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The Transfer Principle holds for definable nonstandard models under Countable Choice

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The Transfer Principle holds for definable nonstandard models under Countable Choice

Frederik S. Herzberg

Abstract. Łoś's theorem for (bounded) D-ultrapowers, D being the ultrafilter introduced by Kanovei and Shelah [*Journal of Symbolic Logic*, 69(1):159–164, 2004], can be established within Zermelo–Fraenkel set theory plus Countable Choice (ZF+AC$_\omega$). Thus, the Transfer Principle for both Kanovei and Shelah's definable nonstandard model of the reals and Herzberg's definable nonstandard enlargement of the superstructure over the reals [*Mathematical Logic Quarterly*, 54(2):167–175; 54(6):666–667, 2008] can be shown in ZF+AC$_\omega$. This establishes a conjecture by Mikhail Katz [personal communication].

1. Introduction

Nonstandard analysis is often viewed as inherently non-constructive. Even at a formal level, nonstandard analysis and constructive analysis are traditionally understood as "antipodes" (Schuster, Berger and Osswald [**13**]). One of the reasons is that nonstandard models are typically "constructed" using non-principal ultrafilters. And while the existence of non-principal ultrafilters, being the dual version of the Boolean Prime Ideal Theorem, is strictly weaker than the Axiom of Choice (Halpern and Levy [**2**]), it is not demonstrable within Zermelo–Fraenkel set theory (ZF) alone. For example, the existence of a non-principal ultrafilter on the set of positive integers implies the existence of non-Lebesgue measurable subsets of the continuum

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(Sierpiński [**14**]), while it is consistent with ZF that the continuum does not have such subsets (Solovay [**15**]).

In fact, as Kanovei and Shelah [**9**, footnote 1] have observed, citing earlier work by Luxemburg [**10**], the following is true: If there is a non-standard model of the reals, then there is a non-principal ultrafilter on the natural numbers (and thus a non-Lebesgue measurable subset of the continuum).

To be sure, the metamathematics of nonstandard analysis has been studied for several decades—and has been surveyed and further developed by Kanovei and Reeken [**8**]. Yet, this literature produced some surprises during the last twelve years: First, Kanovei and Shelah [**9**] established, in Zermelo–Fraenkel set theory plus Axiom of Choice (ZFC), the existence of a definable nonstandard model of the reals. With a similar methodology, it was shown elsewhere [**3, 4**], again in ZFC, that there are even definable nonstandard enlargements of the full superstructure over the reals. Mikhail Katz (personal communication) posed the question whether the Transfer Principle in Kanovei and Shelah [**9**, Theorem 3.2] can be established even in Zermelo–Fraenkel set theory plus Countable Choice (ZF+AC$_\omega$). This short note provides an affirmative answer, for the Transfer Principles of both Kanovei and Shelah's definable nonstandard model of the reals and the definable nonstandard enlargement devised in [**3, 4**].

2. Framework

Let A be a non-empty set. For Theorems 2 and 4, we shall assume that A is the set of all functions $\beth_1 \to 2^{\mathbb{N}}$ whose range is an ultrafilter in the power-set algebra $2^{\mathbb{N}}$, with $\beth_1$ denoting (the cardinality of) the continuum.

Let $\mathcal{H}$ be the set of *finite-support subsets* of $\mathbb{N}^A$: A set $X \subseteq \mathbb{N}^A$ is in $\mathcal{H}$ if and only if there exists some finite $u \subseteq A$ such that

$$\forall g, h \in \mathbb{N}^A \quad (g \restriction u = h \restriction u \Rightarrow (g \in X \Leftrightarrow h \in X)) .$$

There is a smallest such u, called the *support* of X and denoted by $\|X\|$ (as was shown elsewhere [**3**, Lemma 2.1] and already stated by Kanovei and Shelah [**9**, p. 160]). Hence whenever $X \in \mathcal{H}$, membership in X can be decided, uniformly in $\mathbb{N}^A$, by evaluating elements of $\mathbb{N}^A$ only at a finite number of elements of A.

An $\mathbb{N}^A$-indexed sequence of sets $(x_g)_{g\in\mathbb{N}^A}$ is said to be *concentrated on a finite set* if and only if there exists some finite $u \subseteq A$ such that for all $g, h : A \to \mathbb{N}$, if $g \restriction u = h \restriction u$, then also $x_g = x_h$.

As for elements of $\mathcal{H}$, there is a smallest such u, which is called the *support* of $(x_g)_{g\in\mathbb{N}^A}$ and will be denoted by $\left|(x_g)_{g\in\mathbb{N}^A}\right|$.

Let D be an ultrafilter in the algebra $\mathcal{H}$ of finite-support subsets of $\mathbb{N}^A$ (not in the power-set algebra of $\mathbb{N}^A$!). The ultrafilter D induces an equivalence relation $\sim_D$ among $\mathbb{N}^A$-indexed sequences $(x_g)_{g\in\mathbb{N}^A}$ defined by

$$(x_g)_{g\in\mathbb{N}^A} \sim_D (y_g)_{g\in\mathbb{N}^A} \Longleftrightarrow \left\{g \in \mathbb{N}^A \ : \ x_g = y_g\right\} \in D.$$

The equivalence class of a sequence $(x_g)_{g\in\mathbb{N}^A}$ with respect to $\sim_D$ shall be denoted $\left[(x_g)_{g\in\mathbb{N}^A}\right]_D$. We shall later define D-ultrapowers and bounded D-ultrapowers as sets of D-equivalence classes of $\mathbb{N}^A$-indexed sequences

$(x_g)_{g\in\mathbb{N}^A}$ (of elements of the base structure) that are concentrated on a finite set.

Let $V(\mathbb{R})$ be the superstructure over the reals. Let $\mathcal{L}_{V(\mathbb{R})}$ be the language with one binary relation $\dot{\in}$ and card $(V(\mathbb{R}))$ many constant symbols, viz. one symbol $\dot{x}$ for each element $x \in V(\mathbb{R})$.

The *bounded D-ultrapower* $\overline{\mathcal{M}}$ of an $\mathcal{L}_{V(\mathbb{R})}$-structure $\mathcal{M}$ is defined as follows:

$$\overline{M} = \left\{ \left[(x_g)_{g\in\mathbb{N}^A}\right]_D \ : \ \begin{array}{c} \exists n \in \mathbb{N} \quad \left\{g \in \mathbb{N}^A \ : \ \mathcal{M} \models x_g \dot{\in} V_n(\dot{\mathbb{R}})\right\} \in D, \\ (x_g)_g \text{ concentrated on a finite set} \end{array} \right\},$$

$$\in^{\overline{\mathcal{M}}} := \left\{ \left(\left[(x_g)_g\right]_D, \left[(y_g)_g\right]_D\right) \ : \ \begin{array}{c} \left\{g \in \mathbb{N}^A \ : \ \mathcal{M} \models x_g \dot{\in} y_g\right\} \in D, \\ (x_g)_g, (y_g)_g \text{ concentrated on a finite set} \end{array} \right\}$$

Put differently, $\overline{\mathcal{M}} \models \left[(x_g)_{g\in\mathbb{N}^A}\right]_D \dot{\in} \left[(y_g)_{g\in\mathbb{N}^A}\right]_D$ if and only if the set of those $g \in \mathbb{N}^A$ that satisfies $\mathcal{M} \models x_g \dot{\in} y_g$ belongs to D.

In order to define the simpler notion of D-ultrapowers, let $\mathcal{P}$ be the set of all finitary relations on $\mathbb{R}$. Let $\mathcal{L}_{\mathbb{R}}$ be the language containing a symbol for each element of $\mathcal{P}$, and let $\mathcal{R} = (\mathbb{R}, \mathcal{P})$ be the reals understood as an $\mathcal{L}_{\mathbb{R}}$-structure. The D-ultrapower ${}^*\mathcal{R}$ of $\mathcal{R}$ is then the $\mathcal{L}_{\mathbb{R}}$-structure ${}^*\mathcal{R} = ({}^*\mathbb{R}, ({}^*E)_{E\in\mathcal{P}})$ defined as follows:

$${}^*\mathbb{R} = \left\{ \left[(x_g)_{g\in\mathbb{N}^A}\right]_D \ : \ (x_g)_g \in \mathbb{R}^{\mathbb{N}^A} \wedge [(x_g)_g]_D \text{ concentrated on a finite set} \right\}$$

and for all n-ary $E \in \mathcal{P}$ and $\left[\left(x_g^{(1)}\right)_g\right]_D, \ldots, \left[\left(x_g^{(n)}\right)_g\right]_D \in {}^*\mathbb{R}$,

$${}^*E\left(\left[\left(x_g^{(1)}\right)_g\right]_D, \ldots, \left[\left(x_g^{(n)}\right)_g\right]_D\right) \Leftrightarrow \left\{g \in \mathbb{N}^A \ : \ E\left(x_g^{(1)}, \ldots, x_g^{(n)}\right)\right\} \in D.$$

It will often be helpful to use the abbreviation $\overline{x} := \left[(x_g)_{g\in\mathbb{N}^A}\right]_D$ where $(x_g)_{g\in\mathbb{N}^A}$ can be any $\mathbb{N}^A$-indexed sequence.

3. Results

THEOREM 1 (ZF+AC$_\omega$). *Let $\phi(v_1, \ldots, v_n)$ be an $\mathcal{L}_{V(\mathbb{R})}$-formula with bounded quantifiers and n free variables. Then, for all $\overline{x^{(1)}}, \ldots, \overline{x^{(n)}} \in \overline{M}$,*

$$\overline{\mathcal{M}} \models \phi\left[\overline{x^{(1)}}, \ldots, \overline{x^{(n)}}\right] \Longleftrightarrow \left\{g \in \mathbb{N}^A \ : \ \mathcal{M} \models \phi\left[x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \in D.$$

This result has already been shown in ZFC [**3, 4**]. Kanovei and Shelah [**9**, Lemma 3.3] proved a corresponding result for D-ultrapowers of $\mathbb{R}$ (as opposed to bounded D-ultrapowers of $V(\mathbb{R})$).

In Kanovei and Shelah [**9**], the following fact is implicit, as was observed elsewhere [**4**, Lemma 2].

LEMMA 1 (ZF). *Let $\phi(v_1, \ldots, v_n)$ be an $\mathcal{L}_{V(\mathbb{R})}$-formula (with bounded quantifiers and) n free variables. Then, $\left\{g \in \mathbb{N}^A \ : \ \mathcal{M} \models \phi\left[x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \in \mathcal{H}$ for all $\overline{x^{(1)}}, \ldots, \overline{x^{(n)}} \in \overline{M}$.*

The significance of Theorem 1 is that it permits the proof of the following:

Theorem 2 (ZF+AC$_\omega$). *There is a definable set* ${}^*\mathbb{R}$ *and a definable injection* $^* : V(\mathbb{R}) \hookrightarrow V({}^*\mathbb{R})$ *such that* $^* : V_n(\mathbb{R}) \hookrightarrow V_n({}^*\mathbb{R})$ *for all* $n \in \mathbb{N}_0$ *and such that* * *satisfies the following:*

(1) Transfer Principle. *Whenever* $\phi(v_1, \ldots, v_n)$ *is an* $\in$*-formula with bounded quantifiers and* $a_1, \ldots, a_n \in V(\mathbb{R})$,

$$\phi[a_1, \ldots, a_n] \text{ holds in } V(\mathbb{R}) \Leftrightarrow \phi[{}^*a_1, \ldots, {}^*a_n] \text{ holds in } V({}^*\mathbb{R}).$$

(2) Internal Definition Principle. *Whenever* B_0 *is an internal set,* $b_1, \ldots, b_n$ *are internal and* ϕ *is an* $\in$*-formula with* $n+1$ *free variables, then* $\{x \in B_0 \ : \ \phi[x, b_1, \ldots, b_n]\}$ *also is an internal set.*

Moreover, similarly to the proof of Theorem 1, one can show that the analogue of Łoś's theorem for "ordinary" (as opposed to bounded) D-ultrapowers, which Kanovei and Shelah [**9**, Lemma 3.3 (Łoś)] showed in ZFC, is actually provable in ZF+AC$_\omega$.

Theorem 3 (ZF+AC$_\omega$). *If* $\phi(v_1, \ldots, v_n)$ *is any* $\mathcal{L}_\mathbb{R}$*-formulae with* n *free variables and* $\overline{x^{(1)}}, \ldots, \overline{x^{(n)}} \in {}^*\mathbb{R}$, *then*

$${}^*\mathcal{R} \models \phi\left[\overline{x^{(1)}}, \ldots, \overline{x^{(n)}}\right] \Longleftrightarrow \left\{g \in \mathbb{N}^A \ : \ \mathcal{R} \models \phi\left[x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \in D.$$

Thus, Kanovei and Shelah's [**9**, Theorem 3.2] result about the Transfer Principle in their definable nonstandard model of the reals holds in ZF+AC$_\omega$, too:

Theorem 4 (ZF+AC$_\omega$). *There is a definable set* ${}^*\mathbb{R}$ *and a definable injection* $^* : \mathbb{R} \hookrightarrow {}^*\mathbb{R}$ *which is an elementary embedding. In other words, the* Transfer Principle *holds: Whenever* $\phi(v_1, \ldots, v_n)$ *is an* $\mathcal{L}(\mathbb{R})$*-formula* $a_1, \ldots, a_n \in \mathbb{R}$,

$$\mathcal{R} \models \phi[a_1, \ldots, a_n] \Leftrightarrow {}^*\mathcal{R} \models \phi[{}^*a_1, \ldots, {}^*a_n].$$

4. Proofs

It is easiest to follow the proofs when they are studied in reverse order:

Proof of Theorem 4. The proof of Theorem 4 from Theorem 3 is identical to the proof in Kanovei and Shelah [**9**, Proof of Theorem 3.2], which was based on the analogue of Łoś's theorem for D-ultrapowers [**9**, Lemma 3.3 (Łoś)]. □

Proof of Theorem 3. The proof proceeds by induction in the complexity of $\phi(v_1, \ldots, v_n)$:

(1) If ϕ is atomic, then the Theorem is just the definition of truth in the D-ultrapower ${}^*\mathcal{R}$.
(2) If $\phi \equiv \dot{\neg}\psi$, we exploit the ultrafilter property of D, which ensures that for all $X \in \mathcal{H}$, $X \notin D$ if and only if $\mathbb{N}^A \setminus X \in D$. In light of Lemma 1, we may apply this observation to the set $X = \left\{g \in \mathbb{N}^A \ : \ {}^*\mathcal{R} \models \psi\left[x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\}$. Combining this with

the induction hypothesis and Tarski's definition of truth, we get

$$\begin{aligned} & {}^*\mathcal{R} \models \phi\left[\overline{x^{(1)}}, \ldots, \overline{x^{(n)}}\right] \Leftrightarrow {}^*\mathcal{R} \not\models \psi\left[\overline{x^{(1)}}, \ldots, \overline{x^{(n)}}\right] \\ \Leftrightarrow \quad & \left\{g \in \mathbb{N}^A \ : \ \mathcal{R} \models \psi\left[x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \notin D \\ \Leftrightarrow \quad & \left\{g \in \mathbb{N}^A \ : \ \mathcal{R} \not\models \psi\left[x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \in D \\ \Leftrightarrow \quad & \left\{g \in \mathbb{N}^A \ : \ \mathcal{R} \models \phi\left[x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \in D. \end{aligned}$$

(3) Next, let $\phi \equiv \psi \dot{\wedge} \chi$. The closedness of the filter D under intersections and supersets yields that for all $X, Y \in \mathcal{H}$, one has $X \cap Y \in D$ if and only if $X, Y \in D$. Again, Lemma 1 allows us to apply this observation to $X = \left\{g \in \mathbb{N}^A \ : \ \mathcal{R} \models \psi\left[x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\}$ and $Y = \left\{g \in \mathbb{N}^A \ : \ \mathcal{R} \models \chi\left[x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\}$.

(4) Finally, let $\phi \equiv \dot{\exists} y \ \ \psi(\dot{y}, \dot{v}_1, \ldots, \dot{v}_n)$. First, suppose that ${}^*\mathcal{R} \models \phi\left[\overline{x^{(1)}}, \ldots, \overline{x^{(n)}}\right]$. Then there is some $\overline{y} = \left[(y_g)_g\right]_D \in {}^*\mathbb{R}$ such that ${}^*\mathcal{R} \models \psi\left[\overline{y}, \overline{x^{(1)}}, \ldots, \overline{x^{(n)}}\right]$. By induction hypothesis, $\left\{g \in \mathbb{N}^A \ : \ \mathcal{R} \models \psi\left[y_g, x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \in D$, and by the closedness of D under supersets, at last also $\left\{g \in \mathbb{N}^A \ : \ \mathcal{R} \models \dot{\exists} \dot{y} \ \ \psi\left[\dot{y}, x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \in D$.

For the converse, suppose $I_x := \left\{g \in \mathbb{N}^A \ : \ \mathcal{R} \models \dot{\exists} \dot{y} \ \psi\left[\dot{y}, x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \in D$. Define $A_x := \bigcup_{i=1}^n \left|\left(x_g^{(i)}\right)_g\right| \subseteq A$ and first note that

(1)
$$\forall g, h \in \mathbb{N}^A \qquad \left(g \restriction A_x = h \restriction A_x \Longrightarrow \left(x_g^{(1)} = x_h^{(1)} \wedge \cdots \wedge x_g^{(n)} = x_h^{(n)}\right)\right)$$

For $f \in \mathbb{N}^{A_x}$, let $g_f \in \mathbb{N}^A$ be defined by $g_f \restriction A_x = f$ and $g_f(c) = 0$ for all $c \in A \setminus A_x$. Let $\bar{g} := g_{g \restriction A_x}$. Then in light of the implication (1), we have

$$\forall g \in \mathbb{N}^A \qquad x_g^{(1)} = x_{\bar{g}}^{(1)} \wedge \cdots \wedge x_g^{(n)} = x_{\bar{g}}^{(n)} \tag{2}$$

for all $g \in \mathbb{N}^A$. This means in particular that

$$\forall g \in \mathbb{N}^A \qquad (g \in I_x \Longleftrightarrow \bar{g} \in I_x)\,. \tag{3}$$

Furthermore, $\mathbb{N}^{A_x}$ is countable and therefore, $\left\{\bar{g} \ : \ g \in \mathbb{N}^A\right\}$ is countable, too. Thus, the set $\left\{\left\{y \in \mathbb{R} \ : \ \mathcal{R} \models \psi\left[y, x_{\bar{g}}^{(1)}, \ldots, x_{\bar{g}}^{(n)}\right]\right\} \ : \ g \in \mathbb{N}^A\right\}$ is a countable collection of non-empty sets. Therefore, by AC_ω, there is a choice function λ that assigns to every $g \in I_x$ which is constantly $= 0$ on $A \setminus A_x$ some element $\lambda(g) = y_g \in \mathbb{R}$ such that $\mathcal{R} \models \psi\left[y_g, x_{\bar{g}}^{(1)}, \ldots, x_{\bar{g}}^{(n)}\right]$. For arbitrary $g \in I_x$, put

$y_g := \lambda(\bar{g})$. Then, in light of the above equation (2), one has $\mathcal{R} \models \psi\left[y_g, x_g^{(1)}, \ldots, x_g^{(n)}\right]$.

For all $g \notin I_x$, put $y_g = 0$. Note that (through the use of the choice function λ and in light of the implication (1)), the map $g \mapsto y_g$ has been constructed in such a way that $y_g = y_h$ whenever $g, h \in \mathbb{N}^A$ agree on the finite set $A_x = \bigcup_{i=1}^n \left|\left(x_g^{(i)}\right)_g\right| \subseteq A$. This means that $(y_g)_{g\in\mathbb{N}^A}$ is concentrated on a finite set, viz. A_x. Therefore, $\overline{y} \in {}^*\mathbb{R}$. Now by the construction of $(y_g)_{g\in\mathbb{N}^A}$, we also have $\left\{g \in \mathbb{N}^A \,:\, \mathcal{R} \models \psi\left[y_g, x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \supseteq I_x \in D$, hence $\left\{g \in \mathbb{N}^A \,:\, \mathcal{R} \models \psi\left[y_g, x_g^{(1)}, \ldots, x_g^{(n)}\right]\right\} \in D$. By induction hypothesis, this entails ${}^*\mathcal{R} \models \psi\left[\overline{y}, \overline{x^{(1)}}, \ldots, \overline{x^{(n)}}\right]$ and therefore ${}^*\mathcal{R} \models \dot{\exists}\dot{y} \quad \psi\left[\dot{y}, \overline{x^{(1)}}, \ldots, \overline{x^{(n)}}\right]$.

□

Proof of Theorem 2. Apart from the analogue of Łoś's theorem for bounded D-ultrapowers (Theorem 1 of the present paper), the construction of a nonstandard enlargement in the earlier paper [**3**] does not invoke Choice. Therefore, the Transfer Principle (and its consequence, the Internal Definition Principle) for this model hold in ZF+AC$_\omega$, too. □

Proof of Theorem1. The proof proceeds by induction in the complexity of $\phi(v_1, \ldots, v_n)$. Most of the original proof [**4**, Proof of Theorem 1] only uses ZF and can be just copied. The only exception is the last part of the proof, viz. the demonstration that $I_x := \left\{g \in \mathbb{N}^A \,:\, \mathcal{M} \models \dot{\exists}\dot{y}\dot{\in}x_g^{(1)}\ \psi\left[\dot{y}, x_g^{(2)}, \ldots, x_g^{(n)}\right]\right\} \in D$ implies $\overline{\mathcal{M}} \models \dot{\exists}\dot{y}\dot{\in}\overline{x^{(1)}}\ \psi\left[\dot{y}, \overline{x^{(2)}}, \ldots, \overline{x^{(n)}}\right]$.

Suppose $I_x := \left\{g \in \mathbb{N}^A \,:\, \mathcal{M} \models \dot{\exists}\dot{y}\dot{\in}x_g^{(1)}\ \psi\left[\dot{y}, x_g^{(2)}, \ldots, x_g^{(n)}\right]\right\} \in D$. Define $A_x := \bigcup_{i=1}^n \left|\left(x_g^{(i)}\right)_g\right| \subseteq A$ and first note that

(4)
$$\forall g, h \in \mathbb{N}^A \qquad \left(g \restriction A_x = h \restriction A_x \Longrightarrow \left(x_g^{(1)} = x_h^{(1)} \wedge \cdots \wedge x_g^{(n)} = x_h^{(n)}\right)\right)$$

For $f \in \mathbb{N}^{A_x}$, let $g_f \in \mathbb{N}^A$ be defined by $g_f \restriction A_x = f$ and $g_f(c) = 0$ for all $c \in A \setminus A_x$. Let $\bar{g} := g_{g\restriction A_x}$. Then in light of the implication (4), we have

$$\forall g \in \mathbb{N}^A \qquad x_g^{(1)} = x_{\bar{g}}^{(1)} \wedge \cdots \wedge x_g^{(n)} = x_{\bar{g}}^{(n)} \tag{5}$$

for all $g \in \mathbb{N}^A$. This means in particular that

$$\forall g \in \mathbb{N}^A \qquad (g \in I_x \Longleftrightarrow \bar{g} \in I_x)\,. \tag{6}$$

Furthermore, $\mathbb{N}^{A_x}$ is countable and therefore, $\left\{\bar{g} \,:\, g \in \mathbb{N}^A\right\}$ is countable, too. Thus, the set $\left\{x_{\bar{g}}^{(1)} \,:\, g \in \mathbb{N}^A,\ \mathcal{M} \models \dot{\exists}\dot{y}\dot{\in}x_g^{(1)}\ \psi\left[\dot{y}, x_g^{(2)}, \ldots, x_g^{(n)}\right]\right\}$ equals the set $\left\{x_{\bar{g}}^{(1)} \,:\, g \in \mathbb{N}^A,\ \mathcal{M} \models \dot{\exists}\dot{y}\dot{\in}x_{\bar{g}}^{(1)}\ \psi\left[\dot{y}, x_{\bar{g}}^{(2)}, \ldots, x_{\bar{g}}^{(n)}\right]\right\}$ and is a

countable set. Hence, $\left\{\left\{y \in x_{\bar{g}}^{(1)} \ : \ \mathcal{M} \models \psi\left[y, x_{\bar{g}}^{(2)}, \ldots, x_{\bar{g}}^{(n)}\right]\right\} \ : \ g \in \mathbb{N}^A\right\}$ is a countable collection of non-empty sets. Therefore, by AC_ω, there is a choice function λ that assigns to every $g \in I_x$ which is constantly $= 0$ on $A \setminus A_x$ some element $\lambda(g) = y_g \in x_{\bar{g}}^{(1)}$ such that $\mathcal{M} \models \psi\left[y_g, x_{\bar{g}}^{(2)}, \ldots, x_{\bar{g}}^{(n)}\right]$. For arbitrary $g \in I_x$, put $y_g := \lambda(\bar{g})$. Then, in light of the above equation (5), one has both $y_g \in x_g^{(1)}$ and also $\mathcal{M} \models \psi\left[y_g, x_g^{(2)}, \ldots, x_g^{(n)}\right]$.

For all $g \notin I_x$, put $y_g = \emptyset$. Note that (through the use of the choice function λ and in light of the implication (4)), the map $g \mapsto y_g$ has been constructed in such a way that $y_g = y_h$ whenever $g, h \in \mathbb{N}^A$ agree on the finite set $A_x = \bigcup_{i=1}^n \left|\left(x_g^{(i)}\right)_g\right| \subseteq A$. Through this, $(y_g)_{g \in \mathbb{N}^A}$, too, becomes concentrated on the finite set A_x. Furthermore, $\left\{g \in \mathbb{N}^A \ : \ \mathcal{M} \models y_g \dot{\in} x_g^{(1)}\right\} \in D$, hence by transitivity of N, $\overline{y}$ is bounded in the superstructure hierarchy. These two properties of $(y_g)_{g \in \mathbb{N}^A}$ ensure that $\overline{y} \in \overline{M}$ and $\overline{\mathcal{M}} \models \overline{y} \dot{\in} \overline{x^{(1)}}$. However, by the construction of $(y_g)_{g \in \mathbb{N}^A}$, we also have $\left\{g \in \mathbb{N}^A \ : \ \mathcal{M} \models \psi\left[y_g, x_g^{(2)}, \ldots, x_g^{(n)}\right]\right\} \supseteq I_x \in D$, hence $\left\{g \in \mathbb{N}^A \ : \ \mathcal{M} \models \psi\left[y_g, x_g^{(2)}, \ldots, x_g^{(n)}\right]\right\} \in D$. By induction hypothesis, this entails $\overline{\mathcal{M}} \models \overline{y} \dot{\in} \overline{x^{(1)}} \dot{\wedge} \psi\left[\overline{y}, \overline{x^{(2)}}, \ldots, \overline{x^{(n)}}\right]$ and therefore $\overline{\mathcal{M}} \models \dot{\exists} \dot{y} \dot{\in} \overline{x^{(1)}} \ \psi\left[\dot{y}, \overline{x^{(2)}}, \ldots, \overline{x^{(n)}}\right]$. □

5. Discussion and conclusion

Nonstandard analysis presupposes the existence of an enlarged mathematical universe, in the tradition of Robinson and Zakon [**12**] typically understood as an enlarged superstructure over the reals, although for elementary applications an enlargement of the set of reals suffices. Even for certain more sophisticated applications, it is enough that this enlargement of the mathematical universe satisfies the Transfer Principle, which means that it is an elementary extension in the sense of model theory. We have shown that one can find enlargements of both the set of reals and the superstructure over the reals which have the following properties: (i) The enlargements are definable by some set-theoretic class term; (ii) one can prove the Transfer Principle for those enlargements from Zermelo–Fraenkel set theory with merely Countable Choice (ZF+AC_ω); (iii) The countable saturation of those models can be shown in Zermelo–Fraenkel set theory with full Choice (ZFC).

To be sure, much of this is known already, due to Kanovei and Shelah's construction of a definable nonstandard model of the reals [**9**] and a subsequent paper on definable nonstandard enlargements of the superstructure over the reals [**3, 4**]. What is novel is the property (ii) above, i.e. the fact that the demonstration of the Transfer Principle in the definable nonstandard only invokes ZF+AC_ω, rather than full-blown ZFC. Since the Axiom of Countable Choice follows from Bernays' [**1**] Principle of Dependent Choices (e.g. Jech [**6**, Exercise 5.7]) the Transfer Principle

can even be verified in the Solovay [**15**] model! (In fact, AC_ω is strictly weaker than DC, as Jensen [**7**] showed.) Given the widespread reservations against nonstandard analysis as a "non-constructive" approach to analysis, this finding, conjectured by Mikhail Katz, is somewhat unexpected.

The result may be of some interest for practitioners that work with fragments of nonstandard analysis. For instance, the Transfer Principle is all that is required to develop Edward Nelson's [**11**, p. 30] "minimal nonstandard analysis" or the related "minimal Internal Set Theory" [**5**, pp. 3–4, 104]. It has now been shown that there are definable models of these theories, which can be verified using merely Countable Choice. Such fragments of nonstandard analysis have the potential for application in diverse fields, ranging from stochastic calculus and mathematical finance to theoretical quantum mechanics [**5**].

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