium, to perform comparative statics and welfare analysis (from the standpoint of revenue and deadweight loss), and finally to derive implications for policy.

Divisible good auctions are typically populated by heterogeneous participants in a concentrated market, and often we can distinguish a core group of bidders together with a fringe. The former are *strong* in the sense that they have better information, endure lower transaction costs, and are more oligopolistic (or oligopsonistic) than members of the fringe. As examples we discuss Treasury and liquidity auctions in addition to wholesale electricity markets.

Treasury auctions have bidders with significant market shares. That will be the case in most systems featuring a primary dealership, where participation is limited to a fixed number of bidders (this occurs, for example, in 29 out of 39 countries surveyed by Arnone and Iden 2003). A prime example are US Treasury auctions, which are uniform-price auctions.² In these auctions, the top five bidders typically purchase close to half of US Treasury issues (Malvey and Archibald 1998). Armantier and Sbaï (2006) test for whether the bidders in French Treasury auctions are symmetric; these authors conclude that such auction participants can be divided into two distinct groups as a function of (a) their level of risk aversion and (b) the quality of their information about the value of the security to be sold. One small group consists of large financial institutions, which possess better information and are willing to take more risks. Kastl (2011) also finds evidence of two distinct groups of bidders in (uniform-price) Czech Treasury auctions. Other papers that report asymmetries between bidders in Treasury auctions include, among others, Umlauf (1993) for Mexico, Bjonnes (2001) for Norway, and Hortaçsu and McAdams (2010) for Turkey.

¹See Lopomo et al. (2011) for examples of such auctions.

²The relatively small number of primary dealers makes the US Treasury market imperfectly competitive (Bikhchandani and Huang 1993). Uniform-price auctions are often used in Treasury, liquidity and electricity auctions, for example. See Brenner et al. (2009) for Treasury auctions, with the United States a leading example since November 1998. Experimental work has found substantial demand reduction in uniform-price auctions (see e.g. Kagel and Levin 2001; Engelbrecht-Wiggans et al. 2006).

Bindseil et al. (2009) and Cassola et al. (2013) find that the heterogeneity of bidders in liquidity auctions is relevant. Cassola et al. (2013) analyze the evolution of bidding data from the European Central Bank’s weekly refinancing operations before and during the early part of the financial crisis. The authors show that effects of the 2007 subprime market crisis were heterogeneous among European banks, and they conclude that the significant shift in bidding behavior after 9 August 2007 may reflect a change in the cost of short-term funding on the interbank market and/or a strategic response to other bidders. In particular, Cassola et al. (2013) find that one third of bidders experienced no change in their costs of short-term funds from alternative sources; this means that their altered bidding behavior was mainly strategic: bids were increased as a best response to the higher bids of rivals.³

Concentration is high also in other markets, such as wholesale electricity. This issue has attracted attention from academics and policy makers alike. A number of empirical studies have concluded that sellers have exercised significant market power in wholesale electricity markets (see e.g. Green and Newbery 1992; Wolfram 1998; Borenstein et al. 2002; Joskow and Kahn 2002).⁴ Most wholesale electricity markets prefer using a uniform-price auction to using a pay-as-bid auction (Cramton and Stoft 2006, 2007). In several of these markets (e.g., California, Australia), generating companies bid to sell power and wholesale customers bid to buy power. In such markets, asymmetries are prevalent. For example, some generators of wholesale electricity rely heavily on nuclear technology, which has flat marginal costs, whereas others rely mostly on fuel technologies, which have steep marginal costs. Holmberg and Wolak (2015) argue that, in wholesale electricity markets, information on suppliers’ production costs is asymmetric. For evidence on the effect of cost heterogeneity on bidding in wholesale electricity markets, see Crawford et al. (2007) and Bustos-Salvagno (2015).

Our paper makes progress within the linear-Gaussian family of models by incorporating bidders’ asymmetries with regard to payoffs and information. We model a uniform-price auction where asymmetric strategic bidders compete in terms of demand schedules for an inelastic supply (we can easily accommodate supply schedule competition for an inelastic demand). Bidders may differ in their valuations, transaction costs, and/or the precision of their private information.⁵

³Bidder asymmetry has also been found in procurement markets, including school milk (Porter and Zona 1999; Pesendorfer 2000) and public works (Bajari 1998).

⁴European Commission (2007) has asserted that “at the wholesale level, gas and electricity markets remain national in scope, and generally maintain the high level of concentration of the pre-liberalization period. This gives scope for exercising market power” (Inquiry pursuant to Article 17 of Regulation (EC) No 1/2003 into the European gas and electricity sectors (Final Report), Brussels, 10.1.2007).

⁵One reason for differences in private information among bidders may be the presence of both dealers and direct bidders in auctions (such as in US Treasury auctions). Dealers aggregate the information of clients and bid with a higher precision of information (for evidence from Canadian Treasury auctions, see Hortaçsu and

For simplicity and with an empirical basis, we reduce heterogeneity to two groups; within each group, agents are identical.⁶ We seek to identify the conditions under which there exists a linear equilibrium with symmetric treatment of agents in the same group (i.e., we are looking for equilibria such that demand functions are both linear and identical among individuals of the same type). After showing that any such equilibrium must be unique, we derive comparative statics results.

More specifically, our analysis establishes that the number of group members, the transactions costs, the extent to which bidders' valuations are correlated, and the precision of private information affect the sensitivity of traders' demands to private information and prices. When valuations are more correlated, traders react less to the private signal and to the price. We also find that the relative market power of a group increases with the precision of its private information and decreases with its transaction costs. Furthermore, if the transaction costs of a group increase, then the traders of the other group respond strategically by diminishing their reaction to private information and submitting steeper schedules. This result is consistent with the empirical findings of Cassola et al. (2013) in European post-crisis liquidity auctions.

If a group of traders is stronger in the sense described previously (i.e., if its private information is more precise, its transaction costs are lower, and it is more oligopolistic), then the members of that group react more (than do the bidders of the other group) to the private signal and also to the price. This result may help explain the finding of Hortaçsu and Puller (2008) for the Texas balancing market where, there is no accounting for private information on costs that, small firms use steeper schedules than the theory would predict.⁷

We also find when the expected valuations between groups differ that the auction's expected revenue needs *not* be decreasing in the transaction costs of bidders, the noise in their signals, or the correlation of values. These findings contrast with the results obtained when groups are symmetric. We bound the expected revenue of the auction between the revenues of auctions involving extremal yet symmetric groups.

In this paper we consider large markets and find that, if there is both a small and a large group of bidders, then the former (oligopsonistic) group has more market power and yet even

Kastl 2012; for a theoretical model see Boyarchenko et al. 2015).

⁶All traders in a group receive the same signal. In case one group does not receive a signal we reproduce the information structure in Grossman and Stiglitz (1980) of uninformed and informed traders.

⁷Linear supply function models have been used extensively for estimating market power in wholesale electricity auctions. Holmberg et al. (2013) provide a foundation for the continuous approach as an approximation to the discrete supply bids in a spot market. In their experimental work, Brandts et al. (2014) find that observed behavior is more consistent with a supply function model than with a discrete multi-unit auction model. Ciarreta and Espinosa (2010) use Spanish data in finding more empirical support for the smooth supply model than the discrete-bid auction model.

the latter (large) group does not behave competitively since it retains some market power due to incomplete information. We also prove that the equilibrium under imperfect competition converges to a price-taking equilibrium in the limit as the number of traders (of both groups) becomes large.

Finally, we provide a welfare analysis. Toward that end, we characterize the deadweight loss at the equilibrium and show how a subsidy scheme may induce an efficient allocation. We find that if one group is stronger (as previously defined), then it should garner a higher per capita subsidy rate; the reason is that traders in the stronger group will behave more strategically and so must be compensated more to become competitive. The paper also underscores how the bidder heterogeneity in terms of information, preferences, or group size documented in previous work may increase deadweight losses. In particular, when the strong group values the asset at least as much as the weak group, the deadweight loss increases with the quantity auctioned and also with the degree of payoff asymmetries.

Our work is related to the literature on divisible good auctions. Results in symmetric pure common value models have been obtained by Wilson (1979), Back and Zender (1993), and Wang and Zender (2002), among others.⁸

Results in interdependent values models with symmetric bidders are obtained by Vives (2011, 2014) and Ausubel et al. (2014), for example.⁹ Vives (2011), while focusing on the tractable family of linear-Gaussian models, shows how increased correlation in traders' valuations increases the market power of those traders. Bergemann et al. (2015) generalize the information structure in Vives (2011) while retaining the assumption of symmetry. Rostek and Weretka (2012) partially relax that assumption and replace it with a weaker "equicommonality" assumption on the matrix correlation among the agents' values.¹⁰ Du and Zhu (2015) consider a dynamic

⁸Wilson (1979) compares a uniform-price auction for a divisible good with an auction in which the good is treated as an indivisible good; he finds that the price can be significantly lower if bidders are allowed to submit bid schedules rather than a single bid price. That work is extended by Back and Zender (1993), who compare a uniform-price auction with a discriminatory auction. These authors demonstrate the existence of equilibria in which the seller's revenue in a uniform-price auction can be much lower than the revenue obtained in a discriminatory auction. According to Wang and Zender (2002), if supply is uncertain and bidders are risk averse, then there may exist equilibria of a uniform-price auction that yield higher expected revenue than that from a discriminatory auction.

⁹Ausubel et al. (2014) find that, in symmetric auctions with decreasing linear marginal utility, the seller's revenue is greater in a discriminatory auction than in a uniform-price auction. Pycia and Woodward (2016) demonstrate that a discriminatory pay-as-bid auction is revenue-equivalent to the uniform-price auction provided that supply and reserve prices are set optimally.

¹⁰This assumption states that the sum of correlations in each column of this matrix (or, equivalently, in each row) is the same and that the variances of all traders' values are also the same. Unlike our model, Rostek and Weretka's (2012) model maintains the symmetry assumption as regards transaction costs and the precision

auction model with ex post equilibria. For the case of complete information, progress has been made in divisible good auction models by characterizing linear supply function equilibria (e.g., Klemperer and Meyer 1989; Akgün 2004; Anderson and Hu 2008). An exception that incorporates incomplete information is the paper by Kyle (1989), who considers a Gaussian model of a divisible good *double* auction in which some bidders are privately informed and others are uninformed. Sadzik and Andreyanov (2016) study the design of robust exchange mechanisms in a two-type model similar to the one we present here.

Despite the importance of bidder asymmetry, results in multi-unit auctions have been difficult to obtain. As a consequence, most papers that deal with this issue focus on auctions for a single item. In sealed-bid, first-price, single-unit auctions, an equilibrium exists under quite general conditions (Lebrun 1996; Maskin and Riley 2000a; Athey 2001; Reny and Zamir 2004). Uniqueness is explored in Lebrun (1999) and Maskin and Riley (2003). Maskin and Riley (2000b) study asymmetric auctions, and Cantillon (2008) shows that the seller’s expected revenue declines as bidders become less symmetric. On the multi-unit auction front, progress in establishing the existence of monotone equilibria has been made by McAdams (2003, 2006); those papers address uniform-price auctions characterized by multi-unit demand, interdependent values and independent types.¹¹ Reny (2011) stipulates more general existence conditions that allow for infinite-dimensional type and action spaces; these conditions apply to uniform-price, multi-unit auctions with weakly risk-averse bidders and interdependent values (and where bids are restricted to a finite grid).

The rest of our paper is organized as follows. Section 2 outlines the model. Section 3 characterizes the equilibrium, analyzes its existence and uniqueness, and derives comparative statics results. We address large markets in Section 4 and develop the welfare analysis in Section 5. Section 6 concludes. Proofs are gathered in the Appendix.

2 The model

Traders, of whom there are a finite number, face an inelastic supply for a risky asset. Let Q denote the aggregate quantity supplied in the market. In this market there are buyers of two types: type 1 and type 2. Suppose that there are n_i traders of type i , $i = 1, 2$. In that case, if the asset’s price is p , then the profits of a representative type- i trader who buys x_i units of the

of private signals. The equilibrium they derive is therefore still symmetric because all traders use identical strategies.

¹¹McAdams (2006) uses a discrete bid space and atomless types to show that, with risk neutral bidders, monotone equilibria exist. The demonstration is based on checking that the single-crossing condition used by Athey (2001) for the single-object case extends to multi-unit auctions.

asset are given by

$$\pi_i = (\theta_i - p) x_i - \lambda_i x_i^2 / 2.$$

So, for any trader of type i , the marginal benefit of buying x_i units of the asset is $\theta_i - \lambda_i x_i$, where θ_i denotes the valuation of the asset and $\lambda_i > 0$ reflects an adjustment for transaction costs or opportunity costs (or a proxy for risk aversion). Traders maximize expected profits and submit demand schedules, after which the auctioneer selects a price that clears the market. The case of supply schedule competition for inelastic demand is easily accommodated by considering negative demands ($x_i < 0$) and a negative inelastic supply ($Q < 0$). In this case, a producer of type i has a quadratic production cost $-\theta_i x_i + \lambda_i x_i^2 / 2$.

We assume that θ_i is normally distributed with mean $\bar{\theta}_i$ and variance σ_θ^2 , $i = 1, 2$. The random variables θ_1 and θ_2 may be correlated, with correlation coefficient $\rho \in [0, 1]$. Therefore, $\text{cov}(\theta_1, \theta_2) = \rho \sigma_\theta^2$.¹² All type- i traders receive the same noisy signal $s_i = \theta_i + \varepsilon_i$, where ε_i is normally distributed with null mean and variance $\sigma_{\varepsilon_i}^2$. Error terms in the signals are uncorrelated across groups ($\text{cov}(\varepsilon_1, \varepsilon_2) = 0$) and are also uncorrelated with valuations of the asset ($\text{cov}(\varepsilon_i, \theta_j) = 0$, $i, j = 1, 2$).

In our model, two traders of distinct types may differ in several respects:

- different willingness to possess the asset ($\theta_1 \neq \theta_2$),
- different transaction costs ($\lambda_1 \neq \lambda_2$), and/or
- different levels of precision of private information ($\sigma_{\varepsilon_1}^2 \neq \sigma_{\varepsilon_2}^2$).

Applications of this model are Treasury auctions and liquidity auctions. For Treasury auctions, θ_i is the private value of the securities to a bidder of type i ; that value incorporates not only the resale value but also idiosyncratic elements as different liquidity needs of bidders in the two groups. For liquidity auctions, θ_i is the price (or interest rate) that group i commands in the secondary interbank market (which is over-the-counter). Here λ_i reflects the structure of a counterparty's pool of collateral in a repo auction. A bidder bank prefers to offer illiquid collateral to the central bank in exchange for funds; as allotments increase, however, the bidder must offer more liquid types of collateral which have a higher opportunity cost.

3 Equilibrium

Denote by X_i the strategy of a type- i bidder, $i = 1, 2$, which is a mapping from the signal space to the space of demand functions. Thus, $X_i(s_i, \cdot)$ is the demand function of a type- i bidder

¹²The value of ρ will depend of the type of security. In this sense, Bindseil et al. (2009) argue that the common value component is less important in a central bank repo auction than in a T-bill auction.

that corresponds to a given signal s_i . Given her signal s_i , each bidder in a Bayesian equilibrium chooses a demand function that maximizes her conditional profit (while taking as given the other traders' strategies). Our attention will be restricted to anonymous linear Bayesian equilibria in which strategies are linear and identical among traders of the same type (for short, equilibria).

Definition. An *equilibrium* is a linear Bayesian equilibrium such that the demand functions for traders of type i , $i = 1, 2$, are identical and equal to

$$X_i(s_i, p) = b_i + a_i s_i - c_i p,$$

where b_i , a_i , and c_i are constants.

3.1 Equilibrium characterization

Consider a trader of type i . If rival's strategies are linear and identical among traders of the same type and if the market clears, that is, if $(n_i - 1)X_i(s_i, p) + x_i + n_j X_j(s_j, p) = Q$, for $j = 1, 2$ and $j \neq i$, then this trader faces the residual inverse supply $p = I_i + d_i x_i$, where $I_i = ((n_i - 1)(b_i + a_i s_i) + n_j(b_j + a_j s_j) - Q) / ((n_i - 1)c_i + n_j c_j)$ and $d_i = 1 / ((n_i - 1)c_i + n_j c_j)$. The slope (d_i) is an index of the trader's market power.¹³ As a consequence, this trader's information set (s_i, p) is informationally equivalent to (s_i, I_i) . The bidder therefore chooses x_i to maximize

$$\mathbb{E}[\pi_i | s_i, p] = (\mathbb{E}[\theta_i | s_i, I_i] - I_i - d_i x_i) x_i - \lambda_i x_i^2 / 2.$$

The first-order condition (FOC) is given by $\mathbb{E}[\theta_i | s_i, I_i] - I_i - 2d_i x_i - \lambda_i x_i = 0$, or equivalently,

$$X_i(s_i, p) = (\mathbb{E}[\theta_i | s_i, p] - p) / (d_i + \lambda_i). \quad (1)$$

The second-order condition (SOC) that guarantees a maximum is $2d_i + \lambda_i > 0$, which implies that $d_i + \lambda_i > 0$. Using the expression for I_i and assuming that $a_j \neq 0$, we can show that (s_i, p) is informationally equivalent to (s_1, s_2) . Therefore, since $\mathbb{E}[\theta_i | s_i, p] = \mathbb{E}[\theta_i | s_i, I_i]$, it follows that

$$\mathbb{E}[\theta_i | s_i, p] = \mathbb{E}[\theta_i | s_1, s_2]. \quad (2)$$

According to Gaussian distribution theory,

$$\mathbb{E}[\theta_i | s_i, s_j] = \bar{\theta}_i + \Xi_i (s_i - \bar{\theta}_i) + \Psi_i (s_j - \bar{\theta}_j), \quad (3)$$

where

$$\Xi_i = \frac{1 - \rho^2 + \hat{\sigma}_{\varepsilon_j}^2}{(1 + \hat{\sigma}_{\varepsilon_i}^2)(1 + \hat{\sigma}_{\varepsilon_j}^2) - \rho^2} \text{ and } \Psi_i = \frac{\rho \hat{\sigma}_{\varepsilon_i}^2}{(1 + \hat{\sigma}_{\varepsilon_i}^2)(1 + \hat{\sigma}_{\varepsilon_j}^2) - \rho^2},$$

with $\hat{\sigma}_{\varepsilon_i}^2 = \sigma_{\varepsilon_i}^2 / \sigma_\theta^2$ and $\hat{\sigma}_{\varepsilon_j}^2 = \sigma_{\varepsilon_j}^2 / \sigma_\theta^2$. We remark that Equation (3) has the following implications.

¹³We assume that $(n_i - 1)c_i + n_j c_j \neq 0$.

1. The private signal s_i is useful for predicting θ_i whenever $1 - \rho^2 + \hat{\sigma}_{\varepsilon_j}^2 \neq 0$, that is, when either the liquidation values are not perfectly correlated ($\rho \neq 1$) or type- j traders are imperfectly informed about θ_j ($\sigma_{\varepsilon_j}^2 \neq 0$).
2. The private signal s_j is useful for predicting θ_i whenever $\rho \hat{\sigma}_{\varepsilon_i}^2 \neq 0$, that is, when the private liquidation values are correlated ($\rho \neq 0$) and type- i traders are imperfectly informed about θ_i ($\sigma_{\varepsilon_i}^2 \neq 0$).

Our first proposition summarizes the previous results. It shows the relationship between a_i and c_i in equilibrium and also indicates that these coefficients are positive.

Proposition 1. *Let $\rho < 1$. In equilibrium, the demand function of a trader of type i ($i = 1, 2$), $X_i(s_i, p) = b_i + a_i s_i - c_i p$, is given by $X_i(s_i, p) = (\mathbb{E}[\theta_i | s_i, p] - p) / (d_i + \lambda_i)$, with $d_i + \lambda_i > 0$, $d_i = 1 / ((n_i - 1) c_i + n_j c_j)$, and $a_i = \Delta_i c_i > 0$, for $\Delta_i = 1 / (1 + (1 + \rho)^{-1} \hat{\sigma}_{\varepsilon_i}^2)$. The coefficient c_i can be expressed as a function of the ratio $z = c_1 / c_2$, and z is the unique positive solution of a cubic polynomial $p(z) = 0$.*

The equilibrium demand function depends on $\mathbb{E}[\theta_i | s_i, p]$. As for the price coefficient (see Expression (11) in the Appendix), $c_i = (1 - \Psi_i(n_i c_i + n_j c_j)(n_j a_j)^{-1}) / (d_i + \lambda_i)$, note that the term $\Psi_i(n_i c_i + n_j c_j)(n_j a_j)^{-1}$ is the information-sensitivity weight of the price. Note also that, the more informative the price (higher $\Psi_i(n_i c_i + n_j c_j)(n_j a_j)^{-1}$), the lower the price coefficient (lower c_i). Furthermore, this term vanishes when $\Psi_i = 0$, that is, when either the valuations are uncorrelated ($\rho = 0$) or the private signal s_i is perfectly informative ($\sigma_{\varepsilon_i}^2 = 0$) since in those cases the price conveys no additional information to a trader of type i .

Since $a_i > 0$ and $c_i > 0$, for $i = 1, 2$, it follows that in equilibrium the higher the value of the trader's observed private signal (or the lower the price), the higher the quantity she will demand. When $\sigma_{\varepsilon_i}^2 > 0$ we have $a_i < c_i$, since $\Delta_i < 1$ in this case; when $\sigma_{\varepsilon_i}^2 = 0$, we have $\Delta_i = 1$ and $a_i = c_i$. Observe that we can write the demand as $X_i(s_i, p) = b_i + c_i (\Delta_i s_i - p)$.

Because p is a linear function of s_1 and s_2 , for $i = 1, 2$ we have $\mathbb{E}[\theta_i | s_i, p] = \mathbb{E}[\theta_i | s_1, s_2]$ (i.e., Equation (2) holds). The equilibrium price is therefore *privately revealing*, in other words, the private signal and the price enable a type- i trader to learn as much as about θ_i if she had access to all the information available in the market, (s_1, s_2) .

If $\rho = 0$ or if both signals are perfectly informative ($\sigma_{\varepsilon_i}^2 = 0$, $i = 1, 2$), then bidders do not learn about θ_i from prices. Hence, $\mathbb{E}[\theta_i | s_i] = \mathbb{E}[\theta_i | s_i, p] = \mathbb{E}[\theta_i | s_1, s_2]$ for $i = 1, 2$. The demand functions are given by

$$X_i(s_i, p) = (\mathbb{E}[\theta_i | s_i] - p) / (d_i + \lambda_i), \quad i = 1, 2.$$

Hence, $c_i = 1/(d_i + \lambda_i)$ and so, given our expression for d_i , we have

$$d_i = 1 / ((n_i - 1) / (d_i + \lambda_i) + n_j / (d_j + \lambda_j)) \text{ for } i, j = 1, 2 \text{ and } j \neq i.$$

We can show that, this system has a unique solution satisfying the inequality $d_i + \lambda_i > 0$, $i = 1, 2$, iff $n_1 + n_2 \geq 3$. In this case, the equilibrium coincides with the full-information equilibrium (denoted by superscript f).¹⁴ Furthermore, when $\rho = 0$, market power d_i is independent of $\sigma_{\varepsilon_i}^2$, $i = 1, 2$; and when $\sigma_{\varepsilon_i}^2 = 0$, for $i = 1, 2$, market power is independent of ρ .

Our next proposition describes the condition under which an equilibrium exists and shows that, if an equilibrium does exist, then it is unique.

Proposition 2. *There exists a unique equilibrium if and only if $\bar{z}_N > \bar{z}_D$, where \bar{z}_N and \bar{z}_D denote the highest root of, respectively, $q_N(z)$ and $q_D(z)$, with*

$$\begin{aligned} q_N(z) &= n_2^2 \Xi_1 \Delta_1^{-1} + n_2 (\Xi_1 \Delta_1^{-1} (2n_1 - 1) - (n_1 + 1)) z - (n_1 - 1) (1 - \Xi_1 \Delta_1^{-1}) n_1 z^2 \text{ and} \\ q_D(z) &= -n_2 (n_2 - 1) (1 - \Xi_2 \Delta_2^{-1}) + n_1 (\Xi_2 \Delta_2^{-1} (2n_2 - 1) - (n_2 + 1)) z + n_1^2 \Xi_2 \Delta_2^{-1} z^2. \end{aligned}$$

Let $z = c_1/c_2$. Then, in equilibrium, $\bar{z}_D < z < \bar{z}_N$, $\lim_{\lambda_1 \rightarrow 0} z = \bar{z}_N$ and $\lim_{\lambda_2 \rightarrow 0} z = \bar{z}_D$.

For an equilibrium to exist we must have $c_i > 0$ ($i = 1, 2$) and these inequalities hold if and only if (iff) $\bar{z}_D < z < \bar{z}_N$. No equilibrium exists for ρ close to 1 or for low $n_1 + n_2$. Neither does an equilibrium exist when $\rho = 1$. If the price reveals a sufficient statistic for the common valuation, then no trader has an incentive to place any weight on her signal. But if traders put no weight on signals, then the price contains no information about the common valuation. This conundrum is related to the Grossman-Stiglitz (1980) paradox.

Remark 1. If $n_1 = 1$ and $n_2 = 1$, then $\bar{z}_N = 1/(2\Delta_1\Xi_1^{-1} - 1)$ and $\bar{z}_D = 2\Delta_2\Xi_2^{-1} - 1$. Since $\Delta_i\Xi_i^{-1} > 1$, $i = 1, 2$, we can use direct computation to obtain $\bar{z}_N < \bar{z}_D$. Applying Proposition 2, we conclude that no equilibrium exists in this case. Therefore, the inequality $n_1 + n_2 \geq 3$ is a necessary condition for the existence of an equilibrium in our model. This result is in line with Kyle (1989) and Vives (2011).¹⁵

To develop a better understanding of the equilibrium and the condition that guarantees its existence, we consider two particular cases of the model: a monopsony competing with a fringe; and symmetric groups.

¹⁴In the full (shared) information setup, traders can access (s_1, s_2) . In this framework the price does not provide any useful information.

¹⁵Du and Zhu (2016) consider ex post nonlinear equilibria in a bilateral divisible double auction and show that with more than three symmetric traders there are no nonlinear equilibria in the class of smooth demands downward sloping in price and upward sloping in signals.

Monopsony competing with fringe

Corollary 1. *For $n_2 = 1$ the equilibrium exists if $1 - \rho^2 > (2\rho - 1)\hat{\sigma}_{\varepsilon_1}^2$ and $n_1 > \bar{n}_1(\rho, \hat{\sigma}_{\varepsilon_1}^2, \hat{\sigma}_{\varepsilon_2}^2)$, where \bar{n}_1 increases with ρ , $\hat{\sigma}_{\varepsilon_1}^2$, and $\hat{\sigma}_{\varepsilon_2}^2$. If, also, $\lambda_2 = 0$ and $\sigma_{\varepsilon_2}^2 = 0$, then $\bar{n}_1(\rho, \hat{\sigma}_{\varepsilon_1}^2, 0) = 1 + \rho\hat{\sigma}_{\varepsilon_1}^2 / (1 - \rho^2 - (2\rho - 1)\hat{\sigma}_{\varepsilon_1}^2)$ and $x_2 = c_2(\theta_2 - p)$, with $c_2 = n_1 c_1$.*

An equilibrium with linear demand functions exists provided there is a sufficiently competitive trading environment (n_1 high enough). In the particular case where $\lambda_2 = 0$ and $\sigma_{\varepsilon_2}^2 = 0$, expressions for the equilibrium coefficients can be characterized explicitly (see the Appendix). From the expressions for c_i ($i = 1, 2$) it follows that, if $n_1 = \bar{n}_1$, then the equilibrium cannot exist because in this case the demand functions would be completely inelastic ($c_i = 0$, $i = 1, 2$).

Symmetric groups

Consider the following symmetric case: $n_2 = n_1 = n$, $\lambda_1 = \lambda_2 = \lambda$, and $\sigma_{\varepsilon_1}^2 = \sigma_{\varepsilon_2}^2 = \sigma_{\varepsilon}^2$. Here $z = 1$ in equilibrium. From Proposition 2 we know that, if an equilibrium exists, then the value of z is in the interval (\bar{z}_D, \bar{z}_N) . It follows that $\bar{z}_N > 1 > \bar{z}_D$ or, equivalently, that $q_N(1) > 0$ and $q_D(1) > 0$. After performing some algebra, we find that the foregoing inequalities are satisfied iff $n > 1 + \rho\hat{\sigma}_{\varepsilon}^2 / ((1 - \rho)(1 + \rho + \hat{\sigma}_{\varepsilon}^2))$, where $\hat{\sigma}_{\varepsilon}^2 = \sigma_{\varepsilon}^2 / \sigma_{\theta}^2$. Therefore, the equilibrium's existence is guaranteed provided either that n is high enough or that ρ or $\hat{\sigma}_{\varepsilon}^2$ is low enough.

Vives (2011) also analyzes divisible good auctions with symmetric bidders, but in his model the bidders receive different private signals. The condition that guarantees existence of an equilibrium in Vives' setup is $2n > 2 + M$, where $M = 2n\rho\hat{\sigma}_{\varepsilon}^2 / ((1 - \rho)(1 + (2n - 1)\rho + \hat{\sigma}_{\varepsilon}^2))$. Direct computation yields that the condition derived in the model of Vives is more stringent than the condition derived in our setup. The reason is that, in Vives (2011), the degree of asymmetry in information (and induced market power) is greater because each of the $2n$ traders receives a private signal.

The rest of this subsection is devoted to describing some properties that satisfy the equilibrium coefficients and then to comparing the equilibrium quantities.

Informed and uninformed groups

Consider the case where group 1 is uninformed (with no signal) while group 2 is informed ($\sigma_{\varepsilon_2}^2 < \infty$). This is akin to the information structure considered by Grossman and Stiglitz (1980). Then equilibrium exists if and only if $\rho < \frac{n_2(n_1+n_2-2)}{2n_1+n_2-2}$. We have that $a_2 = \left(1 + \frac{\sigma_{\varepsilon_2}^2}{\sigma_{\theta}^2}\right)^{-1} c_2$. This expression is different from the case in which the group 1 is informed since then $a_2 = \left(1 + \frac{\sigma_{\varepsilon_2}^2}{(1+\rho)\sigma_{\theta}^2}\right)^{-1} c_2$. Therefore, the equilibrium that we obtain here is not obtained as a limit of the general case when $\sigma_{\varepsilon_1}^2$ converges to infinity, except in the case in which $\rho = 0$. This is so,

since when $\rho > 0$, no matter how large $\sigma_{\varepsilon_1}^2$ is, but finite, there is learning from the price by the two groups. By contrast, when group 1 does not receive any private signal, there is no learning from the price by group 2. Furthermore, in this last case, we have that c_1 and c_2 (d_1 and d_2) are independent of $\sigma_{\varepsilon_2}^2$, and b_2 is independent of $\bar{\theta}_1$ and Q .

In the case where group receives a noisy signal ($\sigma_{\varepsilon_1}^2 < \infty$) and group 2 is perfectly informed ($\sigma_{\varepsilon_2}^2 = 0$), then $b_2 = 0$ and $a_1 = c_1$.

Comparative statics

We start by considering how the model's underlying parameters affect the equilibrium and, in particular, market power (Proposition 3). We then explore how the equilibrium is affected when there are two distinct groups of traders, that is, a strong group and a weak group (Corollary 2).

Proposition 3. *Let $\rho\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2 > 0$. Then, for $i = 1, 2, i \neq j$, the following statements hold.*

- (i) *An increase in $\bar{\theta}_i$ or Q , or a decrease in $\bar{\theta}_j$, raises the demand intercept b_i .*
- (ii) *An increase in $\lambda_i, \lambda_j, \sigma_{\varepsilon_i}^2, \sigma_{\varepsilon_j}^2$, or ρ makes demand less responsive to private signals and prices (lower a_i and c_i) and increases market power (d_i).*
- (iii) *If $\sigma_{\varepsilon_i}^2$ and/or λ_i increase, then d_i/d_j decreases.*
- (iv) *If n_i and/or n_j increase, then d_i decreases.*

Remark 2. If $\rho = 0$, then: (a) both c_i and d_i (as well as c_j and $a_j, j \neq i$) are independent of $\sigma_{\varepsilon_i}^2$; (b) a_i decreases with $\sigma_{\varepsilon_i}^2$; and (c) b_i is independent of both Q and $\bar{\theta}_j$. If $\sigma_{\varepsilon_i}^2 = 0$ for $i = 1, 2$, then $b_i = 0$, and c_i, c_j, a_i, a_j, d_i , and $d_j, i = 1, 2, j \neq i$, are independent of ρ . That is, for the information parameters to matter for market power, it is necessary that prices convey information. And given that the equilibrium values of d_1 and d_2 when $\rho = 0$ (or when $\sigma_{\varepsilon_i}^2 = 0, i = 1, 2$) are equal to those corresponding to the full-information setup, Proposition 3(ii) implies that, if $\rho\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2 > 0$, then $d_i^f < d_i, i = 1, 2$. Thus, in this case asymmetric information increases the market power of traders in both groups beyond the full-information level.

By Lemma A1, the only equilibrium coefficient affected by the quantity offered in the auction (Q) and by the prior mean of the valuations ($\bar{\theta}_i$ and $\bar{\theta}_j$) is the coefficient b_i . Proposition 3(i) indicates that if Q increases, then all the bidders will increase their demand (higher b_1 and b_2). Moreover, if the prior mean of the valuation of group i increases, then the bidders in this group will demand a greater quantity of the risky asset (higher b_i). Then the intercept of the inverse residual supply for the group j bidder rises in response to a higher $\bar{\theta}_i$. That reaction leads the traders in group j to reduce their demand for the risky asset (lower b_j).

Part (ii) of Proposition 3 shows how the response to private information and price varies with several parameters. If the transaction costs for a bidder increase, then that bidder is less interested in the risky asset and so a_i and c_i are each decreasing in λ_i . Moreover, any increase in

a group's transaction costs also affects the behavior of traders in the other group. If λ_i increases, then c_i decreases, in which case the slope of the inverse residual supply for group j increases (higher d_j). This change leads group- j traders to reduce their demand sensitivity to signals and prices (lower a_j and c_j). We can therefore see how an increase in the transaction costs for group- i traders (say, a deterioration of their collateral in liquidity auctions that raises λ_i) leads not only to steeper demands for bidders in group i but also, as a reaction, to steeper demands for group- j traders. Figure 1 illustrates the case of initially identical groups that become differentiated after a shock induces a higher λ_1 and also raises group's willingness to pay for liquidity as in a crisis situation (both $\bar{\theta}_1$ and $\bar{\theta}_2$, which affect the intercepts of the demand functions).

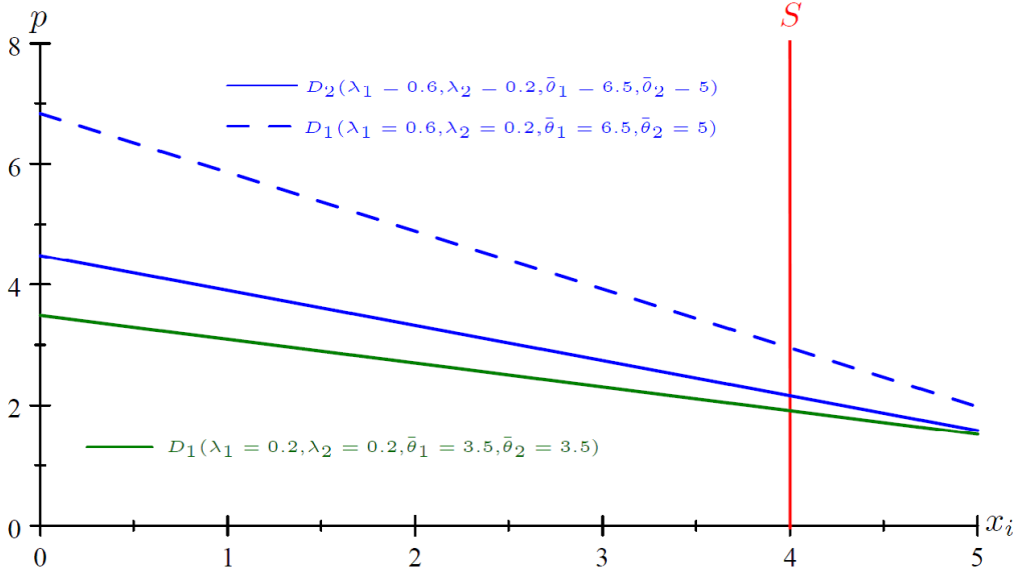


Figure 1: Equilibrium demand functions for $\rho = 0.75$, $\sigma_\theta^2 = 5$, $Q = 4$, $n_i = 5$, $\sigma_{\varepsilon_i}^2 = 1$, and $s_i = \bar{\theta}_i$, $i = 1, 2$.

We also analyze how the response to private information and price varies with a change in the precision of private signals. If the private signal of type- i bidders is less precise (higher $\sigma_{\varepsilon_i}^2$), then their demand is less sensitive to private information and prices. Thus a trader finds it optimal to rely less on her private information when her private signal is less precise. A private signal of reduced precision also gives the type- i bidder more incentive to consider prices when predicting θ_i , which leads in turn to this bidder having a steeper demand function (lower c_i). The same can be said for a bidder of type j because of strategic complementarity in the slopes of demand functions.¹⁶

¹⁶This result (in the supply competition model) may help explain why, in the Texas balancing market, small firms use steeper supply functions than predicted by theory (Hortaçsu and Puller 2008). Indeed, smaller firms may receive lower-quality signals owing to economies of scale in information gathering.

We also find that the more highly the valuations are correlated (higher ρ), the less is trader responsiveness to private signals (lower a_i , $i = 1, 2$) and the steeper are inverse demand functions (lower c_i , $i = 1, 2$). We can explain these results by recalling that, when the valuations are correlated ($\rho > 0$), a type- i trader learns about θ_i from prices. In fact, the price is more informative about θ_i when ρ is larger, in which case demand is less sensitive to private information. The rationale for the relationship between the correlation coefficient (ρ) and the slopes of demand functions is as follows. An increment in the price of the risky asset makes an agent more optimistic about her valuation, which leads to less of a reduction in demand quantity than in the case of uncorrelated valuations.¹⁷

Proposition 3(iii) states that any increase in the signal's noise or in group i 's transaction costs has the effect of reducing its relative market power, since then the ratio d_i/d_j ($i \neq j$) decreases. Finally, part (iv) formalizes the anticipated result that an increase in the number of auction participants (higher n_i or n_j) reduces the market power of traders in both groups.

Corollary 2. *Suppose that $\sigma_{\varepsilon_1}^2 \geq \sigma_{\varepsilon_2}^2$, $\lambda_1 \geq \lambda_2$, and $n_1 \geq n_2$, and suppose that at least one of these inequalities is strict. Then, in equilibrium, the following statements hold.*

- (i) *The stronger group (here, group 2) reacts more both to private information and to prices ($a_1 < a_2$, $c_1 < c_2$) and has more market power ($d_1 < d_2$) than does the weaker group.*
- (ii) *The value of the difference $d_1 + \lambda_1 - (d_2 + \lambda_2)$ is, in general, ambiguous. If*

$$\frac{(1 - \rho) n_1 n_2 (1 + \rho + \hat{\sigma}_{\varepsilon_1}^2)}{n_2 (1 - \rho^2 + \hat{\sigma}_{\varepsilon_1}^2) + n_1 \rho \hat{\sigma}_{\varepsilon_1}^2} + \frac{(1 - \rho) n_1 (n_2 - 1) (1 + \rho + \hat{\sigma}_{\varepsilon_2}^2)}{n_1 (1 - \rho^2 + \hat{\sigma}_{\varepsilon_2}^2) + n_2 \rho \hat{\sigma}_{\varepsilon_2}^2} \leq 1, \quad (4)$$

then $d_1 + \lambda_1 < d_2 + \lambda_2$ always holds. Otherwise, $d_1 + \lambda_1 > d_2 + \lambda_2$ iff λ_1/λ_2 is high enough.

Part (i) of this corollary shows that if a group of traders is less informed, has higher transaction costs, and is more numerous, then it reacts less both to private signals and to prices. Observe in particular that group-1 traders, having less precise private information, rely more on the price for information (higher $\Psi_1 (n_1 c_1 + n_2 c_2) (n_2 a_2)^{-1}$); as a result, their overall price response ($c_1 = (1 - \Psi_1 (n_1 c_1 + n_2 c_2) (n_2 a_2)^{-1}) / (d_1 + \lambda_1)$) is smaller. Similarly, group-1 traders, for whom n_1 is larger, put more information weight on the price (which depends more strongly on s_1).

Corollary 2(ii) is useful for comparing allocations across groups. It indicates that the inequality $d_1 + \lambda_1 > d_2 + \lambda_2$ holds whenever (a) the differences between groups stem mainly from transaction costs;¹⁸ and (b) λ_1/λ_2 is high enough. If signals are perfect ($\sigma_{\varepsilon_i}^2 = 0$, $i = 1, 2$) or if

¹⁷A high price conveys the good news that the private signal received by other group's traders is high. When valuations are positively correlated, a bidder infers from the high private signal of the other group that her own valuation is high.

¹⁸This claim follows because if $n_1 = n_2$ and $\hat{\sigma}_{\varepsilon_1}^2 = \hat{\sigma}_{\varepsilon_2}^2$, then the inequality given in (4) does not hold.

$\rho = 0$, then part (i) holds and $d_1 + \lambda_1 > d_2 + \lambda_2$ iff $\lambda_1 > \lambda_2$.

Equilibrium quantities

Finally, we examine the equilibrium quantities. Let $t_i = \mathbb{E}[\theta_i | s_1, s_2]$, $i = 1, 2$, be the predicted values with full information (s_1, s_2) . After some algebra, it follows that equilibrium quantities are functions of the vector of predicted values $t = (t_1, t_2)$:

$$x_i(t) = \underbrace{\frac{n_j(t_i - t_j)}{n_i(d_j + \lambda_j) + n_j(d_i + \lambda_i)}}_{x_i^I(t)} + \underbrace{\frac{d_j + \lambda_j}{n_i(d_j + \lambda_j) + n_j(d_i + \lambda_i)}}_{x_i^C(t)} Q, \quad i = 1, 2, j \neq i. \quad (5)$$

Observe that, according to these expressions, the equilibrium quantities can be decomposed into two terms: a *valuation* trading term and a *clearing* trading term, which we denote by (respectively) $x_i^I(t)$ and $x_i^C(t)$ for group i , $i = 1, 2$. With regard to the information trading term, it vanishes when $t_1 = t_2$, but has a positive (resp. negative) value for the group with the higher (resp. lower) value of t_i . Moreover, $n_1 x_1^I(t) + n_2 x_2^I(t) = 0$. As for the clearing trading term, we remark that it vanishes when $Q = 0$; otherwise, it is positive for both groups yet lower (resp. higher) for the group with higher (resp. lower) $d_i + \lambda_i$. In addition, $n_1 x_1^C(t) + n_2 x_2^C(t) = Q$.

Taking expectations in Equation (5), we have

$$\mathbb{E}[x_1(t)] - \mathbb{E}[x_2(t)] = \frac{n_1 + n_2}{n_1(d_2 + \lambda_2) + n_2(d_1 + \lambda_1)} (\bar{\theta}_1 - \bar{\theta}_2) + \frac{d_2 + \lambda_2 - (d_1 + \lambda_1)}{n_1(d_2 + \lambda_2) + n_2(d_1 + \lambda_1)} Q.$$

Group 1 trades more when it values the asset more highly ($\bar{\theta}_1 > \bar{\theta}_2$) and when its traders are less cautious ($d_2 + \lambda_2 > d_1 + \lambda_1$) than group 2. By combining Corollary 2 with the equation just displayed, we obtain the following remarks.

Remark 3. If Q is low enough, then $\mathbb{E}[x_1(t)] > \mathbb{E}[x_2(t)]$ whenever $\bar{\theta}_1 > \bar{\theta}_2$. In contrast, if Q is high enough, then $\mathbb{E}[x_1(t)] > \mathbb{E}[x_2(t)]$ whenever $d_2 + \lambda_2 > d_1 + \lambda_1$. Under the assumptions of Corollary 2, this latter inequality is satisfied if (4) holds or if λ_1/λ_2 is sufficiently low.

Remark 4. When $Q = 0$ (i.e., the so-called double auction case), then $\mathbb{E}[x_2(t)] < 0 < \mathbb{E}[x_1(t)]$ iff $\bar{\theta}_1 > \bar{\theta}_2$. Then group 1 consists of buyers and group 2 of sellers.

3.2 Bid shading, expected discount, and expected revenue

Our aim here is to identify factors that affect the magnitudes of bid shading, expected discount, and expected revenue. Let $\tilde{t} = (n_1 t_1 + n_2 t_2) / (n_1 + n_2)$. From the demand of bidders it follows that $p(t) = t_i - (d_i + \lambda_i) x_i(t)$, $i = 1, 2$. Therefore,

$$p(t) = \tilde{t} - ((d_1 + \lambda_1) n_1 x_1(t) + (d_2 + \lambda_2) n_2 x_2(t)) / (n_1 + n_2). \quad (6)$$

Bid shading

For a trader of type i , the expected marginal benefit of buying x_i units of the asset is $t_i - \lambda_i x_i$. Hence, the average marginal benefit is given by $\tilde{t} - (\lambda_1 n_1 x_1 + \lambda_2 n_2 x_2) / (n_1 + n_2)$. The magnitude of bid shading is the difference between the average marginal valuation and the auction price, that is, $(d_1 n_1 x_1 + d_2 n_2 x_2) / (n_1 + n_2)$. We can use Equation (5) to write bid shading as

$$\frac{n_2 d_2 (d_1 + \lambda_1) + n_1 d_1 (d_2 + \lambda_2)}{(n_1 + n_2) (n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))} Q + \frac{(t_2 - t_1) (d_2 - d_1) n_2 n_1}{(n_1 + n_2) (n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))}. \quad (7)$$

At this juncture, some additional remarks are in order.

- Bid shading increases with Q .
- When $d_1 = d_2 = d$ as in the symmetric case, for instance, bid shading consists of only one term (the first one) and it is equal to $dQ / (n_1 + n_2)$.
- When $d_1 \neq d_2$, the second term of (7) is negative and bid shading decreases whenever the group that values the asset more highly ($t_i > t_j$) has less market power ($d_i < d_j$).
- If group 1 has higher transaction costs ($\lambda_1 > \lambda_2$), is more numerous ($n_1 > n_2$), and is less informed ($\sigma_{\varepsilon_1}^2 > \sigma_{\varepsilon_2}^2$) than group 2, then $c_1 < c_2$, and so $d_1 < d_2$. If $t_1 > t_2$, then the second term of (7) is negative and the two terms have opposite signs. Therefore, if Q is low (e.g., $Q = 0$) or if the difference in predicted values of the asset is high, then *negative* bid shading obtains.

Expected discount

The expected discount is defined as $\mathbb{E}[\tilde{t}] - \mathbb{E}[p(t)]$. We can use Equation (6) to write the expected discount as $((d_1 + \lambda_1)n_1 \mathbb{E}[x_1(t)] + (d_2 + \lambda_2)n_2 \mathbb{E}[x_2(t)]) / (n_1 + n_2)$. Now some algebra yields the following expression for the expected discount:

$$\frac{(d_1 + \lambda_1) (d_2 + \lambda_2)}{n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1)} Q + \frac{n_1 n_2 (d_2 + \lambda_2 - d_1 - \lambda_1) (\bar{\theta}_2 - \bar{\theta}_1)}{(n_1 + n_2) (n_1 (d_2 + \lambda_2) + n_2 (d_1 + \lambda_1))}. \quad (8)$$

Here our related comments are as follows.

- When $d_1 + \lambda_1 = d_2 + \lambda_2 = d + \lambda$ (as in the symmetric case), the expected discount is $(d + \lambda)Q / (n_1 + n_2)$.

- The first term is always positive provided $Q > 0$, whereas the second term is positive whenever $(d_2 + \lambda_2 - d_1 - \lambda_1)(\bar{\theta}_2 - \bar{\theta}_1) > 0$. Therefore, the expected discount is lower whenever the group that values the asset more highly ($\bar{\theta}_2 > \bar{\theta}_1$) has a lower "total transaction cost" ($d_2 + \lambda_2 < d_1 + \lambda_1$).
- If group 1 ex ante values the asset more ($\bar{\theta}_1 > \bar{\theta}_2$), has higher transaction costs ($\lambda_1 > \lambda_2$), is more numerous ($n_1 > n_2$), and is less informed ($\sigma_{\varepsilon_1}^2 > \sigma_{\varepsilon_2}^2$), then Corollary 2 shows that $d_1 + \lambda_1 > d_2 + \lambda_2$ whenever (a) the differences between groups are due mostly to transaction costs and (b) λ_1/λ_2 is high enough. In this case, both terms are positive and so the expected discount is positive. Yet, if both groups have similar transactions costs, then the two terms in (8) have opposite signs. In particular, we expect a negative discount when Q is low.

Expected revenue

The expected price is given by

$$\mathbb{E}[p] = \left(\frac{n_1}{d_1 + \lambda_1} \bar{\theta}_1 + \frac{n_2}{d_2 + \lambda_2} \bar{\theta}_2 - Q \right) / \left(\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2} \right).$$

It is worth noting that, in the double auction case ($Q = 0$), $\mathbb{E}[p]$ is a convex combination of $\bar{\theta}_1$ and $\bar{\theta}_2$. Also, for symmetric groups (except possibly with respect to the means) we have $\mathbb{E}[p] = (\bar{\theta}_1 + \bar{\theta}_2)/2$.

The seller's expected revenue is $E[p]Q$ and, provided that he has enough supplies at no cost, the revenue-maximizing supply is given by $Q^* = \frac{1}{2} \left(\frac{n_1}{d_1 + \lambda_1} \bar{\theta}_1 + \frac{n_2}{d_2 + \lambda_2} \bar{\theta}_2 \right)$. That supply Q^* is increasing in the expected valuations and in the number of traders; it is decreasing in those traders' market power and transaction costs.

Proposition 4. *Let $\rho \sigma_{\varepsilon_1}^2 \sigma_{\varepsilon_2}^2 > 0$. Then, in equilibrium, the following statements hold.*

(i) *If $\bar{\theta}_1 = \bar{\theta}_2$, then the expected price is increasing in n_i but is decreasing in λ_i , $\sigma_{\varepsilon_i}^2$, and ρ , $i = 1, 2$. Otherwise, if $|\bar{\theta}_1 - \bar{\theta}_2|$ is large enough, then these results need not hold.*

(ii) *The expected revenue:*

- *increases with $\bar{\theta}_i$ for $i = 1, 2$, and increases with Q for $E[p] > 0$;*

- *is between (a) the larger expected revenue of the auction in which both groups are ex ante identical with a large number of bidders (each group with $\max\{n_1, n_2\}$), high expected valuation ($\max\{\bar{\theta}_1, \bar{\theta}_2\}$), low transaction costs ($\min\{\lambda_1, \lambda_2\}$) and precise signals ($\min\{\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2\}$) and (b) the smaller expected revenue of the auction in which both groups are ex ante identical but with the opposite characteristics (i.e., $\min\{n_1, n_2\}$, $\min\{\bar{\theta}_1, \bar{\theta}_2\}$, $\max\{\lambda_1, \lambda_2\}$, and $\max\{\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2\}$).*

Remark 5. If $\rho = 0$, then $\mathbb{E}[p]$ is independent of $\sigma_{\varepsilon_i}^2$ ($i = 1, 2$), and if $\sigma_{\varepsilon_i}^2 = 0$, $i = 1, 2$, then $\mathbb{E}[p]$ is independent of ρ . The reason is that in both cases, d_i is independent of $\sigma_{\varepsilon_i}^2$ and ρ . If $\sigma_{\varepsilon_1}^2 = \infty$ then $\mathbb{E}[p]$ is independent of $\sigma_{\varepsilon_2}^2$.

Proposition 4 indicates that the relationship between expected price (on the one hand) and λ_i , $\sigma_{\varepsilon_i}^2$, and ρ , $i = 1, 2$ (on the other hand) is potentially ambiguous. For example, if $\bar{\theta}_2 - \bar{\theta}_1$ is high enough, then $\mathbb{E}[p]$ is decreasing in n_1 ; yet, if $\bar{\theta}_1 = \bar{\theta}_2$, then the derived results are in line with those in the symmetric case, where $\mathbb{E}[p] = \bar{\theta} - (d + \lambda)Q/2n$ (see Vives 2010, Prop. 2).

We should like to understand how ex ante differences among bidders affect the seller's expected revenue. Suppose that group 2 is our strong group; it has lower transaction costs ($\lambda_2 < \lambda_1$), is less numerous ($n_2 < n_1$), and is better informed ($\sigma_{\varepsilon_2}^2 < \sigma_{\varepsilon_1}^2$). If this group values the asset less, $\bar{\theta}_2 < \bar{\theta}_1$ (resp., values it more, $\bar{\theta}_2 > \bar{\theta}_1$), then expected revenue is lower (resp., higher) than in the case where $\bar{\theta}_1 = \bar{\theta}_2$. If $\bar{\theta}_1 \approx \bar{\theta}_2$, then Proposition 4(i) suggests that group 2's relatively small size ($n_2 < n_1$) reduces the seller's expected revenue, although both its relatively low transaction costs ($\lambda_2 < \lambda_1$) and its relatively precise signals ($\sigma_{\varepsilon_2}^2 < \sigma_{\varepsilon_1}^2$) have the opposite effect. So in general, the ex ante differences between the two groups have an ambiguous effect on the seller's expected revenue. Nonetheless, part (ii) of Proposition 4 directly follows from part (i).

4 Large markets

Our objective in this section is to determine whether (or not) the equilibrium under imperfect competition converges to a price-taking equilibrium in the limit as the number of traders becomes large. We examine two possible scenarios: in the first, only group 1 is large; in the second, both groups of bidders are large. The *per capita* supply (denoted by q) is assumed to be inelastic, that is, $Q = (n_1 + n_2)q$.

4.1 Oligopsony with competitive fringe

Proposition 5. *Let $\rho\sigma_{\varepsilon_1}^2 > 0$. Suppose that $n_1 \rightarrow \infty$ and $n_2 < \infty$. Then an equilibrium exists iff $n_2 > \bar{n}_2(\rho, \hat{\sigma}_{\varepsilon_1}^2, \hat{\sigma}_{\varepsilon_2}^2)$, where \bar{n}_2 is increasing in ρ and $\hat{\sigma}_{\varepsilon_1}^2$ and where \bar{n}_2 is decreasing in $\hat{\sigma}_{\varepsilon_2}^2$ whenever $(2\rho - 1)\hat{\sigma}_{\varepsilon_1}^2 < 1 - \rho^2$. An agent in the large group absorbs the inelastic per capita supply in the limit ($\lim_{n_1 \rightarrow \infty} b_1 = q$, $\lim_{n_1 \rightarrow \infty} a_1 = \lim_{n_1 \rightarrow \infty} c_1 = 0$) and retains some market power ($\lim_{n_1 \rightarrow \infty} d_1 > 0$), while an agent in the small group commands a higher degree of market power ($\lim_{n_1 \rightarrow \infty} d_2 > \lim_{n_1 \rightarrow \infty} d_1$).*

When $n_2 = 1$, the existence condition stated in Proposition 5 boils down to $(2\rho - 1)\hat{\sigma}_{\varepsilon_1}^2 <$

$1 - \rho^2$ from Corollary 1. Equation (33) shows that, when $n_2 = \bar{n}_2(\rho, \hat{\sigma}_{\varepsilon_1}^2, \hat{\sigma}_{\varepsilon_2}^2)$, the demand functions for bidders in group 2 would be completely inelastic $\left(\lim_{n_1 \rightarrow \infty} c_2 = 0\right)$. This explains why the inequality $n_2 > \bar{n}_2(\rho, \hat{\sigma}_{\varepsilon_1}^2, \hat{\sigma}_{\varepsilon_2}^2)$ is required for the existence of equilibrium. Neither group 1 nor group 2 has flat *aggregate* demand in the limit, and each group has some market power. We see that an agent in the large group just absorbs the inelastic per capita supply, behaving like a "Cournot quantity setter", and keeping some market power ($\lim_{n_1 \rightarrow \infty} d_1 > 0$), while bidders in the small group command relatively more market power ($\lim_{n_1 \rightarrow \infty} d_2 > \lim_{n_1 \rightarrow \infty} d_1$). It is worth to remark that the large group retains market power in the limit only if this group learns from the price (incomplete information and correlation of values, $\rho\sigma_{\varepsilon_1}^2 > 0$). In this case the aggregate demand of group 1 does not become flat, $\lim_{n_1 \rightarrow \infty} n_1 c_1 < \infty$. If $\rho\sigma_{\varepsilon_1}^2 = 0$, $\lim_{n_1 \rightarrow \infty} n_1 c_1 = \infty$ and there is no market power in the limit: $\lim_{n_1 \rightarrow \infty} d_i = 0$, $i = 1, 2$. It is easy to see also that, in the limit, the price depends only on the valuations and market power of agents in the competitive fringe: $\lim_{n_1 \rightarrow \infty} p = \mathbb{E}[\theta_1 | s_1, s_2] - \left(\lim_{n_1 \rightarrow \infty} d_1 + \lambda_1\right) q$ and that the limit expected quantity of a bidder of group 2 is given by $\lim_{n_1 \rightarrow \infty} \mathbb{E}[x_2(t)] = \frac{\bar{\theta}_2 - \bar{\theta}_1 + \left(\lim_{n_1 \rightarrow \infty} d_1 + \lambda_1\right) q}{\lim_{n_1 \rightarrow \infty} d_2 + \lambda_2}$.

If the small group is fully informed ($\sigma_{\varepsilon_2}^2 = 0$) and the large group is entirely uninformed ($\sigma_{\varepsilon_1}^2 \rightarrow \infty$), then: $\bar{n}_2 = 2\rho$; an equilibrium always exists for $n_2 > 2$; and the equilibrium coefficients for group 2 are $\lim_{n_1 \rightarrow \infty} b_2 = 0$, and $\lim_{n_1 \rightarrow \infty} a_2 = \lim_{n_1 \rightarrow \infty} c_2 = \frac{n_2 - 2\rho}{(n_2 - \rho)\lambda_2}$. In this case, the groups' relative market power is given by $\lim_{n_1 \rightarrow \infty} (d_2/d_1) = 1 + \frac{\rho}{n_2 - \rho}$. As ρ increases the relative market power of group 2 increases also.

4.2 A large price-taking market

Consider now the following setup. There is a continuum of bidders along the interval $[0, 1]$, and we let q denote the aggregate (average) quantity supplied in the market. Suppose that a fraction μ_i ($0 < \mu_i < 1$) of these bidders are traders of type i , $i = 1, 2$. Then the following proposition characterizes the equilibrium of this continuum economy and shows that it is the limit of a finite economy's equilibrium.

Proposition 6. *Let $Q = (n_1 + n_2)q$. Suppose that n_1 and n_2 both approach to infinity and that $n_i/(n_1 + n_2)$ converges to μ_i ($0 < \mu_i < 1$) for $i = 1, 2$. Then, the equilibrium coefficients converge to the equilibrium coefficients of the equilibrium of the continuum economy setup, which*

are given by

$$b_i = \frac{\hat{\sigma}_{\varepsilon_i}^2 (\rho \lambda_j q + \mu_j (\bar{\theta}_i - \rho \bar{\theta}_j))}{\mu_i \rho \lambda_j \hat{\sigma}_{\varepsilon_i}^2 + \mu_j \lambda_i (1 - \rho^2 + \hat{\sigma}_{\varepsilon_i}^2)}, \quad a_i = \frac{\mu_j (1 - \rho^2)}{\mu_i \rho \lambda_j \hat{\sigma}_{\varepsilon_i}^2 + \mu_j \lambda_i (1 - \rho^2 + \hat{\sigma}_{\varepsilon_i}^2)}, \quad \text{and}$$

$$c_i = \frac{\mu_j (1 - \rho) (1 + \rho + \hat{\sigma}_{\varepsilon_i}^2)}{\mu_i \rho \lambda_j \hat{\sigma}_{\varepsilon_i}^2 + \mu_j \lambda_i (1 - \rho^2 + \hat{\sigma}_{\varepsilon_i}^2)}, \quad \text{where } i, j = 1, 2, \quad j \neq i.$$

5 Welfare analysis

This section focuses on the welfare loss at the equilibrium. We characterize the equilibrium and efficient allocations in Subsection 5.1 and analyze deadweight losses in Subsection 5.2.

5.1 Characterizing the equilibrium and efficient allocations

Recall that $t_i = \mathbb{E}[\theta_i | s_1, s_2]$, $i = 1, 2$, that is, the predicted values with full information (s_1, s_2) and $t = (t_1, t_2)$. The strategies in the equilibrium induce outcomes as functions of the realized vector of predicted values t and are given in Equation (5). One can easily show that the equilibrium outcome solves the following *distorted* benefit maximization program:¹⁹

$$\max_{x_1, x_2} \mathbb{E} \left[n_1 (\theta_1 x_1 - (d_1 + \lambda_1) x_1^2 / 2) + n_2 (\theta_2 x_2 - (d_2 + \lambda_2) x_2^2 / 2) \mid t \right]$$

$$s.t. \quad n_1 x_1 + n_2 x_2 = Q,$$

where d_1 and d_2 are the equilibrium parameters. The efficient allocation would obtain if we set $d_1 = d_2 = 0$, which corresponds to a price-taking equilibrium (denoted by superscript o). The equilibrium strategy of a type- i bidder ($i = 1, 2$) will be of the form $X_i^o(s_i, p) = b_i^o + a_i^o s_1 - c_i^o p$, $i = 1, 2$, and is derived by maximizing the following program:

$$\max_{x_i} (\mathbb{E}[\theta_i | s_i, p] - p) x_i - \lambda_i x_i^2 / 2,$$

while taking prices as given. The FOC of this optimization problem yields

$$\mathbb{E}[\theta_i | s_i, p] - p - \lambda_i x_i = 0.$$

After identifying coefficients and solving the corresponding system of equations, we find that there exists a unique equilibrium in this setup. The equilibrium coefficients coincide with those in Proposition 6 for the continuum market.

Proposition 7. *Let $Q = (n_1 + n_2)q$ and let $\mu_i = n_i / (n_1 + n_2)$ for $i = 1, 2$. Then there exists a unique price-taking equilibrium, and the equilibrium coefficients coincide with the equilibrium*

¹⁹See Lemma A3 in the Appendix.

coefficients of the continuum setup (whose expressions are given in the statement of Proposition 6).

Our next corollary provides some comparative statics results.

Corollary 3. *Let $\rho\sigma_{\varepsilon_1}^2\sigma_{\varepsilon_2}^2 > 0$. Then the only equilibrium coefficients affected by Q , $\bar{\theta}_i$, and $\bar{\theta}_j$ are the intercepts of the demand functions (with b_i^o increasing in $\bar{\theta}_i$ and Q and decreasing in $\bar{\theta}_j$) for $i, j = 1, 2$ and $i \neq j$. Furthermore, the demands of group i are less sensitive to private signals and prices (lower a_i and c_i) in response to an increase in λ_i , λ_j , ρ , $\sigma_{\varepsilon_i}^2$, and μ_i , and to a decrease in μ_j ; however, group i 's demands are not affected by $\sigma_{\varepsilon_j}^2$.*

Observe that, under competitive behavior, we can derive an additional comparative statics result: the relationship between the equilibrium coefficients and the proportion of individuals in group 1. In particular, increasing the proportion μ_1 of type-1 traders leads, for those traders, to an increased information-sensitivity weight of the price (higher $\Psi_1(n_1c_1^o + n_2c_2^o)(n_2a_2^o)^{-1}$) and so a lower overall response to the price ($c_1^o = \lambda_1^{-1}(1 - \Psi_1(n_1c_1^o + n_2c_2^o)(n_2a_2^o)^{-1})$); the opposite holds for type-2 traders.

Thus the auction outcome can be obtained as the solution to a maximization problem with a more concave objective function than the expected total surplus, which suggests that inefficiency may be eliminated by quadratic subsidies ($\kappa_i x_i^2/2$, $i = 1, 2$) that compensate for the distortions. The per capita subsidy rate (κ_i) to a trader of type i must be such that it compensates for the distortion $d_i(\kappa_i)$ while accounting for the subsidy. Since the aim is to induce competitive behavior, the trader should be led to respond with c_i^o to the price. This means that the exact amount of κ_i must be $d_i(c_1^o, c_2^o)$, since that would be the distortion arising when traders use the competitive linear strategies. The following proposition shows that, if subsidies are selected properly, then bidders behave competitively and so the equilibrium allocation is efficient.

Proposition 8. *Let $i = 1, 2$ and $i \neq j$. Then the efficient allocation is induced by the quadratic subsidies $\kappa_i x_i^2/2$, where $\kappa_i = d_i(c_i^o, c_j^o) = 1/((n_i - 1)c_i^o + n_j c_j^o)$. If $\rho\sigma_{\varepsilon_i}^2\sigma_{\varepsilon_j}^2 > 0$, then the per capita subsidy rates (κ_i , $i = 1, 2$) increase with ρ , $\hat{\sigma}_{\varepsilon_1}^2$, $\hat{\sigma}_{\varepsilon_2}^2$, λ_1 , and λ_2 but decrease with n_1 and n_2 . We have that $\kappa_1 < \kappa_2$ iff $c_1^o < c_2^o$.*

Combining Propositions 7 and 8 now yields closed-form expressions for the optimal subsidy rates:

$$\kappa_i = \frac{1}{n_j(1 - \rho)} \left(\frac{(n_i - 1)(1 + \hat{\sigma}_{\varepsilon_i}^2 + \rho)}{n_i \lambda_j \rho \hat{\sigma}_{\varepsilon_i}^2 + n_j \lambda_i (1 - \rho^2 + \hat{\sigma}_{\varepsilon_i}^2)} + \frac{n_i (1 + \hat{\sigma}_{\varepsilon_j}^2 + \rho)}{n_i \lambda_j (1 - \rho^2 + \hat{\sigma}_{\varepsilon_j}^2) + n_j \lambda_i \rho \hat{\sigma}_{\varepsilon_j}^2} \right)^{-1},$$

$i = 1, 2$, $i \neq j$. If $\rho = 0$ (or, with full information, if $\sigma_{\varepsilon_i}^2 = 0$, $i = 1, 2$), then $\kappa_i^f = 1/((n_i - 1)\lambda_i^{-1} + n_j \lambda_j^{-1})$, $i = 1, 2$. Proposition 8 implies that the optimal subsidy rates with

incomplete information and learning from prices are higher than with full information: $\kappa_i > \kappa_i^f$ if (a) $\rho > 0$ and (b) at least one of $\hat{\sigma}_{\varepsilon_1}^2$ or $\hat{\sigma}_{\varepsilon_2}^2$ is strictly positive. If $\sigma_{\varepsilon_1}^2 = \infty$ then κ_i , $i = 1, 2$, is independent of $\sigma_{\varepsilon_2}^2$.

Note that the optimal subsidy rates are decreasing in the number of traders because, when there are many agents, competitive behavior is already being approached in the market without subsidies. Moreover, $\text{sgn}\{\kappa_1 - \kappa_2\} = \text{sgn}\{c_1^o - c_2^o\}$. Hence $\kappa_1 < \kappa_2$ iff $c_1^o < c_2^o$. The implication is that the bidders who require a higher per capita subsidy rate are the ones whose demands are more sensitive to price. Corollary 3 allows us to conclude that, if there is a group with more precise private information, with lower transaction costs, and that is less numerous, then it is the group meriting a higher per capita subsidy rate. The reason is that the stronger group's strategic behavior is more pronounced and so it must receive more compensation in order to become competitive. If $\bar{\theta}_2 \geq \bar{\theta}_1$, then $\mathbb{E}[(x_2^o(t))^2] > \mathbb{E}[(x_1^o(t))^2]$, from which it follows that the bidders from the stronger group (group 2) should receive the higher expected subsidy.²⁰ However, if $\bar{\theta}_1 > \bar{\theta}_2$, then there are parameter configurations under which bidders from the weaker group (group 1) should receive the higher expected subsidy, even though $\kappa_1 < \kappa_2$. These conclusions would have to be revised if redistributive considerations come into play.²¹

Our result has policy implications. It implies, for example, that a central bank seeking an efficient distribution of liquidity among banks should relax collateral requirements (i.e., provide a larger subsidy) to the strong group. This prescription sounds counterintuitive because the efficiency motive may conflict with the central bank's function as lender of last resort, which often involves shoring up weak banks (e.g., the European Central Bank relaxing the collateral requirements for Greek banks to avoid a meltdown of that country's banking system). Another example is that of a wholesale electricity market characterized by a small (oligopolistic) group and a fringe; in this case, a regulator looking to improve productive efficiency should set a higher subsidy rate for the oligopolistic group. This could be accomplished by offering differential subsidies to renewable energy technologies, for instance, that lower the marginal cost of production.

²⁰The expected optimal subsidy for group i is $\kappa_i \mathbb{E}[(x_i^o(t))^2]/2$, where $x_i^o(t) = (n_j(t_i - t_j) + \lambda_j Q) / (n_i \lambda_j + n_j \lambda_i)$.

²¹Athey et al. (2013) find with regard to US Forest Service timber auctions that restricting entry increases small business participation but substantially reduces efficiency and revenue. In contrast, subsidizing small bidders directly increases revenue and the profits of small bidders without much cost in efficiency. See also Loertscher and Marx (2016) and Pai and Vohra (2012).

5.2 Deadweight loss

The expected deadweight loss, $\mathbb{E}[DWL]$, at an anonymous allocation $(x_1(t), x_2(t))$ is the difference between expected total surplus at the efficient allocation, ETS^o , and at the baseline allocation, denoted simply by ETS . Lemma A4 in the Appendix shows that

$$\mathbb{E}[DWL] = \frac{1}{2}\lambda_1 n_1 \mathbb{E}[(x_1(t) - x_1^o(t))^2] + \frac{1}{2}\lambda_2 n_2 \mathbb{E}[(x_2(t) - x_2^o(t))^2],$$

where $(x_1^o(t), x_2^o(t))$ corresponds to the price-taking equilibrium. We can use the equilibrium expressions for $(x_1(t), x_2(t))$ and $(x_1^o(t), x_2^o(t))$ to show that

$$\begin{aligned} \mathbb{E}[DWL] = & \frac{n_2 n_1 (n_2 d_1 + n_1 d_2)^2}{2(n_2 \lambda_1 + n_1 \lambda_2)(n_2(d_1 + \lambda_1) + n_1(d_2 + \lambda_2))^2} \mathbb{E}(t_1 - t_2)^2 + \\ & + \frac{n_2 n_1 (n_2 d_1 + n_1 d_2)(\lambda_2 d_1 - \lambda_1 d_2)}{(n_2 \lambda_1 + n_1 \lambda_2)(n_2(d_1 + \lambda_1) + n_1(d_2 + \lambda_2))^2} Q(\bar{\theta}_1 - \bar{\theta}_2) + \\ & \frac{n_2 n_1 (\lambda_1 d_2 - \lambda_2 d_1)^2}{2(n_2 \lambda_1 + n_1 \lambda_2)(n_2(d_1 + \lambda_1) + n_1(d_2 + \lambda_2))^2} Q^2, \end{aligned}$$

where $\mathbb{E}[(t_1 - t_2)^2] = (\bar{\theta}_1 - \bar{\theta}_2)^2 + (1 - \rho)^2 \sigma_\theta^2 (2(1 + \rho) + \hat{\sigma}_{\varepsilon_1}^2 + \hat{\sigma}_{\varepsilon_2}^2) / ((1 + \hat{\sigma}_{\varepsilon_1}^2)(1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2)$.

The expected deadweight loss consists of three terms. The first term is the only one present in a double auction ($Q = 0$). This term, which is due to uncertainty and information, is the product of two factors. One factor

$$\frac{n_2 n_1 (n_2 d_1 + n_1 d_2)^2}{2(n_2 \lambda_1 + n_1 \lambda_2)(n_2(d_1 + \lambda_1) + n_1(d_2 + \lambda_2))^2}$$

increases with d_1 and d_2 . Since d_1 and d_2 are each increasing in ρ , it follows that this multiplier also increases with ρ . The other factor, $\mathbb{E}[(t_1 - t_2)^2]$, decreases with ρ and with $\hat{\sigma}_{\varepsilon_i}^2$ and increases with $(\bar{\theta}_1 - \bar{\theta}_2)^2$; it vanishes when $\rho = 1$ or when there is no uncertainty ($\sigma_\theta^2 = 0$) provided that $\bar{\theta}_1 = \bar{\theta}_2$. As a result, the first term of $\mathbb{E}[DWL]$ may be either increasing or decreasing in ρ . Note that if $\sigma_{\varepsilon_1}^2 = \infty$, then d_1 and d_2 are independent of $\sigma_{\varepsilon_2}^2$, and $\mathbb{E}[DWL]$ decreases with $\sigma_{\varepsilon_2}^2$.

The third term derives from the absorption of Q by the traders, and it increases with the quantity offered (Q) as well as with the difference (in absolute terms) between d_1/d_2 and λ_1/λ_2 . The second term is an interaction term that is positive for $Q > 0$ iff $(\lambda_2 d_1 - \lambda_1 d_2)(\bar{\theta}_1 - \bar{\theta}_2) > 0$, that is, when the relative distortion between groups (d_i/d_j) is large whenever $\bar{\theta}_i > \bar{\theta}_j$. When $d_1/d_2 = \lambda_1/\lambda_2$, the expected deadweight loss consists of the first term only. That is because, in this case, the non-informational trading term corresponding to the equilibrium with imperfect competition coincides with the one corresponding to the competitive equilibrium. Note that if we interpret the traders as producers competing to supply a fixed demand Q , then the condition

$d_1/d_2 = \lambda_1/\lambda_2$ means that the ratio of the production of the two types of firms is aligned with the ratio of the slopes of their respective marginal costs. This condition guarantees productive efficiency provided that $\bar{\theta}_1 = \bar{\theta}_2$ and $\rho = 1$ and, since demand is fixed, this coincides with overall efficiency.

Furthermore, if group 1 has higher transaction costs ($\lambda_1 > \lambda_2$), is more numerous ($n_1 > n_2$), and is less informed ($\sigma_{\varepsilon_1}^2 > \sigma_{\varepsilon_2}^2$) than group 2, then $d_1/d_2 < \lambda_1/\lambda_2$ (and therefore $\lambda_2 d_1 - \lambda_1 d_2 < 0$). In this case, the third term in our expression for $\mathbb{E}[DWL]$ is not null. In addition, if group 1 ex ante values the asset less ($\bar{\theta}_1 \leq \bar{\theta}_2$), then the second term in our expression for $\mathbb{E}[DWL]$ is positive. The expected deadweight loss increases with Q and $|\bar{\theta}_1 - \bar{\theta}_2|$ when the stronger group values the asset no less than does the weaker group.

Under full information (i.e., $\sigma_{\varepsilon_1}^2 = \sigma_{\varepsilon_2}^2 = 0$), both d_1 and d_2 are independent of ρ ; in this case, then $\mathbb{E}[DWL]$ decreases with ρ . Similarly, if $\rho = 0$, then d_1 and d_2 are independent of $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$, from which it follows that $\mathbb{E}[DWL]$ decreases with $\sigma_{\varepsilon_1}^2$ and $\sigma_{\varepsilon_2}^2$. Some of these results are summarized in our last proposition.

Proposition 9.

(i) *The expected deadweight loss may be increasing or decreasing in the information parameters (ρ , $\hat{\sigma}_{\varepsilon_1}^2$, and $\hat{\sigma}_{\varepsilon_2}^2$) and, therefore, market power (d_1, d_2) and the $\mathbb{E}[DWL]$ may be negatively associated.*

(ii) *The $\mathbb{E}[DWL]$ increases with payoff asymmetry and with Q whenever the stronger group (say group 2, with $\lambda_1 > \lambda_2$, $n_1 > n_2$, and $\sigma_{\varepsilon_1}^2 > \sigma_{\varepsilon_2}^2$) values the asset no less than does the weaker group (i.e., $\bar{\theta}_1 \leq \bar{\theta}_2$).*

(iii) *When groups are symmetric, the expected deadweight loss is independent of Q , and market power d and the $\mathbb{E}[DWL|t]$ are positively associated, given predicted values t , for changes in information parameters. This need not be the case with asymmetric groups (e.g., for Q large, $d_i/d_j > \lambda_i/\lambda_j$ implies that $\mathbb{E}[DWL|t]$ increases in d_i and decreases in d_j).*

6 Concluding remarks

We analyze a divisible good uniform-price auction, where two types of bidders compete. Each of these two groups contains a finite number of identical bidders. At the unique equilibrium, a group's relative market power increases with the precision of private information and decreases with the group's transaction costs. Consistently with the empirical evidence, we find that an increase in the transaction costs of a group of bidders induces a strategic response from the other group, whose members then submit steeper schedules. The group that is stronger (because it has more precise private information, faces lower transaction costs, and is more oligopsonistic)

has more market power and must therefore receive a higher subsidy to behave competitively. The expected deadweight loss increases with the quantity auctioned and with the degree of payoff asymmetry provided the stronger group values the asset no less than does the weaker group.

Our findings have policy implications. Consider a regulator who wants to reduce inefficiency in an industry with two groups of firms (e.g., a small oligopolistic group and a competitive fringe). This regulator must bear in mind that any intervention directed toward one group will also affect the other's behavior. In addition, the regulator should set a higher subsidy rate for the group that has better information, is more oligopsonistic, and has lower transaction costs. The framework developed here can be adapted to study competition policy analyzing the effects of merger and industry capacity redistribution.

Appendix

Proposition 1 will follow from Lemmata A1 and A2.

Lemma A1. *Let $\rho < 1$. In equilibrium, the demand function for a trader of type i , $i = 1, 2$, is given by $X_i(s_i, p) = (\mathbb{E}[\theta_i|s_i, p] - p) / (d_i + \lambda_i)$, with $d_i + \lambda_i > 0$. The equilibrium coefficients satisfy the following system of equations:*

$$b_i = \left((1 - \Xi_i) \bar{\theta}_i - \Psi_i \bar{\theta}_j - \frac{\Psi_i (n_i b_i + n_j b_j - Q)}{n_j a_j} \right) / (d_i + \lambda_i), \quad (9)$$

$$a_i = \left(\Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i \right) / (d_i + \lambda_i), \text{ and} \quad (10)$$

$$c_i = \left(1 - \frac{\Psi_i (n_i c_i + n_j c_j)}{n_j a_j} \right) / (d_i + \lambda_i), \quad (11)$$

where $i, j = 1, 2, j \neq i$. Moreover, in equilibrium, $a_i > 0$, $i = 1, 2$.

Proof: Consider a trader of type i . Recall that at the beginning of Subsection 3.1 we obtain $X_i(s_i, p) = (\mathbb{E}[\theta_i|s_i, p] - p) / (d_i + \lambda_i)$ and $\mathbb{E}[\theta_i|s_i, p] = \mathbb{E}[\theta_i|s_i, s_j]$. Since we are looking for strategies of the form $X_i(s_i, p) = b_i + a_i s_i - c_i p$, from the market clearing condition we have $p = (n_i (b_i + a_i s_i) + n_j (b_j + a_j s_j) - Q) / (n_i c_i + n_j c_j)$ and, hence,

$$s_j = ((n_i c_i + n_j c_j) p + Q - n_i (b_i + a_i s_i) - n_j b_j) / (n_j a_j).$$

Thus, from Expression (3), it follows that

$$\mathbb{E}[\theta_i|s_i, s_j] = (1 - \Xi_i) \bar{\theta}_i - \Psi_i \bar{\theta}_j + \Psi_i \left(\frac{Q - n_i b_i - n_j b_j}{n_j a_j} \right) + \left(\Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i \right) s_i + \Psi_i \left(\frac{n_i c_i + n_j c_j}{n_j a_j} \right) p.$$

Substituting the foregoing expression in (1), and then identifying coefficients, we obtain the expressions for the demand coefficients given in (9)-(11).

Finally, we show the positiveness of the coefficients a_i , $i = 1, 2$. From Expression (10), we have $a_i = \Xi_i / (d_i + \lambda_i + n_i \Psi_i / (n_j a_j))$, $i, j = 1, 2, j \neq i$. Combining the previous expressions, we have that

$$a_i = \frac{n_j (\Xi_i \Xi_j - \Psi_i \Psi_j)}{n_i \Psi_i (d_j + \lambda_j) + \Xi_j n_j (d_i + \lambda_i)}, \quad i, j = 1, 2, j \neq i. \quad (12)$$

Direct computation yields $\Xi_i \Xi_j - \Psi_i \Psi_j = (1 - \rho^2) / ((1 + \hat{\sigma}_{\varepsilon_1}^2) (1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2) > 0$, whenever $\rho < 1$. Moreover, using the positiveness of $d_i + \lambda_i$, Ξ_i , and Ψ_i , $i = 1, 2$, we conclude that, in equilibrium, the coefficients a_i , $i = 1, 2$, are strictly positive. ■

Lemma A2. *Let $z = c_1/c_2$. In equilibrium,*

$$b_i = \frac{\Psi_i}{n_i n_j} \frac{n_i \Xi_j \frac{a_i}{a_j} - n_j \Psi_j}{\Xi_i \Xi_j - \Psi_i \Psi_j} Q + a_i \left(\frac{\Xi_j \bar{\theta}_i - \Psi_i \bar{\theta}_j}{\Xi_i \Xi_j - \Psi_i \Psi_j} - \bar{\theta}_i \right), \quad (13)$$

$$a_i = \Delta_i c_i, \quad (14)$$

$$c_1 = \left(\Xi_1 \Delta_1^{-1} - \frac{n_1}{n_2} (1 - \Xi_1 \Delta_1^{-1}) z - \frac{z}{(n_1 - 1) z + n_2} \right) / \lambda_1, \text{ and} \quad (15)$$

$$c_2 = \left(\Xi_2 \Delta_2^{-1} - \frac{n_2}{n_1} (1 - \Xi_2 \Delta_2^{-1}) \frac{1}{z} - \frac{1}{n_1 z + n_2 - 1} \right) / \lambda_2, \quad (16)$$

where $\Delta_i = (\Xi_i \Xi_j - \Psi_i \Psi_j) / (\Xi_j - \Psi_i) = 1 / (1 + (1 + \rho)^{-1} \hat{\sigma}_{\varepsilon_i}^2)$, $i, j = 1, 2, j \neq i$. Moreover, z is the unique positive solution to the cubic polynomial $p(z) = p_3 z^3 + p_2 z^2 + p_1 z + p_0$, with

$$\begin{aligned} p_3 &= n_1^2 (n_1 - 1) (n_2 \Xi_2 \Delta_2^{-1} \lambda_1 + n_1 (1 - \Xi_1 \Delta_1^{-1}) \lambda_2), \\ p_2 &= n_1 ((3n_2 n_1 - n_1 - 2n_2 + 1) (n_2 \Xi_2 \Delta_2^{-1} \lambda_1 - n_1 \Xi_1 \Delta_1^{-1} \lambda_2) + \\ &\quad + \lambda_2 n_1 (2n_2 n_1 - n_1 + 1) - (n_1 - 1) (n_2 + 1) n_2 \lambda_1), \\ p_1 &= n_2 ((3n_2 n_1 - 2n_1 - n_2 + 1) (n_2 \Xi_2 \Delta_2^{-1} \lambda_1 - n_1 \Xi_1 \Delta_1^{-1} \lambda_2) + \\ &\quad + \lambda_2 n_1 (n_2 - 1) (n_1 + 1) - (2n_2 n_1 - n_2 + 1) n_2 \lambda_1), \text{ and} \\ p_0 &= -n_2^2 (n_2 - 1) (n_2 (1 - \Xi_2 \Delta_2^{-1}) \lambda_1 + n_1 \Xi_1 \Delta_1^{-1} \lambda_2). \end{aligned}$$

Proof: In relation to the expression for b_i , notice that (10) implies

$$d_i + \lambda_i = \left(\Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i \right) / a_i, \quad i, j = 1, 2, j \neq i. \quad (17)$$

Substituting these expressions in (9), it follows that

$$b_i = a_i \frac{(1 - \Xi_i) \bar{\theta}_i - \Psi_i \bar{\theta}_j - \frac{\Psi_i (n_i b_i + n_j b_j - Q)}{n_j a_j}}{\Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i}, \quad i, j = 1, 2, j \neq i. \quad (18)$$

Thus,

$$n_i b_i + n_j b_j = n_i a_i \frac{(1 - \Xi_i) \bar{\theta}_i - \Psi_i \bar{\theta}_j - \frac{\Psi_i(n_i b_i + n_j b_j - Q)}{n_j a_j}}{\Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i} + n_j a_j \frac{(1 - \Xi_j) \bar{\theta}_j - \Psi_j \bar{\theta}_i - \frac{\Psi_j(n_i b_i + n_j b_j - Q)}{n_i a_i}}{\Xi_j - \frac{n_j a_j}{n_i a_i} \Psi_j},$$

Isolating $n_i b_i + n_j b_j$ in the previous formula and substituting the resulting expression in (18), Expression (13) is obtained.

Concerning the expression for a_i , substituting (17) in (11), it follows that

$$c_i = a_i \left(1 - \frac{\Psi_i(n_i c_i + n_j c_j)}{n_j a_j} \right) / \left(\Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i \right), \quad i, j = 1, 2, j \neq i. \quad (19)$$

Hence, $n_i c_i + n_j c_j = n_i a_i \frac{n_j a_j - \Psi_i(n_i c_i + n_j c_j)}{n_j a_j \Xi_i - n_i a_i \Psi_i} + n_j a_j \frac{n_i a_i - \Psi_j(n_i c_i + n_j c_j)}{n_i a_i \Xi_j - n_j a_j \Psi_j}$, which implies that $n_i c_i + n_j c_j = (n_i a_i (\Xi_j - \Psi_i) + n_j a_j (\Xi_i - \Psi_j)) / (\Xi_i \Xi_j - \Psi_i \Psi_j)$. Then, substituting the previous expression in Equation (19), we obtain $c_i = (\Xi_j - \Psi_i) a_i / (\Xi_i \Xi_j - \Psi_i \Psi_j)$, which is equivalent to (14).

In relation to c_1 and c_2 , using the expression for d_i and (14), (17) implies that

$$\lambda_i = \left(\frac{\Xi_i}{\Delta_i} - \frac{n_i \Psi_i c_i}{n_j \Delta_j c_j} \right) c_i^{-1} - ((n_i - 1) c_i + n_j c_j)^{-1}, \quad i, j = 1, 2, j \neq i.$$

or, since

$$\Psi_i \Delta_j^{-1} = 1 - \Xi_i \Delta_i^{-1}, \quad (20)$$

$\lambda_i = \left(\Xi_i \Delta_i^{-1} - \frac{n_i}{n_j} (1 - \Xi_i \Delta_i^{-1}) \frac{c_i}{c_j} \right) c_i^{-1} - ((n_i - 1) c_i + n_j c_j)^{-1}$, $i, j = 1, 2, j \neq i$, which imply (15) and (16) since $z = c_1/c_2$. Moreover, dividing the previous two equalities, it follows that

$$\frac{\lambda_1}{\lambda_2} = \frac{\Xi_1 \Delta_1^{-1} - \frac{n_1}{n_2} (1 - \Xi_1 \Delta_1^{-1}) z - z ((n_1 - 1) z + n_2)^{-1}}{\Xi_2 \Delta_2^{-1} z - \frac{n_2}{n_1} (1 - \Xi_2 \Delta_2^{-1}) - z (n_1 z + n_2 - 1)^{-1}}. \quad (21)$$

After some algebra, (21) is equivalent to $p(z) = 0$, where $p(z) = p_3 z^3 + p_2 z^2 + p_1 z + p_0$, as stated in this lemma. Notice that $p(0) < 0$ and $\lim_{z \rightarrow \infty} p(z) = \infty$. Consequently, there exists $z \in (0, \infty)$ such that $p(z) = 0$. Furthermore, we have that $p_2/n_1 > p_1/n_2$. This property implies that there exists only one sign change in the coefficients of $p(z)$. Applying the Descartes' rule, we conclude that there exists a unique positive root of $p(z)$. ■

Proof of Proposition 2: (Necessity). From Proposition 1 we know that $a_i > 0$, $i = 1, 2$. Combining this property with expressions given in (14), we have that, in equilibrium, the coefficients c_1 and c_2 are strictly positive. Moreover, (15) and (16) can be rewritten as

$$c_1 = \frac{q_N(z)}{((n_1 - 1) z + n_2) n_2 \lambda_1} \text{ and } c_2 = \frac{q_D(z)}{(n_1 z + n_2 - 1) n_1 z \lambda_2},$$

where $q_N(z) = n_2^2 \Xi_1 \Delta_1^{-1} + n_2 (\Xi_1 \Delta_1^{-1} (2n_1 - 1) - (n_1 + 1)) z - (n_1 - 1) (1 - \Xi_1 \Delta_1^{-1}) n_1 z^2$ and $q_D(z) = -n_2 (n_2 - 1) (1 - \Xi_2 \Delta_2^{-1}) + n_1 (\Xi_2 \Delta_2^{-1} (2n_2 - 1) - (n_2 + 1)) z + n_1^2 \Xi_2 \Delta_2^{-1} z^2$.

Let \bar{z}_N and \bar{z}_D denote the highest root of $q_N(z)$ and $q_D(z)$, respectively. Notice that the positiveness of c_1 and c_2 is equivalent to $\bar{z}_N > z > \bar{z}_D$. Therefore, $\bar{z}_N > \bar{z}_D$.

(Sufficiency). Suppose that $\bar{z}_N > \bar{z}_D$. Recall that Lemma A2 shows that there exists a unique positive value of z that solves (21), which can be rewritten as $\frac{\lambda_1}{\lambda_2} = \frac{n_1(n_2-1+n_1z)q_N(z)}{(n_2+(n_1-1)z)n_2q_D(z)}$. This implies that $\bar{z}_N > z > \bar{z}_D$. Notice that these inequalities guarantee the positiveness of c_1 and c_2 . Therefore, d_1 and d_2 are strictly positive, and consequently, the SOC of the optimization problems are satisfied. Thus, we conclude that whenever $\bar{z}_N > \bar{z}_D$ there exists a unique equilibrium. ■

Proof of Corollary 1: Notice that if $n_2 = 1$, then $\bar{z}_D = (2\Delta_2\Xi_2^{-1} - 1)/n_1$. Thus, the condition that guarantees the existence of an equilibrium is equivalent to $q_N((2\Delta_2\Xi_2^{-1} - 1)/n_1) > 0$. Direct computation yields that the last inequality holds iff

$$(2 - \Xi_2\Delta_2^{-1})(1 - \Xi_1\Delta_1^{-1} - \Xi_2\Delta_2^{-1}) + (\Xi_2\Delta_2^{-1} - 2(1 - \Xi_1\Delta_1^{-1}))n_1 > 0.$$

As $1 < \Xi_1\Delta_1^{-1} + \Xi_2\Delta_2^{-1}$, we conclude that an equilibrium exists iff $\Xi_2\Delta_2^{-1} > 2(1 - \Xi_1\Delta_1^{-1})$ and $n_1 > (2 - \Xi_2\Delta_2^{-1})(\Xi_1\Delta_1^{-1} + \Xi_2\Delta_2^{-1} - 1) / (\Xi_2\Delta_2^{-1} - 2(1 - \Xi_1\Delta_1^{-1}))$. Using the expressions of Ξ_i and Δ_i , the previous two inequalities are equivalent to $1 - \rho^2 > (2\rho - 1)\hat{\sigma}_{\varepsilon_1}^2$ and $n_1 > \bar{n}_1(\rho, \hat{\sigma}_{\varepsilon_1}^2, \hat{\sigma}_{\varepsilon_2}^2)$, where

$$\bar{n}_1(\rho, \hat{\sigma}_{\varepsilon_1}^2, \hat{\sigma}_{\varepsilon_2}^2) = 1 + \frac{\rho(1 - \rho^2 + \hat{\sigma}_{\varepsilon_1}^2)((1 + \rho)(\hat{\sigma}_{\varepsilon_1}^2 + \hat{\sigma}_{\varepsilon_2}^2) + 2\hat{\sigma}_{\varepsilon_1}^2\hat{\sigma}_{\varepsilon_2}^2)}{(1 + \rho)((1 + \hat{\sigma}_{\varepsilon_1}^2)(1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2)(1 - \rho^2 - (2\rho - 1)\hat{\sigma}_{\varepsilon_1}^2)}. \quad (22)$$

It can be shown that \bar{n}_1 increases with ρ , $\hat{\sigma}_{\varepsilon_1}^2$, and $\hat{\sigma}_{\varepsilon_2}^2$. In particular, if $\sigma_{\varepsilon_2}^2 = 0$, then (22) can be rewritten as $\bar{n}_1(\rho, \hat{\sigma}_{\varepsilon_1}^2, 0) = 1 + \rho\hat{\sigma}_{\varepsilon_1}^2 / (1 - \rho^2 - (2\rho - 1)\hat{\sigma}_{\varepsilon_1}^2)$.

Consider the case that one group is formed by a unique trader perfectly informed and with no transaction costs ($n_2 = 1$, $\lambda_2 = 0$ and $\sigma_{\varepsilon_2}^2 = 0$). Then, $z = \bar{z}_D = 1/n_1$, $\Xi_1 = (1 - \rho^2)/(1 - \rho^2 + \hat{\sigma}_{\varepsilon_1}^2)$, $\Psi_1 = \rho\hat{\sigma}_{\varepsilon_1}^2/(1 - \rho^2 + \hat{\sigma}_{\varepsilon_1}^2)$, $\Xi_2 = 1$, and $\Psi_2 = 0$. From Lemma A2, the coefficients of the demand functions are given by: $b_1 = \frac{\rho\hat{\sigma}_{\varepsilon_1}^2}{n_1(1-\rho)(1+\rho+\hat{\sigma}_{\varepsilon_1}^2)}Q + a_1\frac{\hat{\sigma}_{\varepsilon_1}^2}{1-\rho^2}(\bar{\theta}_1 - \rho\bar{\theta}_2)$, $a_1 = \Delta_1c_1$, $c_1 = \frac{2(n_1-\bar{n}_1(\rho,\hat{\sigma}_{\varepsilon_1}^2,\hat{\sigma}_{\varepsilon_2}^2))(1-\rho^2-\hat{\sigma}_{\varepsilon_1}^2(2\rho-1))}{\lambda_1(1-\rho^2+\hat{\sigma}_{\varepsilon_1}^2)(2n_1-1)}$, $b_2 = 0$, and $a_2 = c_2 = \frac{c_1}{z} = n_1c_1$. ■

Proof of Proposition 3: In what follows we prove the following comparative statics results for $i, j = 1, 2$, $i \neq j$:

- a) $\frac{\partial b_i}{\partial \theta_i} > 0$, $\frac{\partial a_i}{\partial \theta_i} = 0$ and $\frac{\partial c_i}{\partial \theta_i} = 0$,
- b) $\frac{\partial b_i}{\partial \theta_j} < 0$, $\frac{\partial a_i}{\partial \theta_j} = 0$ and $\frac{\partial c_i}{\partial \theta_j} = 0$,
- c) $\frac{\partial b_i}{\partial Q} > 0$, $\frac{\partial a_i}{\partial Q} = 0$ and $\frac{\partial c_i}{\partial Q} = 0$,
- d) $\frac{\partial a_i}{\partial \lambda_i} < 0$ and $\frac{\partial c_i}{\partial \lambda_i} < 0$,

- e) $\frac{\partial a_i}{\partial \lambda_j} < 0$ and $\frac{\partial c_i}{\partial \lambda_j} < 0$,
- f) $\frac{\partial a_i}{\partial \rho} < 0$ and $\frac{\partial c_i}{\partial \rho} < 0$,
- g) $\frac{\partial}{\partial \sigma_{\varepsilon_i}^2} (d_i/d_j) < 0$, $\frac{\partial}{\partial \sigma_{\varepsilon_j}^2} (d_i/d_j) > 0$, $\frac{\partial}{\partial \lambda_i} (d_i/d_j) < 0$, and $\frac{\partial}{\partial \lambda_j} (d_i/d_j) > 0$,
- h) $\frac{\partial a_i}{\partial \sigma_{\varepsilon_i}^2} < 0$ and $\frac{\partial c_i}{\partial \sigma_{\varepsilon_i}^2} < 0$,
- i) $\frac{\partial a_i}{\partial \sigma_{\varepsilon_j}^2} < 0$ and $\frac{\partial c_i}{\partial \sigma_{\varepsilon_j}^2} < 0$, and
- k) $\frac{\partial d_i}{\partial n_i} < 0$ and $\frac{\partial d_j}{\partial n_i} < 0$.

From Lemma A1, we know that the equilibrium coefficients that depend on $\bar{\theta}_i$, $\bar{\theta}_j$ and Q are b_1 and b_2 . Using Lemma A2 and after some algebra, the results given in a), b) and c) are obtained. In what follows, without any loss of generality, let $i = 1$. First, we prove that $\frac{\partial z}{\partial \lambda_1} < 0$. From Lemma A2, we know that z is the unique positive solution of the following equation:

$$\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} = 0, \quad (23)$$

where $N(z) = \Xi_1 \Delta_1^{-1} - \frac{n_1}{n_2} (1 - \Xi_1 \Delta_1^{-1}) z - z ((n_1 - 1) z + n_2)^{-1}$ and

$$D(z) = \Xi_2 \Delta_2^{-1} z - \frac{n_2}{n_1} (1 - \Xi_2 \Delta_2^{-1}) - z (n_1 z + n_2 - 1)^{-1},$$

with $\Xi_i \Delta_i^{-1} = (1 - \rho^2 + \hat{\sigma}_{\varepsilon_j}^2) (1 + \rho + \hat{\sigma}_{\varepsilon_i}^2) \left(\left((1 + \hat{\sigma}_{\varepsilon_i}^2) (1 + \hat{\sigma}_{\varepsilon_j}^2) - \rho^2 \right) (1 + \rho) \right)^{-1}$, $i, j = 1, 2, i \neq j$. Applying the Implicit Function Theorem, $\frac{\partial z}{\partial \lambda_i} = - \left(\frac{\partial}{\partial \lambda_i} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right) \right) / \frac{\partial}{\partial z} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right)$, $i = 1, 2$. As $\frac{\partial}{\partial \lambda_1} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right) = \frac{1}{\lambda_2} > 0$, $\frac{\partial}{\partial \lambda_2} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right) = -\frac{\lambda_1}{\lambda_2^2} < 0$, and $\frac{\partial}{\partial z} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right) > 0$, because of $z \in (\bar{z}_D, \bar{z}_N)$, we conclude that $\frac{\partial z}{\partial \lambda_1} < 0$ and $\frac{\partial z}{\partial \lambda_2} > 0$.

Next, we study the relationship between c' s and λ_1 . Differentiating (16), we have

$$\frac{\partial c_2}{\partial \lambda_1} = \frac{\partial c_2}{\partial z} \frac{\partial z}{\partial \lambda_1} = \frac{1}{\lambda_2} \left(\frac{n_2}{n_1} (1 - \Xi_2 \Delta_2^{-1}) \frac{1}{z^2} + \frac{n_1}{(n_2 + n_1 z - 1)^2} \right) \frac{\partial z}{\partial \lambda_1} < 0,$$

since $\frac{\partial z}{\partial \lambda_1} < 0$. Moreover, as $c_1 = z c_2$, it follows that $\frac{\partial c_1}{\partial \lambda_1} = \frac{\partial z}{\partial \lambda_1} c_2 + z \frac{\partial c_2}{\partial \lambda_1} < 0$, because of the positiveness of c_2 and z , and the negativeness of $\frac{\partial z}{\partial \lambda_1}$ and $\frac{\partial c_2}{\partial \lambda_1}$. In relation to a_1 and a_2 , from (14), direct computation yields $\frac{\partial a_1}{\partial \lambda_1} < 0$ and $\frac{\partial a_2}{\partial \lambda_1} < 0$, since $\frac{\partial c_1}{\partial \lambda_1} < 0$ and $\frac{\partial c_2}{\partial \lambda_1} < 0$.

Now, we study how the correlation coefficient ρ affects a_1 . Let $y = a_1/a_2$. As $a_1 = \Delta_1 c_1$ and $a_2 = \Delta_2 c_2$, then $z = \Delta_2 y / \Delta_1$. Substituting this expression in (21), and after some algebra, we have that

$$\frac{\lambda_1}{\lambda_2} y = \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)}, \quad (24)$$

where $\tilde{N}(y, \rho) = \frac{1-\rho^2+\hat{\sigma}_{\varepsilon_2}^2-\frac{n_1}{n_2}\hat{\sigma}_{\varepsilon_1}^2\rho y}{(1+\hat{\sigma}_{\varepsilon_1}^2)(1+\hat{\sigma}_{\varepsilon_2}^2)-\rho^2} - \left((n_1-1) \frac{1+\rho+\hat{\sigma}_{\varepsilon_1}^2}{1+\rho} + n_2 \frac{1+\rho+\hat{\sigma}_{\varepsilon_2}^2}{1+\rho} \frac{1}{y} \right)^{-1}$ and

$$\tilde{D}(y, \rho) = \frac{1-\rho^2+\hat{\sigma}_{\varepsilon_1}^2-\frac{n_2}{n_1}\hat{\sigma}_{\varepsilon_2}^2\rho\frac{1}{y}}{(1+\hat{\sigma}_{\varepsilon_1}^2)(1+\hat{\sigma}_{\varepsilon_2}^2)-\rho^2} - \left(n_1 \frac{1+\rho+\hat{\sigma}_{\varepsilon_1}^2}{1+\rho} y + (n_2-1) \frac{1+\rho+\hat{\sigma}_{\varepsilon_2}^2}{1+\rho} \right)^{-1}.$$

Moreover, $a_1 = \tilde{N}(y, \rho)/\lambda_1$ and $a_2 = \tilde{D}(y, \rho)/\lambda_2$. Hence, $\frac{\partial a_1}{\partial \rho} = \left(\frac{\partial}{\partial y} \tilde{N}(y, \rho) \frac{\partial y}{\partial \rho} + \frac{\partial}{\partial \rho} \tilde{N}(y, \rho) \right) / \lambda_1$. Thus, in order to show $\frac{\partial a_1}{\partial \rho} < 0$, it suffices to prove that

$$\frac{\partial}{\partial y} \tilde{N}(y, \rho) \frac{\partial y}{\partial \rho} + \frac{\partial}{\partial \rho} \tilde{N}(y, \rho) < 0. \quad (25)$$

Direct computation yields $\frac{\partial}{\partial y} \tilde{N}(y, \rho) < 0$. Then, (25) is equivalent to

$$\frac{\partial y}{\partial \rho} > - \frac{\frac{\partial}{\partial \rho} \tilde{N}(y, \rho)}{\frac{\partial}{\partial y} \tilde{N}(y, \rho)}. \quad (26)$$

Moreover, recall that y in equilibrium is the unique positive value that satisfies (24). Thus, applying the Implicit Function Theorem, it follows that

$$\frac{\partial y}{\partial \rho} = - \frac{\frac{\partial}{\partial \rho} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right)}{\frac{\partial}{\partial y} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right)}.$$

Then, (26) can be rewritten as

$$- \frac{\frac{\partial}{\partial \rho} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right)}{\frac{\partial}{\partial y} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right)} > - \frac{\frac{\partial}{\partial \rho} \tilde{N}(y, \rho)}{\frac{\partial}{\partial y} \tilde{N}(y, \rho)},$$

or using the fact that in equilibrium $\frac{\partial}{\partial y} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right) > 0$, (26) is satisfied iff

$$- \frac{\partial}{\partial \rho} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right) > - \frac{\frac{\partial}{\partial \rho} \tilde{N}(y, \rho)}{\frac{\partial}{\partial y} \tilde{N}(y, \rho)} \frac{\partial}{\partial y} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right). \quad (27)$$

Notice that

$$\frac{\partial}{\partial \rho} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right) = - \frac{\left(\frac{\partial}{\partial \rho} \tilde{N}(y, \rho) \right) \tilde{D}(y, \rho) - \tilde{N}(y, \rho) \left(\frac{\partial}{\partial \rho} \tilde{D}(y, \rho) \right)}{\tilde{D}^2(y, \rho)},$$

or using (23),

$$\frac{\partial}{\partial \rho} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right) = - \frac{\frac{\partial}{\partial \rho} \tilde{N}(y, \rho) - \frac{\lambda_1}{\lambda_2} y \frac{\partial}{\partial \rho} \tilde{D}(y, \rho)}{\tilde{D}(y, \rho)}.$$

Analogously,

$$\frac{\partial}{\partial y} \left(\frac{\lambda_1}{\lambda_2} y - \frac{\tilde{N}(y, \rho)}{\tilde{D}(y, \rho)} \right) = \frac{\lambda_1}{\lambda_2} - \frac{\left(\frac{\partial}{\partial y} \tilde{N}(y, \rho) \right) - \frac{\lambda_1}{\lambda_2} y \left(\frac{\partial}{\partial y} \tilde{D}(y, \rho) \right)}{\tilde{D}(y, \rho)}.$$

Therefore, (27) is equivalent to

$$\frac{\frac{\partial}{\partial \rho} \tilde{N}(y, \rho) - \frac{\lambda_1}{\lambda_2} y \frac{\partial}{\partial \rho} \tilde{D}(y, \rho)}{\tilde{D}(y, \rho)} > - \frac{\frac{\partial}{\partial \rho} \tilde{N}(y, \rho)}{\frac{\partial}{\partial y} \tilde{N}(y, \rho)} \left(\frac{\lambda_1}{\lambda_2} - \frac{\left(\frac{\partial}{\partial y} \tilde{N}(y, \rho) \right) - \frac{\lambda_1}{\lambda_2} y \left(\frac{\partial}{\partial y} \tilde{D}(y, \rho) \right)}{\tilde{D}(y, \rho)} \right),$$

or,

$$- \frac{y \frac{\partial}{\partial \rho} \tilde{D}(y, \rho)}{\tilde{D}(y, \rho)} > - \frac{\frac{\partial}{\partial \rho} \tilde{N}(y, \rho)}{\frac{\partial}{\partial y} \tilde{N}(y, \rho)} \left(1 + \frac{y \frac{\partial}{\partial y} \tilde{D}(y, \rho)}{\tilde{D}(y, \rho)} \right). \quad (28)$$

Moreover, recall that $a_2 = \tilde{D}(y, \rho)/\lambda_2$. The positiveness of a_2 tells us that $\tilde{D}(y, \rho) > 0$. After some algebra, we have that $\frac{\partial}{\partial \rho} \tilde{D}(y, \rho) < 0$, $\frac{\partial}{\partial \rho} \tilde{N}(y, \rho) < 0$ and $\frac{\partial}{\partial y} \tilde{D}(y, \rho) > 0$. Hence, we conclude that the *LHS* of (28) is positive, whereas the *RHS* of (28) is negative since $\frac{\partial}{\partial y} \tilde{N}(y, \rho) < 0$. Consequently, the fact that (28) is satisfied allows us to conclude that $\frac{\partial a_1}{\partial \rho} < 0$.

Concerning the effect of ρ on c_1 , recall that $c_1 = a_1/\Delta_1 = (1 + \rho + \hat{\sigma}_{\varepsilon_1}^2) a_1/(1 + \rho)$. This expression tells us that c_1 is the product of two decreasing positive functions in ρ . Therefore, $\frac{\partial c_1}{\partial \rho} < 0$.

Next, we prove that $\frac{\partial}{\partial \sigma_{\varepsilon_1}^2} (d_1/d_2) < 0$ and $\frac{\partial}{\partial \sigma_{\varepsilon_2}^2} (d_1/d_2) > 0$. From the expressions of d_1 , d_2 , and z , it follows that $d_1/d_2 = (n_1 z + (n_2 - 1)) / ((n_1 - 1)z + n_2)$. Applying the chain rule, we get $\frac{\partial}{\partial \sigma_{\varepsilon_i}^2} (d_1/d_2) = \frac{\partial}{\partial z} (d_1/d_2) \frac{\partial z}{\partial \sigma_{\varepsilon_i}^2}$. As $\frac{\partial}{\partial z} (d_1/d_2) > 0$, we know that the sign of $\frac{\partial}{\partial \sigma_{\varepsilon_i}^2} (d_1/d_2)$ is the same as the sign of $\frac{\partial z}{\partial \sigma_{\varepsilon_i}^2}$. Applying the Implicit Function Theorem, $\frac{\partial z}{\partial \sigma_{\varepsilon_i}^2} = - \frac{\frac{\partial}{\partial \sigma_{\varepsilon_i}^2} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right)}{\frac{\partial}{\partial z} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right)}$. Direct computation yields that $\frac{\partial}{\partial \sigma_{\varepsilon_1}^2} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right) > 0$ and $\frac{\partial}{\partial \sigma_{\varepsilon_2}^2} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right) < 0$. In addition, since $\frac{\partial}{\partial z} \left(\frac{\lambda_1}{\lambda_2} - \frac{N(z)}{D(z)} \right) > 0$, we obtain $\frac{\partial z}{\partial \sigma_{\varepsilon_1}^2} < 0$ and $\frac{\partial z}{\partial \sigma_{\varepsilon_2}^2} > 0$, and hence, we conclude that $\frac{\partial}{\partial \sigma_{\varepsilon_1}^2} (d_1/d_2) < 0$ and $\frac{\partial}{\partial \sigma_{\varepsilon_2}^2} (d_1/d_2) > 0$. Analogously, the negativeness of $\frac{\partial z}{\partial \lambda_1}$ and the positiveness of $\frac{\partial z}{\partial \lambda_2}$ allows us to conclude that $\frac{\partial}{\partial \lambda_1} (d_1/d_2) < 0$ and $\frac{\partial}{\partial \lambda_2} (d_1/d_2) > 0$.

Now, we study how a_1 and c_1 vary with a change in $\sigma_{\varepsilon_i}^2$, $i = 1, 2$. In order to do that first we analyze the effect of $\sigma_{\varepsilon_i}^2$ on d_1 and d_2 . From Proposition 1, we know that $d_i = ((n_i - 1) c_i + n_j c_j)^{-1}$ and $a_i = \Delta_i c_i > 0$, $i = 1, 2$. Therefore, $d_i = ((n_i - 1) \Delta_i^{-1} a_i + n_j \Delta_j^{-1} a_j)^{-1}$, $i, j = 1, 2$, $j \neq i$. Substituting the expressions of (12) and the expression for Δ_i given in Lemma A2, it follows that

$$d_i = \left(\frac{(n_i - 1) n_j}{n_j \Upsilon_i (d_i + \lambda_i) + n_i (\Upsilon_i - 1) (d_j + \lambda_j)} + \frac{n_j n_i}{n_i \Upsilon_j (d_j + \lambda_j) + n_j (\Upsilon_j - 1) (d_i + \lambda_i)} \right)^{-1},$$

where $\Upsilon_i = \Xi_j / (\Xi_j - \Psi_i) = (1 - \rho^2 + \widehat{\sigma}_{\varepsilon_i}^2) / ((1 - \rho)(1 + \rho + \widehat{\sigma}_{\varepsilon_i}^2)) > 1$, $i, j = 1, 2$, $j \neq i$. Therefore, we derive the following equations that are satisfied in equilibrium: $F_i(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) = 0$, $i = 1, 2$, where

$$F_i(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) = \frac{(n_i - 1)n_j d_i}{n_j \Upsilon_i (d_i + \lambda_i) + n_i (\Upsilon_i - 1)(d_j + \lambda_j)} + \frac{n_i n_j d_i}{n_i \Upsilon_j (d_j + \lambda_j) + n_j (\Upsilon_j - 1)(d_i + \lambda_i)} - 1,$$

$i, j = 1, 2$, $j \neq i$. Let $DF_{d_1, d_2}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2)$ denote the following matrix:

$$\begin{pmatrix} \frac{\partial F_1}{\partial d_1}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) & \frac{\partial F_1}{\partial d_2}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) \\ \frac{\partial F_2}{\partial d_1}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) & \frac{\partial F_2}{\partial d_2}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) \end{pmatrix}.$$

Direct computation yields

$$\begin{aligned} \frac{\partial F_i}{\partial d_i}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) = & \frac{n_j (n_i - 1)(n_i (\lambda_j + d_j)(\Upsilon_i - 1) + \Upsilon_i \lambda_i n_j)}{(n_j \Upsilon_i (d_i + \lambda_i) + n_i (\Upsilon_i - 1)(d_j + \lambda_j))^2} + \frac{n_i n_j (\lambda_i n_j (\Upsilon_j - 1) + \Upsilon_j n_i (\lambda_j + d_j))}{(n_i \Upsilon_j (d_j + \lambda_j) + n_j (\Upsilon_j - 1)(d_i + \lambda_i))^2}, \text{ and} \\ \frac{\partial F_i}{\partial d_j}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) = & - \left(\frac{d_i n_i n_j (n_i - 1)(\Upsilon_i - 1)}{(n_j \Upsilon_i (d_i + \lambda_i) + n_i (\Upsilon_i - 1)(d_j + \lambda_j))^2} + \frac{d_i n_i^2 n_j \Upsilon_j}{(n_i \Upsilon_j (d_j + \lambda_j) + n_j (\Upsilon_j - 1)(d_i + \lambda_i))^2} \right), \end{aligned}$$

$i, j = 1, 2$, $j \neq i$. After some tedious algebra, the determinant of $DF_{d_1, d_2}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2)$ is given by

$$\begin{aligned} & n_1 n_2^2 (n_1 - 1) \frac{\lambda_1 \Upsilon_1 n_2 + \lambda_2 n_1 (\Upsilon_1 - 1)}{(n_2 \Upsilon_1 (d_1 + \lambda_1) + n_1 (\Upsilon_1 - 1)(d_2 + \lambda_2))^3} + \\ & n_1^2 n_2 (n_2 - 1) \frac{\lambda_1 n_2 (\Upsilon_2 - 1) + \lambda_2 \Upsilon_2 n_1}{(n_1 \Upsilon_2 (d_2 + \lambda_2) + n_2 (\Upsilon_2 - 1)(d_1 + \lambda_1))^3} + \\ & \frac{n_1 n_2}{(n_2 \Upsilon_1 (d_1 + \lambda_1) + n_1 (\Upsilon_1 - 1)(d_2 + \lambda_2))^2 (n_1 \Upsilon_2 (d_2 + \lambda_2) + n_2 (\Upsilon_2 - 1)(d_1 + \lambda_1))^2} \times \\ & ((n_1 n_2 + (n_1 - 1)(n_2 - 1))(\lambda_1 n_2 (\Upsilon_2 - 1) + \lambda_2 \Upsilon_2 n_1)(\lambda_1 \Upsilon_1 n_2 + \lambda_2 n_1 (\Upsilon_1 - 1)) + \\ & n_2 (d_2 n_1 (\Upsilon_1 - 1) + \Upsilon_1 d_1 n_2)(\lambda_1 (\Upsilon_2 - 1)(n_1 - 1)(n_2 - 1) + \lambda_2 \Upsilon_2 n_1^2) + \\ & n_1 (d_1 n_2 (\Upsilon_2 - 1) + \Upsilon_2 d_2 n_1)(\lambda_1 \Upsilon_1 n_2^2 + \lambda_2 (\Upsilon_1 - 1)(n_1 - 1)(n_2 - 1))). \end{aligned}$$

Notice that all the summands in the previous expression are positive. In particular, the determinant of $DF_{d_1, d_2}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2)$ is not null and, therefore, this matrix is invertible. Hence, we can apply the Implicit Function Theorem, we have

$$\begin{pmatrix} \frac{\partial d_1}{\partial \sigma_{\varepsilon_1}^2} & \frac{\partial d_1}{\partial \sigma_{\varepsilon_2}^2} \\ \frac{\partial d_2}{\partial \sigma_{\varepsilon_1}^2} & \frac{\partial d_2}{\partial \sigma_{\varepsilon_2}^2} \end{pmatrix} =$$

$$- \left(DF_{d_1, d_2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) \right)^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial \sigma_{\varepsilon_1}^2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) & \frac{\partial F_1}{\partial \sigma_{\varepsilon_2}^2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) \\ \frac{\partial F_2}{\partial \sigma_{\varepsilon_1}^2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) & \frac{\partial F_2}{\partial \sigma_{\varepsilon_2}^2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) \end{pmatrix}. \quad (29)$$

Notice that

$$\left(DF_{d_1, d_2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) \right)^{-1} = \begin{pmatrix} \frac{\frac{\partial F_2}{\partial d_2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2)}{\det(DF_{d_1, d_2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2))} & -\frac{\frac{\partial F_1}{\partial d_2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2)}{\det(DF_{d_1, d_2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2))} \\ -\frac{\frac{\partial F_2}{\partial d_1} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2)}{\det(DF_{d_1, d_2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2))} & \frac{\frac{\partial F_1}{\partial d_1} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2)}{\det(DF_{d_1, d_2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2))} \end{pmatrix}.$$

Hence, we know that all the elements of $\left(DF_{d_1, d_2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) \right)^{-1}$ are positive. Moreover,

$$\begin{aligned} \frac{\partial F_i}{\partial \sigma_{\varepsilon_i}^2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) &= -\frac{d_i n_j (n_i - 1) (n_j (d_i + \lambda_i) + n_i (d_j + \lambda_j))}{(n_j \Upsilon_i (d_i + \lambda_i) + n_i (\Upsilon_i - 1) (d_j + \lambda_j))^2} \frac{\rho (1 + \rho)}{(1 - \rho) \sigma_{\theta}^2 (1 + \rho + \hat{\sigma}_{\varepsilon_i}^2)^2}, \text{ and} \\ \frac{\partial F_i}{\partial \sigma_{\varepsilon_j}^2} (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2) &= -\frac{d_i n_i n_j (n_j (d_i + \lambda_i) + n_i (d_j + \lambda_j))}{(n_i \Upsilon_j (d_j + \lambda_j) + n_j (\Upsilon_j - 1) (d_i + \lambda_i))^2} \frac{\rho (1 + \rho)}{(1 - \rho) \sigma_{\theta}^2 (1 + \rho + \hat{\sigma}_{\varepsilon_j}^2)^2}, \end{aligned}$$

$i, j = 1, 2, j \neq i$. Using all these expressions, (29) implies that $\frac{\partial d_i}{\partial \sigma_{\varepsilon_i}^2} > 0$ and $\frac{\partial d_i}{\partial \sigma_{\varepsilon_j}^2} > 0$.

Next, we study the comparative statics of c_1 and c_2 with respect to $\sigma_{\varepsilon_1}^2$. Recall that

$$\begin{aligned} c_1 &= a_1 / \Delta_1 = n_2 / (n_2 \Upsilon_1 (d_1 + \lambda_1) + n_1 (\Upsilon_1 - 1) (d_2 + \lambda_2)) \text{ and} \\ c_2 &= a_2 / \Delta_2 = n_1 / (n_1 \Upsilon_2 (d_2 + \lambda_2) + n_2 (\Upsilon_2 - 1) (d_1 + \lambda_1)). \end{aligned}$$

Using the fact that Υ_1 , d_1 , and d_2 are increasing in $\sigma_{\varepsilon_1}^2$ and that Υ_2 is independent of $\sigma_{\varepsilon_1}^2$, we have the denominators of the previous expressions are increasing in $\sigma_{\varepsilon_1}^2$, which allows us to conclude that c_1 and c_2 are decreasing in $\sigma_{\varepsilon_1}^2$. Combining these results with the fact that Δ_1 is decreasing in $\sigma_{\varepsilon_1}^2$ and Δ_2 is independent of $\sigma_{\varepsilon_1}^2$, it follows that a_1 and a_2 are decreasing in $\sigma_{\varepsilon_1}^2$, since $a_1 = \Delta_1 c_1$ and $a_2 = \Delta_2 c_2$.

Finally, concerning h), notice that doing a similar reasoning as before we derive the following equations that are satisfied in equilibrium: $F_i (n_1, n_2, d_1, d_2) = 0$, $i = 1, 2$, where $F_i (n_1, n_2, d_1, d_2) = F_i (\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2)$, $i, j = 1, 2, j \neq i$. Hence,

$$\begin{pmatrix} \frac{\partial d_1}{\partial n_1} & \frac{\partial d_1}{\partial n_2} \\ \frac{\partial d_2}{\partial n_1} & \frac{\partial d_2}{\partial n_2} \end{pmatrix} = - \left(DF_{d_1, d_2} (n_1, n_2, d_1, d_2) \right)^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial n_1} (n_1, n_2, d_1, d_2) & \frac{\partial F_1}{\partial n_2} (n_1, n_2, d_1, d_2) \\ \frac{\partial F_2}{\partial n_1} (n_1, n_2, d_1, d_2) & \frac{\partial F_2}{\partial n_2} (n_1, n_2, d_1, d_2) \end{pmatrix}.$$

In addition,

$$\begin{aligned} \frac{\partial F_i}{\partial n_i} (n_1, n_2, d_1, d_2) &= \\ d_i n_j \frac{(\Upsilon_i - 1) (d_j + \lambda_j) + \Upsilon_i n_j (d_i + \lambda_i)}{(n_j \Upsilon_i (d_i + \lambda_i) + n_i (\Upsilon_i - 1) (d_j + \lambda_j))^2} &+ \frac{d_i n_j^2 (\Upsilon_j - 1) (d_i + \lambda_i)}{(n_i \Upsilon_j (d_j + \lambda_j) + n_j (\Upsilon_j - 1) (d_i + \lambda_i))^2}, \text{ and} \end{aligned}$$

$$\frac{\partial F_i}{\partial n_j}(n_1, n_2, d_1, d_2) =$$

$$\frac{d_i n_i (\Upsilon_i - 1) (n_i - 1) (d_j + \lambda_j)}{(n_j \Upsilon_i (d_i + \lambda_i) + n_i (\Upsilon_i - 1) (d_j + \lambda_j))^2} + \frac{\Upsilon_j d_i n_i^2 (d_j + \lambda_j)}{(n_i \Upsilon_j (d_j + \lambda_j) + n_j (\Upsilon_j - 1) (d_i + \lambda_i))^2},$$

$i, j = 1, 2, j \neq i$. Therefore, all these partial derivatives are positive. Combining this result with the positiveness of all the elements of $(DF_{d_1, d_2}(n_1, n_2, d_1, d_2))^{-1}$,²² we conclude that $\frac{\partial d_i}{\partial n_i} < 0$ and $\frac{\partial d_i}{\partial n_j} < 0, i, j = 1, 2, j \neq i$. ■

Proof of Corollary 2: (i) Suppose that $\sigma_{\varepsilon_1}^2 \geq \sigma_{\varepsilon_2}^2$, $\lambda_1 \geq \lambda_2$, and $n_1 \geq n_2$. Using the expressions of Ξ_i and $\Delta_i, i = 1, 2$, it is easy to see that in this case $\Xi_2 \Delta_2^{-1} > \Xi_1 \Delta_1^{-1}$. Next, we distinguish two cases:

Case 1: $(n_1 + n_2 - 2) n_1 / ((n_1 + n_2) (n_1 + n_2 - 1)) \geq 1 - \Xi_2 \Delta_2^{-1}$. Evaluating the polynomial $p(z)$, stated in the proof of Lemma A2, at $z = 1$, we have that

$$p(1) = (n_1 + n_2) (n_1 + n_2 - 1)^2 \times \left(n_2 \left(\frac{(n_1 + n_2 - 2) n_1}{(n_1 + n_2) (n_1 + n_2 - 1)} - (1 - \Xi_2 \Delta_2^{-1}) \right) \lambda_1 - n_1 \left(\frac{(n_1 + n_2 - 2) n_2}{(n_1 + n_2) (n_1 + n_2 - 1)} - (1 - \Xi_1 \Delta_1^{-1}) \right) \lambda_2 \right).$$

As $(n_1 + n_2 - 2) n_1 / ((n_1 + n_2) (n_1 + n_2 - 1)) \geq 1 - \Xi_2 \Delta_2^{-1}$ and $\lambda_1 \geq \lambda_2$,

$$p(1) \geq (n_1 + n_2) (n_1 + n_2 - 1)^2 ((1 - \Xi_1 \Delta_1^{-1}) n_1 - n_2 (1 - \Xi_2 \Delta_2^{-1})) \lambda_2,$$

and since $n_1 \geq n_2$, $p(1) \geq (n_1 + n_2) (n_1 + n_2 - 1)^2 \lambda_2 n_2 (\Xi_2 \Delta_2^{-1} - \Xi_1 \Delta_1^{-1}) \geq 0$. This implies that $z \leq 1$, and therefore, $c_1 \leq c_2$. In addition, using the expressions of d_1 and d_2 , we get $\text{sgn}\{d_1 - d_2\} = \text{sgn}\{c_1 - c_2\}$, which implies $d_1 \leq d_2$. Finally, notice that $\Delta_1 \leq \Delta_2$ whenever $\sigma_{\varepsilon_1}^2 \geq \sigma_{\varepsilon_2}^2$. Hence, $a_1/a_2 = z \Delta_1/\Delta_2 \leq 1$.

Case 2: $(n_1 + n_2 - 2) n_1 / ((n_1 + n_2) (n_1 + n_2 - 1)) < 1 - \Xi_2 \Delta_2^{-1}$. Notice that

$$\frac{(n_1 + n_2 - 2) n_2}{(n_1 + n_2) (n_1 + n_2 - 1)} - (1 - \Xi_1 \Delta_1^{-1}) \leq \frac{(n_1 + n_2 - 2) n_1}{(n_1 + n_2) (n_1 + n_2 - 1)} - (1 - \Xi_2 \Delta_2^{-1}),$$

since $\Xi_2 \Delta_2^{-1} > \Xi_1 \Delta_1^{-1}$ and $n_1 \geq n_2$. Thus, in this case we have that

$$\begin{aligned} q_N(1) &= (n_1 + n_2 - 1) (n_1 + n_2) \left(\frac{(n_1 + n_2 - 2) n_2}{(n_1 + n_2) (n_1 + n_2 - 1)} - (1 - \Xi_1 \Delta_1^{-1}) \right) < 0 \text{ and} \\ q_D(1) &= (n_1 + n_2 - 1) (n_1 + n_2) \left(\frac{(n_1 + n_2 - 2) n_1}{(n_1 + n_2) (n_1 + n_2 - 1)} - (1 - \Xi_2 \Delta_2^{-1}) \right) < 0. \end{aligned}$$

Taking into account the shape of these polynomials, the previous two inequalities imply that $\bar{z}_D > 1 > \bar{z}_N$. However, Proposition 2 indicates that in this case there is no equilibrium.

²²Notice that $(DF_{d_1, d_2}(n_1, n_2, d_1, d_2))^{-1} = (DF_{d_1, d_2}(\sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2, d_1, d_2))^{-1}$.

(ii) By virtue of (17), the inequality $d_1 + \lambda_1 > d_2 + \lambda_2$ is equivalent to

$$\left(\Xi_1 - \frac{n_1 a_1}{n_2 a_2} \Psi_1 \right) / a_1 > \left(\Xi_2 - \frac{n_2 a_2}{n_1 a_1} \Psi_2 \right) / a_2.$$

Using (14) and (20), and after some algebra, the previous inequality is equivalent to

$$z < \frac{\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})}{\Xi_2 \Delta_2^{-1} + n_1 n_2^{-1} (1 - \Xi_1 \Delta_1^{-1})}. \quad (30)$$

We distinguish two cases:

Case I: $\frac{\Xi_2 \Delta_2^{-1} + \Xi_1 \Delta_1^{-1} - 1}{(1 - \Xi_1 \Delta_1^{-1}) n_1 n_2^{-1} + \Xi_2 \Delta_2^{-1}} \leq \frac{\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})}{n_1 (\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})) + (n_2 - 1) ((1 - \Xi_1 \Delta_1^{-1}) n_1 n_2^{-1} + \Xi_2 \Delta_2^{-1})}$. Using

the expressions of Ξ_i and Δ_i , we get that the previous inequality is equivalent to

$$\frac{(1 - \rho) n_1 n_2 (1 + \rho + \hat{\sigma}_{\varepsilon_1}^2)}{n_2 (1 - \rho^2 + \hat{\sigma}_{\varepsilon_1}^2) + n_1 \rho \hat{\sigma}_{\varepsilon_1}^2} + \frac{(1 - \rho) n_1 (n_2 - 1) (1 + \rho + \hat{\sigma}_{\varepsilon_2}^2)}{n_1 (1 - \rho^2 + \hat{\sigma}_{\varepsilon_2}^2) + n_2 \rho \hat{\sigma}_{\varepsilon_2}^2} \leq 1.$$

Moreover, after some algebra, we have that $q_D \left(\frac{\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})}{\Xi_2 \Delta_2^{-1} + n_1 n_2^{-1} (1 - \Xi_1 \Delta_1^{-1})} \right) \leq 0$. Consequently,

$\frac{\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})}{\Xi_2 \Delta_2^{-1} + n_1 n_2^{-1} (1 - \Xi_1 \Delta_1^{-1})} < \bar{z}_D < z$, which implies that in this case $d_1 + \lambda_1 < d_2 + \lambda_2$ holds.

Case II: $\frac{\Xi_2 \Delta_2^{-1} + \Xi_1 \Delta_1^{-1} - 1}{(1 - \Xi_1 \Delta_1^{-1}) n_1 n_2^{-1} + \Xi_2 \Delta_2^{-1}} > \frac{\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})}{n_1 (\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})) + (n_2 - 1) ((1 - \Xi_1 \Delta_1^{-1}) n_1 n_2^{-1} + \Xi_2 \Delta_2^{-1})}$. In this

case, taking into account that z is the unique positive solution of (21), (30) is equivalent to

$$\frac{\lambda_1}{\lambda_2} > \frac{\frac{\Xi_2 \Delta_2^{-1} + \Xi_1 \Delta_1^{-1} - 1}{(1 - \Xi_1 \Delta_1^{-1}) n_1 n_2^{-1} + \Xi_2 \Delta_2^{-1}} - \frac{(\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1}))}{(n_1 - 1) (\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})) + n_2 ((1 - \Xi_1 \Delta_1^{-1}) n_1 n_2^{-1} + \Xi_2 \Delta_2^{-1})}}{\frac{\Xi_2 \Delta_2^{-1} + \Xi_1 \Delta_1^{-1} - 1}{(1 - \Xi_1 \Delta_1^{-1}) n_1 n_2^{-1} + \Xi_2 \Delta_2^{-1}} - \frac{\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})}{n_1 (\Xi_1 \Delta_1^{-1} + n_2 n_1^{-1} (1 - \Xi_2 \Delta_2^{-1})) + (n_2 - 1) ((1 - \Xi_1 \Delta_1^{-1}) n_1 n_2^{-1} + \Xi_2 \Delta_2^{-1})}}.$$

Taking into account that $\sigma_{\varepsilon_1}^2 \geq \sigma_{\varepsilon_2}^2$, $\lambda_1 \geq \lambda_2$, and $n_1 \geq n_2$, and after some algebra, we get that both sides of the inequality are higher than (or equal to) 1. Therefore, if the value of λ_1/λ_2 is high enough, then we obtain $d_1 + \lambda_1 > d_2 + \lambda_2$. Otherwise, the opposite inequality holds. ■

Proof of Proposition 4: (i) First, suppose that $\bar{\theta}_1 = \bar{\theta}_2$. Then $E[p] = \bar{\theta}_1 - Q / \left(\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2} \right)$. From Proposition 3 we know that d_i and d_j decrease with n_i , and increase with $\sigma_{\varepsilon_i}^2$, λ_i , and ρ . Using these results in the previous expression, we conclude that the expected price is increasing in n_i and decreasing in λ_i , $\sigma_{\varepsilon_i}^2$, and ρ .

Now, suppose that $\bar{\theta}_1 \neq \bar{\theta}_2$. The results we have just derived may not hold if $|\bar{\theta}_1 - \bar{\theta}_2|$ is large enough. For example, let us focus on the relationship between the expected price and

n_1 . To study this relationship, we first show that $n_2 (d_1 + \lambda_1) / (n_1 (d_2 + \lambda_2))$ decreases with n_1 . Recall that

$$d_2 = 1 \left/ \left(\frac{n_1 n_2}{n_2 \Upsilon_1 (d_1 + \lambda_1) + n_1 (\Upsilon_1 - 1) (d_2 + \lambda_2)} + \frac{(n_2 - 1) n_1}{n_1 \Upsilon_2 (d_2 + \lambda_2) + n_2 (\Upsilon_2 - 1) (d_1 + \lambda_1)} \right) \right.$$

Hence,

$$1 = \left(\frac{n_2}{\Upsilon_1 \frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)} + \Upsilon_1 - 1} + \frac{n_2 - 1}{\Upsilon_2 + (\Upsilon_2 - 1) \frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}} \right)^{-1} + \frac{\lambda_2}{d_2 + \lambda_2}.$$

The fact that d_2 decreases with n_1 implies that $\lambda_2 / (d_2 + \lambda_2)$ increases with n_1 . Then, the previous inequality tells us that $\frac{n_2}{\Upsilon_1 \frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)} + \Upsilon_1 - 1} + \frac{n_2 - 1}{\Upsilon_2 + (\Upsilon_2 - 1) \frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}}$ increases with n_1 . For this to be possible, $\frac{n_2(d_1+\lambda_1)}{n_1(d_2+\lambda_2)}$ needs to be decreasing in n_1 . Given that the expected price satisfies

$$E[p] = \left(1 + \frac{n_2 (d_1 + \lambda_1)}{n_1 (d_2 + \lambda_2)} \right)^{-1} \bar{\theta}_1 + \left(1 - \left(1 + \frac{n_2 (d_1 + \lambda_1)}{n_1 (d_2 + \lambda_2)} \right)^{-1} \right) \bar{\theta}_2 - \left(\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2} \right)^{-1} Q,$$

we have that the relationship between the expected price and n_1 is ambiguous. For instance, if $\bar{\theta}_2$ is low enough, then the fact that d_1 , d_2 , and $n_2 (d_1 + \lambda_1) / (n_1 (d_2 + \lambda_2))$ are decreasing in n_1 allows us to conclude that the expected price increases with n_1 . However, if $\bar{\theta}_2$ is large and $\bar{\theta}_1$ and Q are low enough, then the expected price decreases with n_1 .

(ii) From the expression for the expected revenue it follows that $QE[p]$ increases with $\bar{\theta}_i$, $i = 1, 2$, and with Q , whenever $Q < \frac{n_1}{d_1 + \lambda_1} \bar{\theta}_1 + \frac{n_2}{d_2 + \lambda_2} \bar{\theta}_2$, or equivalently, $E[p] > 0$. In addition, using the expression for the expected price, it follows that

$$\left(\min \{ \bar{\theta}_1, \bar{\theta}_2 \} - \frac{Q}{\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2}} \right) \leq E[p] \leq \left(\max \{ \bar{\theta}_1, \bar{\theta}_2 \} - \frac{Q}{\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2}} \right).$$

Notice that left-hand side (*LHS*) and the right-hand side (*RHS*) of this expression corresponds to the expected revenue in an auction where all participants have an expected valuation of $\min \{ \bar{\theta}_1, \bar{\theta}_2 \}$ and of $\max \{ \bar{\theta}_1, \bar{\theta}_2 \}$, respectively. Using Proposition 4(i), we know that both *LHS* and *RHS* increase with n_i but decrease with λ_i and $\sigma_{\varepsilon_i}^2$. Hence, we obtain that $QE[p]$ is lower than the expected revenue of the symmetric auction in which both groups are ex-ante identical, with large size (each group with $\max \{ n_1, n_2 \}$ bidders), with high expected valuation ($\max \{ \bar{\theta}_1, \bar{\theta}_2 \}$), low transaction costs ($\min \{ \lambda_1, \lambda_2 \}$), and precise signals ($\min \{ \sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2 \}$), and larger than the expected revenue of the symmetric auction in which both groups are ex-ante identical but with the opposite characteristics (i.e., $\min \{ n_1, n_2 \}$, $\min \{ \bar{\theta}_1, \bar{\theta}_2 \}$, $\max \{ \lambda_1, \lambda_2 \}$, and $\max \{ \sigma_{\varepsilon_1}^2, \sigma_{\varepsilon_2}^2 \}$). ■

Proof of Proposition 5: Direct computation yields

$$\bar{z}_N = \frac{n_2 \left((n_1 - 1) (2\Xi_1 \Delta_1^{-1} - 1) - (2 - \Xi_1 \Delta_1^{-1}) + \sqrt{(2 - \Xi_1 \Delta_1^{-1})^2 + (n_1 - 1) (n_1 + 3 - 6\Xi_1 \Delta_1^{-1})} \right)}{2n_1 (n_1 - 1) (1 - \Xi_1 \Delta_1^{-1})}$$

and

$$\bar{z}_D = \frac{n_2 + 1 - \Xi_2 \Delta_2^{-1} (2n_2 - 1) + \sqrt{(2 - \Xi_2 \Delta_2^{-1})^2 + (n_2 - 1)(n_2 + 3 - 6\Xi_2 \Delta_2^{-1})}}{2\Xi_2 \Delta_2^{-1} n_1}.$$

Thus, when n_1 tends to infinity, note that $\lim_{n_1 \rightarrow \infty} \bar{z}_N = \lim_{n_1 \rightarrow \infty} \bar{z}_D = 0$. Furthermore, using the previous expressions and after some algebra, the necessary and sufficient condition for the existence of an equilibrium (i.e., $\lim_{n_1 \rightarrow \infty} \frac{\bar{z}_N}{\bar{z}_D} > 1$) is equivalent to $n_2 > \bar{n}_2(\rho, \hat{\sigma}_{\varepsilon_1}^2, \hat{\sigma}_{\varepsilon_2}^2)$, where

$$\bar{n}_2(\rho, \hat{\sigma}_{\varepsilon_1}^2, \hat{\sigma}_{\varepsilon_2}^2) = \frac{((2 - \rho)\hat{\sigma}_{\varepsilon_2}^2 + 2(1 - \rho^2))\hat{\sigma}_{\varepsilon_1}^2 \rho}{(1 - \rho^2)((1 + \hat{\sigma}_{\varepsilon_1}^2)(1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2)}.$$

Moreover, taking the limit in (21), it follows that $\lim_{n_1 \rightarrow \infty} z = 0$ and

$$\lim_{n_1 \rightarrow \infty} n_1 z = n_2 \Xi_1 \Delta_1^{-1} / (1 - \Xi_1 \Delta_1^{-1}). \quad (31)$$

Using the expressions included in the statement of Lemma A2, and after some tedious algebra, we get $\lim_{n_1 \rightarrow \infty} b_1 = q$, $\lim_{n_1 \rightarrow \infty} a_1 = 0$, $\lim_{n_1 \rightarrow \infty} c_1 = 0$, $\lim_{n_1 \rightarrow \infty} a_2 = \Delta_2 \lim_{n_1 \rightarrow \infty} c_2$,

$$\begin{aligned} \lim_{n_1 \rightarrow \infty} b_2 = & \frac{\hat{\sigma}_{\varepsilon_2}^2 \left(\frac{(n_2 - 1)(1 - \rho^2)}{(1 - \rho^2 + \hat{\sigma}_{\varepsilon_2}^2)} + \frac{(1 - \rho^2 + \hat{\sigma}_{\varepsilon_1}^2)(1 - 2\rho)}{(1 + \hat{\sigma}_{\varepsilon_1}^2)(1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2} \right) (q\rho\lambda_1 + \bar{\theta}_2 - \rho\bar{\theta}_1)}{(1 - \rho)\lambda_2 \left(n_2(1 + \rho) - \frac{\hat{\sigma}_{\varepsilon_1}^2 \rho(1 + \rho + \hat{\sigma}_{\varepsilon_2}^2)}{(1 + \hat{\sigma}_{\varepsilon_1}^2)(1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2} \right)} + \\ & + q \frac{\rho^2 \hat{\sigma}_{\varepsilon_2}^2 \hat{\sigma}_{\varepsilon_1}^2}{n_2(1 - \rho^2)((1 + \hat{\sigma}_{\varepsilon_1}^2)(1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2)}, \text{ and} \end{aligned} \quad (32)$$

$$\lim_{n_1 \rightarrow \infty} c_2 = \frac{n_2 - \bar{n}_2(\rho, \hat{\sigma}_{\varepsilon_1}^2, \hat{\sigma}_{\varepsilon_2}^2)}{\lambda_2 \frac{1 - \rho^2 + \hat{\sigma}_{\varepsilon_2}^2}{1 - \rho} \left(\frac{n_2}{(1 + \rho + \hat{\sigma}_{\varepsilon_2}^2)} - \rho \frac{\hat{\sigma}_{\varepsilon_1}^2}{(1 + \rho)((1 + \hat{\sigma}_{\varepsilon_1}^2)(1 + \hat{\sigma}_{\varepsilon_2}^2) - \rho^2)} \right)}. \quad (33)$$

Next, in relation to the expressions for d_1 and d_2 , we have that

$$\lim_{n_1 \rightarrow \infty} d_1 = \lim_{n_1 \rightarrow \infty} \frac{1}{(n_1 - 1)c_1 + n_2 c_2} = \lim_{n_1 \rightarrow \infty} \frac{1}{\left(\frac{(n_1 - 1)}{n_1} n_1 z + n_2 \right) c_2} = \frac{1}{\left(\lim_{n_1 \rightarrow \infty} n_1 z + n_2 \right) \lim_{n_1 \rightarrow \infty} c_2} > 0.$$

The fact that $n_1 z$ and c_2 converge to a positive finite number (see (31) and (33)) implies that d_1 does not converge to zero (provided that $\rho \hat{\sigma}_{\varepsilon_1}^2 > 0$; if $\rho \hat{\sigma}_{\varepsilon_1}^2 = 0$, then it is easy to see that $\lim_{n_1 \rightarrow \infty} n_1 z = \infty$). A similar result is obtained with the limit of d_2 . In particular,

$$\lim_{n_1 \rightarrow \infty} d_2 = \left(\left(\lim_{n_1 \rightarrow \infty} n_1 z + n_2 - 1 \right) \lim_{n_1 \rightarrow \infty} c_2 \right)^{-1} > \lim_{n_1 \rightarrow \infty} d_1 > 0. \blacksquare$$

Proof of Proposition 6: Suppose that n_1 and n_2 go to infinity and that $n_1/(n_1 + n_2)$ converges to μ_1 . Taking limits in the equation that characterizes z (i.e., Equation (21)) and operating, we have

$$z = \frac{\Xi_1 \Delta_1^{-1} + \mu_2 \mu_1^{-1} (1 - \Xi_2 \Delta_2^{-1}) \lambda_1 \lambda_2^{-1}}{\mu_1 \mu_2^{-1} (1 - \Xi_1 \Delta_1^{-1}) + \Xi_2 \Delta_2^{-1} \lambda_1 \lambda_2^{-1}}. \quad (34)$$

Moreover, taking the limit in the expressions of the equilibrium coefficients given in Lemma A2, it follows that

$$\begin{aligned} b_i &= \frac{\Psi_i}{\mu_i \mu_j} \frac{\mu_i \Xi_j \frac{a_i}{a_j} - \mu_j \Psi_j}{(\Xi_i \Xi_j - \Psi_i \Psi_j)} q + a_i \left(\frac{\Xi_j \bar{\theta}_i - \Psi_i \bar{\theta}_j}{\Xi_i \Xi_j - \Psi_i \Psi_j} - \bar{\theta}_i \right), \quad a_i = \Delta_i c_i, \quad i, j = 1, 2, \quad j \neq i, \\ c_1 &= \frac{\Xi_1 \Delta_1^{-1} - \mu_1 \mu_2^{-1} (1 - \Xi_1 \Delta_1^{-1}) z}{\lambda_1}, \quad \text{and} \quad c_2 = \frac{\Xi_2 \Delta_2^{-1} - \mu_2 \mu_1^{-1} (1 - \Xi_2 \Delta_2^{-1}) z^{-1}}{\lambda_2}. \end{aligned}$$

Substituting (34) in the previous expressions and after some algebra, we get

$$\begin{aligned} b_i &= \frac{\lambda_j \Psi_i}{\mu_j \lambda_i \Xi_j + \mu_i \lambda_j \Psi_i} q + a_i \left(\frac{\Xi_j \bar{\theta}_i - \Psi_i \bar{\theta}_j}{\Xi_i \Xi_j - \Psi_i \Psi_j} - \bar{\theta}_i \right), \quad a_i = \frac{\mu_j (\Xi_i \Xi_j - \Psi_i \Psi_j)}{\mu_j \lambda_i \Xi_j + \mu_i \lambda_j \Psi_i}, \quad \text{and} \\ c_i &= \frac{\mu_j (\Xi_j - \Psi_i)}{\mu_j \lambda_i \Xi_j + \mu_i \lambda_j \Psi_i}, \quad i, j = 1, 2, \quad j \neq i. \end{aligned}$$

Next, we derive the equilibrium in the following continuous setup: Consider now that there is a continuum of bidders $[0, 1]$. Let q denote the aggregate quantity supplied in the market. Suppose that a fraction μ_1 of these bidders are traders of type 1 and the remainder fraction, μ_2 , are bidders of type 2.

Consider a trader of type i . This bidder chooses to maximize

$$\mathbb{E}[\pi_i | s_i, p] = (\mathbb{E}[\theta_i | s_i, p] - p) x_i - \lambda_i x_i^2 / 2.$$

The FOC is given by $\mathbb{E}[\theta_i | s_i, p] - p - \lambda_i x_i = 0$, or equivalently,

$$X_i(s_i, p) = (\mathbb{E}[\theta_i | s_i, p] - p) / \lambda_i. \quad (35)$$

Positing linear strategies, the market clearing condition implies that

$$p = \frac{\mu_i (b_i + a_i s_i) + \mu_j (b_j + a_j s_j) - q}{\mu_i c_i + \mu_j c_j}, \quad (36)$$

provided that $\mu_i c_i + \mu_j c_j \neq 0$. Using the expression for p and assuming that $a_i \neq 0$, $i = 1, 2$, it follows that $\mathbb{E}[\theta_i | s_i, p] = \mathbb{E}[\theta_i | s_i, s_j]$. Hence, $\mathbb{E}[\theta_i | s_i, p] = \bar{\theta}_i + \Xi_i (s_i - \bar{\theta}_i) + \Psi_i (s_j - \bar{\theta}_j)$. Using (36), $s_j = (q - \mu_i b_i - \mu_j b_j - \mu_i a_i s_i + p (\mu_i c_i + \mu_j c_j)) / (\mu_j a_j)$. Therefore,

$$\mathbb{E}[\theta_i | s_i, p] = \bar{\theta}_i + \Xi_i (s_i - \bar{\theta}_i) + \Psi_i \left(\frac{q - \mu_i b_i - \mu_j b_j - \mu_i a_i s_i + p (\mu_i c_i + \mu_j c_j)}{\mu_j a_j} - \bar{\theta}_j \right).$$

Substituting this expression in (35), and identifying coefficients, it follows that

$$b_i = \left((1 - \Xi_i) \bar{\theta}_i - \Psi_i \bar{\theta}_j + \frac{\Psi_i (q - (\mu_i b_i + \mu_j b_j))}{a_j \mu_j} \right) / \lambda_i, \quad (37)$$

$$a_i = \left(\Xi_i - \Psi_i \frac{\mu_i a_i}{\mu_j a_j} \right) / \lambda_i, \text{ and} \quad (38)$$

$$c_i = \left(1 - \frac{\Psi_i}{\mu_j a_j} (\mu_i c_i + c_j \mu_j) \right) / \lambda_i, \quad i, j = 1, 2, \quad j \neq i. \quad (39)$$

Note that (38) implies that $a_i/a_j = \lambda_j (\Xi_i - \Psi_i \mu_i a_i / (\mu_j a_j)) / (\lambda_i (\Xi_j - \Psi_j \mu_j a_j / (\mu_i a_i)))$. Hence, $a_i/a_j = \mu_j (\Psi_j \lambda_i \mu_j + \Xi_i \lambda_j \mu_i) / (\mu_i (\Psi_i \lambda_j \mu_i + \lambda_i \Xi_j \mu_j))$. Then, plugging the previous expression into (38), we get

$$a_i = \mu_j (\Xi_i \Xi_j - \Psi_i \Psi_j) / (\mu_i \Psi_i \lambda_j + \mu_j \lambda_i \Xi_j). \quad (40)$$

Using (37), and after some algebra, we get

$$\mu_i b_i + \mu_j b_j = \frac{\frac{\mu_i}{\lambda_i} \left(\bar{\theta}_i (1 - \Xi_i) - \Psi_i \bar{\theta}_j + q \frac{\Psi_i}{\mu_j a_j} \right) + \frac{\mu_j}{\lambda_j} \left(\bar{\theta}_j (1 - \Xi_j) - \Psi_j \bar{\theta}_i + q \frac{\Psi_j}{\mu_i a_i} \right)}{\frac{\Psi_i}{\lambda_i} \frac{\mu_i}{\mu_j a_j} + \frac{\Psi_j}{\lambda_j} \frac{\mu_j}{\mu_i a_i} + 1}.$$

Substituting (40) and the last expression in (37),

$$b_i = \frac{\lambda_j \Psi_i}{\mu_i \lambda_j \Psi_i + \mu_j \lambda_i \Xi_j} q + \frac{\mu_j (\Xi_j \Xi_i - \Psi_j \Psi_i)}{\mu_i \lambda_j \Psi_i + \mu_j \lambda_i \Xi_j} \left(\frac{\Xi_j \bar{\theta}_i - \Psi_i \bar{\theta}_j}{\Xi_j \Xi_i - \Psi_j \Psi_i} - \bar{\theta}_i \right).$$

Moreover, from (39), and after some algebra, we get $\mu_i c_i + c_j \mu_j = \left(\frac{\mu_i}{\lambda_i} + \frac{\mu_j}{\lambda_j} \right) / \left(\frac{\mu_i}{\mu_j a_j} \frac{\Psi_i}{\lambda_i} + \frac{\mu_j}{\mu_i a_i} \frac{\Psi_j}{\lambda_j} + 1 \right)$. Using (40) and the last expression in (39), it follows that $c_i = \mu_j (\Xi_j - \Psi_i) / (\mu_j \lambda_i \Xi_j + \mu_i \lambda_j \Psi_i)$. Comparing the equilibrium coefficients of the limiting case with the ones of the continuous setup, we conclude that the equilibrium coefficients of the limiting case converge to the equilibrium coefficients of the continuous setup. Finally, taking into account the expressions for Ξ_i , Ξ_j , Ψ_i , and Ψ_j , we obtain the expressions stated in Proposition 6. ■

Lemma A3. *The equilibrium quantities solves the following distorted benefit maximization program:*

$$\begin{aligned} \max_{x_1, x_2} \mathbb{E} \left[n_1 \left(\theta_1 x_1 - (d_1 + \lambda_1) \frac{x_1^2}{2} \right) + n_2 \left(\theta_2 x_2 - (d_2 + \lambda_2) \frac{x_2^2}{2} \right) \middle| t \right] \\ \text{s.t. } n_1 x_1 + n_2 x_2 = Q, \end{aligned}$$

taking as given the equilibrium parameters d_1 and d_2 .

Proof: The Lagrangian function of the maximization program is given by

$$\mathcal{L}(x_1, x_2, \mu) = n_1 \left(t_1 x_1 - (d_1 + \lambda_1) \frac{x_1^2}{2} \right) + n_2 \left(t_2 x_2 - (d_2 + \lambda_2) \frac{x_2^2}{2} \right) - \mu (n_1 x_1 + n_2 x_2 - Q),$$

where μ denotes the Lagrange multiplier. Differentiating, we obtain the following FOCs:

$$n_1(t_1 - (d_1 + \lambda_1)x_1) - \mu n_1 = 0, \quad (41)$$

$$n_2(t_2 - (d_2 + \lambda_2)x_2) - \mu n_2 = 0, \text{ and} \quad (42)$$

$$n_1x_1 + n_2x_2 = Q. \quad (43)$$

From (41) and (42), it follows that $x_i = (t_i - \mu) / (d_i + \lambda_i)$, $i = 1, 2$. Substituting these expressions in (43) and operating, we have $\mu = \left(n_1 \frac{t_1}{d_1 + \lambda_1} + n_2 \frac{t_2}{d_2 + \lambda_2} - Q\right) \left(\frac{n_1}{d_1 + \lambda_1} + \frac{n_2}{d_2 + \lambda_2}\right)^{-1}$. Hence,

$$x_i = \frac{n_j(t_i - t_j)}{n_i(d_j + \lambda_j) + n_j(d_i + \lambda_i)} + \frac{d_j + \lambda_j}{n_i(d_j + \lambda_j) + n_j(d_i + \lambda_i)}Q, \quad i = 1, 2,$$

i.e., the equilibrium quantities. In addition, since the objective function is concave and the constraint is a linear equation, we conclude that the critical point is a global maximum. Hence, we conclude that the equilibrium quantities are the solutions of the optimization problem stated in Lemma A3. ■

Proof of Proposition 7: Recall that in the competitive setup, the FOC of the two optimization problems are given by $\mathbb{E}[\theta_i | s_i, p] - p - \lambda_i x_i = 0$, $i = 1, 2$. Doing similar computations as in the proof of Lemma A1, we derive the following system of equations:²³

$$\begin{aligned} b_i &= \frac{(1 - \Xi_i)\bar{\theta}_i - \Psi_i\bar{\theta}_j + \Psi_i\left(\frac{Q - n_i b_i - n_j b_j}{n_j a_j}\right)}{\lambda_i}, \quad a_i = \frac{\Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i}{\lambda_i}, \text{ and} \\ c_i &= \frac{1 - \Psi_i\left(\frac{n_i c_i + n_j c_j}{n_j a_j}\right)}{\lambda_i}, \quad i, j = 1, 2, j \neq i. \end{aligned}$$

Taking into account that $Q = (n_i + n_j)q$ and $\mu_i = n_i / (n_i + n_j)$, we have that the previous system is identical to the system of equations given in (37)-(39). As a consequence, the equilibrium coefficients given in the statement of Proposition 6 coincide with the equilibrium coefficients in the competitive setup. ■

Proof of Proposition 8: Performing similar computations as in the proof of Lemma A1, we obtain that the equilibrium coefficients with subsidies $\kappa_i = d_i(c_1^o, c_2^o)$ satisfy

$$\begin{aligned} b_i &= \frac{(1 - \Xi_i)\bar{\theta}_i - \Psi_i\bar{\theta}_j - \frac{\Psi_i(n_i b_i + n_j b_j - Q)}{n_j a_j}}{d_i + \lambda_i - d_i(c_1^o, c_2^o)}, \quad a_i = \frac{\Xi_i - \frac{n_i a_i}{n_j a_j} \Psi_i}{d_i + \lambda_i - d_i(c_1^o, c_2^o)} > 0, \text{ and} \\ c_i &= \frac{1 - \frac{\Psi_i(n_i c_i + n_j c_j)}{n_j a_j}}{d_i + \lambda_i - d_i(c_1^o, c_2^o)}, \quad i, j = 1, 2, j \neq i. \end{aligned}$$

²³To ease the notation the superscript o is omitted in this proof.

Comparing this system of equations and the one derived in the proof of Proposition 7, we obtain that the equilibrium coefficients of the price-taking equilibrium solves this system. Therefore, we conclude that the quadratic subsidies $\kappa_i x_i^2/2$, with $\kappa_i = d_i(c_1^o, c_2^o)$, $i = 1, 2$, induce an efficient allocation. ■

Lemma A4. *The expected deadweight loss at an anonymous allocation $(x_1(t), x_2(t))$ satisfies*

$$\mathbb{E}[DWL] = \frac{1}{2}\lambda_1 n_1 \mathbb{E}[(x_1(t) - x_1^o(t))^2] + \frac{1}{2}\lambda_2 n_2 \mathbb{E}[(x_2(t) - x_2^o(t))^2]. \quad (44)$$

Proof: Notice that $ETS = \mathbb{E}[\mathbb{E}[TS|t]]$, where

$$\begin{aligned} \mathbb{E}[TS|t] &= \mathbb{E}\left[n_1(\theta_1 x_1(t) - \lambda_1(x_1(t))^2/2) + n_2(\theta_2 x_2(t) - \lambda_2(x_2(t))^2/2) \mid t\right] = \\ &= n_1(t_1 x_1(t) - \lambda_1(x_1(t))^2/2) + n_2(t_2 x_2(t) - \lambda_2(x_2(t))^2/2). \end{aligned}$$

A Taylor series expansion of $\mathbb{E}[TS|t]$ around the price-taking equilibrium $(x_1^o(t), x_2^o(t))$, stopping at the second term due to the fact that $\mathbb{E}[TS|t]$ is quadratic, yields

$$\begin{aligned} \mathbb{E}[TS|t](x(t)) &= \mathbb{E}[TS|t](x^o(t)) + \nabla \mathbb{E}[TS|t](x^o(t))(x(t) - x^o(t)) + \\ &+ \frac{1}{2}(x(t) - x^o(t))' D^2 \mathbb{E}[TS|t](x^o(t))(x(t) - x^o(t)), \end{aligned}$$

where $\nabla \mathbb{E}[TS|t](x^o(t))$ and $D^2 \mathbb{E}[TS|t](x^o(t))$ are, respectively, the gradient and the Hessian matrix of $\mathbb{E}[TS|t]$ evaluated at $x^o(t)$. Notice that we know

$$\nabla \mathbb{E}[TS|t](x^o(t)) = (n_1(t_1 - \lambda_1 x_1^o(t)), n_2(t_2 - \lambda_2 x_2^o(t))).$$

Using the expressions of $x_1^o(t)$ and $x_2^o(t)$,

$$\nabla \mathbb{E}[TS|t](x^o(t))(x(t) - x^o(t)) = (n_1(t_1 - \lambda_1 x_1^o(t)), n_2(t_2 - \lambda_2 x_2^o(t))) \begin{pmatrix} x_1(t) - x_1^o(t) \\ x_2(t) - x_2^o(t) \end{pmatrix} =$$

$$\left(\frac{n_1 \frac{t_1}{\lambda_1} + n_2 \frac{t_2}{\lambda_2}}{\frac{n_1}{\lambda_1} + \frac{n_2}{\lambda_2}} - \frac{Q}{\frac{n_1}{\lambda_1} + \frac{n_2}{\lambda_2}} \right) (n_1(x_1 - x_1^o) + n_2(x_2 - x_2^o)) = \left(\frac{n_1 \frac{t_1}{\lambda_1} + n_2 \frac{t_2}{\lambda_2}}{\frac{n_1}{\lambda_1} + \frac{n_2}{\lambda_2}} - \frac{Q}{\frac{n_1}{\lambda_1} + \frac{n_2}{\lambda_2}} \right) (Q - Q) = 0.$$

In addition, $D^2 \mathbb{E}[TS|t](x^o(t)) = \begin{pmatrix} -\lambda_1 n_1 & 0 \\ 0 & -\lambda_2 n_2 \end{pmatrix}$. Hence,

$$\mathbb{E}[TS|t](x(t)) - \mathbb{E}[TS|t](x^o(t)) = -\frac{1}{2}\lambda_1 n_1 (x_1(t) - x_1^o(t))^2 - \frac{1}{2}\lambda_2 n_2 (x_2(t) - x_2^o(t))^2$$

and, therefore, (44) is satisfied. ■

Proof of Proposition 9: i) Suppose that $Q = 0$. Then, $\mathbb{E}[DWL]$ is given by

$$\mathbb{E}[DWL] = \frac{n_2 n_1 (n_2 d_1 + n_1 d_2)^2}{2(n_2 \lambda_1 + n_1 \lambda_2)(n_2(d_1 + \lambda_1) + n_1(d_2 + \lambda_2))^2} \mathbb{E}(t_1 - t_2)^2.$$

Hence, $\frac{d}{d\sigma_{\varepsilon_1}^2}\mathbb{E}[DWL] = \frac{\partial}{\partial d_1}\mathbb{E}[DWL]\frac{d}{d\sigma_{\varepsilon_1}^2}d_1 + \frac{\partial}{\partial d_2}\mathbb{E}[DWL]\frac{d}{d\sigma_{\varepsilon_1}^2}d_2 + \frac{\partial}{\partial \sigma_{\varepsilon_1}^2}\mathbb{E}[DWL]$. It is easy to see that in this case $\frac{\partial}{\partial d_i}\mathbb{E}[DWL] > 0$, $i = 1, 2$, and $\frac{\partial}{\partial \sigma_{\varepsilon_1}^2}\mathbb{E}[DWL] < 0$. Combining these results with Proposition 3, we have that the first two terms of $\frac{d}{d\sigma_{\varepsilon_1}^2}\mathbb{E}[DWL]$ are positive, while the last one is negative.

We know that d_1 and d_2 are independent of $\sigma_{\varepsilon_1}^2$ when $\rho = 0$. By continuity, we know that for very low values of ρ is $\frac{d}{d\sigma_{\varepsilon_1}^2}d_1$ and $\frac{d}{d\sigma_{\varepsilon_1}^2}d_2$ are positive, but very low. Hence, we conclude that the last term in $\frac{d}{d\sigma_{\varepsilon_1}^2}\mathbb{E}[DWL]$ dominates and, hence, in this case we have that $\frac{d}{d\sigma_{\varepsilon_1}^2}\mathbb{E}[DWL] < 0$ although $\frac{d}{d\sigma_{\varepsilon_1}^2}d_i > 0$, $i = 1, 2$. By contrast, if we consider the case in which ρ is not low and $(\bar{\theta}_1 - \bar{\theta}_2)^2$ is high enough, then the first two terms in $\frac{d}{d\sigma_{\varepsilon_1}^2}\mathbb{E}[DWL]$ dominate, which implies that $\frac{d}{d\sigma_{\varepsilon_1}^2}\mathbb{E}[DWL] > 0$, $i = 1, 2$.

ii) Omitted since it is trivial.

iii) When groups are symmetric ($n_2 = n_1 = n$, $\lambda_1 = \lambda_2 = \lambda$, and $\sigma_{\varepsilon_1}^2 = \sigma_{\varepsilon_2}^2 = \sigma_{\varepsilon}^2$) $d_1 = d_2 = d$ and $\lambda_2 d_1 - \lambda_1 d_2 = 0$. Therefore, the expected deadweight loss consists of only one term, the first one, that is independent of Q . Moreover, after some computations, we get that $\mathbb{E}[DWL|t] = \frac{nd^2}{4(d+\lambda)^2\lambda}(t_2 - t_1)^2$. Using this expression and the fact that $\frac{\partial}{\partial d}\mathbb{E}[DWL|t] > 0$ allows us to conclude that an increase in an information parameter (ρ or $\hat{\sigma}_{\varepsilon}^2$) raises both d and $\mathbb{E}[DWL|t]$.

However, with asymmetric groups the previous results may not hold. In this case, suppose that Q is large enough. Then, for $i = 1, 2$, $j \neq i$,

$$\begin{aligned} \text{sign}\left(\frac{\partial}{\partial d_i}E[DWL|t]\right) &= \text{sign}\left(\frac{\partial}{\partial d_i}\left(\frac{1}{2}n_1n_2\frac{(\lambda_1d_2 - \lambda_2d_1)^2}{(\lambda_1n_2 + \lambda_2n_1)(n_1(d_2 + \lambda_2) + n_2(d_1 + \lambda_1))^2}Q^2\right)\right) \\ &= \text{sign}\left(-Q^2n_1n_2(\lambda_j + d_j)\frac{\lambda_id_j - \lambda_jd_i}{(n_1(d_2 + \lambda_2) + n_2(d_1 + \lambda_1))^3}\right) \end{aligned}$$

When Q is large enough we have that if $\frac{d_i}{d_j} > \frac{\lambda_i}{\lambda_j}$, then $\frac{\partial}{\partial d_i}E[DWL|t] > 0$ and $\frac{\partial}{\partial d_j}E[DWL|t] < 0$. Thus, with asymmetric groups market power (d_1, d_2) and the $E[DWL|t]$ are not always positively associated, given predicted values t , for changes in information parameters. ■

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