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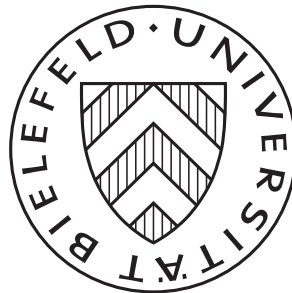
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## Preemptive Investment under Uncertainty

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Jan-Henrik Steg



# Preemptive Investment under Uncertainty

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## Abstract

This paper provides a general characterization of subgame-perfect equilibria for a strategic timing problem, where two firms have the (real) option to invest irreversibly in some market. Profit streams are uncertain and depend on the market structure. The analysis of the problem emphasizes its dynamic nature and exploits only its economic structure. In particular, the determination of equilibria with preemption is reduced to solving a single class of constrained stopping problems. The general results are applied to typical state-space models from the literature, to point out common deficits in equilibrium arguments and to suggest alternative equilibria that are Pareto improvements.

*Keywords:* Preemption, real options, irreversible investment, equilibrium, optimal stopping.

*JEL subject classification:* C61, C73, D21, D43, L12, L13

## 1 Introduction

Preemption is a well-known phenomenon in the context of irreversible investment. In their seminal paper, Fudenberg and Tirole (1985) argue that the commitment power of irreversibility and subgame-perfectness together imply that any firm which is the first to adopt a new technology in some industry can deter adoption by another firm; the benefits from adoption for the second firm will already be reduced by competition and thus not worth the immediate adoption cost. In consequence, the firms try to preempt each other to secure the (temporary) monopoly profit.<sup>1</sup>

Such preemption is particularly interesting when it is costly. In their deterministic model, Fudenberg and Tirole (1985) assume that the adoption cost decreases over time, which generates an incentive to delay adoption and thus a conflict with the preemption impulse. Another possibility is to introduce uncertainty, so that the real-option effect would induce the firms to wait for an optimal adoption time. There is already a sizable literature on similar real-option games, aiming to identify a drastic impact of competition on the valuation of real options and most of it using principles as in Fudenberg and Tirole (1985).

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<sup>1</sup>This effect does not appear in simple Nash equilibria as studied by Reinganum (1981), where firms pre-commit to adoption times.

With uncertainty, the analysis of the models changes, which typically causes some problems to be addressed in this paper. The standard model of uncertainty in the literature on real options is a Markovian shock process (most frequently its growth rates follow a Brownian motion), such that it is natural to work with value functions and also strategies as functions of the state space. However, the stochastic state does not evolve linearly like time in general, so one needs to be careful when borrowing arguments from deterministic models. Indeed, many arguments in the existing literature rely on analytic properties of value functions and often remain incomplete or obscure, because they mask the dynamic nature of the involved problems and partly neglect the inherent economic structure.

In the following, a general model of preemptive investment under uncertainty based on revenue streams is proposed and used to establish important principles for subgame-perfect equilibria. The analysis only uses arguments in terms of comparing revenue streams, which thus have a direct economic meaning, but which are fully formal at the same time. In a first step, the determination of subgame-perfect equilibria with preemption is reduced to solving a single class of constrained optimal stopping problems. Then some verification problems for alternative equilibria avoiding preemption are formulated.

Alongside, important general questions for equilibria of real-option games are addressed, such as:

- At what times is there a first-mover advantage for both firms that they fight for by trying to preempt each other?
- Does anyone want to stop and invest as leader to escape expected preemption at a later point in time?
- In particular, is it always optimal to wait while one has a second-mover advantage?
- If someone takes the lead, when and how is that decision affected by a threat of preemption?

Answers to these question will be found by studying appropriate optimal stopping problems.

Afterwards, the general principles will be applied to two typical state-space models from the literature, those of Grenadier (1996) and Pawlina and Kort (2006), to point out that commonly not all relevant stopping problems are considered for equilibrium verification in similar models, and to actually provide complete subgame-perfect equilibria. We further identify and analyze additional equilibria, some Pareto dominating the equilibrium outcomes proposed in the original papers.

More generally, some examples that are covered by the present general model are the deterministic ones of Reinganum (1981) and Fudenberg and Tirole (1985), the stochastic model of Mason and Weeds (2010), where revenue is linear in a geometric Brownian motion, as in the model of Pawlina and Kort (2006), who add asymmetry in investment costs, which is further extended to an exponential Lévy process by Boyarchenko and Levendorskiĭ (2014); the model of Weeds (2002) additionally includes Poisson arrivals of R&D success and the model

of Grenadier (1996) includes a construction delay, but they are both formally equivalent to a symmetric setting with geometric Brownian motion again.

The paper is organized as follows. The general model is presented in Section 2. Section 3 characterizes equilibria with and without preemption, first providing some equilibrium verification problems and then identifying conditions when investment cannot be delayed in equilibrium. Different versions of the problem when to become optimally the leader play a key role therein. The applications in Section 4 first illustrate how the general results solve common equilibrium verification issues in the literature. Then additional equilibrium effects neglected by the literature are studied. Section 5 concludes and the Appendix contains some technical results and most proofs.

## 2 Strategic investment timing problem

Consider two firms  $i \in \{1, 2\}$  that both have the possibility to make an irreversible investment in the same market, either to enter the market or to improve their operations if they are already present (e.g., technology or production capacity). Before any investment occurs, each firm  $i$  earns a discounted revenue stream given by the stochastic process  $(\pi_t^{0i})_{t \geq 0}$ . If firm  $i$  is the first to invest, it switches to a new revenue stream, net of (capitalized) investment costs, given by the stochastic process  $(\pi_t^{Li})_{t \geq 0}$ . Firm  $i$ 's investment potentially also affects the revenue stream of the other firm  $j \in \{1, 2\} \setminus i$ , which switches to the process  $(\pi_t^{Fj})_{t \geq 0}$  as long as  $j$  has not invested itself. Once both firms  $i = 1, 2$  have invested, each finally earns a permanent net revenue stream given by the process  $(\pi_t^{Bi})_{t \geq 0}$ .

To set the formal basis, all processes are assumed to be product-measurable w.r.t. a given probability space  $(\Omega, \mathcal{F}, P)$  and continuous time  $t \in \mathbb{R}_+$ , and indeed  $P \otimes dt$ -integrable to ensure finite expectations throughout (i.e.,  $E[\int_0^\infty |\pi_t^{0i}| dt] < \infty$  and analogously for all others). All revenue streams are further assumed to be adapted to a given filtration  $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$  satisfying the usual conditions<sup>2</sup>, which captures the dynamic information about the state of the world.

As a standing economic assumption, the following orders among the revenue processes are imposed. To reflect a monopoly premium as long as some firm is the only one having invested, let  $\pi_t^{Li} \geq \pi_t^{Bi}$   $P \otimes dt$ -a.e.,  $i = 1, 2$ . Correspondingly, to have also the first investment by some firm rather harm the revenue of the other (e.g., due to business stealing), let  $\pi_t^{0i} \geq \pi_t^{Fi}$   $P \otimes dt$ -a.e.,  $i = 1, 2$ . The important special case  $\pi_t^{0i} \equiv \pi_t^{Fi}$  would typically be assumed in market entry models. Some of the subsequent results will be shown to become stronger and/or simpler in that case.

Finally, firm 2 is allowed to be handicapped relative to firm 1 in the sense of smaller gains over being laggard, letting  $\pi_t^{B2} - \pi_t^{F2} \leq \pi_t^{B1} - \pi_t^{F1}$  and  $\pi_t^{L2} - \pi_t^{F2} \leq \pi_t^{L1} - \pi_t^{F1}$   $P \otimes dt$ -a.e. The disadvantage may stem, e.g., from a higher capitalized investment cost. Given the first condition, i.e., that firm 2 has less to gain from a follow-up investment, the second condition would be implied by the revenue loss due to an opponent's follow-up investment,  $\pi_t^{Li} - \pi_t^{Bi}$ , being higher for firm 1.

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<sup>2</sup>That is, the filtration is right-continuous and complete.

Both firms can decide when to invest in continuous time  $t \in \mathbb{R}_+$  (or not at all), taking into account information about the exogenous uncertainty and whether the respective other has already invested or not. In particular, if the opponent of firm  $i \in \{1, 2\}$  is the first to invest at time  $t \geq 0$ , then firm  $i$  will decide when to follow optimally, to attain the follower value

$$\begin{aligned} F_t^i &= \int_0^t \pi_s^{0i} ds + \operatorname{ess\,sup}_{\tau \geq t} E \left[ \int_t^\tau \pi_s^{Fi} ds + \int_\tau^\infty \pi_s^{Bi} ds \mid \mathcal{F}_t \right] \\ &= \int_0^t \pi_s^{0i} ds + E \left[ \int_t^{\tau_{F^i}^i(t)} \pi_s^{Fi} ds + \int_{\tau_{F^i}^i(t)}^\infty \pi_s^{Bi} ds \mid \mathcal{F}_t \right]. \end{aligned} \quad (2.1)$$

The supremum is over all feasible timing rules  $\tau$ , which are the  $\mathbb{F}$ -stopping times. Let  $\mathcal{T}$  denote the set of all stopping times. The constraint  $\tau \geq t$  is understood to hold a.s., like all (in-)equalities between random variables in the following. By continuity and integrability of the process  $(\int_t^\cdot \pi_s^{Fi} ds + \int_\cdot^\infty \pi_s^{Bi} ds)$  to be stopped, there exists a *latest* optimal (thus uniquely defined) stopping time  $\tau_{F^i}^i(t) \in \mathcal{T}$  attaining the value  $F_t^i$ .

Now suppose on the contrary that firm  $i$  is the first to invest at some time  $t$ . Then the other firm  $j$  is assumed to follow suit at  $\tau_{F^j}^j(t)$  to realize  $F_t^j$ , thus yielding  $i$  the instantaneous expected leader payoff

$$L_t^i = \int_0^t \pi_s^{0i} ds + E \left[ \int_t^{\tau_{F^j}^j(t)} \pi_s^{Li} ds + \int_{\tau_{F^j}^j(t)}^\infty \pi_s^{Bi} ds \mid \mathcal{F}_t \right]. \quad (2.2)$$

Finally, if both firms happen to invest simultaneously at time  $t$ , each firm  $i = 1, 2$  realizes

$$M_t^i = \int_0^t \pi_s^{0i} ds + E \left[ \int_t^\infty \pi_s^{Bi} ds \mid \mathcal{F}_t \right] \leq \min\{F_t^i, L_t^i\}. \quad (2.3)$$

Note that if no firm invests in finite time, then each firm  $i$  earns

$$F_\infty^i = L_\infty^i = M_\infty^i = \int_0^\infty \pi_s^{0i} ds.$$

*Remark 2.1* (Regularity of the payoff processes). Investment will typically occur at stopping times, say  $\vartheta \in \mathcal{T}$ , not only inside the follower's reaction problem (2.1), but also for the first investor. Thus the relations (2.1), (2.2) and (2.3) should still hold if one replaces  $t$  by any stopping time  $\vartheta \in \mathcal{T}$ . By Lemma A.1 in Appendix A.1 there are indeed processes  $(L_t^i)_{t \geq 0}$ ,  $(F_t^i)_{t \geq 0}$  and  $(M_t^i)_{t \geq 0}$ , such that the value of each process at any  $\vartheta \in \mathcal{T}$  corresponds to the right-hand side of (2.1), (2.2) or (2.3), respectively, if one replaces  $t$  by  $\vartheta$  therein, where  $\tau_{F^j}^j(\vartheta) \in \mathcal{T}$  is still the latest stopping time attaining  $F_\vartheta^j$ . It is much more convenient to work with payoff processes than families like  $\{F_\vartheta^i \mid \vartheta \in \mathcal{T}\}$ . Indeed, by Lemma A.1 we may assume all payoff processes to be right-continuous and sufficiently integrable, precisely of class (D).<sup>3</sup>

<sup>3</sup>This ensures that each process, if evaluated at stopping times, is bounded in expectation and that pointwise limits at stopping times induce the corresponding limits in expectation. All regularity properties are easier to verify for  $M^i$ , as it is the difference of the martingale  $(E[\int_0^\infty \pi_s^{Bi} ds \mid \mathcal{F}_t])_{t \geq 0}$  and the continuous process  $(\int_0^t (\pi_s^{Bi} - \pi_s^{0i}) ds)_{t \geq 0}$ .

Since each firm's investment timing also affects the respective other firm's payoff, the investment decisions are strategic. For instance, from the point of view of time  $t = 0$ , if firm  $j$  plans to invest at the random time  $\tau^j \in \mathcal{T}$ , then firm  $i$  faces the optimal stopping problem

$$\begin{aligned} & \sup_{\tau^i \in \mathcal{T}} E \left[ L_{\tau^i}^i \mathbf{1}_{\tau^i < \tau^j} + F_{\tau^i}^i \mathbf{1}_{\tau^i > \tau^j} + M_{\tau^i}^i \mathbf{1}_{\tau^i = \tau^j} \mid \mathcal{F}_0 \right] \\ &= \sup_{\tau^i \leq \tau^j} E \left[ L_{\tau^i}^i \mathbf{1}_{\tau^i < \tau^j} + F_{\tau^i}^i \mathbf{1}_{\tau^i = \tau^j} \mid \mathcal{F}_0 \right]. \end{aligned} \quad (2.4)$$

Obviously firm  $i$  then can only consider to become leader before  $\tau^j$ ; otherwise it will (at most) become follower at  $\tau^j$ . We will analyze the timing game between the players firm 1 and firm 2 of when to invest, played as long as *no* firm has invested; the first investment by some firm settles the payoffs by the processes  $(L_t^i)$ ,  $(F_t^i)$  and  $(M_t^i)$ .

### 3 Equilibrium characterization

The assumptions on the relation between the different revenue processes have important consequences for equilibria of the timing game, independently of any more specific model for the uncertainty. The aim of this section is to illuminate the structure of possible equilibria just by comparing payoff streams. We will show that it suffices to solve one particular class of constrained optimal stopping problems to construct subgame-perfect equilibria with preemption. Then some verification problems for equilibria avoiding preemption will be formulated. Finally we will determine times when investment cannot be delayed in equilibrium.

The formal notion of subgame-perfect equilibrium for timing games under uncertainty used here is that of Riedel and Steg (2014). In particular, we consider arbitrary stopping times  $\vartheta \in \mathcal{T}$  as the possible beginnings of subgames in which no firm has invested before. The players' strategies must form a Nash equilibrium in every subgame, independently of whether it is actually reached by equilibrium play or not, and the strategies must be time consistent across subgames. A pure strategy of player  $i$  in any subgame is a stopping time  $\tau_{\vartheta}^i \geq \vartheta$ , such that according to any pair of pure strategies, investment by some firm at  $\min\{\tau_{\vartheta}^1, \tau_{\vartheta}^2\}$  effectively terminates the game, with the corresponding expected payoffs

$$E \left[ L_{\tau_{\vartheta}^i}^i \mathbf{1}_{\tau_{\vartheta}^i < \tau_{\vartheta}^j} + F_{\tau_{\vartheta}^i}^i \mathbf{1}_{\tau_{\vartheta}^i > \tau_{\vartheta}^j} + M_{\tau_{\vartheta}^i}^i \mathbf{1}_{\tau_{\vartheta}^i = \tau_{\vartheta}^j} \mid \mathcal{F}_{\vartheta} \right]$$

at  $\vartheta$ . Time consistency requires that a family of pure strategies  $\{\tau_{\vartheta}^i \mid \vartheta \in \mathcal{T}\}$  satisfies  $\tau_{\vartheta}^i = \tau_{\vartheta'}^i$  a.s. on the event  $\{\vartheta' \leq \tau_{\vartheta}^i\}$  for any two  $\vartheta \leq \vartheta' \in \mathcal{T}$ .

All necessary equilibrium conditions in the following also hold for mixed strategies, even if we do not repeat their formal definition here; we will only make use of the fact that mixed strategies imply certain conditional investment probabilities at any stopping time.<sup>4</sup>

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<sup>4</sup>See Riedel and Steg (2014) for the formal definition of mixed strategies. They specify for any subgame a distribution function over the remaining time  $(G_i^{\vartheta}(t))$ , taking into account the dynamic information about the state of the world, and some extensions  $(\alpha_i^{\vartheta}(t))$  to model preemption in continuous time.

### 3.1 Sufficient equilibrium conditions

In order to construct subgame-perfect equilibria with preemption, we first establish two classes of equilibria for certain subgames where some immediate investment occurs.

#### 3.1.1 Simultaneous investment

The followers' reaction times  $\tau_F^i(\cdot)$  are central for any equilibrium analysis, as they enter also the leader payoff. As a first observation, the assumption  $\pi_s^{B2} - \pi_s^{F2} \leq \pi_s^{B1} - \pi_s^{F1}$  implies the reverse order for the reaction times and allows to identify simultaneous investment equilibria.

**Lemma 3.1.**  $\tau_F^1(\vartheta) \leq \tau_F^2(\vartheta)$  and  $F_\vartheta^1 - M_\vartheta^1 \leq F_\vartheta^2 - M_\vartheta^2$  a.s. for any  $\vartheta \in \mathcal{T}$ .

*Proof.* The follower problem (2.1) (with  $\vartheta$  replacing  $t$ ) is equivalent to minimizing the opportunity cost of waiting  $E[\int_\vartheta^\tau (\pi_s^{Bi} - \pi_s^{Fi}) ds | \mathcal{F}_\vartheta]$ ,  $\tau \geq \vartheta$ . By optimality of  $\tau_F^i(\vartheta)$ , it holds that  $E[\int_\tau^{\tau_F^i(\vartheta)} (\pi_s^{Bi} - \pi_s^{Fi}) ds | \mathcal{F}_\tau] \leq 0$  for all  $\tau \in [\vartheta, \tau_F^i(\vartheta)]$  and  $E[\int_{\tau_F^i(\vartheta)}^\tau (\pi_s^{Bi} - \pi_s^{Fi}) ds | \mathcal{F}_{\tau_F^i(\vartheta)}] \geq 0$  for all  $\tau \geq \tau_F^i(\vartheta)$ , strictly on  $\{\tau > \tau_F^i(\vartheta)\}$ , since  $\tau_F^i(\vartheta)$  is the latest time attaining (2.1).

Thus, with  $\vartheta' = \min\{\tau_F^1(\vartheta), \tau_F^2(\vartheta)\}$  and  $\pi_s^{B2} - \pi_s^{F2} \leq \pi_s^{B1} - \pi_s^{F1}$  we have

$$0 \leq E\left[\int_{\vartheta'}^{\tau_F^1(\vartheta)} (\pi_s^{B2} - \pi_s^{F2}) ds \middle| \mathcal{F}_{\vartheta'}\right] \leq E\left[\int_{\vartheta'}^{\tau_F^1(\vartheta)} (\pi_s^{B1} - \pi_s^{F1}) ds \middle| \mathcal{F}_{\vartheta'}\right] \leq 0.$$

The first inequality is strict on  $\{\tau_F^2(\vartheta) < \tau_F^1(\vartheta)\}$ , so  $\tau_F^1(\vartheta) \leq \tau_F^2(\vartheta)$  a.s.

Finally,  $F_\vartheta^i - M_\vartheta^i = \text{ess sup}_{\tau \geq \vartheta} E[\int_\vartheta^\tau (\pi_s^{Fi} - \pi_s^{Bi}) ds | \mathcal{F}_\vartheta]$  is less for firm 1 than for firm 2.  $\square$

Lemma 3.1 shows that where  $\tau = \vartheta$  attains  $F_\vartheta^2$ , it also attains  $F_\vartheta^1$ , such that  $F_\vartheta^i = M_\vartheta^i$  for both  $i \in \{1, 2\}$ . Then it is an equilibrium that both firms invest immediately. Indeed, given that the opponent plans to invest immediately, each firm  $i$  can either invest by choosing  $\tau_\vartheta^i = \vartheta$ , or become follower from any strategy  $\tau_\vartheta^i > \vartheta$  of investing later *if no one invests before*, which implies the same expected payoff, but with actual investment at  $\tau_F^i(\vartheta)$ . If firm  $i$  chose  $\tau_\vartheta^i > \vartheta$ , however, the other firm would obtain more options and might not want to invest immediately anymore, so *proactive* investment by both firms is important for the equilibrium.

This principle also applies at  $\vartheta' = \tau_F^2(\vartheta)$ , which satisfies  $\vartheta' = \tau_F^2(\vartheta')$  due to  $\tau_F^i(\tau_F^i(\vartheta)) = \tau_F^i(\vartheta)$  by definition. Then Lemma 3.1 implies that it would be mandatory to follow immediately for firm 1, too,  $\vartheta' = \tau_F^1(\vartheta') = \tau_F^2(\vartheta')$ , and thus  $L_{\vartheta'}^i = F_{\vartheta'}^i = M_{\vartheta'}^i$  for both  $i \in \{1, 2\}$ . In this case any unilateral deviation still implies immediate investment, but as a *reaction* to the opponent's investment. Even if all strategies of firm  $i$  now lead to the same *physical* outcome given the strategy of its opponent, it is important to distinguish strategies to actually support the equilibrium.

#### 3.1.2 Preemption

Preemption is to be expected where both firms have a strict first-mover advantage  $L_\vartheta^i > F_\vartheta^i$  (which requires  $\vartheta < \tau_F^1(\vartheta)$ ), such that both firms may try to invest although they want to avoid simultaneous investment.



Given the assumption  $\pi_s^{L1} - \pi_s^{F1} \geq \pi_s^{L2} - \pi_s^{F2}$ , also firm 1's first-mover advantage is not less than that of firm 2.

**Lemma 3.2.**  $L_\vartheta^1 - F_\vartheta^1 \geq L_\vartheta^2 - F_\vartheta^2$  a.s. for any  $\vartheta \in \mathcal{T}$ .

*Proof.* We have

$$L_\vartheta^2 - F_\vartheta^2 = E \left[ \int_\vartheta^{\tau_F^1(\vartheta)} (\pi_s^{L2} - \pi_s^{F2}) ds + \int_{\tau_F^1(\vartheta)}^{\tau_F^2(\vartheta)} (\pi_s^{B2} - \pi_s^{F2}) ds \middle| \mathcal{F}_\vartheta \right] \quad (3.1)$$

and

$$L_\vartheta^1 - F_\vartheta^1 = E \left[ \int_\vartheta^{\tau_F^1(\vartheta)} (\pi_s^{L1} - \pi_s^{F1}) ds + \int_{\tau_F^1(\vartheta)}^{\tau_F^2(\vartheta)} (\pi_s^{L1} - \pi_s^{B1}) ds \middle| \mathcal{F}_\vartheta \right],$$

where  $\tau_F^1(\vartheta) \leq \tau_F^2(\vartheta)$  by Lemma 3.1. By the optimality of  $\tau_F^2(\vartheta)$  for stopping the stream  $(\pi_s^{B2} - \pi_s^{F2})$ , the second integral on the RHS of (3.1) has non-positive conditional expectation, cf. the proof of Lemma 3.1. The claim now follows from the assumptions  $\pi_s^{L1} - \pi_s^{F1} \geq \pi_s^{L2} - \pi_s^{F2}$  and  $\pi_s^{L1} \geq \pi_s^{B1}$ .  $\square$

By Lemma 3.2, the *preemption region* – where both firms have a strict first-mover advantage – is  $\mathcal{P} := \{L^2 > F^2\}$ .

Let  $\tau_{\mathcal{P}}(\vartheta) := \inf\{t \geq \vartheta \mid L_t^2 > F_t^2\} \in \mathcal{T}$  denote the first hitting time of the preemption region from  $\vartheta \in \mathcal{T}$ . At  $\vartheta' = \tau_{\mathcal{P}}(\vartheta)$  there exists a preemption equilibrium given by Proposition 3.1 of Riedel and Steg (2014), in which both firms plan to invest immediately. Some strategy extensions allow the firms to coordinate endogenously to a certain degree about the roles of leader and follower. In that equilibrium, any player can realize a strict first-mover advantage iff the opponent is indifferent between roles. By Lemma 3.2, here only firm 1's preemption equilibrium payoff can be  $L_{\tau_{\mathcal{P}}(\vartheta)}^1 > F_{\tau_{\mathcal{P}}(\vartheta)}^1$  where  $L_{\tau_{\mathcal{P}}(\vartheta)}^2 = F_{\tau_{\mathcal{P}}(\vartheta)}^2$ ; otherwise it is only  $F_{\tau_{\mathcal{P}}(\vartheta)}^1$ . Firm 2's preemption payoff is always  $F_{\tau_{\mathcal{P}}(\vartheta)}^2$ .<sup>5</sup>

The location of the preemption region  $\mathcal{P}$  will be characterized in Section 3.1.5. By (3.1) one can already see that  $\mathcal{P} = \emptyset$  if, e.g., even  $\pi_s^{B1} - \pi_s^{F1} \geq \pi_s^{L2} - \pi_s^{F2}$ , because  $E[\int_\vartheta^{\tau_F^i(\vartheta)} (\pi_s^{Bi} - \pi_s^{Fi}) ds \mid \mathcal{F}_\vartheta] \leq 0$  by the optimality of  $\tau_F^i(\vartheta)$ ,  $i = 1, 2$ .

### 3.1.3 Subgame-perfect equilibrium with preemption

The subsequent equilibrium construction is facilitated by the fact that independently of what happens in the preemption region, no firm ever wants to invest with a current second-mover advantage under the present assumptions.

**Lemma 3.3.** *Investment is never optimal for any firm  $i \in \{1, 2\}$  where  $F^i > L^i$ . Further, waiting until  $\min\{\tau_{\mathcal{P}}(\vartheta), \tau_F^2(\vartheta)\}$  does not restrict firm 2's payoff in the subgame at  $\vartheta \in \mathcal{T}$  for any (mixed) strategy of firm 1.*

<sup>5</sup>These payoffs can of course also simply be assumed if both firms invest at  $\tau_{\mathcal{P}}(\vartheta)$ .

*Proof:* See Appendix A.2.

It is crucial for Lemma 3.3 that  $(F_t^i)$  here is a *submartingale* while it strictly exceeds  $(L_t^i)$ ; the fact  $F^i > L^i$  alone does not suffice to delay investment – in contrast to the customary suggestion throughout the literature (like the examples in the Introduction).

By Lemma 3.3 we may assume that firm 2 is inactive from any  $\vartheta$  until  $\min\{\tau_{\mathcal{P}}(\vartheta), \tau_F^2(\vartheta)\}$ , where preemption or simultaneous investment is an equilibrium. Assuming that firm 2 invests by the corresponding strategy at that point, it remains to determine an optimal time for firm 1 to invest up to  $\min\{\tau_{\mathcal{P}}(\vartheta), \tau_F^2(\vartheta)\}$ , which, unlike firm 2, may have a strict first-mover advantage before  $\tau_{\mathcal{P}}(\vartheta)$  and be willing to exploit it.

If firm 1 waits until  $\min\{\tau_{\mathcal{P}}(\vartheta), \tau_F^2(\vartheta)\}$ , its local equilibrium payoff from preemption or simultaneous investment derived before is  $L^1 \geq F^1$  where  $\tau_F^2(\vartheta) < \tau_{\mathcal{P}}(\vartheta)$  or where  $L^2 = F^2$  (in particular where  $\tau_{\mathcal{P}}(\vartheta) = \tau_F^2(\vartheta)$ ); it is  $F^1 < L^1$  where  $L^2 > F^2$  at  $\tau_{\mathcal{P}}(\vartheta) < \tau_F^2(\vartheta)$ .

Thus the “equilibrium stopping problem” for firm 1 at any  $\vartheta \in \mathcal{T}$  is

$$\begin{aligned} & \operatorname{ess\,sup}_{\vartheta \leq \tau \leq \tau_{\mathcal{P}}(\vartheta) \wedge \tau_F^2(\vartheta)} E \left[ L_{\tau}^1 \mathbf{1}_{\{\tau < \tau_{\mathcal{P}}(\vartheta)\} \cup \{L_{\tau}^2 = F_{\tau}^2\}} + F_{\tau}^1 \mathbf{1}_{\{\tau = \tau_{\mathcal{P}}(\vartheta)\} \cap \{L_{\tau}^2 > F_{\tau}^2\}} \middle| \mathcal{F}_{\vartheta} \right] \\ = & \operatorname{ess\,sup}_{\vartheta \leq \tau \leq \tau_{\mathcal{P}}(\vartheta) \wedge \tau_F^2(\vartheta)} E \left[ L_{\tau}^1 \mathbf{1}_{\{L_{\tau}^2 \leq F_{\tau}^2\}} + F_{\tau}^1 \mathbf{1}_{\{L_{\tau}^2 > F_{\tau}^2\}} \middle| \mathcal{F}_{\vartheta} \right]. \end{aligned} \quad (3.2)$$

Note that firm 1 realizes  $L_{\tau}^1$  on  $\{\tau < \tau_{\mathcal{P}}(\vartheta)\} \cup \{L_{\tau}^2 = F_{\tau}^2\} = \{L_{\tau}^2 \leq F_{\tau}^2\}$ . If it has a solution  $\tau_*^1(\vartheta)$ , the value of problem (3.2) is firm 1’s equilibrium payoff at  $\vartheta$ , and that of firm 2 is  $E[F_{\tau_*^1(\vartheta)}^2 | \mathcal{F}_{\vartheta}]$ , getting the local equilibrium payoff  $F^2$  also where  $\tau_*^1(\vartheta) = \min\{\tau_{\mathcal{P}}(\vartheta), \tau_F^2(\vartheta)\}$ .

We can summarize as follows.

**Theorem 3.4.** *If there is a family of solutions  $\{\tau_*^1(\vartheta) | \vartheta \in \mathcal{T}\}$  to (3.2) satisfying the time-consistency condition  $\tau_*^1(\vartheta) = \tau_*^1(\vartheta')$  a.s. on the event  $\{\vartheta' \leq \tau_*^1(\vartheta)\}$  for any two  $\vartheta \leq \vartheta' \in \mathcal{T}$ , then there is the following subgame-perfect equilibrium. In the subgame beginning at  $\vartheta \in \mathcal{T}$ , firm 1’s strategy is to invest at  $\tau_*^1(\vartheta)$  and firm 2’s to invest at  $\tau_*^2(\vartheta) = \min\{\tau_{\mathcal{P}}(\vartheta), \tau_F^2(\vartheta)\}$ , with the mixed strategy extensions from Proposition 3.1 of Riedel and Steg (2014) governing preemption at  $\tau_{\mathcal{P}}(\vartheta)$ .*

Time consistency can easily be ensured whenever there exist optimal stopping times  $\tau_*^1(\vartheta)$ ,  $\vartheta \in \mathcal{T}$ , by choosing always the respective first or last ones.<sup>6</sup> It holds automatically if the  $\tau_*^1(\vartheta)$  are of threshold-type in a state-space model.

Existence of a solution  $\tau_*^1(\vartheta)$  to (3.2) is generally not clear, however, because of a possible discontinuity of the payoff process at  $\tau_{\mathcal{P}}(\vartheta)$  where  $\vartheta < \tau_{\mathcal{P}}(\vartheta) < \tau_F^2(\vartheta)$  and  $L_{\tau_{\mathcal{P}}(\vartheta)}^2 > F_{\tau_{\mathcal{P}}(\vartheta)}^2$ , which then implies also  $L_{\tau_{\mathcal{P}}(\vartheta)}^1 > F_{\tau_{\mathcal{P}}(\vartheta)}^1$ .

A sufficient condition for existence of a solution  $\tau_*^1(\vartheta)$  is that the process  $(L_t^2 - F_t^2)$  is

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<sup>6</sup>The families  $\{\tau_{\mathcal{P}}(\vartheta) | \vartheta \in \mathcal{T}\}$  and  $\{\tau_F^2(\vartheta) | \vartheta \in \mathcal{T}\}$  satisfy the time-consistency condition by construction and thus also  $\{\tau_*^2(\vartheta) | \vartheta \in \mathcal{T}\}$ . As the latter are the constraints in (3.2), any family of earliest or latest solutions  $\{\tau_*^1(\vartheta) | \vartheta \in \mathcal{T}\}$  will then be time consistent, respectively. The strategy extensions for preemption from Proposition 3.1 of Riedel and Steg (2014) are time consistent by construction.

lower semi-continuous, since then  $L_{\tau_{\mathcal{P}}(\vartheta)}^2 = F_{\tau_{\mathcal{P}}(\vartheta)}^2$  on  $\{\vartheta < \tau_{\mathcal{P}}(\vartheta)\}$ , where (3.2) reduces to

$$\operatorname{ess\,sup}_{\vartheta \leq \tau \leq \tau_{\mathcal{P}}(\vartheta) \wedge \tau_F^2(\vartheta)} E \left[ L_{\tau}^1 \mid \mathcal{F}_{\vartheta} \right]. \quad (3.3)$$

**Proposition 3.5.** *Assume that  $L_t^2 - F_t^2$  is lower semi-continuous from the left. Then there exists a subgame-perfect equilibrium as described in Theorem 3.4, with each  $\tau_*^1(\vartheta)$  the respective latest solution of*

$$\operatorname{ess\,sup}_{\vartheta \leq \tau \leq \tau_{\mathcal{P}}(\vartheta) \wedge \tau_F^2(\vartheta)} E \left[ \int_0^{\tau} \pi_s^{01} ds + \int_{\tau}^{\infty} \pi_s^{L1} ds \mid \mathcal{F}_{\vartheta} \right]. \quad (3.4)$$

That the solutions of problem (3.3) are the (existing) solutions of the conceptually much simpler constrained permanent monopoly problem (3.4) follows from the fact that the follower reaction time  $\tau_F^2(\tau)$  in  $L_{\tau}^1$  remains constant for  $\tau \in [\vartheta, \tau_F^2(\vartheta)]$ , cf. Lemma 3.10 below. One can even ignore the constraint  $\tau \leq \tau_F^2(\vartheta)$  in (3.4) if  $\pi_s^{L1} - \pi_s^{01} \geq \pi_s^{B1} - \pi_s^{F1}$  (as in market entry with  $\pi_s^{0i} \equiv \pi_s^{Fi}$ ), because then the solution is to stop no later than at  $\tau_F^1(\vartheta) \leq \tau_F^2(\vartheta)$ , see the discussion after Lemma 3.10. Of course it is optimal to stop in (3.4) when it is so for  $i = 1$  in the completely unconstrained monopoly problem

$$\operatorname{ess\,sup}_{\tau \geq \vartheta} E \left[ \int_0^{\tau} \pi_s^{0i} ds + \int_{\tau}^{\infty} \pi_s^{Li} ds \mid \mathcal{F}_{\vartheta} \right]. \quad (3.5)$$

Without (lower semi-)continuity of  $L_t^2 - F_t^2$ , the simpler problem (3.4) still provides a sufficient condition for stopping in (3.2), because the continuation value in (3.2) is at most that in (3.3). Indeed, firm 1 receives a local payoff  $F^1$  in (3.2) only at the terminal date and where it has a first-mover advantage (at  $\tau_{\mathcal{P}}(\vartheta)$ , precisely if  $L^1 - F^1 \geq L^2 - F^2 > 0$ ). That means, if the local payoff  $L^1$  is optimal in (3.3), it is so in (3.2), or, using (3.4), whenever it is optimal to switch from the stream  $\pi_s^{01}$  to  $\pi_s^{L1}$  immediately (with or without constraint), one also has  $\tau_*^1(\vartheta) = \vartheta$ .

### 3.1.4 Equilibria without preemption

There can be other equilibria without preemption, even if the preemption region is non-empty. For instance, joint investment at a future stopping time  $\tau_J$  can be an equilibrium in the subgame starting at  $\vartheta \in \mathcal{T}$  if no firm wants to deviate and become leader before. The firms can also plan to invest sequentially if one accepts to become follower when the other invests. Such equilibria depend on the relative magnitudes of the revenue processes, however, so existence cannot be ensured by simple regularity properties like continuity sufficing for the equilibria of Theorem 3.4. On the contrary, if  $\pi_s^{Fi} \equiv \pi_s^{0i}$  and  $L_{\vartheta}^i > F_{\vartheta}^i$ , then firm  $i$  prefers investing immediately over waiting until firm  $j$  invests at some  $\tau_*^j > \vartheta$ , because waiting would yield at most  $E[F_{\tau_*^j}^i \mid \mathcal{F}_{\vartheta}]$  and  $(F_t^i)$  is a supermartingale now.

If  $\pi_s^{Fi} < \pi_s^{0i}$  occurs (e.g., due to the first investment stealing business), then the following proposition helps to reduce the search for times at which firm  $i$  may still want to preempt firm  $j$  and thus to verify a best reply  $\tau_*^i \geq \tau_*^j$ . It avoids to maximize the leader payoff directly,

which is a complex problem due to the follower reaction. Applied to state-space models, it may suffice to consider deviations at a single threshold, like in the examples in Section 4.

**Proposition 3.6.** *Consider any given  $\vartheta \in \mathcal{T}$  and  $i, j \in \{1, 2\}$ ,  $i \neq j$ . If firm  $j$  plans to invest at the stopping time  $\tau_*^j \geq \vartheta$ , then  $\tau_*^i \geq \tau_*^j$  is a best reply for firm  $i$  if  $F_{\tau_*^j}^i = M_{\tau_*^j}^i$  on  $\{\tau_*^i = \tau_*^j\}$  and*

$$(i) \ E[F_{\tau_*^j}^i | \mathcal{F}_\vartheta] \geq \text{ess sup}_{\tau \in [\vartheta, \tau_*^j]} E[M_\tau^i | \mathcal{F}_\vartheta] \text{ and}$$

(ii) *for each stopping time  $\vartheta' \geq \vartheta$ , on  $\{\vartheta' < \tau_*^j\}$  one of the solutions  $\tau_D^i(\vartheta') \in \mathcal{T}$  of the problem*

$$\text{ess sup}_{\tau \in [\vartheta', \tau_*^j \vee \vartheta']} E \left[ \int_0^\tau \pi_s^{0i} ds + \int_\tau^\infty \pi_s^{Li} ds \mid \mathcal{F}_{\vartheta'} \right] \quad (3.6)$$

*satisfies either  $\tau_D^i(\vartheta') \geq \tau_F^j(\vartheta')$  or  $L_{\tau_D^i(\vartheta')}^i \leq E[F_{\tau_*^j}^i | \mathcal{F}_{\tau_D^i(\vartheta')}]$ .*

*Where  $\vartheta'$  attains (3.6), it holds that  $L_{\vartheta'}^i - E[F_{\tau_*^j}^i | \mathcal{F}_{\vartheta'}] \geq E[L_\tau^i - F_{\tau_*^j}^i | \mathcal{F}_{\vartheta'}]$  for all stopping times  $\tau \in [\vartheta', \tau_F^j(\vartheta')]$ .*

*Further, if  $\pi^{L1} - \pi^{01} \geq \pi^{L2} - \pi^{02}$ ,  $\pi^{B1} - \pi^{01} \geq \pi^{B2} - \pi^{02}$ ,  $F_{\tau_*^2}^2 = M_{\tau_*^2}^2$  and (i), (ii) hold for  $i = 1$ , then  $\tau_*^1 = \tau_*^2$  are mutual best replies.*

*Proof:* See Appendix A.2.

Condition (i) is obviously also necessary, since the terminal payoff is at most  $F_{\tau_*^j}^i$  (without preemption modeled as in Section 3.1.2) and  $L^i \geq M^i$ . Condition (ii) says that it suffices to check for deviations by firm  $i$  at solutions  $\tau^i(\vartheta') < \tau_F^j(\vartheta')$  of (3.6), so there is nothing to check where  $\vartheta' = \tau_F^j(\vartheta')$ . Note that the joint investment problem in condition (i) and the constrained monopoly problem (3.6) involve no follower reactions  $\tau_F^j(\tau)$  and are thus conceptually simpler than determining an optimal deviation time to become leader. In threshold-type models, it is typically enough to consider  $\vartheta' = \tau_D^i(\vartheta)$ : if firm  $i$  does not want to become leader there, it does not at any value that the state process will attain before crossing firm  $j$ 's follower threshold that determines  $\tau_F^j(\vartheta)$ . For states above that threshold, no deviations need to be considered.

Proposition 3.6 immediately allows to identify equilibria of joint investment at some time  $\tau_J = \tau_*^1 = \tau_*^2 \geq \vartheta$ . Therefore on the one hand  $F_{\tau_J}^2 = M_{\tau_J}^2$  is necessary, which automatically implies  $F_{\tau_J}^1 = M_{\tau_J}^1$  by Lemma 3.1. On the other hand, (i) is then the clearly necessary condition that  $\tau_J$  must be an (at least constrained) optimal time for maximizing the expected joint investment payoff  $E[M_{\tau_J}^i | \mathcal{F}_\vartheta]$ . Given such  $\tau_J$ , an equilibrium can be verified by condition (ii), where it suffices to consider firm 1 if the additional revenue order holds. The problem of maximizing  $E[M_{\tau_J}^i | \mathcal{F}_\vartheta]$  is considered in Lemma 3.11 below.

If delayed joint investment is not feasible because firm 1, say, would want to become leader before, then preemption may still be avoidable in an equilibrium with sequential investment. In the equilibria of Theorem 3.4 for an empty preemption region, firm 1 becomes leader at an optimal time before simultaneous investment would happen at  $\tau_F^2(\vartheta)$ . Simply ignoring

preemption in a non-empty preemption region, but keeping simultaneous investment at  $\tau_F^2(\vartheta)$ , firm 1's problem becomes

$$\operatorname{ess\,sup}_{\tau \in [\vartheta, \tau_F^2(\vartheta)]} E[L_\tau^1 | \mathcal{F}_\vartheta], \quad (3.7)$$

since  $F^1 = M^1 = L^1$  at  $\tau_F^2(\vartheta)$ . Problem (3.7) is again equivalent to a constrained monopoly problem by Lemma 3.10 below and has a solution  $\tau_S \in \mathcal{T}$ . Any such solution is a best reply for firm 1 against firm 2's strategy  $\tau_*^2 = \tau_F^2(\vartheta)$ . Optimality of the latter for firm 2 against  $\tau_*^1 = \tau_S \leq \tau_F^2(\vartheta)$  can be verified by Proposition 3.6, which now simplifies as follows.

**Corollary 3.7.** *Consider any  $\vartheta \in \mathcal{T}$  and let  $\tau_S \in \mathcal{T}$  solve (3.7). It is an equilibrium in the subgame beginning at  $\vartheta$  that firm 1 plans to invest at  $\tau_*^1 = \tau_S$  and firm 2 at  $\tau_*^2 = \tau_F^2(\vartheta)$  if condition (ii) of Proposition 3.6 is satisfied for firm  $i = 2$ .*

*Further, if  $\pi^{L1} - \pi^{O1} \geq \pi^{L2} - \pi^{O2}$ , then  $\tau_D^2(\vartheta') = \tau_S$  attains (3.6) where  $\vartheta' \leq \tau_*^1 = \tau_S$ .*

*Proof:* See Appendix A.2.

In the setting of Corollary 3.7, condition (ii) of Proposition 3.6 holds if firm 2 will not have a local first-mover advantage where  $\tau_D^2(\vartheta') < \tau_F^1(\vartheta')$  attains (3.6), since  $(F_t^2)$  is a submartingale on  $[\vartheta', \tau_F^2(\vartheta')]$ . Under the additional revenue order in the corollary, this simply amounts to  $\tau_S$  not being in the preemption region  $\mathcal{P}$ .

### 3.1.5 Preemption region

Concerning the location of the preemption region  $\mathcal{P}$ , we can say the following, which will be particularly helpful in state-space models, in which the unilateral stopping problems (like the follower reaction problem) have threshold-type solutions, like in the examples in Section 4. First, we already noted that we are never inside the preemption region when firm 1 would follow immediately:  $\vartheta = \tau_F^1(\vartheta) \Rightarrow L_\vartheta^2 = M_\vartheta^2 \leq F_\vartheta^2$ . Indeed, we are not even on the boundary of the preemption region where  $\vartheta = \tau_F^1(\vartheta)$  and where it would not be optimal for firm 2 to follow immediately, since then  $L_\vartheta^2 = M_\vartheta^2 < F_\vartheta^2$  and hence  $\vartheta < \tau_{\mathcal{P}}(\vartheta)$  by right-continuity of the processes.

Second, to see if the preemption region is empty, it suffices to consider certain simple optimal stopping times, which are the solutions of firm  $i$ 's permanent monopoly problem (3.5) if  $\pi^{O_i} \equiv \pi^{F_i}$  (like in a market entry model).

**Lemma 3.8.** *For any  $\vartheta \in \mathcal{T}$ ,  $L_\vartheta^2 > F_\vartheta^2$  only if  $E[L_\tau^2 - F_\tau^2 | \mathcal{F}_\vartheta] > 0$  for all times  $\tau_\Delta^i \in \mathcal{T}$  attaining*

$$\operatorname{ess\,sup}_{\tau \geq \vartheta} E \left[ \int_0^\tau \pi_s^{F_i} ds + \int_\tau^\infty \pi_s^{L_i} ds \mid \mathcal{F}_\vartheta \right] \quad (3.8)$$

*for some firm  $i \in \{1, 2\}$ . Where  $\tau_\Delta^2 = \vartheta$  attains (3.8) for  $i = 2$ , it holds that  $L_\vartheta^2 - F_\vartheta^2 \geq E[L_\tau^2 - F_\tau^2 | \mathcal{F}_\vartheta]$  for all  $\tau \in [\vartheta, \tau_F^1(\vartheta)]$ .*

*Proof:* See Appendix A.2.

The implications of  $L^2 \leq F^2$  at any  $\tau_F^i(\vartheta)$  as observed before and Lemma 3.8 for state-space models like in Section 4 are the following. First, a follower threshold for either firm  $i$ , say  $x_F^i \in \mathbb{R}$ , is never contained in the preemption region<sup>7</sup>, not even in its closure if investment at  $x_F^1$  is not optimal for firm 2. As  $L^2 \leq F^2$  for all states above such  $x_F^i$ , the latter must lie above any non-empty preemption region. Second, any non-empty preemption region must intersect the stopping regions from (3.8) for both  $i = 1, 2$ ; a threshold solving that problem, say  $x_\Delta^i \in \mathbb{R}$ , cannot lie above the preemption region. In particular, if  $x_\Delta^2 \geq x_F^1$ , the preemption region must be empty. Third, if firm 2 has no first-mover advantage at  $x_\Delta^2$ , then it has none at any value that the state will attain before crossing  $x_F^1$ . Thus, if the state, starting from some  $x_\Delta^2 < x_F^1$ , will attain any intermediate value before reaching  $x_F^1$ , then it is sufficient to check whether there is a first-mover advantage for firm 2 at  $x_\Delta^2$ ; otherwise the preemption region is empty, because  $x_\Delta^2$  cannot lie above it.

## 3.2 Necessary equilibrium conditions

Lemma 3.3 has established that investment is never optimal with a current strict second-mover advantage  $F_\vartheta^i > L_\vartheta^i$ , given the standing assumptions  $\pi^{Li} \geq \pi^{Bi}$  and  $\pi^{0i} \geq \pi^{Fi}$ . This section presents some counterparts: conditions when investment is unavoidable in equilibrium.

### 3.2.1 The leader stopping problem

Already from (2.4) it has been evident that equilibria are related to optimally stopping the leader payoff process, subject to certain constraints. Now we consider the *unconstrained* problem of when to become leader, to obtain a necessary condition for any equilibrium: the assumptions  $\pi^{Li} \geq \pi^{Bi}$  and  $\pi^{0i} \geq \pi^{Fi}$  imply the following “terminal” points of the game, where some investment must occur at the latest.

**Lemma 3.9.** *In any equilibrium for the subgame starting at  $\vartheta \in \mathcal{T}$ , the game cannot continue past any last optimal time for any firm  $i = 1, 2$  to become leader, i.e., past any maximal  $\tau_l^i \in \mathcal{T}$  attaining*

$$\text{ess sup}_{\tau \geq \vartheta} E \left[ L_\tau^i \mid \mathcal{F}_\vartheta \right]. \quad (3.9)$$

*At any  $\tau_l^i \in \mathcal{T}$  that attains (3.9), it is also optimal for a permanent monopolist to invest immediately, i.e.,  $\tau_l^i$  also attains*

$$\text{ess sup}_{\tau \geq \tau_l^i} E \left[ \int_0^\tau \pi_s^{0i} ds + \int_\tau^\infty \pi_s^{Li} ds \mid \mathcal{F}_{\tau_l^i} \right].$$

*Proof:* See Appendix A.2.

Lemma 3.9 rests on the observation that if it is optimal to become leader immediately in (3.9), then there is no superior future follower payoff, either: if firm  $i$  had the choice when to

<sup>7</sup>Here “the preemption region” refers to an area in the same state space in which the thresholds are defined, which is of course an abuse of terminology regarding the previous definition of  $\mathcal{P}$ .

become follower, it would generally prefer times  $\tau_F^i(\cdot)$  to avoid the low revenue  $\pi^{Fi}$ . At any  $\tau_F^i(\cdot)$ , however, becoming follower is not better than becoming leader by  $\pi^{Li} \geq \pi^{Bi}$ .

The stopping problem (3.9) of when to become optimally the leader implicitly assumes that the respective opponent only ever invests by reacting optimally after firm  $i$ 's investment, whereas the permanent monopoly problem assumes that the opponent never invests at all. When looking for solutions of the former (in particular latest ones), it suffices to consider solutions of the latter. The reverse is not true, however, due to the dependence of  $L^i$  on the follower's reaction. When a permanent monopolist finds it optimal to invest, it may be that the opponent would follow immediately in the leader problem; but when only  $\pi^{Bi}$  can be realized, it may be better to wait for a time when the follower will react with a lag.<sup>8</sup>

Therefore (3.9) is a difficult problem in general. It becomes much easier with a state-space structure, like in the applications in Section 4, or by considering certain "continuation" equilibria, like simultaneous investment at  $\tau_F^2(\vartheta)$ . Then any earlier investment by firm 1 does not affect the follower investment timing by firm 2.

**Lemma 3.10.** *Suppose that firm 2's strategy in the subgame at  $\vartheta \in \mathcal{T}$  is such that the game ends no later than at  $\tau_F^2(\vartheta)$ . Then it is uniquely optimal for firm 1 to invest immediately where  $\vartheta$  uniquely solves*

$$\operatorname{ess\,sup}_{\tau \in [\vartheta, \tau_F^2(\vartheta)]} E \left[ L_\tau^1 \mid \mathcal{F}_\vartheta \right],$$

which has the same solutions as

$$\operatorname{ess\,sup}_{\tau \in [\vartheta, \tau_F^2(\vartheta)]} E \left[ \int_0^\tau \pi_s^{01} ds + \int_\tau^\infty \pi_s^{L1} ds \mid \mathcal{F}_\vartheta \right]. \quad (3.10)$$

*Proof:* See Appendix A.2.

The observation behind Lemma 3.10 is that if firm 2 will invest (no later than) when  $\tau_F^2(\vartheta)$  is reached, then earlier investment by 1 does not "trigger" a response at  $\tau_F^2(\vartheta)$  that might otherwise have come later, i.e., it does not cannibalize any monopoly revenue  $\pi^{L1}$  after  $\tau_F^2(\vartheta)$ . Then only the *constrained* problem of becoming leader up to  $\tau_F^2(\vartheta)$  is relevant for the reasoning of Lemma 3.9, which indeed has the same solutions as (3.10). It is important that firm 1 will not regret to receive  $\pi^{B1}$  from  $\tau_F^2(\vartheta)$  on by having invested before.<sup>9</sup>

In particular, if a monopolist's investment gain  $\pi^{L1} - \pi^{01}$  is not less than a follower's,  $\pi^{B1} - \pi^{F1}$  (as in typical market entry with  $\pi^{0i} \equiv \pi^{Fi}$ ), then the latest solution of (3.10) does not exceed  $\tau_F^1(\vartheta)$ , where any delay only means foregone revenue for a follower in (2.1), and firm 1 would now lose at least as much as prospective leader. Then immediate investment is dominant at  $\tau_F^1(\vartheta)$ , irrespective of when exactly firm 2 plans to invest on  $[\tau_F^1(\vartheta), \tau_F^2(\vartheta)]$ . In this case the constraint  $\tau \leq \tau_F^2(\vartheta)$  becomes irrelevant in (3.10), which reduces to the unconstrained problem (3.5).

<sup>8</sup>See Remark A.3 in Appendix A.1 on the relation between the monopolist and leader problems for standard diffusion models.

<sup>9</sup>Firm 2, on the contrary, might forego some revenue  $\pi^{L2}$  on  $[\tau_F^1(\vartheta), \tau_F^2(\vartheta)]$  by investing before  $\tau_F^1(\vartheta)$ , or, if it can only become leader up to  $\tau_F^1(\vartheta)$ , it may prefer to become follower there and effectively invest later.

Another “continuation” equilibrium that potentially induces earlier investment is preemption at  $\tau_{\mathcal{P}}(\vartheta)$  as in Section 3.1.2. Given preemption in  $\mathcal{P}$  (or an empty preemption region), firm 2 can never realize local payoffs exceeding  $F_t^2$ , whence the game has to end immediately at all latest optimal times to stop the process ( $F_t^2$ ). Such times have to satisfy  $\tau = \tau_F^2(\tau)$  (since  $F_\tau^2 \leq E[F_{\tau_F^2(\tau)}^2 | \mathcal{F}_\tau]$  by  $\pi^{02} \geq \pi^{F2}$ ), where simultaneous investment is an equilibrium with payoffs  $L_\tau^i = F_\tau^i = M_\tau^i$ ,  $i = 1, 2$ , as observed before, so firm 2 can also enforce these payoffs by investing.

However, given the assumption  $\pi^{02} \geq \pi^{F2}$ , a stopping time satisfying  $\tau = \tau_F^2(\tau)$  only maximizes firm 2’s follower payoff if it is also optimal to switch to  $\pi^{B2}$  directly from  $\pi^{02}$ , which is what happens under simultaneous investment.

**Lemma 3.11.** *Every stopping time  $\tau_M^i \geq \vartheta$  that attains*

$$\operatorname{ess\,sup}_{\tau \geq \vartheta} E \left[ M_\tau^i \middle| \mathcal{F}_\vartheta \right] = \operatorname{ess\,sup}_{\tau \geq \vartheta} E \left[ \int_0^\tau \pi_s^{0i} ds + \int_\tau^\infty \pi_s^{Bi} ds \middle| \mathcal{F}_\vartheta \right] \quad (3.11)$$

for some given  $\vartheta \in \mathcal{T}$  and  $i \in \{1, 2\}$  also attains

$$\operatorname{ess\,sup}_{\tau \geq \vartheta} E \left[ F_\tau^i \middle| \mathcal{F}_\vartheta \right]. \quad (3.12)$$

If  $\tau_M^i \geq \vartheta$  attains (3.12), then  $\tau_F^i(\tau_M^i)$  also attains (3.11). In particular, the latest solution of (3.11) is the latest solution of (3.12).

*Proof:* See Appendix A.2.

If  $\pi^{0i} \equiv \pi^{Fi}$ , as in typical market entry models, then (3.11) equals  $F_\vartheta^i$  and  $(F_t^i)$  is a supermartingale and indeed  $\tau_F^i(\vartheta)$  the latest time attaining (3.12). In particular, any delay at  $\tau_F^i(\vartheta)$  then means foregone revenue for firm  $i = 2$  and immediate investment must occur (conditional on preemption if  $\mathcal{P}$  was reached). Firm 1 may then want to invest even earlier in response by Lemma 3.10.

In general, however, it need not be optimal for firm 2 to secure the follower payoff at  $\tau_F^2(\vartheta)$  by proactive investment. It may be better to become follower later if possible, to benefit from a high pre-investment revenue: where  $\vartheta = \tau_F^2(\vartheta) \leq \tau$ ,  $E[F_\tau^2 | \mathcal{F}_\vartheta] - F_\vartheta^2 \geq E[M_\tau^2 | \mathcal{F}_\vartheta] - M_\vartheta^2 = E[\int_\vartheta^\tau (\pi_s^{02} - \pi_s^{B2}) ds | \mathcal{F}_\vartheta]$ , which may be positive if  $\pi^{02} > \pi^{F2}$ ; but here it depends on firm 1’s strategy, of course, whether  $F_\tau^2$  can be realized (e.g., by delayed joint investment).

## 4 Applications

As an illustration, the previous general results will now be applied to two typical models from the strategic real options literature, in order to provide complete proofs for basic equilibrium outcomes that are discussed extensively in the literature, to develop additional equilibria that may constitute Pareto improvements, and to point out that some equilibria analyzed in the literature only exist under additional restrictions, if at all. The model of Pawlina and Kort



(2006) first serves as the main vehicle; then the results of Grenadier (1996) will be put into perspective by the same arguments, although that model is quite different.

#### 4.1 Irreversible investment with asymmetric costs

The model of Pawlina and Kort (2006) is quite prototypic for the real options literature. However, the preemption equilibrium outcome proposed in that paper is not supported by the strategies described therein.<sup>10</sup> Theorem 3.4 yields even subgame-perfect equilibria, which we will analyze in more detail, including some behavior not treated by Pawlina and Kort (2006). Their revenue streams for firm  $i \in \{1, 2\}$  are

$$\left. \begin{aligned} \pi_t^{0i} &= e^{-rt} x_t D_{00}, & \pi_t^{Li} &= e^{-rt} (x_t D_{10} - rI^i), \\ \pi_t^{Fi} &= e^{-rt} x_t D_{01}, & \pi_t^{Bi} &= e^{-rt} (x_t D_{11} - rI^i), \end{aligned} \right\} \quad (4.1)$$

with discount factor  $r > 0$  and demand uncertainty reflected by a geometric Brownian motion  $(x_t)_{t \geq 0}$  satisfying

$$dx_t = \mu x_t dt + \sigma x_t dB_t, \quad (4.2)$$

where  $(B_t)_{t \geq 0}$  is Brownian noise,  $\mu < r$  the expected growth rate and  $\sigma > 0$  the volatility. The constants  $D_{10} \geq D_{11}$  and  $D_{00} \geq D_{01}$  capture a negative impact of investment on the opponent's revenue, and  $I^2 \geq I^1 > 0$  are the constant investment costs, here capitalized. The leader and follower processes are then continuous (as functions of the state  $x_t$ ), and the present instances of the follower problems (2.1) and the monopoly problems (3.5) are solved by stopping when  $x_t$  exceeds some thresholds  $x_F^i$  and  $x_L^i$ , respectively.<sup>11</sup> Thus, simultaneous investment is an equilibrium for all states  $x_\vartheta \geq x_F^2$ .

If the preemption region in this model is non-empty, it is characterized by an open interval  $(\underline{x}, \bar{x})$  of the state space  $\mathbb{R}_+$  with  $\bar{x} \leq x_F^1 \leq x_F^2$  (where both inequalities are strict if  $I^2 > I^1$  and  $D_{10} > D_{11} > D_{01}$ ), such that one can simply call  $(\underline{x}, \bar{x})$  preemption region. The proof of the following proposition generalizes to other models driven by a continuous Markov process that affects revenues monotonically.

**Proposition 4.1.** *Consider the specification (4.1). There are two numbers  $\underline{x} \leq \bar{x} \in (0, x_F^1]$  such that  $L_t^2 > F_t^2 \Leftrightarrow x_t \in (\underline{x}, \bar{x})$  for all  $t \in \mathbb{R}_+$ .*

*Proof:* See Appendix A.2.

By Lemma 3.8 and the discussion thereafter it is enough to check if  $L_0^2 - F_0^2 > 0$  for  $x_0 = x_\Delta^2$ , the threshold solving (3.8), which is the case iff the cost-disadvantage  $I^2/I^1$  is not

<sup>10</sup>Pawlina and Kort (2006) do not model preemption and just state that the high cost firm 2 invests at its follower threshold  $x_F^2$ . Knowing that, firm 1 could decide when to become optimally the leader up to that point and would *not* be willing to invest already at the preemption point. Even taking preemption as given, it is not verified that both firms are willing to wait until the preemption point; the argument that there is a second-mover advantage is not sufficient.

<sup>11</sup>If  $D_{11} > D_{01}$ , then  $x_F^i = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I^i(r - \mu)}{D_{11} - D_{01}}$ , where  $\beta_1 > 1$  is the positive root of  $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0$ . If  $D_{11} \leq D_{01}$ , then  $x_F^i = \infty$ . Analogously,  $x_L^i = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I^i(r - \mu)}{(D_{10} - D_{00})^+}$ .

too large; otherwise firm 2 prefers to invest much later than firm 1 and the preemption region is empty (in particular if  $x_\Delta^2 \geq x_F^1$ , where firm 1 would follow immediately).<sup>12</sup>

We can now characterize the equilibria of Theorem 3.4 for this model, which also have outcomes not captured in Pawlina and Kort (2006). Continuity ensures existence and it suffices to solve the simpler constrained monopoly problems (3.4) from Proposition 3.5. By the strong Markov property, this amounts to finding the region in the state space  $\mathbb{R}_+$  where immediate stopping is optimal for the problem at  $t = 0$ ,

$$\sup_{\tau \leq \tau_{\mathcal{P}}(0) \wedge \tau_F^2(0)} E \left[ \int_{\tau}^{\infty} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right]. \quad (4.3)$$

The constraint here takes the form  $\min\{\tau_{\mathcal{P}}(0), \tau_F^2(0)\} = \inf\{t \geq 0 \mid x_t \in (\underline{x}, \bar{x}) \cup [x_F^2, \infty)\} = \inf\{t \geq 0 \mid x_t \in [\underline{x}, \bar{x}] \cup [x_F^2, \infty)\}$  a.s. Problem (3.4) is then solved by stopping once the state  $x_t$  hits the stopping region  $\{x \in \mathbb{R}_+ \mid \tau = 0 \text{ attains (4.3) for } x_0 = x\}$  from time  $\vartheta$ .

First consider a non-empty preemption region  $(\underline{x}, \bar{x})$  that is connected to the unconstrained monopoly stopping set  $[x_L^1, \infty)$ , which is the case, e.g., for the market entry variant of the model with  $D_{01} = D_{00}$ , see Lemma 3.8. Then immediate stopping is optimal in (4.3) for any state  $x_0 \geq \bar{x} \geq x_L^1$ , as it is in the unconstrained problem. Since the preemption constraint in (4.3) is a constant upper threshold for states  $x_0 < \underline{x}$ , it is optimal to wait there until  $x_t$  exceeds either the constraint  $\underline{x}$  or the unconstrained threshold  $x_L^1$ , see Lemma A.4 in Appendix A.1.

The subgame-perfect equilibrium is complete in this case: no investment for states strictly below  $\min\{\underline{x}, x_L^1\}$ , preemptive investment in  $[\underline{x}, \bar{x}]$  as described in Section 3.1.2, firm 1 investing as the leader in  $[x_L^1, x_F^2) \setminus [\underline{x}, \bar{x}]$ , and simultaneous investment for all states in  $[x_F^2, \infty)$ .

If  $D_{00} = D_{01}$ , we can also conclude that preemption cannot be avoided and consequently neither simultaneous investment in  $[x_F^2, \infty)$  by Lemma 3.11, such that the equilibrium is unique: If the preemption region is non-empty, it certainly contains the optimal stopping region for the continuous process  $L_t^2 - F_t^2$ , which takes positive values only there. Then one also has to stop  $L_t^2$  in that stopping region, the problem considered in Lemma 3.9, because  $L_t^2 = (L_t^2 - F_t^2) + F_t^2$  and  $F_t^2$  is a supermartingale now.

Next, if the preemption region is empty, then firm 2 simply plans to invest as soon as the state hits  $[x_F^2, \infty)$ , such that for states below it, firm 1 only faces the upper constraint  $x_F^2$  in (4.3). Again by Lemma A.4, it is then optimal for firm 1 to invest as soon as  $x_t$  exceeds either the constraint  $x_F^2$  or the unconstrained monopoly threshold  $x_L^1$ . Note that in the market entry variant with  $D_{00} = D_{01} < D_{11}$ ,  $x_L^1 \leq x_F^1 < x_F^2 < \infty$ .

However, even if firm 1 uses the unconstrained monopoly threshold in this case, and thus acts *as if* it ignored firm 2 completely, this does not mean that firm 1 is able to maximize

<sup>12</sup>The precise condition  $(I^2/I^1)^{\beta_1-1} < ((1+c)^{\beta_1} - 1)/(\beta_1 c)$  if  $c := (D_{10} - D_{11})/(D_{11} - D_{01}) \in (0, \infty)$  is obtained by plugging  $x_\Delta^2 = \frac{\beta_1}{\beta_1-1} \cdot \frac{I^2(r-\mu)}{(D_{10}-D_{01})^+}$  (cf. fn. 11) into the explicit functional expressions for the leader and follower values (8) and (9) in Pawlina and Kort (2006), who identify the same condition by a graphical argument. This condition indeed implies the weaker  $x_\Delta^2 < x_F^1$ . The constraint on the cost ratio strictly exceeds 1 and is strictly increasing in  $c$  to  $\infty$ , since  $\beta_1 > 1$ . If  $D_{10} > D_{01} \geq D_{11}$ , then  $x_F^1 = \infty$  and the preemption region is non-empty for all  $I^2 \geq I^1$ . Finally, if  $D_{10} \leq \max\{D_{11}, D_{01}\}$ , then  $x_\Delta^2 \geq x_F^1$  and the preemption region is empty.

the unconstrained leader payoff, as it could if it had the exclusive right to invest first (like a Stackelberg leader). Firm 1 only maximizes the leader payoff subject to firm 2 investing also *proactively* in  $[x_F^2, \infty)$ . The latter is unavoidable in the market entry variant with an empty preemption region, since the threshold  $x_F^2$  then uniquely solves problem (3.11) for firm 2.

#### 4.1.1 Preemption when demand falls

So far, if  $x_L^1 \leq \bar{x}$  or  $\mathcal{P} = \emptyset$ , there has been immediate investment by some firm in any state above  $\min\{\underline{x}, x_L^1\}$  and no investment below it (also in case  $\mathcal{P} = \emptyset$ , then setting  $\underline{x} = \bar{x} = x_F^2$ ). To complete the explicit description of the equilibria from Theorem 3.4, it remains to consider a monopoly threshold lying above a non-empty preemption region,  $x_L^1 > \bar{x} > \underline{x}$ , which requires a sufficiently high pre-investment revenue level  $D_{00} > D_{01}$ . Firm 1 may then remain inactive even where it would invest immediately as follower (in states above  $x_F^1$ ), because it has higher opportunity costs as prospective leader. This case is not addressed by Pawlina and Kort (2006), who only consider states below  $\underline{x}$ , where the same observation as before applies: firm 1 waits until  $x_t$  hits the constraint  $\underline{x} < x_L^1$ . Problem (4.3) becomes more complicated for  $x_0 \in (\bar{x}, x_F^2)$ , where both constraints may become binding if that interval intersects the continuation region  $[0, x_L^1)$  of the unconstrained problem.

A lower constraint like the present  $\bar{x}$  has a much stronger effect than any upper constraint considered before. Two cases can be distinguished for the problem of delaying the revenue change  $\pi_t^{L1} - \pi_t^{01} = e^{-rt}(x_t(D_{10} - D_{00}) - rI^1)$  in  $[\bar{x}, x_F^2]$ . The easier one is that  $x(D_{10} - D_{00}) > rI^1$  on all of  $(\bar{x}, x_F^2)$ . Then it is optimal to stop immediately everywhere, since any delay is a loss of revenue. The more difficult case is that  $x(D_{10} - D_{00}) < rI^1$  near the preemption region. Firm 1 must wait where this inequality holds, in order not to start with running losses, so one has to determine the stopping region towards the upper constraint  $x_F^2$ .

**Proposition 4.2.** *Consider the specification (4.1) and suppose the corresponding preemption region  $(\underline{x}, \bar{x}) \subset (0, x_F^1]$  from Proposition 4.1 is non-empty. If  $\bar{x}(D_{10} - D_{00}) \geq rI^1$ , then the solution of problem (4.3) for all states  $x_0$  in  $(\bar{x}, x_F^2)$  is to stop immediately, while if  $D_{10} - D_{00} \leq 0$ , the solution is to wait until the state exits  $(\bar{x}, x_F^2)$ .*

*If  $0 < \bar{x}(D_{10} - D_{00}) < rI^1$ , then there is a unique threshold  $\hat{x} \in [rI^1/(D_{10} - D_{00}), x_L^1)$  solving*

$$(\beta_1 - 1)A(x)x^{\beta_1} + (\beta_2 - 1)B(x)x^{\beta_2} = I^1 \quad (4.4)$$

with

$$\begin{pmatrix} A(x) \\ B(x) \end{pmatrix} = [\bar{x}^{\beta_1}x^{\beta_2} - x^{\beta_1}\bar{x}^{\beta_2}]^{-1} \begin{pmatrix} x^{\beta_2} & -\bar{x}^{\beta_2} \\ -x^{\beta_1} & \bar{x}^{\beta_1} \end{pmatrix} \begin{pmatrix} \bar{x} \frac{D_{10} - D_{00}}{r - \mu} - I^1 \\ x \frac{D_{10} - D_{00}}{r - \mu} - I^1 \end{pmatrix} \quad (4.5)$$

and  $\beta_1 > 1$  and  $\beta_2 < 0$  the roots of  $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0$ , and the solution of problem (4.3) for all states  $x_0$  in  $(\bar{x}, x_F^2)$  is to stop when  $(x_t)$  exits  $(\bar{x}, \hat{x} \wedge x_F^2)$ .

*Proof:* See Appendix A.2.

In particular, if  $x_F^2(D_{10} - D_{00}) \leq rI^1$ , then  $\hat{x} \geq x_F^2$  and the solution is to wait until the state exits  $(\bar{x}, x_F^2)$ . It is easy to calculate the solutions  $\hat{x}$  of (4.4), which are typically much

lower than the upper constraint  $x_F^2$  or the unconstrained solution  $x_L^1$ . Thus, the risk of getting trapped at  $\bar{x}$  by preemption implies very early stopping, as illustrated in Section 4.1.4 below.

#### 4.1.2 Joint investment equilibria

If  $D_{00} > D_{01}$ , then there are potentially many more equilibria than those from Theorem 3.4, since one can now drop the premise that preemption occurs in the preemption region, and/or that simultaneous investment occurs everywhere above  $x_F^2$ .

We will first apply Proposition 3.6 to joint investment at some threshold, which cannot be below  $x_F^2$  for simultaneous investment to be an equilibrium. The highest expected value of joint investment can be achieved by solving (3.11), which yields a maximal threshold, say  $x_M^1$  for firm 1. But we can also consider constrained versions of that problem, with some investment threshold  $x_J \in [x_F^2, x_M^1]$ . Joint investment when  $x_t \geq x_J$  now is an equilibrium if firm 1 does not want to become leader at the threshold solving (3.6), which is  $\min\{x_J, x_L^1\}$  again by Lemma A.4. Specifically, the cost difference cannot be too large, such that firm 1 cannot enjoy a leader's monopoly revenue for very long, which bounds  $L^1$ .

**Proposition 4.3.** *Consider the specification (4.1) and let  $x_M^1 \geq x_L^1 \in [0, \infty]$  denote the threshold solving problem (3.11) for firm 1.<sup>13</sup> Suppose  $x_M^1 \geq x_F^2$ . Then there exists a subgame-perfect equilibrium with simultaneous investment above the threshold  $x_J \in [x_F^2, x_M^1]$  iff*

$$x_L^1 \geq x_F^2 \quad \Leftrightarrow \quad \frac{I^2}{I^1} \leq \frac{D_{11} - D_{01}}{(D_{10} - D_{00})^+}$$

or

$$\left(\frac{I^2}{I^1}\right)^{\beta_1-1} \left[1 + \left(\frac{x_L^1}{x_J}\right)^{\beta_1} \left(\beta_1 - 1 - \frac{x_J}{x_L^1} \beta_1 \frac{D_{11} - D_{00}}{D_{10} - D_{00}}\right)\right] \leq \beta_1 \frac{D_{10} - D_{11}}{D_{10} - D_{00}} \left(\frac{(D_{11} - D_{01})^+}{D_{10} - D_{00}}\right)^{\beta_1-1} \quad (4.6)$$

with  $\beta_1 > 1$  from Proposition 4.2. The LHS of (4.6) is strictly positive and strictly decreasing in  $x_J \in [x_L^1, x_M^1]$  if  $x_L^1 < x_F^2$ .

*Proof:* See Appendix A.2.

Note that  $x_L^1 < x_F^2$  implies  $D_{10} > D_{00}$ . Then the second restriction on  $I^2/I^1$  in the proposition is weaker than the first if setting  $x_J = x_L^1$ , and it is further relaxed if  $x_J$  increases. If  $x_J = x_M^1 < \infty$ , then (4.6) coincides with the bound on  $I^2/I^1$  identified by a graphical argument in Pawlina and Kort (2006), who impose  $D_{11} > D_{00}$ .<sup>14</sup> Proposition 4.3 also applies for  $D_{11} \leq D_{00}$ , when the firms end up worse after both having invested than before. Even then it can be an equilibrium to invest simultaneously at some threshold  $x_J \in \mathbb{R}_+$ , although it would be more favorable that both firms never invest at all.

Indeed, there may be many equilibria with “inefficient” joint investment in states above  $x_F^2$  and where the expected joint investment payoff could be improved. If  $(D_{11} - D_{00})x_F^2 < rI^1$ ,

<sup>13</sup>  $x_M^1 = \frac{\beta_1}{\beta_1-1} \cdot \frac{I^1(r-\mu)}{D_{11}-D_{00}}$  if  $D_{11} > D_{00}$  and  $x_M^1 = \infty$  else, cf. fn. 11.

<sup>14</sup>  $x_M^1 < \infty \Leftrightarrow D_{11} > D_{00}$  and then  $x_J = x_M^1$  implies  $x_J/x_L^1 = (D_{10} - D_{00})/(D_{11} - D_{00})$ .

then the drift of  $M_t^i$  is positive for states in the interval  $[x_F^2, rI^1/(D_{11} - D_{00})^+)$ , and hence it is optimal to wait in any constrained version of problem (3.11). Therefore one can partition the latter interval into arbitrary subintervals of alternating joint investment and waiting.

### 4.1.3 Sequential investment equilibria

Sequential investment may also be an equilibrium if the preemption region is non-empty, which is a Pareto improvement compared to the equilibria of Pawlina and Kort (2006) if delayed joint investment as in Section 4.1.2 is not feasible. Such an equilibrium can be verified by Corollary 3.7 and it exists for the current specification iff firm 2 does not have a strict first-mover advantage at  $x_L^1$ , where firm 1 invests.

**Proposition 4.4.** *Consider the specification (4.1) and suppose  $x_L^1 < x_F^2$  (whence  $D_{10} > D_{00}$ ). Then there exists a subgame-perfect equilibrium with firm 1 investing as soon as  $x_t$  exceeds  $x_L^1$  and firm 2 planning to invest when  $x_t$  exceeds  $x_F^2$  iff  $x_L^1 \notin (\underline{x}, \bar{x})$  from Proposition 4.1, which is iff*

$$x_L^1 \geq x_F^1 \quad \Leftrightarrow \quad D_{10} - D_{00} \leq D_{11} - D_{01}$$

or

$$(\beta_1 - 1) \frac{I^2}{I^1} + \left( \frac{I^2}{I^1} \right)^{1 - \beta_1} \left( \frac{(D_{11} - D_{01})^+}{D_{10} - D_{00}} \right)^{\beta_1} \geq \beta_1 \left[ \frac{D_{10} - D_{01}}{D_{10} - D_{00}} - \frac{D_{10} - D_{11}}{D_{10} - D_{00}} \left( \frac{(D_{11} - D_{01})^+}{D_{10} - D_{00}} \right)^{\beta_1 - 1} \right] \quad (4.7)$$

with  $\beta_1 > 1$  from Proposition 4.2. The LHS of (4.7) is strictly increasing in  $I^2/I^1$  and the RHS strictly positive if  $x_L^1 < x_F^1$ .

*Proof:* See Appendix A.2.

Finally, there can be equilibria with sequential investment as in Proposition 4.4 or preemption as in Proposition 4.2 where the joint investment is delayed to some threshold  $x_J > x_F^2$ , such that firm 1 can optimize over larger intervals when to become leader. This may separate the investment regions in the sequential equilibria into one where firm 1 invests as leader and one where simultaneous investment occurs, with a gap in between. Such equilibria are more difficult to characterize explicitly. If  $x_F^2$  is between two investment regions, the follower reaction has to be taken into account without the simplifications used in the previous propositions.

### 4.1.4 Comparison of leader stopping regions

To illustrate the potentially strong impact of preemption on states in  $(\bar{x}, x_F^2)$  for varying parameter values in Figure 1, the model is re-parameterized as follows. First,  $r$ ,  $\mu$  and  $\sigma$  determine  $\beta_{1,2}$  and together with the ratio  $I^1/(D_{11} - D_{01})$  also firm 1's follower threshold  $x_F^1$ , which we fix and which is an upper bound on  $\bar{x}$ .

The distance between  $\bar{x}$  and  $x_F^2$ , which is the region where firm 1 can invest as leader, is growing in  $I^2$ . Indeed,  $x_F^2$  obviously grows with  $I^2$ , and if the preemption region  $(\underline{x}, \bar{x})$  is

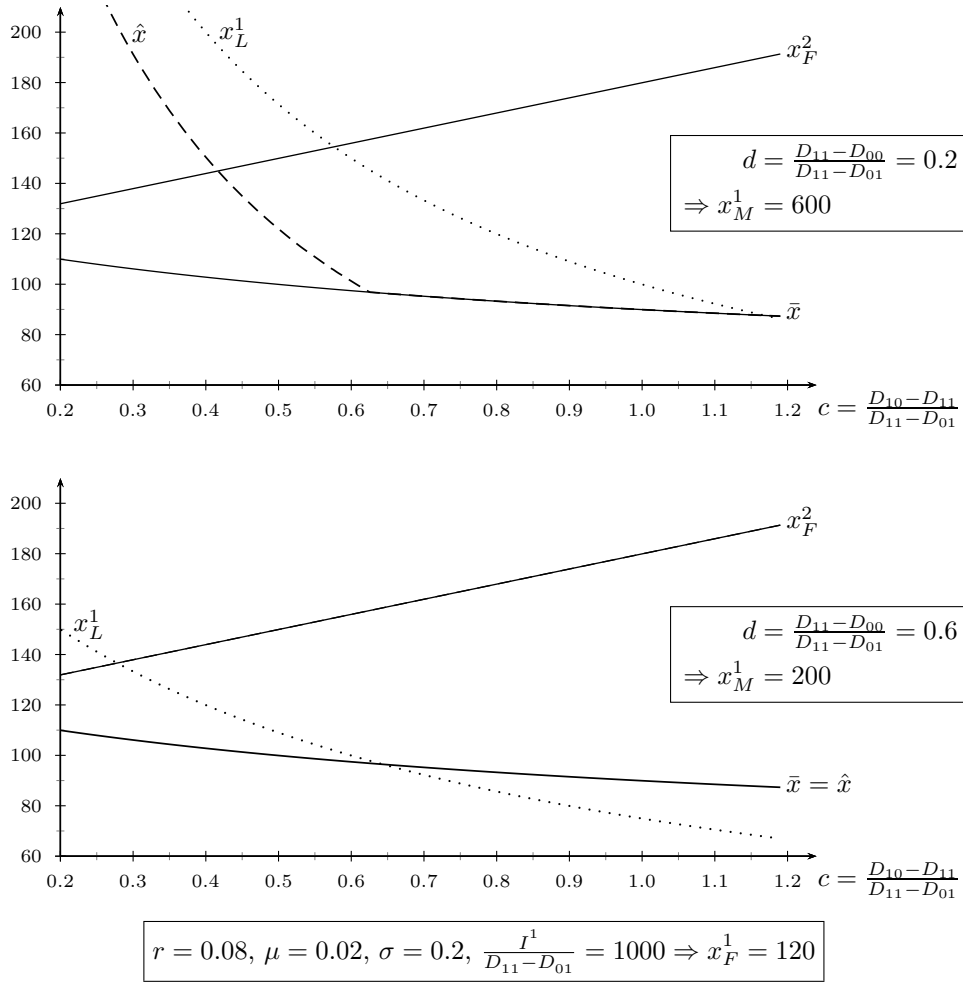


Figure 1: Constrained leader stopping regions.

non-empty, it is strictly shrinking if  $I^2$  grows<sup>15</sup>;  $(x, \bar{x})$  collapses when  $I^2/I^1 = x_F^2/x_F^1$  reaches a bound given in fn. 12 in terms of  $c = (D_{10} - D_{11})/(D_{11} - D_{01})$ , the loss of a monopolist relative to the gain of the follower when the latter invests. We pick those limit values for  $I^2$  and  $x_F^2$  for simplicity, thus making both functions of  $c$ , although then just  $\underline{x} = \bar{x} = x_\Delta^2$ , the threshold solving (3.8). Now  $c$  directly determines  $\bar{x}$  by  $x_\Delta^2 = x_F^2/(1+c)$ .

Equation (4.4) for  $\hat{x}$  can be reduced to the parameters  $\beta_{1,2}$  and  $x_L^1$ , the unconstrained monopoly threshold, which is an upper bound on  $\hat{x}$  and itself satisfies  $x_L^1 = x_F^1/(c+d)$  with  $d := (D_{11} - D_{00})/(D_{11} - D_{01})$ . The latter ratio comes close to 1 if the leader's investment has not much influence on the follower's revenue, like in a market entry situation; it becomes small when the leader steals considerable business from the follower, like by a drastic innovation.  $d$

<sup>15</sup>Suppose  $x_0 < x_F^2$ , such that firm 2's first-mover advantage  $L_0^2 - F_0^2$  is non-trivial. If  $I^2$  is increased, that has two negative effects on  $L_0^2 - F_0^2$ . First, it increases the investment cost stream  $e^{-rt}rI^2$  up to firm 2's former follower investment time  $\tau_F^2(0)$ , which reduces  $L_0^2$ . Second, it delays  $\tau_F^2(0)$ . The new revenue stream difference  $e^{-rt}(x_t(D_{11} - D_{01}) - rI^2)$  (with increased  $I^2$ ) between the former and the new  $\tau_F^2(0)$  has non-positive expectation by optimality of the new  $\tau_F^2(0)$ , and thus reduces  $L_0^2 - F_0^2$ .

controls the best simultaneous investment threshold by  $x_M^1 = x_F^1/d$ .

In the equilibria from Theorem 3.4, firm 1 can freely decide when to invest in the interval  $(\bar{x}, x_F^2)$ . Without the threat of preemption, it would not invest below  $\min\{x_L^1, x_F^2\}$ . However, given the threat of preemption, firm 1 already invests when the state exceeds  $\hat{x}$ , which may be much earlier as Figure 1 shows. In the upper panel with a low value of  $d$ , the threat of preemption strongly matters for  $c \geq 0.45$ . Firm 1 never chooses to wait at all in the lower panel with a moderate value of  $d$ . Joint investment at  $x_M^1$  is an equilibrium avoiding preemption if  $x_L^1 \geq x_F^2$ ; it is not an equilibrium for  $d = 0.6$  and  $c \geq 0.45$ .

## 4.2 Strategic real estate development with construction time

Similar reasoning as before shows on the one hand that equilibria discussed in Grenadier (1996) only exist under certain parameter restrictions, while on the other hand there exist additional equilibria that are Pareto improvements.

Grenadier (1996) models a real option game between two symmetric real estate owners, who may each invest to redevelop their property in order to earn higher rents. His model needs a slight translation to fit into the current framework, since it includes a delay of construction: if an owner invests, it takes  $\delta \geq 0$  time units until the new building yields any revenues. Before investment by any owner, both earn the deterministic rent  $R \geq 0$ . Investment at cost  $I > 0$  terminates that rent, reduces the rent of the opponent to  $(1 - \gamma)R$  with  $\gamma \in [0, 1]$  and initiates own new rent  $D_1 x_t$  after the delay  $\delta$ .  $(x_t)$  is a geometric Brownian motion as in (4.2). Once both new buildings are completed, each owner earns the rent  $D_2 x_t$  with  $D_2 \in (0, D_1]$ .

Grenadier's model is strategically equivalent to specifying

$$\begin{aligned} \pi_t^{0i} &= e^{-rt} R, & \pi_t^{Li} &= e^{-rt} (D_1 e^{-(r-\mu)\delta} x_t - rI), \\ \pi_t^{Fi} &= e^{-rt} (1 - \gamma)R, & \pi_t^{Bi} &= e^{-rt} (D_2 e^{-(r-\mu)\delta} x_t - rI) \end{aligned}$$

in the general framework. The equilibria proposed in Grenadier (1996) are justified by the insufficient argument that waiting is optimal if the current follower payoff exceeds the current leader payoff. Nevertheless there exists a subgame-perfect equilibrium as in Theorem 3.4, since Proposition 3.5 applies thanks to continuity; it can be characterized as follows. The follower problems (2.1) are again solved by investing once  $x_t$  exceeds a threshold  $x_F > 0$ , whence simultaneous investment is an equilibrium for all states  $x_\vartheta \geq x_F$ .<sup>16</sup> Problem (3.8) is solved by a threshold  $x_\Delta = x_F D_2 / D_1$  and the preemption region  $\mathcal{P}$  is non-empty iff  $D_2 < D_1$ .  $\mathcal{P}$  can be represented by an interval  $(\underline{x}, \bar{x})$  of the state space by the same arguments as in the proof of Proposition 4.1, where now  $\bar{x} = x_F$ .

Thus problem (3.4) only needs to be solved for states  $x_\vartheta < \underline{x}$  (also in case  $\mathcal{P} = \emptyset$ , then setting  $\underline{x} = \bar{x} = x_F$ ). Here  $\pi^{L1} - \pi^{01}$  has the same structure as under the specification (4.1), making Lemma A.4 apply again. It is now optimal to wait until the upper constraint  $\underline{x}$  is reached, since the present instance of the unconstrained problem (3.5) is solved by a threshold

<sup>16</sup>  $x_F = \frac{\beta_1(r-\mu)}{(\beta_1-1)D_2} e^{(r-\mu)\delta} (I + (1-\gamma)R/r)$  with  $\beta_1 > 1$  from fn. 11.

$x_L \geq x_\Delta \in [x, \bar{x}]$  (with strict inequality iff  $\gamma > 0$ ).<sup>17</sup>

#### 4.2.1 Qualification of further equilibria

There may be other equilibria with delayed simultaneous investment and/or no preemption. For states above  $\bar{x} = x_F$ , any investment will induce the simultaneous investment payoff. Contrarily to the claim made in Grenadier (1996), simultaneous investment cannot be delayed past the threshold  $x_M = x_L D_1 / D_2 \geq x_F$  solving problem (3.11). Indeed, in any equilibrium with preemption in  $\mathcal{P}$ , by symmetry now neither firm ever gets more than the follower payoff at the time of investment. The same holds for any equilibrium with only joint investment. In either case investment must occur as soon as the state exceeds  $x_M$ , since any delay would be a loss by Lemma 3.11.

With preemption occurring in the preemption region, one can only consider to delay simultaneous investment in the interval  $[\bar{x}, x_M]$ , i.e., to delay the revenue change  $\pi_t^{Bi} - \pi_t^{0i} = e^{-rt}(D_2 e^{-(r-\mu)\delta} x_t - rI - R)$ . This has the same form as the problem with two-sided constraint considered in Proposition 4.2 (recall also the illustration in Section 4.1.4), with  $D_2 e^{-(r-\mu)\delta}$  replacing  $D_{10} - D_{00}$ ,  $I + R/r$  replacing  $I^1$  and  $x_M$  replacing  $x_F^2$ . Thus, given now  $\bar{x} = x_F$ , if  $D_2 e^{-(r-\mu)\delta} x_F \geq rI + R$ , which means if

$$\gamma \leq \left( \frac{rI}{R} + 1 \right) \left( 1 - \frac{\beta_1 - 1}{\beta_1(r - \mu)} \right), \quad (4.8)$$

then investment cannot be delayed at all for states above  $x_F$ , which is ignored in Grenadier (1996). In this case the preemption region extends to such high states that any foregone revenue above it is a loss. Note that the RHS of (4.8) is strictly positive.

Only if (4.8) fails, there will exist a solution  $\hat{x} \in [(rI + R)e^{(r-\mu)\delta} / D_2, x_M)$  to the current version of (4.4), such that investment can be held back in  $(x_F, \hat{x})$ . Only then the phenomenon discussed extensively in Section V of Grenadier (1996) can arise, that preemption occurs when demand *falls* to  $x_F$ .

However, if  $\gamma$  is sufficiently large to violate (4.8), then delayed joint investment may be attractive enough to avoid preemption altogether, which will be a Pareto improvement w.r.t. Grenadier (1996). By Proposition 3.6, preemption can be avoided in an equilibrium of joint investment with the threshold  $x_M \geq x_L$  iff the latter, which now solves problem (3.6), satisfies

$$x_L \geq x_F \quad \Leftrightarrow \quad \gamma \geq \left( \frac{rI}{R} + 1 \right) \left( 1 - \frac{D_2}{D_1} \right) \quad (4.9)$$

or if  $L_0^i \leq E[M_{\tau_M}^i]$  with  $\tau_M := \inf\{t \geq 0 \mid x_t \geq x_M\}$  holds for  $x_0 = x_L < x_F$ <sup>18</sup>, which is iff

$$\gamma \geq \left( \frac{rI}{R} + 1 \right) \left( 1 - D_2 \left( \beta_1 \frac{D_1 - D_2}{D_1^{\beta_1} - D_2^{\beta_1}} \right)^{\frac{1}{\beta_1 - 1}} \right).$$

<sup>17</sup>  $x_L = \frac{\beta_1(r-\mu)}{(\beta_1-1)D_1} e^{(r-\mu)\delta} (I + R/r)$  with  $\beta_1 > 1$  from fn. 11. This should not be confused with  $X_L$  in Grenadier (1996), which corresponds to the present  $\bar{x}$ .

<sup>18</sup> For details, cf. the proof of Proposition 4.3 in Appendix A.2.



The last restriction on  $\gamma$  is indeed weaker than the one in (4.9).

## 5 Conclusion

The equilibrium analysis of the general model in Section 3 was based directly on the comparison of revenue streams and not on derived analytic properties of value functions, as it frequently happens in the growing literature on real option games. By that more general perspective, there is on the one hand less risk to neglect some verification problems for equilibria and on the other hand the economic structure of equilibria becomes clearer. For models that satisfy the general assumptions made here, the number of equilibrium verification problems to be solved has been greatly reduced. By Theorem 3.4 it remains to solve a single class of optimal stopping problems for one firm. It applies to many more examples from the literature than the ones analyzed in Section 4 (e.g., to those listed in the Introduction). The presented applications, which have quite distinctive economic properties, illustrate how the general results act in typical state-space models, to answer possibly neglected verification questions and to identify equilibria without preemption that may be Pareto improvements.

Therefore the general model presented here provides a foundation for a more complete analysis of (existing) preemptive investment models and a guideline for the analysis of further models that do not satisfy the revenue orders assumed here.

## A Appendix

### A.1 Technical results

**Lemma A.1.** *Let  $\pi^0$ ,  $\pi^L$ ,  $\pi^F$  and  $\pi^B$  be adapted processes in  $L^1(dt \otimes P)$ , and  $\{\tau_F(\vartheta) \mid \vartheta \in \mathcal{T}\}$  a family of stopping times satisfying  $\vartheta \leq \tau_F(\vartheta) \leq \tau_F(\tau)$  a.s. for all  $\vartheta, \tau \in \mathcal{T}$  with  $\vartheta \leq \tau$  a.s. Then there exist optional processes  $L = (L_t)_{t \geq 0}$  and  $F = (F_t)_{t \geq 0}$  that are of class (D) and which satisfy*

$$L_\vartheta = L(\vartheta) := \int_0^\vartheta \pi^0(u) du + E \left[ \int_\vartheta^{\tau_F(\vartheta)} \pi^L(u) du + \int_{\tau_F(\vartheta)}^\infty \pi^B(u) du \middle| \mathcal{F}_\vartheta \right]$$

and

$$F_\vartheta = F(\vartheta) := \int_0^\vartheta \pi^0(u) du + \operatorname{ess\,sup}_{\tau \geq \vartheta} E \left[ \int_\vartheta^\tau \pi^F(u) du + \int_\tau^\infty \pi^B(u) du \middle| \mathcal{F}_\vartheta \right]$$

a.s. for every  $\vartheta \in \mathcal{T}$ . In particular, the process  $F$  can be chosen right-continuous. If  $\lim \tau_F(\vartheta^n) = \tau_F(\vartheta)$  a.s. for any  $\vartheta \in \mathcal{T}$  and sequence  $(\vartheta^n)_{n \in \mathbb{N}} \subset \mathcal{T}$  with  $\vartheta^n \searrow \vartheta$  a.s., then also  $L$  can be chosen right-continuous.

All conditions are met when letting each  $\tau_F(\vartheta)$  be the latest stopping time attaining the value of  $F(\vartheta)$ .

*Proof.* First rewrite  $F(\vartheta)$  as

$$\begin{aligned} F(\vartheta) &= \int_0^{\vartheta} (\pi^0(u) - \pi^F(u)) du + E \left[ \int_0^{\infty} \pi^B(u) du \middle| \mathcal{F}_{\vartheta} \right] \\ &\quad + \operatorname{ess\,sup}_{\tau \geq \vartheta} E \left[ \int_0^{\tau} (\pi^F(u) - \pi^B(u)) du \middle| \mathcal{F}_{\vartheta} \right]. \end{aligned} \tag{A.1}$$

The first term on the RHS is obviously a continuous, adapted process evaluated at  $\vartheta$  which is bounded by  $\int_0^{\infty} (|\pi^0(u)| + |\pi^F(u)|) du \in L^1(P)$ , hence optional and of class (D). The second and third terms are (super-)martingale-systems (cf. El Karoui (1981), Proposition 2.26) of class (D), particularly the latter bounded by the family  $\{E[\int_0^{\infty} (|\pi^F(u)| + |\pi^B(u)|) du \mid \mathcal{F}_{\vartheta}] \mid \vartheta \in \mathcal{T}\}$  of class (D). Thus there exist optional processes of class (D) that aggregate the two (super-)martingale-systems, respectively. The former, being a martingale, may be chosen right-continuous. For the latter, we identify in fact the Snell envelope  $U_Y$  of the continuous process  $Y := (\int_0^t (\pi^F(u) - \pi^B(u)) du)_{t \geq 0}$ , whence  $U_Y$  is (right-)continuous in expectation and thus may be assumed to have right-continuous paths, a.s.

$L(\vartheta)$  has a decomposition analogous to (A.1) with a third term

$$X(\vartheta) := E \left[ \int_0^{\tau_F(\vartheta)} (\pi^L(u) - \pi^B(u)) du \middle| \mathcal{F}_{\vartheta} \right].$$

Suppose first that  $\pi^L(u) - \pi^B(u) \geq 0$  for all  $u \in \mathbb{R}_+$ , a.s. In this case

$$\begin{aligned} E[X(\tau) \mid \mathcal{F}_{\vartheta}] &= X(\vartheta) + E \left[ \int_{\tau_F(\vartheta)}^{\tau_F(\tau)} (\pi^L(u) - \pi^B(u)) du \middle| \mathcal{F}_{\vartheta} \right] \\ &\geq X(\vartheta) \end{aligned}$$

for all stopping times  $\tau \geq \vartheta$  (whence  $\tau_F(\vartheta) \leq \tau_F(\tau)$ ), so  $X := \{X(\vartheta) \mid \vartheta \in \mathcal{T}\}$  is a submartingale-system.  $X$  is bounded by  $\{E[\int_0^{\infty} (|\pi^L(u)| + |\pi^B(u)|) du \mid \mathcal{F}_{\vartheta}] \mid \vartheta \in \mathcal{T}\}$ , hence of class (D). In general, the last argument applies separately to  $(\pi^L(u) - \pi^B(u))^+$  and  $(\pi^L(u) - \pi^B(u))^-$ , showing that  $X$  is the difference of two submartingale-systems which can be aggregated by two optional processes of class (D).

If  $\lim \tau_F(\vartheta^n) = \tau_F(\vartheta)$  a.s. for any sequence  $(\vartheta^n)_{n \in \mathbb{N}} \subset \mathcal{T}$  with  $\vartheta^n \searrow \vartheta$  a.s., then  $X$  – being of class (D) – is right-continuous in expectation and the aggregating submartingales can be chosen with right-continuous paths.

As the process  $Y$  defined above is continuous, the latest stopping time after  $\vartheta$  that attains  $F(\vartheta)$  is the first time the monotone part of the Snell envelope  $U_Y$  increases. That monotone part inherits continuity from  $Y$ . Thus chosen,  $\vartheta \leq \tau_F(\vartheta) \leq \tau_F(\tau)$  on  $\{\vartheta \leq \tau\}$  for all  $\vartheta, \tau \in \mathcal{T}$ . Now consider a sequence of stopping times  $\vartheta^n \searrow \vartheta$  a.s., whence also  $\tau_F(\vartheta^n)$  decreases in  $n$ . By construction we can only have  $\lim \tau_F(\vartheta^n) > \tau_F(\vartheta) \geq \vartheta$  where the monotone part of  $U_Y$  is constant on  $(\tau_F(\vartheta), \lim \tau_F(\vartheta^n)]$ . By continuity it must then be constant on  $[\tau_F(\vartheta), \lim \tau_F(\vartheta^n)]$ . The monotone part of  $U_Y$  increases at  $\tau_F(\vartheta)$  by definition, however, so we must have  $\tau_F(\vartheta) = \lim \tau_F(\vartheta^n)$  a.s.  $\square$

*Remark A.2.* Since the proof of Lemma A.1 relies on the aggregation of supermartingales of class (D), we may further assume that the processes  $L$  and  $F$  have left (and right) limits at any time  $t$  a.s.; see El Karoui (1981), Proposition 2.27.

*Remark A.3.* The solutions – and in particular the stopping regions – for the monopoly problem (3.5) and the problem (3.9) of when to become optimally the leader typically differ. Consider a model in which the profit streams are driven by a diffusion  $Y$  such that each firm  $i$  has a follower threshold, say  $y_F^i$  solving (2.1) with  $\tau_F^i(\vartheta) = \inf\{t \geq \vartheta \mid Y_t \geq y_F^i\}$ , and firm 1 also has a monopoly threshold, say  $y_L^1 \leq y_F^1$  solving (3.5), and where  $L^1$  can be represented as a continuous function of the state process  $Y$ . Now one can apply arguments of Jacka (1993) relying on the semi-martingale property of  $L^1$ , which the proof of Lemma A.1 actually establishes. Denote the finite-variation part of  $L^1$  by  $A$ . The Snell envelope  $S$  of  $L^1$ , i.e., the value process of optimally stopping  $L^1$ , now is continuous (as a function of the state) as well and its monotone decreasing part  $B$  is given by  $dB_t = \mathbf{1}_{S=L^1} dA_t + \frac{1}{2} dL_t^0(S - L^1)$ . The last term is the local time of  $S - L^1$  spent at 0 (i.e., in the stopping region), which is absolutely continuous w.r.t.  $\mathbf{1}_{S=L^1} dA_t \leq 0$ .

Now suppose the stopping region  $\{S = L^1\}$  is that of the monopoly problem,  $\{Y \geq y_L^1\}$ , whence  $dL_t^0(S - L^1)$  lives on the boundary  $\{Y = y_L^1\}$ . For  $Y_t \in [y_L^1, y_F^2)$ ,  $L^1$  has a drift given by the foregone monopoly profit stream,  $dA_t = -\pi_t^{L1} dt$ , whence  $dL_t^0(S - L^1) \equiv 0$  if  $Y$  has a transition density, cf. Theorem 6 of Jacka (1993).

As  $L^1$  is of class (D), so is  $S$ , which thus converges to  $S_\infty = L_\infty^1 = 0$  in  $L^1(P)$  as  $t \rightarrow \infty$ . Therefore the martingale part of  $S$  is simply  $E[-B_\infty \mid \mathcal{F}_t]$  and  $S_t = E[-\int_t^\infty \mathbf{1}_{S=L^1} dA_s \mid \mathcal{F}_t]$ . Noting further that for  $Y_t > y_F^2$ ,  $L^1$  has a drift given by the foregone duopoly stream,  $dA_t = -\pi_t^{B1} dt$ , we then obtain

$$S_t = E \left[ \int_t^\infty \left( \mathbf{1}_{Y_s \in [y_L^1, y_F^2)} \pi_s^{L1} + \mathbf{1}_{Y_s > y_F^2} \pi_s^{B1} \right) ds - \int_t^\infty \mathbf{1}_{Y_s = y_F^2} dA_s \mid \mathcal{F}_t \right]. \quad (\text{A.2})$$

By applying similar reasoning to firm 1's monopoly problem (3.5), which is solved by  $\tau_L^1(t) = \inf\{s \geq t \mid Y_s \geq y_L^1\}$ , its value is  $E[\int_{\tau_L^1(t)}^\infty \pi_s^{L1} ds \mid \mathcal{F}_t] = E[\int_t^\infty \mathbf{1}_{Y_s \geq y_L^1} \pi_s^{L1} ds \mid \mathcal{F}_t]$ , i.e.,  $E[\int_{\tau_L^1(t)}^\infty \mathbf{1}_{Y_s < y_L^1} \pi_s^{L1} ds \mid \mathcal{F}_t] = 0$ . Thus, if  $Y_t \geq y_L^1$ , (A.2) can be rewritten as

$$S_t = E \left[ \int_t^\infty \left( \mathbf{1}_{Y_s < y_F^2} \pi_s^{L1} + \mathbf{1}_{Y_s > y_F^2} \pi_s^{B1} \right) ds - \int_t^\infty \mathbf{1}_{Y_s = y_F^2} dA_s \mid \mathcal{F}_t \right].$$

In this hypothesized stopping region for  $L^1$ , also  $S_t = L_t^1$ , in particular for  $Y_t \geq y_F^2 \geq y_L^1$ ,

$$S_t = E \left[ \int_t^\infty \pi_s^{B1} ds \mid \mathcal{F}_t \right].$$

With  $y_F^2$  in the stopping region,  $-\mathbf{1}_{Y_s = y_F^2} dA_s \geq 0$ , and by assumption  $\pi_s^{L1} \geq \pi_s^{B1}$ . Further,  $\mathbf{1}_{Y_s = y_F^2}$  is a  $P \otimes dt$  nullset if  $Y$  has a transition density, such that equating the two last

expressions for  $S_t$  implies indeed

$$E \left[ \int_t^\infty \mathbf{1}_{Y_s < y_F^2} \left( \pi_s^{L1} - \pi_s^{B1} \right) ds \middle| \mathcal{F}_t \right] = 0$$

(and  $E[-\int_t^\infty \mathbf{1}_{Y_s = y_F^2} dA_s | \mathcal{F}_t] = 0$ ). This contradicts the typical strict ordering  $\pi^{L1} > \pi^{B1}$ .

**Lemma A.4.** *Let  $(x_t)_{t \geq 0}$  be a geometric Brownian motion on  $(\Omega, \mathcal{F}, P)$ , satisfying*

$$dx_t = \mu x_t dt + \sigma x_t dB_t$$

for a Brownian motion  $(B_t)_{t \geq 0}$  adapted to  $(\mathcal{F}_t)_{t \geq 0}$ . Further let  $\tau_{\tilde{x}} := \inf\{t \geq 0 \mid x_t \geq \tilde{x}\}$  for any given constant  $\tilde{x} \in \mathbb{R}_+$ . Then the problem

$$\sup_{\tau \in \mathcal{T}, \tau \leq \tau_{\tilde{x}}} E \left[ \int_\tau^\infty e^{-rt} (Dx_t - rI) dt \right] \quad (\text{A.3})$$

with  $r > \max\{\mu, 0\}$ ,  $D \in \mathbb{R}$  and  $I > 0$  is solved by  $\tau^* := \inf\{t \geq 0 \mid x_t \geq \tilde{x} \wedge x^*\}$ , where

$$x^* = \frac{\beta_1}{\beta_1 - 1} \cdot \frac{I(r - \mu)}{D^+}$$

and  $\beta_1 > 1$  is the positive root of  $\frac{1}{2}\sigma^2\beta(\beta - 1) + \mu\beta - r = 0$ .

*Proof.* If  $D \leq 0$ , then the integrand in (A.3) is always negative and the latest feasible stopping time is optimal, which indeed satisfies  $\tau_{\tilde{x}} = \tau^*$  since now  $x^* = \infty$ . For  $D > 0$ , Lemma A.4 is a special case of Proposition 4.6 in Steg and Thijssen (2015), setting their  $Y_0 = Dx_0$ ,  $\mu_Y = \mu$ ,  $\sigma_Y = \sigma$ ,  $X_0 = c_0 = c_B = 0$  and  $y_P = (r - \mu_Y)(I - c_A/r) = \tilde{x}$ .  $\square$

## A.2 Proofs

**Proof of Lemma 3.3.** We only use the assumptions  $\pi^{Li} \geq \pi^{Bi}$  and  $\pi^{0i} \geq \pi^{Fi}$  (except for the representation with  $\tau_P(\vartheta)$ ). Let  $\tau_{1st}^i(\vartheta) = \inf\{t \geq \vartheta \mid L_t^i > F_t^i\}$  ( $= \tau_P(\vartheta)$  for  $i = 2$ ), such that by right-continuity  $L^i \geq F^i$  at  $\tau_{1st}^i(\vartheta)$ . Also  $L^i \geq F^i$  at  $\tau_F^i(\vartheta)$  by  $\pi^{Li} \geq \pi^{Bi}$ . Next, by the optimality of  $\tau_F^i(\vartheta)$  in  $F_\vartheta^i$  and  $\pi^{0i} \geq \pi^{Fi}$ ,  $(F_t^i)$  is a *submartingale* on  $[\vartheta, \tau_F^i(\vartheta)]$ . Hence  $i$  prefers to become follower as late as possible on that interval. On  $[\vartheta, \tau_{1st}^i(\vartheta))$ ,  $M_t^i \leq L_t^i \leq F_t^i$ , so stopping is nowhere better, but inferior if the last inequality is strict. All claims now follow from the follower value at  $\min\{\tau_{1st}^i(\vartheta), \tau_F^i(\vartheta)\}$  being indeed attainable (in expectation) for any strategy of the opponent  $j$ : At  $\min\{\tau_{1st}^i(\vartheta), \tau_F^i(\vartheta)\}$ , the limit payoff to  $i$  of stopping ever quicker after is at least  $F^i$  since  $L^i \geq F^i$  and  $(L_t^i)$  is right-continuous (in the limit,  $i$  obtains  $F^i$  with the probability that  $j$  stops immediately and  $L^i$  with the probability that  $j$  does not stop immediately).  $\square$

**Proof of Proposition 3.6.** Firm  $i$ 's payoff from  $\tau^i \equiv \infty$  is  $E[F_{\tau_*^i}^i | \mathcal{F}_{\vartheta'}] \geq E[M_{\tau_*^i}^i \mathbf{1}_{\tau_*^i = \tau_*^j} + F_{\tau_*^i}^i \mathbf{1}_{\tau_*^i > \tau_*^j} | \mathcal{F}_{\vartheta'}]$  for any  $\tau_*^i \geq \tau_*^j$ , with equality iff  $F_{\tau_*^i}^i = M_{\tau_*^i}^i$  on  $\{\tau_*^i = \tau_*^j\}$ . Thus  $\tau_*^i$  is a best

reply to  $\tau_*^j$  iff the latter condition holds and  $\tau = \tau_*^j$  attains

$$\operatorname{ess\,sup}_{\vartheta \leq \tau \leq \tau_*^j} E \left[ L_\tau^i \mathbf{1}_{\tau < \tau_*^j} + F_{\tau_*^j}^i \mathbf{1}_{\tau \geq \tau_*^j} \mid \mathcal{F}_\vartheta \right].$$

By iterated expectations, this is equivalent to  $L_{\vartheta'}^i - E[F_{\tau_*^j}^i \mid \mathcal{F}_{\vartheta'}] \leq 0$  on  $\{\vartheta' < \tau_*^j\}$  for all stopping times  $\vartheta' \geq \vartheta$ . To establish the latter under conditions (i) and (ii), fix arbitrary  $\vartheta' \geq \vartheta$  and let  $\tau_D^i(\vartheta') \in \mathcal{T}$  attain (3.6) (such  $\tau_D^i(\vartheta')$  exists by continuity and integrability of the process to be stopped), whence  $E[\int_{\vartheta'}^{\tau_D^i(\vartheta')} (\pi_s^{Li} - \pi_s^{0i}) ds \mid \mathcal{F}_{\vartheta'}] \leq 0$ . Then, on  $\{\vartheta' < \tau_*^j\}$  we have

$$\begin{aligned} L_{\vartheta'}^i - E \left[ M_{\tau_*^j}^i \mid \mathcal{F}_{\vartheta'} \right] &= E \left[ \int_{\vartheta'}^{\tau_F^j(\vartheta')} (\pi_s^{Li} - \pi_s^{0i}) ds + \int_{\tau_F^j(\vartheta')}^{\tau_*^j} (\pi_s^{Bi} - \pi_s^{0i}) ds \mid \mathcal{F}_{\vartheta'} \right] \quad (\text{A.4}) \\ &\leq E \left[ \int_{\vartheta'}^{\tau_F^j(\vartheta') \vee \tau_D^i(\vartheta')} (\pi_s^{Li} - \pi_s^{0i}) ds + \int_{\tau_F^j(\vartheta') \vee \tau_D^i(\vartheta')}^{\tau_*^j} (\pi_s^{Bi} - \pi_s^{0i}) ds \mid \mathcal{F}_{\vartheta'} \right] \\ &\leq E \left[ \int_{\tau_D^i(\vartheta')}^{\tau_F^j(\vartheta') \vee \tau_D^i(\vartheta')} (\pi_s^{Li} - \pi_s^{0i}) ds + \int_{\tau_F^j(\vartheta') \vee \tau_D^i(\vartheta')}^{\tau_*^j} (\pi_s^{Bi} - \pi_s^{0i}) ds \mid \mathcal{F}_{\vartheta'} \right] \\ &= E \left[ \mathbf{1}_{\tau_D^i(\vartheta') < \tau_F^j(\vartheta')} \left( L_{\tau_D^i(\vartheta')}^i - M_{\tau_*^j}^i \right) \right. \\ &\quad \left. + \mathbf{1}_{\tau_D^i(\vartheta') \geq \tau_F^j(\vartheta')} \left( M_{\tau_D^i(\vartheta')}^i - M_{\tau_*^j}^i \right) \mid \mathcal{F}_{\vartheta'} \right]. \end{aligned}$$

The first equality uses the convention  $\int_a^b \cdot ds = -\int_b^a \cdot ds$  for  $a < b$ . The first inequality is due to  $\pi_s^{Li} \geq \pi_s^{Bi}$  and the second due to the optimality of  $\tau_D^i(\vartheta')$ . The last equality is analogous to the first, using iterated expectations and  $\tau_D^i(\vartheta') < \tau_F^j(\vartheta') \Rightarrow \tau_F^j(\tau_D^i(\vartheta')) = \tau_F^j(\vartheta')$ . After replacing  $M_{\tau_*^j}^i$  by  $F_{\tau_*^j}^i$  in the first and last terms of (A.4), conditions (i) and (ii) make the last non-positive (taking iterated expectations at  $\tau_D^i(\vartheta')$ ) and thus also  $L_{\vartheta'}^i - E[F_{\tau_*^j}^i \mid \mathcal{F}_{\vartheta'}] \leq 0$ .

To prove the next claim, note that for any stopping time  $\tau \in [\vartheta', \tau_F^j(\vartheta')]$  we have  $\tau_F^j(\tau) = \tau_F^j(\vartheta')$  and thus  $L_{\vartheta'}^i - E[L_\tau^i \mid \mathcal{F}_{\vartheta'}] = E[\int_{\vartheta'}^\tau (\pi_s^{Li} - \pi_s^{0i}) ds \mid \mathcal{F}_{\vartheta'}] \geq 0$  where  $\vartheta'$  attains (3.6).

For the final claim consider any stopping time  $\tau_*^2 \geq \vartheta$  such that  $F_{\tau_*^2}^2 = M_{\tau_*^2}^2$ ; then also  $F_{\tau_*^2}^1 = M_{\tau_*^2}^1$  by Lemma 3.1. Suppose further that (i), (ii) hold for  $i = 1$ , so  $\tau_*^1 = \tau_*^2$  is a best reply for firm 1. To prove that  $\tau_*^2$  is a best reply for firm 2 to  $\tau_*^1 = \tau_*^2$  if  $\pi_s^{L1} - \pi_s^{01} \geq \pi_s^{L2} - \pi_s^{02}$  and  $\pi_s^{B1} - \pi_s^{01} \geq \pi_s^{B2} - \pi_s^{02}$ , we show that now (A.4) for firm 2 is not greater than for firm 1. Therefore first note that for each  $i = 1, 2$ ,  $F_{\tau_*^i}^i = M_{\tau_*^i}^i$  implies  $E[\mathbf{1}_A \int_{\tau_*^i}^{\tau_F^i(\vartheta')} (\pi_s^{Bi} - \pi_s^{Fi}) ds \mid \mathcal{F}_{\vartheta'}] = 0$  for any set  $A \subset \{\tau_F^i(\vartheta') \geq \tau_*^i\}$  (taking iterated expectations at  $\tau_*^i$ ), so in particular for  $A = \{\tau_F^1(\vartheta') > \tau_*^2\}$  as  $\tau_F^2(\vartheta') \geq \tau_F^1(\vartheta')$ . Since further  $E[\mathbf{1}_{\tau_F^1(\vartheta') > \tau_*^2} \int_{\tau_F^1(\vartheta')}^{\tau_F^2(\vartheta')} (\pi_s^{B2} - \pi_s^{F2}) ds \mid \mathcal{F}_{\vartheta'}] \leq 0$  by optimality of  $\tau_F^2(\vartheta')$  (and taking iterated expectations at  $\tau_F^1(\vartheta')$ ), we also have  $E[\mathbf{1}_{\tau_F^1(\vartheta') > \tau_*^2} \int_{\tau_*^2}^{\tau_F^1(\vartheta')} (\pi_s^{B2} - \pi_s^{F2}) ds \mid \mathcal{F}_{\vartheta'}] \geq 0$ .

Now, rewriting (A.4) for  $i = 2$ , we obtain

$$\begin{aligned}
& E \left[ \int_{\vartheta'}^{\tau_F^1(\vartheta') \wedge \tau_*^2} (\pi_s^{L2} - \pi_s^{02}) ds + \mathbf{1}_{\tau_F^1(\vartheta') \leq \tau_*^2} \int_{\tau_F^1(\vartheta')}^{\tau_*^2} (\pi_s^{B2} - \pi_s^{02}) ds \right. \\
& \quad \left. + \mathbf{1}_{\tau_F^1(\vartheta') > \tau_*^2} \int_{\tau_*^2}^{\tau_F^1(\vartheta')} (\pi_s^{L2} - \pi_s^{B2}) ds \middle| \mathcal{F}_{\vartheta'} \right] \\
& \leq E \left[ \int_{\vartheta'}^{\tau_F^1(\vartheta') \wedge \tau_*^2} (\pi_s^{L1} - \pi_s^{01}) ds + \mathbf{1}_{\tau_F^1(\vartheta') \leq \tau_*^2} \int_{\tau_F^1(\vartheta')}^{\tau_*^2} (\pi_s^{B1} - \pi_s^{01}) ds \right. \\
& \quad \left. + \mathbf{1}_{\tau_F^1(\vartheta') > \tau_*^2} \int_{\tau_*^2}^{\tau_F^1(\vartheta')} (\pi_s^{L2} - \pi_s^{F2}) ds \middle| \mathcal{F}_{\vartheta'} \right] \\
& \leq E \left[ \int_{\vartheta'}^{\tau_F^1(\vartheta') \wedge \tau_*^2} (\pi_s^{L1} - \pi_s^{01}) ds + \mathbf{1}_{\tau_F^1(\vartheta') \leq \tau_*^2} \int_{\tau_F^1(\vartheta')}^{\tau_*^2} (\pi_s^{B1} - \pi_s^{01}) ds \right. \\
& \quad \left. + \mathbf{1}_{\tau_F^1(\vartheta') > \tau_*^2} \int_{\tau_*^2}^{\tau_F^1(\vartheta')} (\pi_s^{L1} - \pi_s^{F1}) ds \right. \\
& \quad \left. + \int_{\tau_F^1(\vartheta')}^{\tau_F^2(\vartheta')} (\pi_s^{L1} - \pi_s^{B1}) ds \middle| \mathcal{F}_{\vartheta'} \right] \quad (\text{A.5})
\end{aligned}$$

The last inequality uses the standing assumption  $\pi_s^{L1} - \pi_s^{F1} \geq \pi_s^{L2} - \pi_s^{F2}$ , as well as  $\tau_F^1(\vartheta') \leq \tau_F^2(\vartheta')$  and  $\pi_s^{L1} \geq \pi_s^{B1}$ . Rearranging (A.5) using  $E[\mathbf{1}_{\tau_F^1(\vartheta') > \tau_*^2} \int_{\tau_*^2}^{\tau_F^1(\vartheta')} (\pi_s^{Bi} - \pi_s^{Fi}) ds \middle| \mathcal{F}_{\vartheta'}] = 0$  yields (A.4) for  $i = 1$ .  $\square$

**Proof of Corollary 3.7.** We only need to verify optimality for firm  $i = 2$  by applying Proposition 3.6 with  $\tau_*^1 = \tau_S \leq \tau_F^2(\vartheta) = \tau_*^2$ . Then indeed  $F_{\tau_*^2}^2 = M_{\tau_*^2}^2$ . Further, condition (i) is satisfied since  $M^2 \leq F^2$  and  $(F_t^2)$  is a submartingale on  $[\vartheta, \tau_F^2(\vartheta)]$  by  $\pi_s^{F2} \leq \pi_s^{02}$ . Hence  $\tau_*^2$  is optimal if the remaining condition (ii) is satisfied.

For the second claim note that if  $\pi_s^{L1} - \pi_s^{01} \geq \pi_s^{L2} - \pi_s^{02}$ , then  $E[\int_{\tau}^{\tau_S} (\pi_s^{L2} - \pi_s^{02}) ds \middle| \mathcal{F}_{\tau}] \leq E[\int_{\tau}^{\tau_S} (\pi_s^{L1} - \pi_s^{01}) ds \middle| \mathcal{F}_{\tau}] \leq 0$  for any stopping time  $\tau \in [\vartheta, \tau_S]$  by the optimality of  $\tau_S$ , cf. Lemma 3.10, and thus  $\tau_D^2(\vartheta') = \tau_S \vee \vartheta'$  attains the current instance of (3.6).  $\square$

**Proof of Lemma 3.8.** First note that there are solutions  $\tau_{\Delta}^i \leq \tau_F^i(\vartheta) \leq \tau_F^2(\vartheta)$  to (3.8) for  $i = 1, 2$ , since the respective process to be stopped is continuous and integrable. The estimate follows from the assumption  $\pi_s^{Li} - \pi_s^{Fi} \geq \pi_s^{Bi} - \pi_s^{Fi}$ , cf. the proof of Lemma 3.1.

By the optimality of  $\tau_{\Delta}^i$  in (3.8),  $E[\int_{\vartheta}^{\tau_{\Delta}^i} (\pi_s^{Li} - \pi_s^{Fi}) ds \middle| \mathcal{F}_{\vartheta}] \leq 0$ . Therefore, as  $\pi_s^{L2} - \pi_s^{F2} \leq \pi_s^{Li} - \pi_s^{Fi}$ , (3.1) can only be strictly positive if

$$E \left[ \int_{\tau_{\Delta}^i}^{\tau_F^1(\vartheta)} (\pi_s^{L2} - \pi_s^{F2}) ds + \int_{\tau_F^1(\vartheta)}^{\tau_F^2(\vartheta)} (\pi_s^{B2} - \pi_s^{F2}) ds \middle| \mathcal{F}_{\vartheta} \right] > 0$$

(which can in fact only be the case if  $P[\tau_\Delta^i < \tau_F^1(\vartheta)] > 0$ ), and which implies

$$E\left[L_{\tau_\Delta^i}^2 - F_{\tau_\Delta^i}^2 \mid \mathcal{F}_\vartheta\right] = E\left[\int_{\tau_\Delta^i}^{\tau_F^1(\tau_\Delta^i)} (\pi_s^{L2} - \pi_s^{F2}) ds + \int_{\tau_F^1(\tau_\Delta^i)}^{\tau_F^2(\vartheta)} (\pi_s^{B2} - \pi_s^{F2}) ds \mid \mathcal{F}_\vartheta\right] > 0$$

as  $\tau_F^1(\tau_\Delta^i) \geq \tau_F^1(\vartheta)$ ,  $\tau_F^2(\tau_\Delta^i) = \tau_F^2(\vartheta)$  and  $\pi_s^{L2} \geq \pi_s^{B2}$ .

For all stopping times  $\tau \in [\vartheta, \tau_F^1(\vartheta)]$ , indeed  $\tau_F^i(\tau) = \tau_F^i(\vartheta)$ ,  $i = 1, 2$ , and thus  $L_\vartheta^2 - F_\vartheta^2 - E[L_\tau^2 - F_\tau^2 \mid \mathcal{F}_\vartheta] = E[\int_\vartheta^\tau (\pi_s^{L2} - \pi_s^{F2}) ds \mid \mathcal{F}_\vartheta] \geq 0$  if  $\tau_\Delta^2 = \vartheta$  attains (3.8).  $\square$

**Proof of Lemma 3.9.** For the first claim suppose wlog. that  $\tau_l^i = \vartheta$  is attaining (3.9) for some given  $\vartheta \in \mathcal{T}$  and maximal, i.e.,  $L_\vartheta^i > E[L_\tau^i \mid \mathcal{F}_\vartheta]$  (up to a  $P$ -nullset) on  $\{\tau > \vartheta\}$  for any other  $\tau \in \mathcal{T}$ . Then we must also have  $L_\vartheta^i \geq E[F_\tau^i \mid \mathcal{F}_\vartheta]$  on  $\{\tau \geq \vartheta\}$ , strictly on  $\{\tau > \vartheta\}$ , as follows. First note that  $F_\tau^i - E[F_{\tau_F^i(\tau)}^i \mid \mathcal{F}_\tau] = E[\int_\tau^{\tau_F^i(\tau)} (\pi_s^{Fi} - \pi_s^{0i}) \mid \mathcal{F}_\tau] \leq 0$ , since  $\tau_F^i(\tau_F^i(\tau)) = \tau_F^i(\tau)$ . Furthermore, note that  $L_{\tau_F^i(\tau)}^i \geq F_{\tau_F^i(\tau)}^i$  by  $\pi_s^{Li} \geq \pi_s^{Bi}$ . Together with the hypothesis it must thus hold that  $L_\vartheta^i > E[F_\tau^i \mid \mathcal{F}_\vartheta] \geq E[M_\tau^i \mid \mathcal{F}_\vartheta]$  on  $\{\tau > \vartheta\}$  for any  $\tau \in \mathcal{T}$ , and  $L_\vartheta^i \geq F_\vartheta^i \geq M_\vartheta^i$  using  $\tau = \vartheta$ .

Then, in case that the opponent  $j$ 's strategy does not imply immediate stopping with probability 1 (else there is nothing to prove),  $i$  cannot achieve a higher payoff than  $L_\vartheta^i$  with the probability that  $j$  does not stop immediately and  $F_\vartheta^i$  with the probability that  $j$  stops immediately.<sup>19</sup> This upper bound is the limit of  $i$  stopping ever quicker after  $\vartheta$  (say, at  $\vartheta + 1/n$ , with  $n \nearrow \infty$ ) since  $(L_t^i)$  is right-continuous (cf. fn. 3), but it is not attainable by any strategy not inducing immediate stopping with probability 1.

For the second claim suppose now by way of contradiction that  $\tau_l^i = \vartheta$  attains (3.9), but that there exists a stopping time  $\tau \geq \vartheta$  with

$$E\left[\int_\vartheta^\infty \pi_s^{Li} ds \mid \mathcal{F}_\vartheta\right] < E\left[\int_\vartheta^\tau \pi_s^{0i} ds + \int_\tau^\infty \pi_s^{Li} ds \mid \mathcal{F}_\vartheta\right] \Leftrightarrow E\left[\int_\vartheta^\tau (\pi_s^{Li} - \pi_s^{0i}) ds \mid \mathcal{F}_\vartheta\right] < 0$$

with positive probability. Then

$$\begin{aligned} L_\vartheta^i &= \int_0^\vartheta \pi_s^{0i} ds + E\left[\int_\vartheta^{\tau_F^j(\vartheta)} \pi_s^{Li} ds + \int_{\tau_F^j(\vartheta)}^\infty \pi_s^{Bi} ds \mid \mathcal{F}_\vartheta\right] \\ &< \int_0^\vartheta \pi_s^{0i} ds + E\left[\int_\vartheta^\tau \pi_s^{0i} ds + \int_\tau^{\tau_F^j(\vartheta)} \pi_s^{Li} ds + \int_{\tau_F^j(\vartheta)}^\infty \pi_s^{Bi} ds \mid \mathcal{F}_\vartheta\right] \\ &\leq E\left[L_\tau^i \mid \mathcal{F}_\vartheta\right], \end{aligned}$$

since  $\tau_F^j(\tau) \geq \tau_F^j(\vartheta)$  and  $\pi_s^{Li} \geq \pi_s^{Bi}$ , contradicting the hypothesized optimality of  $\vartheta$  for (3.9). Hence  $\vartheta$  must also be optimal for the permanent monopoly problem.  $\square$

<sup>19</sup>This phrasing accounts for a possible coordination device for immediate stopping. With an extended mixed strategy that is used to model preemption in Section 3.1.2,  $j$  can induce the game to end immediately with probability 1 for all strategies of  $i$ , but  $i$  may obtain a different payoff than  $F_\vartheta^i$ .

**Proof of Lemma 3.10.** First note that there exists an optimal stopping time for (3.10) (and also a latest one), because the process to be stopped is continuous and integrable. For any stopping time  $\tau \in [\vartheta, \tau_F^2(\vartheta)]$ ,  $\tau_F^2(\tau) = \tau_F^2(\vartheta)$  and thus  $L_\vartheta^1 - E[L_\tau^1 | \mathcal{F}_\vartheta] = E[\int_\vartheta^\tau (\pi_s^{L1} - \pi_s^{01}) ds | \mathcal{F}_\vartheta]$  is the same payoff difference as that between  $\vartheta$  and  $\tau$  in (3.10). Thus, where  $\vartheta$  is uniquely optimal in (3.10), there also  $L_\vartheta^1 > E[L_\tau^1 | \mathcal{F}_\vartheta]$  on  $\{\tau > \vartheta\}$ . Regarding the other possible payoffs, as argued in the proof of Lemma 3.9,  $M_\tau^1 \leq F_\tau^1 \leq E[F_{\tau_F^1(\tau)}^1 | \mathcal{F}_\tau] \leq E[L_{\tau_F^1(\tau)}^1 | \mathcal{F}_\tau]$ , where now  $\tau_F^1(\tau) \leq \tau_F^2(\tau) = \tau_F^2(\vartheta)$  for  $\tau \in [\vartheta, \tau_F^2(\vartheta)]$ . Hence  $L_\vartheta^1$  is strictly superior to any future payoff on  $(\vartheta, \tau_F^2(\vartheta)]$  and the game has to end by the same arguments as in the proof of Lemma 3.9.  $\square$

**Proof of Lemma 3.11.** First note that there exists an optimal stopping time  $\tau_M^i \geq \vartheta$  for (3.11) and also a latest one, because the process to be stopped is continuous and integrable. An optimal  $\tau_M^i$  satisfies the necessary and sufficient conditions  $E[\int_\tau^{\tau_M^i} (\pi_s^{0i} - \pi_s^{Bi}) ds | \mathcal{F}_\tau] \geq 0$  on  $\{\tau \leq \tau_M^i\}$  and  $E[\int_{\tau_M^i}^\tau (\pi_s^{0i} - \pi_s^{Bi}) ds | \mathcal{F}_{\tau_M^i}] \leq 0$  on  $\{\tau \geq \tau_M^i\}$  for all stopping times  $\tau \geq \vartheta$ , the last inequality being strict on  $\{\tau > \tau_M^i\}$  if  $\tau_M^i$  is the latest solution. We will derive the analogous properties for the process  $(F_t^i)$ ; thus consider an arbitrary stopping time  $\tau \geq \vartheta$ .

For the first property, note that on  $\{\tau \leq \tau_M^i\}$  we have

$$E[F_{\tau_M^i \wedge \tau_F^i(\tau)}^i | \mathcal{F}_\tau] - F_\tau^i = E\left[\int_\tau^{\tau_M^i \wedge \tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Fi}) ds \middle| \mathcal{F}_\tau\right] \geq 0$$

by  $\pi_s^{0i} \geq \pi_s^{Fi}$  and  $\tau_F^i(\tau_M^i \wedge \tau_F^i(\tau)) = \tau_F^i(\tau)$ . Further, on the subset  $\{\tau_M^i > \tau_F^i(\tau)\}$  we have

$$E[F_{\tau_M^i}^i | \mathcal{F}_{\tau_F^i(\tau)}] - F_{\tau_F^i(\tau)}^i = E\left[\int_{\tau_F^i(\tau)}^{\tau_M^i} (\pi_s^{0i} - \pi_s^{Bi}) ds + \int_{\tau_M^i}^{\tau_F^i(\tau_M^i)} (\pi_s^{Fi} - \pi_s^{Bi}) ds \middle| \mathcal{F}_{\tau_F^i(\tau)}\right] \geq 0$$

by the optimality of  $\tau_M^i$  and the definition of  $\tau_F^i(\tau_M^i)$ , cf. the proof of Lemma 3.1. Together,  $E[F_{\tau_M^i}^i | \mathcal{F}_\tau] - F_\tau^i = E[F_{\tau_M^i}^i - F_{\tau_M^i \wedge \tau_F^i(\tau)}^i | \mathcal{F}_\tau] + E[F_{\tau_M^i \wedge \tau_F^i(\tau)}^i | \mathcal{F}_\tau] - F_\tau^i \geq 0$ .

For the second property, note that  $E[F_{\tau_F^i(\tau)}^i | \mathcal{F}_\tau] - F_\tau^i = E[\int_\tau^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Fi}) ds | \mathcal{F}_\tau] \geq 0$ , again by  $\pi_s^{0i} \geq \pi_s^{Fi}$  and  $\tau_F^i(\tau_F^i(\tau)) = \tau_F^i(\tau)$ , hence it is sufficient to show  $E[F_{\tau_F^i(\tau)}^i | \mathcal{F}_{\tau_M^i}] \leq F_{\tau_M^i}^i$  on  $\{\tau \geq \tau_M^i\}$ . There, where  $\tau_F^i(\tau) \geq \tau_F^i(\tau_M^i)$ , it holds that

$$\begin{aligned} E[F_{\tau_F^i(\tau)}^i | \mathcal{F}_{\tau_M^i}] - F_{\tau_M^i}^i &= E\left[\int_{\tau_M^i}^{\tau_F^i(\tau_M^i)} (\pi_s^{0i} - \pi_s^{Fi}) ds + \int_{\tau_F^i(\tau_M^i)}^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Bi}) ds \middle| \mathcal{F}_{\tau_M^i}\right] \\ &\leq E\left[\int_{\tau_M^i}^{\tau_F^i(\tau_M^i)} (\pi_s^{0i} - \pi_s^{Bi}) ds + \int_{\tau_F^i(\tau_M^i)}^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Bi}) ds \middle| \mathcal{F}_{\tau_M^i}\right] \leq 0, \end{aligned}$$

where we have used the definition of  $\tau_F^i(\tau_M^i)$  in the first estimate, and the optimality of  $\tau_M^i$  in the last. The last inequality is strict on  $\{\tau > \tau_M^i\}$  if  $\tau_M^i$  is the latest solution of (3.11).

Now suppose that the stopping time  $\tau_M^i \geq \vartheta$  optimally stops  $(F_t^i)$  from  $\vartheta \in \mathcal{T}$ , i.e., it satisfies  $E[F_{\tau_M^i}^i | \mathcal{F}_\tau] \geq F_\tau^i$  on  $\{\tau \leq \tau_M^i\}$  and  $E[F_\tau^i | \mathcal{F}_{\tau_M^i}] \leq F_{\tau_M^i}^i$  on  $\{\tau \geq \tau_M^i\}$  for all stopping



times  $\tau \geq \vartheta$ . Since  $E[F_{\tau_F^i(\tau_M)}^i | \mathcal{F}_{\tau_M}^i] \geq F_{\tau_M}^i$  as noted above, we must then have equality, i.e.,  $\tau_F^i(\tau_M^i)$  is optimal, too, and we may set  $\tau_M^i = \tau_F^i(\tau_M^i)$  for simplicity to show optimality of  $\tau_F^i(\tau_M^i)$  in (3.11). Therefore, consider again an arbitrary stopping time  $\tau \geq \vartheta$ .

On  $\{\tau \leq \tau_M^i\}$ , where  $\tau_F^i(\tau) \leq \tau_F^i(\tau_M^i) = \tau_M^i$ , it then holds that

$$\begin{aligned} 0 &\leq E[F_{\tau_M^i}^i | \mathcal{F}_{\tau}^i] - F_{\tau}^i = E\left[\int_{\tau}^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Fi}) ds + \int_{\tau_F^i(\tau)}^{\tau_M^i} (\pi_s^{0i} - \pi_s^{Bi}) ds \middle| \mathcal{F}_{\tau}^i\right] \\ &\leq E\left[\int_{\tau}^{\tau_F^i(\tau)} (\pi_s^{0i} - \pi_s^{Bi}) ds + \int_{\tau_F^i(\tau)}^{\tau_M^i} (\pi_s^{0i} - \pi_s^{Bi}) ds \middle| \mathcal{F}_{\tau}^i\right] \end{aligned}$$

by the definition of  $\tau_F^i(\tau)$ , which yields the first optimality property for  $\tau_M^i$  in (3.11).

On  $\{\tau \geq \tau_M^i\}$ , where  $\tau_F^i(\tau) \geq \tau_M^i$ , we have

$$\begin{aligned} 0 &\geq E[F_{\tau}^i | \mathcal{F}_{\tau_M^i}^i] - F_{\tau_M^i}^i = E\left[\int_{\tau_M^i}^{\tau} (\pi_s^{0i} - \pi_s^{Bi}) ds + \int_{\tau}^{\tau_F^i(\tau)} (\pi_s^{Fi} - \pi_s^{Bi}) ds \middle| \mathcal{F}_{\tau_M^i}^i\right] \\ &\geq E\left[\int_{\tau_M^i}^{\tau} (\pi_s^{0i} - \pi_s^{Bi}) ds \middle| \mathcal{F}_{\tau_M^i}^i\right] \end{aligned}$$

again by the definition of  $\tau_F^i(\tau)$ , which yields the second optimality property for  $\tau_M^i$  in (3.11).  $\square$

**Proof of Proposition 4.1.** By the strong Markov property it suffices to consider  $t = 0$ . If the preemption region is empty, one can set  $x = \bar{x}$  and pick any number in  $(0, x_F^1]$ . The upper and lower bounds for a non-empty preemption region are obtained as follows. First note that  $L_0^2 = M_0^2 \leq F_0^2$  for all  $x_0 \geq x_F^1$ . Second, for all  $x_0 > 0$ ,  $L_0^2 \leq E[\int_0^{\infty} e^{-rs} (x_s D_{10} - rI^2) ds] = x_0 D_{10}/(r - \mu) - I^2$  by  $D_{10} \geq D_{11}$  and  $F_0^2 \geq E[\int_0^{\infty} e^{-rs} x_s D_{01} ds] = x_0 D_{01}/(r - \mu)$ , the value of never investing as follower. Thus,  $L_0^2 - F_0^2 \leq x_0(D_{10} - D_{01})/(r - \mu) - I^2 \leq 0$  on the non-empty interval  $(0, (r - \mu)I^2/(D_{10} - D_{01})^+)$ .

Now suppose  $L_0^2 > F_0^2$  for some  $x_0 = \hat{x} \in (0, x_F^1)$  and also for some  $x_0 = \check{x} < \hat{x}$ , and assume by way of contradiction that  $L_0^2 \leq F_0^2$  for  $x_0 = x' \in (\check{x}, \hat{x})$ . Then we must have  $x' > rI^2/(D_{10} - D_{01})^+$ , because otherwise  $L_0^2 - F_0^2 = E[\int_0^{\tau'} e^{-rs} (x_s(D_{10} - D_{01}) - rI^2) ds] + E[L_{\tau'}^2 - F_{\tau'}^2] \leq 0$  if  $x_0 = \check{x}$  and  $x' \in (\check{x}, rI^2/(D_{10} - D_{01})^+ \wedge x_F^1]$ , where  $\tau' := \inf\{s \geq 0 | x_s \geq x'\} \leq \tau_F^1(0)$ . By the same argument, we must also have  $L_0^2 > F_0^2$  for  $x_0 = \check{x} \vee rI^2/(D_{10} - D_{01}) < x'$ . But then, if we set  $x_0 = x'$  and  $\hat{\tau} := \inf\{s \geq 0 | x_s \notin (\check{x} \vee rI^2/(D_{10} - D_{01}), \hat{x})\} \leq \tau_F^1(0)$ , we obtain  $L_0^2 - F_0^2 = E[\int_0^{\hat{\tau}} e^{-rs} (x_s(D_{10} - D_{01}) - rI^2) ds] + E[L_{\hat{\tau}}^2 - F_{\hat{\tau}}^2] > 0$ , whence the set  $\{x > 0 | L_0^2 > F_0^2 \text{ given } x_0 = x\}$  is convex. Further, that set is open since  $L_0^2 - F_0^2$  is continuous in  $x_0$ .  $\square$

**Proof of Proposition 4.2.**  $\bar{x} < x_F^2$  can be any two numbers from  $[0, \infty]$  in this proof, i.e., we only assume  $\bar{x}$  finite. For initial states  $x_0 \in (\bar{x}, x_F^2)$ , the constraint  $\tau_{\mathcal{P}}(0) \wedge \tau_F^2(0)$  in problem

(4.3) is the exit time from the given interval and (4.3) is equivalent to

$$\sup_{\tau \leq \inf\{s \geq 0 \mid x_s \notin (\bar{x}, x_F^2)\}} E \left[ \int_{\tau}^{\infty} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right]. \quad (\text{A.6})$$

If  $\bar{x}(D_{10} - D_{00}) \geq rI^1$ , the expected payoff difference between stopping at time 0 and any feasible  $\tau \geq 0$  is  $E \left[ \int_0^{\tau} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right] \geq 0$ , such that immediate stopping is optimal. If  $D_{10} - D_{00} \leq 0$ , also  $E \left[ \int_{\tau}^{\tau_{\mathcal{P}}(0) \wedge \tau_F^2(0)} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right] \leq 0$  for any  $\tau \leq \tau_{\mathcal{P}}(0) \wedge \tau_F^2(0)$ , such that waiting until the constraint is optimal.

Now suppose  $0 < \bar{x}(D_{10} - D_{00}) < rI^1$ , whence  $D_{10} > D_{00}$  and  $x_L^1 < \infty$ . Note that

$$E \left[ \int_0^{\infty} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right] = x_0 \frac{D_{10} - D_{00}}{r - \mu} - I^1$$

is the value of stopping immediately in (A.6). Letting  $x_0 = x$ , we will first verify that the value function of problem (A.6) is

$$V(x) := \begin{cases} A(\hat{x})x^{\beta_1} + B(\hat{x})x^{\beta_2} & \text{if } x \in (\bar{x}, \hat{x}), \\ x \frac{D_{10} - D_{00}}{r - \mu} - I^1 & \text{else,} \end{cases} \quad (\text{A.7})$$

and thus  $(\bar{x}, \hat{x})^c$  the sought stopping region, under the hypothesis that either  $\hat{x} \in [rI^1/(D_{10} - D_{00}), x_F^2]$  solves (4.4) or “ $\leq$ ” holds for  $\hat{x} = x_F^2$ . Afterwards we will establish existence of a unique such  $\hat{x}$ .

$V(x)$  as defined in (A.7) is continuous because  $A(\hat{x})$  and  $B(\hat{x})$  given by (4.5) are the solution to the continuity conditions

$$\begin{aligned} A\bar{x}^{\beta_1} + B\bar{x}^{\beta_2} &= \bar{x} \frac{D_{10} - D_{00}}{r - \mu} - I^1, \\ A\hat{x}^{\beta_1} + B\hat{x}^{\beta_2} &= \hat{x} \frac{D_{10} - D_{00}}{r - \mu} - I^1. \end{aligned} \quad (\text{A.8})$$

$V(x)$  is also twice continuously differentiable on  $(\bar{x}, x_F^2)$ , except possibly at  $\hat{x}$ . At  $\hat{x} < x_F^2$ , the first derivative of  $V$  is continuous, however, because (4.4) is the differentiability condition  $\beta_1 A \hat{x}^{\beta_1 - 1} + \beta_2 B \hat{x}^{\beta_2 - 1} = (D_{10} - D_{00})/(r - \mu)$  multiplied by  $\hat{x}$ , minus the second continuity condition in (A.8). Therefore one can apply Itô's lemma to see that  $(e^{-rt}V(x_t))$  is a continuous, bounded supermartingale until  $\tau = \inf\{t \geq 0 \mid x_t \notin (\bar{x}, x_F^2)\}$ , with zero drift for  $x_t \in (\bar{x}, \hat{x})$  and drift  $e^{-rt}(rI^1 - x_t(D_{10} - D_{00})) dt < 0$  for  $x_t \in (\hat{x}, x_F^2)$ . As that supermartingale coincides with the payoff process at  $\tau = \inf\{t \geq 0 \mid x_t \notin (\bar{x}, x_F^2)\}$ , it remains to show that  $V(x)$  dominates the payoff process for  $x \in (\bar{x}, x_F^2)$ , which it does by construction for  $x \in [\hat{x}, x_F^2]$ .

For  $x \in (\bar{x}, \hat{x})$ ,  $V''(x) = x^{\beta_2 - 2} [\beta_1(\beta_1 - 1)A(\hat{x})x^{\beta_1 - \beta_2} + \beta_2(\beta_2 - 1)B(\hat{x})]$ . As  $\beta_k(\beta_k - 1) > 0$ ,  $k = 1, 2$ , the difference  $V(x) - x(D_{10} - D_{00})/(r - \mu) + I^1$  would be convex if  $A(\hat{x}), B(\hat{x}) \geq 0$ , and it vanishes at both ends  $\bar{x}, \hat{x}$ . By (4.4), that difference's derivative is non-positive at  $\hat{x}$ , where the difference would thus take its minimum. Hence it would vanish on all of  $[\bar{x}, \hat{x}]$ , but  $V(x)$

cannot be affine on non-empty  $(\bar{x}, \hat{x})$ . So we must have  $A(\hat{x}) \wedge B(\hat{x}) < 0$ . If we had  $B(\hat{x}) \geq 0$ , then  $A(\hat{x}) < 0$  and  $V(x)$  would be strictly decreasing on  $(\bar{x}, \hat{x})$ , contradicting  $V(\hat{x}) \geq V(\bar{x})$ ; thus  $B(\hat{x}) < 0$ . Going back to  $V''(x)$ , which can switch sign at most once, it must start strictly negative at  $\bar{x}$ . If it stays non-positive, the difference  $V(x) - x(D_{10} - D_{00})/(r - \mu) + I^1$  is concave and thus non-negative on  $(\bar{x}, \hat{x})$ . If  $V''(x)$  eventually becomes positive, then the convex part of  $V(x) - x(D_{10} - D_{00})/(r - \mu) + I^1$  takes its minimum 0 at  $\hat{x}$  as argued before, such that the difference is non-negative at the transition, and thus non-negative for the first, concave part. In summary,  $(e^{-rt}V(x_t))$  is a supermartingale until  $x_t$  leaves  $(\bar{x}, x_F^2)$ , dominating the payoff  $e^{-rt}(x_t(D_{10} - D_{00})/(r - \mu) - I^1)$ , which it coincides with for  $x_t \in \{\bar{x}\} \cup [\hat{x}, x_F^2]$ , so the latter is the stopping set in  $[\bar{x}, x_F^2]$ .

Next, we show that there is a unique threshold  $\hat{x} \in [rI^1/(D_{10} - D_{00}), x_L^1]$  solving (4.4), and then finally consider the constraint  $x_F^2$ .

As the first step, note that  $B(x) < 0$  in (4.5) for all  $x \in (\bar{x}, x_L^1]$ . Indeed, as the first term  $[\bar{x}^{\beta_1} x^{\beta_2} - x^{\beta_1} \bar{x}^{\beta_2}]^{-1} < 0$  for  $x > \bar{x}$  by  $\beta_1 > 1$  and  $\beta_2 < 0$ , we have  $B(x) < 0 \Leftrightarrow x^{-\beta_1} [x(D_{10} - D_{00})/(r - \mu) - I^1] > \bar{x}^{-\beta_1} [\bar{x}(D_{10} - D_{00})/(r - \mu) - I^1]$ . The derivative of the latter function of  $x$  can be written as  $x^{-\beta_1 - 1} [\beta_1 I^1 - (\beta_1 - 1)x(D_{10} - D_{00})/(r - \mu)] > 0$  for all  $x < x_L^1 = \beta_1(r - \mu)I^1/((\beta_1 - 1)(D_{10} - D_{00}))$ .

As the second step, note that with  $A = A(x_L^1)$  and  $B = B(x_L^1)$ , we have  $A \cdot (x_L^1)^{\beta_1} + B \cdot (x_L^1)^{\beta_2} = I^1/(\beta_1 - 1)$  by using the definition of  $x_L^1$  in (A.8), and thus  $(\beta_1 - 1)A \cdot (x_L^1)^{\beta_1} + (\beta_2 - 1)B \cdot (x_L^1)^{\beta_2} = I^1 + (\beta_2 - \beta_1)B \cdot (x_L^1)^{\beta_2} > I^1$  compared to “=” in (4.4).

The third step is to show that “ $\leq$ ” holds in (4.4) for the candidate  $\hat{x} = rI^1/(D_{10} - D_{00}) \in (\bar{x}, x_F^2)$ , where the inclusion is exactly the current considered case. By similar arguments as above, using the continuity condition (A.8),  $V(x)$  then satisfies

$$V(x) = E \left[ \int_{\hat{\tau}}^{\infty} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right], \quad x_0 = x \in [\bar{x}, \hat{x}],$$

where we let  $\hat{\tau} := \inf\{s \geq 0 \mid x_s \notin (\bar{x}, \hat{x})\}$ . For  $\hat{x} = rI^1/(D_{10} - D_{00})$ , the integrand would be strictly negative until  $\hat{\tau}$ , so  $V(x) > x(D_{10} - D_{00})/(r - \mu) - I^1$  for all  $x \in (\bar{x}, \hat{x})$ . At  $x = \hat{x}$ , however, equality holds by (A.8) and thus  $V'(\hat{x}-) = \beta_1 A(\hat{x})\hat{x}^{\beta_1 - 1} + \beta_2 B(\hat{x})\hat{x}^{\beta_2 - 1} \leq (D_{10} - D_{00})/(r - \mu)$ . Together with (A.8), the latter inequality implies also “ $\leq$ ” in (4.4).

As the last step, since the function  $(\beta_1 - 1)A(x)x^{\beta_1} + (\beta_2 - 1)B(x)x^{\beta_2}$  is continuous, it must attain  $I^1$  at some  $\hat{x} \in [rI^1/(D_{10} - D_{00}), x_L^1]$  by the second and third steps. The latter interval is non-empty by the estimate for  $x_L^1$  at the beginning of the proof.

Concerning uniqueness, suppose  $\hat{x}_1, \hat{x}_2 \in [rI^1/(D_{10} - D_{00}), x_L^1]$  solve (4.4). With either solution, as we have proved above,  $V(x)$  is the value function of problem (A.6) for any  $x_F^2 \geq x_L^1$ , and (A.6) is solved by both  $\hat{\tau}_k := \inf\{s \geq 0 \mid x_s \notin (\bar{x}, \hat{x}_k)\}$ ,  $k = 1, 2$ . In particular, for any  $x_0 \in [x_1, x_2]$ ,

$$V(x_0) = x_0 \frac{D_{10} - D_{00}}{r - \mu} - I^1 = E \left[ \int_{\hat{\tau}_2}^{\infty} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right]$$

$$\Rightarrow \quad 0 = E \left[ \int_0^{\hat{\tau}_2} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right].$$

Thus, letting  $\check{\tau}_1 := \inf\{s \geq 0 \mid x_s \leq \hat{x}_1\} \leq \hat{\tau}_2$  and still  $x_0 \in [x_1, x_2]$ ,

$$\begin{aligned} 0 &= E \left[ \int_0^{\hat{\tau}_2} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right] \\ &= E \left[ \int_0^{\check{\tau}_1 \wedge \hat{\tau}_2} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds + \int_{\check{\tau}_1 \wedge \hat{\tau}_2}^{\hat{\tau}_2} e^{-rs} (x_s(D_{10} - D_{00}) - rI^1) ds \right]. \end{aligned}$$

The second integral vanishes itself in expectation, while the first integrand is strictly positive for  $x_s \in (\hat{x}_1, \hat{x}_2)$ . Therefore the latter interval must be empty.

The proof is complete for  $\hat{x} \leq x_F^2$ . Finally, if  $rI^1/(D_{10} - D_{00}) < x_F^2 < \hat{x}$ , then the “ $\leq$ ” in (4.4) that we derived above for the candidate  $x = rI^1/(D_{10} - D_{00})$  must be strict, and thus also “ $<$ ” must hold in (4.4) for  $x_F^2$ , because otherwise  $\hat{x} \leq x_F^2$  by continuity of  $(\beta_1 - 1)A(x)x^{\beta_1} + (\beta_2 - 1)B(x)x^{\beta_2}$ . Now the verification argument above applies if we consider instead  $\hat{x} := x_F^2$  with “ $\leq$ ” in (4.4).  $\square$

**Proof of Proposition 4.3.** The stopping times  $\tau_J(\vartheta) := \inf\{t \geq \vartheta \mid x_t \geq x_J\}$ ,  $\vartheta \in \mathcal{T}$ , satisfy time consistency  $\vartheta' \leq \tau_J(\vartheta) \Rightarrow \tau_J(\vartheta') = \tau_J(\vartheta)$  for any two  $\vartheta \leq \vartheta' \in \mathcal{T}$  by construction.  $\tau_J(\vartheta)$  is a mutual best reply at  $\vartheta$  if the conditions from Proposition 3.6 hold. By  $x_J \geq x_F^2$ ,  $F_{\tau_J(\vartheta)}^2 = M_{\tau_J(\vartheta)}^2$ . Under the current specification it suffices to verify conditions (i) and (ii) for firm 1.

Condition (i) holds since waiting until the threshold  $x_J \leq x_M^1$  is optimal for the constrained problem of stopping  $M_t^1$  up to it by Lemma A.4; cf. the unconstrained problem (3.11). Analogously, the threshold  $\min\{x_J, x_L^1\}$  solves problem (3.6). Thus condition (ii) holds if  $x_L^1 \geq x_F^2$  or, using the strong Markov property, if  $0 \geq D_J(x) := L_0^1 - E[M_{\tau_J(0)}^1]$  given  $x_0 = x \in [x_L^1, x_F^2]$ .

By Proposition 3.6, if  $x_L^1 < x_F^2$  solves (3.6) and we let  $\tau(x) = \inf\{t \geq 0 \mid x_t \geq x\} \leq \tau_F^2(0)$  for any  $x \in [x_L^1, x_F^2]$ , then  $D_J(x_L^1) \geq E[L_{\tau(x)}^1 - M_{\tau_J(0)}^1] = E[D_J(x)]$ , where the last identity is due to  $x_{\tau(x)} = x$ . Therefore it remains to verify  $D_J(x_L^1) \leq 0$  for  $x_L^1 < x_F^2$ .

If  $x_L^1 < x_F^2$ , the former is finite and we can write  $\lambda := x_J/x_L^1 \in [1, \infty]$ . Then also  $x_L^1 < x_J$  and thus (cf. equations (9), (10) in Pawlina and Kort (2006), accounting for possibly  $x_F^2 = \infty$ )

$$\begin{aligned} D_J(x_L^1) &= \frac{x_L^1 D_{10}}{r - \mu} - I^1 - \frac{x_F^2 (D_{10} - D_{11})}{r - \mu} \left( \frac{x_L^1}{x_F^2} \right)^{\beta_1} \\ &\quad - \frac{x_L^1 D_{00}}{r - \mu} - \left( \frac{x_J (D_{11} - D_{00})}{r - \mu} - I^1 \right) \left( \frac{x_L^1}{x_J} \right)^{\beta_1} \end{aligned}$$

$$\begin{aligned}
&= \frac{\beta_1}{\beta_1 - 1} I^1 - I^1 - \frac{\beta_1}{\beta_1 - 1} I^1 \frac{D_{10} - D_{11}}{D_{10} - D_{00}} \left( \frac{I^1 (D_{11} - D_{01})^+}{I^2 (D_{10} - D_{00})} \right)^{\beta_1 - 1} \\
&\quad - \left( \lambda \frac{\beta_1}{\beta_1 - 1} I^1 \frac{D_{11} - D_{00}}{D_{10} - D_{00}} - I^1 \right) \lambda^{-\beta_1} \\
&\stackrel{!}{\leq} 0.
\end{aligned}$$

Rearranging yields the condition (4.6). The derivative of the square bracket in (4.6) w.r.t.  $\lambda$  is strictly negative for  $\lambda \in (0, x_M^1/x_L^1)$  given  $\beta_1 > 1$ , where it is important to note that  $\lambda(D_{11} - D_{00}) < D_{10} - D_{00}$ , because  $D_{10} > D_{00}$  for  $x_L^1 < x_F^2$  and  $(D_{10} - D_{00})/(D_{11} - D_{00}) = x_M^1/x_L^1 > \lambda$  if  $D_{11} > D_{00}$ . Using the latter fact also shows that for  $\lambda = x_M^1/x_L^1$ , the square bracket is either  $1 - (x_L^1/x_M^1)^{\beta_1} \geq 0$  or 1, if  $x_M^1$  is finite or not, respectively.

Finally, necessity of  $D_J(x_L^1) \leq 0$  for  $x_L^1 < x_F^2 \leq x_J$  is obvious.  $\square$

**Proof of Proposition 4.4.** By the hypothesis  $x_L^1 < x_F^2$  and Lemmata 3.10 and A.4, problem (3.7) is solved by  $\tau_S(\vartheta) := \tau_L^1(\vartheta) = \inf\{t \geq \vartheta \mid x_t \geq x_L^1\} \in \mathcal{T}$  for any  $\vartheta \in \mathcal{T}$ . These stopping times for firm 1 satisfy time consistency  $\vartheta' \leq \tau_S(\vartheta) \Rightarrow \tau_S(\vartheta') = \tau_S(\vartheta)$  for any two  $\vartheta \leq \vartheta' \in \mathcal{T}$  by construction, as do firm 2's stopping times  $\tau_F^2(\vartheta) = \inf\{t \geq \vartheta \mid x_t \geq x_F^2\}$ .

To verify the equilibrium at  $\vartheta \in \mathcal{T}$  by Corollary 3.7, note that now  $\pi^{L1} - \pi^{01} \geq \pi^{L2} - \pi^{02}$ , whence problem (3.6) is solved by  $\tau_D^2(\vartheta') = \tau_S(\vartheta) \vee \vartheta'$ . Thus we have an equilibrium if  $x_L^1 \geq x_F^1$  ( $\geq \bar{x}$ ) or, using the strong Markov property, if  $0 \geq D_S(x) := L_0^2 - E[F_{\tau_S(0)}^2]$  given  $x_0 = x \in [x_L^1, x_F^1]$ .

By Proposition 3.6, if  $x_L^1 < x_F^1$  and we let  $\tau(x) = \inf\{t \geq 0 \mid x_t \geq x\} \leq \tau_F^1(0)$  for any  $x \in [x_L^1, x_F^1]$ , then  $D_S(x_L^1) \geq E[L_{\tau(x)}^2 - F_{\tau_S(0)}^2] = E[D_S(x)]$ , where the last identity is due to  $x_{\tau(x)} = x$ . Therefore it remains to verify  $D_S(x_L^1) \leq 0$  for  $x_L^1 < x_F^1$ , i.e.,  $x_L^1 \notin (x, \bar{x})$ . The latter condition is (cf. equations (8), (9) in Pawlina and Kort (2006), accounting for possibly  $x_F^1 = x_F^2 = \infty$ )

$$\begin{aligned}
D_S(x_L^1) &= \frac{x_L^1 D_{10}}{r - \mu} - I^2 - \frac{x_F^1 (D_{10} - D_{11})}{r - \mu} \left( \frac{x_L^1}{x_F^1} \right)^{\beta_1} \\
&\quad - \frac{x_L^1 D_{01}}{r - \mu} - \left( \frac{x_F^2 (D_{11} - D_{01})}{r - \mu} - I^2 \right) \left( \frac{x_L^1}{x_F^2} \right)^{\beta_1} \\
&= \frac{\beta_1}{\beta_1 - 1} I^1 \frac{D_{10} - D_{01}}{D_{10} - D_{00}} - I^2 - \frac{\beta_1}{\beta_1 - 1} I^1 \frac{D_{10} - D_{11}}{D_{10} - D_{00}} \left( \frac{(D_{11} - D_{01})^+}{D_{10} - D_{00}} \right)^{\beta_1 - 1} \\
&\quad - \frac{1}{\beta_1 - 1} I^2 \left( \frac{I^1 (D_{11} - D_{01})^+}{I^2 (D_{10} - D_{00})} \right)^{\beta_1} \\
&\stackrel{!}{\leq} 0.
\end{aligned}$$

Rearranging yields the condition (4.7). The derivative of its LHS w.r.t.  $I^2/I^1$  is strictly positive for  $x_L^1 < x_F^1$  given  $\beta_1 > 1$ , because then  $(D_{11} - D_{00})^+/(D_{10} - D_{00}) < 1$ . By the same fact the RHS of (4.7) is strictly positive.

To show necessity of  $x_L^1 \notin (\underline{x}, \bar{x})$ , suppose the contrary, whence  $x_L^1 < x_F^1$  and  $D_S(x_L^1) > 0$  by definition. For any  $x \leq x_L^1$ ,

$$\begin{aligned} D_S(x) &= E\left[D_S(x_L^1)\right] + L_0^2 - E\left[L_{\tau_S(0)}^2\right] = D_S(x_L^1) + E\left[\int_0^{\tau_S(0)} (\pi_s^{L2} - \pi_s^{02}) ds\right] \\ &= D_S(x_L^1) + \frac{x(D_{10} - D_{00})}{r - \mu} - I^2 - \frac{x_L^1(D_{10} - D_{00})}{r - \mu} \left(\frac{x}{x_L^1}\right)^{\beta_1}, \end{aligned}$$

which converges continuously to  $D_S(x_L^1) > 0$  as  $x \rightarrow x_L^1$ . Thus  $D_S(x) > 0$  for some  $x < x_L^1$ .  $\square$

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