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Working Paper

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IHS Economics Series, No. 323

Provided in Cooperation with:
Institute for Advanced Studies (IHS), Vienna

Suggested Citation: Candogan, Ozan; Epitropou, Markos; Vohra, Rakesh V. (2016) : Competitive equilibrium and trading networks: A network flow approach, IHS Economics Series, No. 323, Institute for Advanced Studies (IHS), Vienna

This Version is available at:
https://hdl.handle.net/10419/148268

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Competitive Equilibrium and Trading Networks: A Network Flow Approach

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August 2016

Competitive Equilibrium and Trading Networks: A Network Flow Approach

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Under full substitutability of preferences, it has been shown that a competitive equilibrium exists in trading networks, and is equivalent (after a restriction to equilibrium trades) to (chain) stable outcomes. In this paper, we formulate the problem of finding an efficient outcome as a generalized submodular flow problem on a suitable network. Equivalence with seemingly weaker notions of stability follows directly from the optimality conditions, in particular the absence of improvement cycles in the flow problem. Our formulation yields strongly polynomial algorithms for finding competitive equilibria in trading networks, and testing (chain) stability.

CCS Concepts: •**Applied computing**→ **Economics;** •**Theory of computation**→ **Market equilibria;** •**Mathematics of computing**→ **Discrete mathematics;**

Additional Key Words and Phrases: Trading Networks, Competitive Equilibrium, Stability, Submodular Flow Problems, Discrete Convexity.

1. INTRODUCTION

In many economic settings, trades are based on bilateral contracts which can be represented by a trading network ([Hatfield et al. 2013]). Nodes of the network correspond to agents. Edges represent the non-price elements of a bilateral trade and their orientation identifies which agent is the "buyer" and which the "seller". Agents have quasi-linear preferences over the set of trades and associated prices. The model is rich enough to allow an agent to be a buyer in some trades and a seller in others. In particular, it subsumes the classic assignment model [Shapley and Shubik 1971], as well as the model of supply chain networks, [Ostrovsky 2008], where the underlying directed graph is acyclic.

In trading networks where agents exchange indivisible goods (or indivisible contracts), [Hatfield et al. 2013, 2015b] has established that under a *full substitutability* condition on agents' preferences, a competitive equilibrium exists. The full substitutability condition generalizes the well-known *gross substitutability* condition, which is used to establish existence of a competitive equilibrium in two-sided markets [Gul and Stacchetti 1999; Kelso and Crawford 1982; Sun and Yang 2006]. Thus, the trading networks model extends the competitive equilibrium existence results to multi-sided settings. Competitive equilibria of trading networks are also *stable outcomes* in that they cannot be blocked by any coalition of agents and trades. A *blocking set* is a set of (feasible) trades and corresponding prices such that all agents who can participate in these trades (strictly) prefer them (while possibly declining some of their equilibrium contracts) [Hatfield et al. 2013]. Conversely, in any stable outcome it is possible to set prices for trades not involved in this outcome, to support the outcome as a competitive equilibrium. In fact, the stability condition is equivalent to the seemingly weaker *chain stability* condition [Hatfield et al. 2015a]. The latter condition restricts blocking sets to be paths/cycle of trades in the underlying trading network.

This paper's contribution is to show that under the full substitutability assumption, all these results can be obtained simply and directly from the optimality conditions of a *generalized submodular flow problem* in a suitable network. The optimal solutions to this flow problem (and its dual) yield a competitive equilibrium outcome and supporting prices. Moreover, in generalized submodular flow problems, a feasible flow is optimal if and only if there does not exist an *improvement cycle*. This optimality con-

dition yields the equivalence between a competitive equilibrium outcome and (chain) stability. A consequence of this is a strongly polynomial algorithm to find a competitive equilibrium as well as to identify a blocking chain when an outcome is not stable.

Our starting point is to express the problem of identifying the set of trades that maximize welfare, as a network flow problem on an appropriately defined *flow network*. The flow network is related to, but distinct from the underlying trading network. In the flow network each node corresponds to an agent-trade pair of the trading network. Since exactly two agents are involved in each trade, the flow network has two nodes for each feasible trade (one associated with the buyer the other associated with the seller in this trade). These nodes are connected by an edge in the flow network. However, this network is not connected in general.

Full substitutability of agents' preferences corresponds to $M^\natural$-concavity of the value functions [Hatfield et al. 2015b]. This observation allows us to represent the problem of finding the set of welfare-maximizing trades as a generalized submodular flow problem on the flow network. In this problem, we do not impose flow conservation at all nodes. Instead, we associate an M-convex penalty term with the net flow at nodes associated with the same agent in the flow network. Intuitively, the net flow encodes the trades where an agent participates as a buyer/seller, and the penalty term captures the total value the agent enjoys for these trades. Minimum cost flows in this network correspond to trades in the original network that maximize total welfare. The optimal dual solution to this problem are competitive equilibrium prices that support this set of trades. Thus, our approach generates the equilibrium trades and prices through the solution of an optimization problem. In contrast, [Hatfield et al. 2013], construct an auxiliary two-sided market and invoke the competitive equilibrium existence results (based on fixed point arguments due to [Kelso and Crawford 1982]) for that market.

We establish the equivalence between stability, chain stability, and competitive equilibrium outcomes directly from the fact that a given flow is optimal if and only if it admits no improvement cycles. Our proof technique also provides an algorithm that (i) checks whether an outcome is (chain) stable, and (ii) identifies a blocking chain if it is not. In particular, given a set of trades and associated prices, we first consider a (reduced) trading network that consists of the remaining trades (after an appropriate modification of the payoff functions), and the corresponding flow network. The algorithm starts with the (trivial) flow which does not use any edge of the flow network that is associated with the trades in the (reduced) trading network. Then, the algorithm searches for an improvement cycle. If such a cycle is not found, we conclude that the initial set of trades/prices constitute a (chain) stable outcome. Otherwise, the shortest such cycle reveals a blocking chain. The computational complexity of this approach is equivalent to that of constructing the flow network, and identifying the smallest negative cycle in this network. The overall complexity is polynomial in the number of nodes/edges of the underlying trading network. Thus, the network flow approach presented in this paper not only gives simpler existence proofs of the properties of trading networks (e.g., existence of competitive equilibrium, and its equivalence to stability), but also provides a tractable algorithm for determining competitive equilibria, testing (chain) stability, and identifying blocking sets of trades whenever they exist.

[Hatfield et al. 2015a] observed an equivalence between stability and chain stability which resembles an analogous equivalence result in classical network flows. Those authors argued that there are important differences between the two settings:

> "...in the 'network flows' environment, there is a single type of good 'flowing' through the network, and the objective function is the maximization or minimization of the aggregate flow, whereas in our setting many different types

of goods may be present, and the preferences of agents in the market may be more complex."

Our paper shows that these differences are superficial. An outcome is not stable if the corresponding flow is suboptimal. In the generalized submodular flow problems, suboptimality implies the existence of an improvement cycle. This indicates that whenever the initial outcome is not stable, it can be blocked by relying on a "simple" set of trades, which correspond to a chain in the underlying trading network.

The related literature is discussed below. Section 2 introduces notation and the model. Section 3 describes the submodular flow problem and its optimality properties. Section 4 describes the transformation of the problem of finding an efficient outcome into an instance of the submodular flow problem. Section 5 discusses the equivalence of various stability notions.

Related literature:. Gross substitutes of preferences is a sufficient condition for the existence of competitive equilibrium with indivisible goods ([Gul and Stacchetti 1999; Kelso and Crawford 1982]). It is also equivalent to $M^\natural$-concavity of the valuations [Fujishige and Yang 2003; Murota and Tamura 2003b; Paes Leme 2014; Shioura and Tamura 2015]. $M^\natural$-concavity has found applications in mathematical economics; such as direct proofs of the competitive equilibrium existence results, and algorithms for computing the competitive equilibrium outcome [Danilov et al. 2001, 2003; Murota and Tamura 2003a,b].

[Gul and Stacchetti 1999; Kelso and Crawford 1982] were concerned with a two-sided market of buyers and sellers. The trading networks literature ([Hatfield et al. 2013, 2015a,b; Ostrovsky 2008]) generalized the gross substitutes property to full substitutability. This extended the existence of competitive equilibrium result beyond the two-sided setting. These papers established that full substitutability corresponds to $M^\natural$-concavity of preferences. It suggests that the desirable properties of trading networks (under the full substitutability assumption) could be directly obtained by leveraging the rich literature on discrete convexity. This paper does just this. It shows that the results obtained in the recent literature on trading networks can be deduced from optimality conditions in (generalized submodular) network flow problems, where the cost functions (which are obtained by a transformation of valuation functions) are M-convex.

In [Murota 2003; Murota and Tamura 2003a] it was shown that the efficient allocation problem for a two-sided economy with multiple buyers and sellers, could be formulated as a generalized submodular flow problem on a bipartite network. We follow a similar approach in the more general setting of trading networks. The presence of agents who participate as buyers for some trades and sellers for others renders the reduction in [Murota 2003; Murota and Tamura 2003a] inapplicable. We provide an alternative network flow formulation for identifying the set of efficient trades in this more general setup. Additionally, it shows the equivalence of competitive equilibrium to (chain) stable outcomes can be characterized using a generalized submodular flow formulation. Thus, together with the results of [Murota 2003; Murota and Tamura 2003a], our paper indicates that a generalized submodular flow formulation provides a unifying framework for the study of various competitive equilibrium results in the literature.

2. THE MODEL

A trading network is represented by a directed multigraph $G = (N, E)$ where N is the set of vertices and E the set of arcs. Each vertex corresponds to an agent and each arc corresponds to a trade that can take place between the incident pair of vertices. For each $e \in E$, the source vertex e^+ corresponds to the seller and the sink vertex e^-

corresponds to the buyer in the trade. Let $\delta_+(i)$ and $\delta_-(i)$ be the outgoing and incoming arcs incident to vertex $i \in N$, and $\delta(i) = \delta_+(i) \cup \delta_-(i)$. An outcome of the market is a set of trades i.e. $X \subset E$. We define a price vector $p \in \mathbb{R}^E$, where p_e is the price associated with the trade that corresponds to the arc e. Denote by p^X the price vector restricted to the arcs in X.

Denote agent i's value function for any set of trades involving agent i by $w_i : 2^{\delta(i)} \to \mathbb{R}$. Agent i's utility function is $u_i : 2^{\delta(i)} \times \mathbb{R}^{\delta(i)} \to \mathbb{R}$. For each $S \subset \delta(i)$ and $p \in \mathbb{R}^E$

$$u_i(S, p) = w_i(S) + \sum_{e \in S \cap \delta_+(i)} p_e - \sum_{e \in S \cap \delta_-(i)} p_e.$$

The demand correspondence for agent $i \in N$, given a price vector $p \in \mathbb{R}^{\delta(i)}$, is

$$D_i(p) = \arg\max\{u_i(Y, p) : Y \subset \delta(i)\}$$

Definition 2.1. An outcome $X \subset E$ along with a price vector $p \in \mathbb{R}^E$ is a competitive equilibrium (X, p) if, for all $i \in N$,

$$X \cap \delta(i) \in D_i(p).$$

Definition 2.2. An outcome $X \subset E$ is efficient if

$$X \in \arg\max_{S \subset E} \sum_{i \in N} w_i(S \cap \delta(i)).$$

3. THE M-CONVEX SUBMODULAR FLOW PROBLEM

Here we introduce the M-convex submodular flow problem which generalizes the standard network flow problems (see Chapter 9 of [Murota 2003]). We are given a directed graph (V, A), where V is the set of vertices and A is the set of arcs. For each $v \in V$ denote by $\delta_+(v)$ and $\delta_-(v)$ respectively the set of outgoing and incoming arcs incident to vertex v.

As in the standard network flow problem each arc $a \in A$ has a cost c_a per unit of flow, and lower and upper capacities $\underline{k}_a$, $\overline{k}_a$. Denote by x_a the amount flowing through $a \in A$. Given flows in the arcs, denote by y_v the net outflow (that can be positive or negative) from vertex v, and let y denote the vector of $\{y_v\}_{v \in V}$. The added feature of the M-convex submodular flow problem (MSFP) is a term, $f(y)$, in the objective function which is M-convex (defined below). the MSFP can be formulated as follows:

$$\begin{aligned} &\min_{x,y} \sum_{a \in A} c_a x_a + f(y) \\ &\text{s.t.} \sum_{a \in \delta_+(v)} x_a - \sum_{a \in \delta_-(v)} x_a = y_v \ \forall v \in V \\ &\quad \underline{k}_a \le x_a \le \overline{k}_a \ \forall a \in A. \end{aligned}$$

In the standard network flow problem, f's domain is just a single point ($y = 0$). In our case f is M-convex. To define M-convexity let $\chi^j \in \mathbb{Z}^n$ denote the 0-1 vector with exactly one non-zero entry in component j. A function $f : \mathbb{Z}^n \to \mathbb{R} \cup \{\infty\}$ on the integer lattice is M-convex if it satisfies the following exchange axiom:

(M-**EXC[$\mathbb{Z}$]**): For all $x, y \in \mathbb{Z}^n$ and for all $u \in supp^+(x - y)$,
$f(x) + f(y) \ge \min_{v \in supp^-(x-y)} f(x - \chi^u + \chi^v) + f(y + \chi^u - \chi^v)$

where $supp^+(x-y)$ ($supp^-(x-y)$) is the set of all indices in $\{1, \ldots, n\}$ such that $x_i - y_i > 0$ ($x_i - y_i < 0$). A function g is called M-concave if $-g$ is M-convex. An M-convex function

f's *effective domain* (i.e., $\mathrm{dom} f \triangleq \{x \in \mathbb{Z}^n | -\infty < f(x) < \infty\}$) as well as its maximizers form the basis family of a matroid.

The domain of an M-convex function $f(\cdot)$ can be extended to $\mathbb{R}^n$ by computing the convex closure of $f(\cdot)$, i.e., by finding

$$\bar{f}(x) = \sup_{p \in \mathbb{R}^n, \alpha \in R} \left\{ \sum_i p_i x_i + \alpha \Big| \sum_i p_i y_i + \alpha \leq f(y) \ \forall y \in \mathbb{Z}^n \right\} \quad \text{for all } x \in \mathbb{R}^n.$$

We assume that the flow cost associated with the MSFP is the closure of an M-convex function $f(\cdot)$ defined on the integer lattice.[1] Theorem 9.15 in [Murota 2003], guarantees the existence of an optimal integer flow when the capacities are integer valued – a result analogous to integrality of optimal solution in classical network flow problems. Thus, in our exposition we focus on $f(\cdot)$ defined on the integer lattice but implicitly considering its convex closure when we formulate MSFP. Note that for optimal solutions only the values of $f(\cdot)$ on the integer lattice matter. Our theoretical and algorithmic results do not require explicitly constructing this convex closure.

One can generalize the optimality conditions of the standard flow problem with a linear objective function to the MSFP (see [Murota 2003]). In particular, the optimality of a flow is characterized by the nonexistence of a negative cycle in an auxiliary network as well as in terms of a set of potentials associated with the nodes of the network.

First, define an auxiliary network G^{aux} which is an extension of the idea of a residual network used in the standard network flow problem to account for the non-linearities in f. Let x be a feasible flow in G and y be the associated vector of net flows at each vertex. Let $G^{aux}(x,y) = (V, A^{aux}(x,y) \cup B^{aux}(x,y) \cup C^{aux}(x,y))$ where

(1) $A^{aux}(x,y) = \{a | a \in A, x_a < \overline{k}_a\}$,
(2) $B^{aux}(x,y) = \{-a | a \in A, x_a > \underline{k}_a\}$($-a$ denotes arc a with its orientation reversed),
(3) and $C^{aux}(x,y) = \{(u,v) | u, v \in V, f(y - \chi^u + \chi^v) < +\infty\}$.

The auxiliary network has no arc capacities. The cost for each arc in $G^{aux}(x,y)$ is given by

$$c_a^{aux}(x,y) = \begin{cases} c_a & \text{if } a \in A^{aux}(x,y) \\ -c_a & \text{if } a \in B^{aux}(x,y) \\ f(y - \chi^u + \chi^v) - f(y) & \text{otherwise.} \end{cases}$$

The sum of the edge costs associated with a directed cycle of the auxiliary network can be interpreted as the "length" of this cycle. A directed cycle of negative length is referred to as a *negative cycle*. The optimality criteria are listed in the following theorem.

THEOREM 3.1. *(Theorems 3.1, 3.2 in [Murota 1999]) The following three conditions are equivalent:*

(1) (x,y) *is an optimal solution to MSFP.*
(2) There does not exist a negative cycle in $G^{aux}(x,y)$.
(3) There exists a potential $p : V \to \mathbb{R}$ *such that*
  (a) for each $(u,v) \in A$,
    (i) $c_{(u,v)} + p(u) - p(v) > 0 \Rightarrow x_{(u,v)} = \underline{k}_{(u,v)}$
    (ii) $c_{(u,v)} + p(u) - p(v) < 0 \Rightarrow x_{(u,v)} = \overline{k}_{(u,v)}$
  (b) $f(y) - p \cdot y \leq f(y') - p \cdot y'$ *for all* $y' \in \mathbb{Z}^V$.

[1] Such functions are also referred to as integral polyhedral M-convex functions. See [Murota 2003], Section 6.11.

If the $\{\underline{k}_a, \overline{k}_a\}_{a \in A}$ are all integral, there is an optimal solution (x, y) to MFSP that is integral. Unlike the standard network flow problem, a negative cycle is only a certificate of non-optimality. It is not the case that augmenting flow along *any* negative cycle can improve the objective function. However, there exists a particular negative cycle that does correspond to an improving direction. This cycle and how it is to be found is described in greater detail later.

3.1. $M^\natural$-Concave Valuation Functions

An $M^\natural$-convex function $f : \mathbb{Z}^n \to \mathbb{R}$ is a function satisfying the following exchange axiom

($M^\natural$-**EXC[$\mathbb{Z}$]**): For all $x, y \in \mathbb{Z}^n$ and for all $u \in supp^+(x - y)$,
$f(x) + f(y) \geq \min[f(x - \chi^u) + f(y + \chi^u), \min_{v \in supp^-(x-y)} f(x - \chi^u + \chi^v) + f(x + \chi^u - \chi^v)]$.

An $M^\natural$-convex function is supermodular. A function f is $M^\natural$-concave if $-f$ is $M^\natural$-convex.

Any $M^\natural$-convex function $f : \mathbb{Z}^n \to \mathbb{R} \cup \infty$ can be represented as an M-convex function $f' : \mathbb{Z}^{n+1} \to \mathbb{R} \cup \infty$ where

$$f'(x_0, x) = \begin{cases} f(x) & \text{if } x_0 = -\sum_{i=1}^{n} x_i \\ +\infty & \text{otherwise.} \end{cases}$$

3.2. Full Substitutability

Earlier, we defined each w_i as a function over subsets of $\delta(i)$. If we represent sets by their characteristic vectors, we can treat each w_i as a function over $\{0, -1\}^{\delta_-(i)} \times \{0, 1\}^{\delta_+(i)}$. We extend the domain of w_i to $\mathbb{Z}^{\delta(i)}$ by following the convention that $w_i(x) = -\infty$ for $x \in \mathbb{Z}^{\delta(i)}$ such that $x \notin \{0, -1\}^{\delta_-(i)} \times \{0, 1\}^{\delta_+(i)}$. An analogous convention applies to the utility functions u_i. Next, assume that each w_i for each $i \in N$ is $M^\natural$-concave. This is equivalent to the property that an agent's demand correspondence satisfies the full substitutes property (see [Hatfield et al. 2013, 2015b], and Theorem 7 of [Murota and Tamura 2003b]).

Definition 3.2. Agent i's preferences are fully substitutable if:

(1) For all $p, \tilde{p} \in \mathbb{R}^{\delta(i)}$ such that $p_e = \tilde{p}_e$ for all $e \in \delta_+(i)$ and $\tilde{p}_e \geq p_e$ for all $e \in \delta_-(i)$, for every $Y^i \in D_i(p)$ there exists $\tilde{Y}^i \in D_i(\tilde{p})$ such that $(Y^i \cap \{e | p_e = \tilde{p}_e\}) \cap \delta_-(i) \subset \tilde{Y}^i \cap \delta_-(i)$ and $\tilde{Y}^i \cap \delta_+(i) \subset Y^i \cap \delta_+(i)$.
(2) For all $p, \tilde{p} \in \mathbb{R}^{\delta(i)}$ such that $p_e = \tilde{p}_e$ for all $e \in \delta_-(i)$ and $\tilde{p}_e \leq p_e$ for all $e \in \delta_+(i)$, for every $Y^i \in D_i(p)$ there exists $\tilde{Y}^i \in D_i(\tilde{p})$ such that $(Y^i \cap \{e | p_e = \tilde{p}_e\}) \cap \delta_+(i) \subset \tilde{Y}^i \cap \delta_+(i)$ and $\tilde{Y}^i \cap \delta_-(i) \subset Y^i \cap \delta_-(i)$.

4. TRANSFORMATION TO MSFP

We use the optimality conditions of the MSFP to show that a competitive equilibrium exists. To do this we transform the problem of finding an efficient outcome into an instance of the MSFP.

We introduce a flow network $G' = (V, A)$, associated with the trading network G. Recall, there is an $M^\natural$-concave function $w_i : \mathbb{Z}^{\delta(i)} \to \mathbb{R}$ associated with each vertex $i \in N$. We represent the set of trades agent i is involved in by a characteristic vector y^i, where for each trade $e \in \delta(i)$ which occurs, we set $y^i_e = 1$ if $e \in \delta_+(i)$ and $y^i_e = -1$ if $e \in \delta_-(i)$. With this representation we can replace each w_i by an M-concave function

$w_i' : \mathbb{Z} \times \mathbb{Z}^{\delta(i)} \to \mathbb{R}$, such that

$$w_i'(z_0, z) = \begin{cases} w_i(z) \text{ if } z_0 = -\sum_{r=1}^{n} z_r \\ -\infty \quad \text{otherwise} \end{cases} \tag{1}$$

For $y = \{y^i\}_{i \in N}$, the social welfare of the trading network is given by $-f(y) = \sum_{i \in N} w_i'(y^i)$. M-concavity of $\{w_i'\}$ implies implies that $f(\cdot)$ is M-convex as the arguments of the M-convex functions in the summand are disjoint.[2]

Now, each w_i is a function of the characteristic vector of arcs incident to i that carry a positive amount of flow. To account for this we represent each $i \in N$ by a set V^i of vertices associated with the arguments of each $M^\natural$-concave function w_i', i.e. $|V^i| = |\delta(i)| + 1$. Formally,

$$V = \bigcup_{i \in N} V^i = \{v_e^i | i \in N, e \in \{0\} \cup \delta(i)\}.$$

We refer to vertices of the form v_0^i as **special vertices**. We add a set of (directed) arcs A_0 between every pair of special vertices. Additionally, for each $e \in E$ with $e = (i, k)$ we introduce an arc $a = (v_e^i, v_e^k)$. Intuitively, one unit of flow on this arc represents that both agent i and agent k participate in trade e (and since $y_e^i = -y_e^k$ the corresponding flow cost reflects values of both agents for this trade). These arcs form set A_1.

Formally,

$$A = A_0 \cup A_1 = \{(v_0^i, v_0^k) : i, k \in N\} \cup \{(v_e^i, v_e^k) : e = (i, k) \in E\}.$$

Figure 1 displays an example.

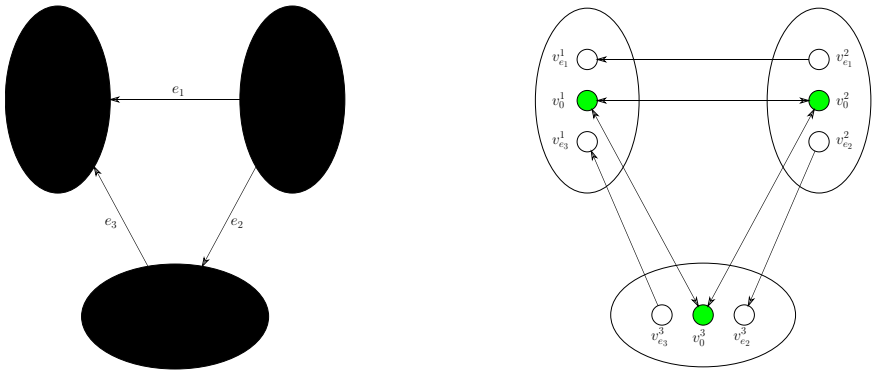

Fig. 1. (a) A trading network $G = (N, E)$ (b) Corresponding flow network $G' = (V, A)$

We define the following instance of MSFP on (V, A):

$$\begin{aligned} \min_{x,y} \quad & f(y) \\ \text{s.t.} \quad & \sum_{a \in \delta_+(v)} x_a - \sum_{a \in \delta_-(v)} x_a = y_v \quad \forall v \in V \\ & \underline{k}_a \le x_a \le \overline{k}_a \quad \forall a \in A \end{aligned}$$

[2]In general, the sum of M-convex functions is not M-convex. However, this property trivially holds when M-convex functions with disjoint arguments are considered.

Notice here that all arc costs are zero, i.e. $c_a = 0$, and lower and upper capacities are set as follows :

$$\underline{k}_a = -\infty, \overline{k}_a = +\infty.$$

Additionally, by construction $-f(y)$ is M-convex. We can associate an auxiliary network with a given feasible solution (x, y) of this problem, as demonstrated in Figure 2.

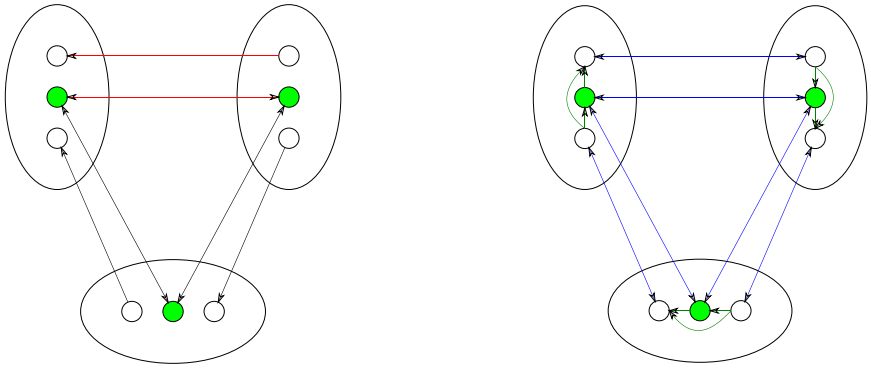

Fig. 2. (a) Feasible Solution (b) Auxiliary Graph (blue: $A^{aux} \cup B^{aux}$, green: C^{aux})

Consider a set of trades $S \subset E$ in the trading network $G = (N, E)$. A corresponding flow in $G' = (V, A)$ can be obtained by sending one unit of flow on each arc in A_1 associated with these trades, and choosing the flow through arcs between special vertices to keep the total net flow into vertices in V^i equal to zero (for all i). Observe that the absolute value of the associated flow cost is equal to the welfare corresponding to S. Conversely, by construction of $f(\cdot)$, it can be seen that any flow with bounded cost is such that the net flow into vertices in V^i is equal to zero for all i (see (1)), and each arc in A_1 carries at most one unit of flow. Moreover, the absolute value of cost of any such flow is equivalent to the total welfare associated with the trades that correspond to arcs in A_1 with nonzero flow. Hence, integer flows with bounded cost in G' correspond to feasible sets of trades in G. The MSFP on $G' = (V, A)$ is guaranteed to have an integer optimal solution. Thus, this solution corresponds to an efficient outcome for the trading network $G = (N, E)$.

Consider an optimal solution (x, y) to the MSFP on G'. According to Theorem 3.1, the corresponding auxiliary graph does not have negative cycles. Thus, as in the classical network flow problem, one can associate a potential function with the auxiliary graph, satisfying the following constraints (where the right hand sides correspond to edge costs in the auxiliary graph):

$$p(v) - p(u) \leq 0 \qquad \forall (u, v) \in A^{aux}(x, y) \cup B^{aux}(x, y), \tag{2}$$

$$p(v) - p(u) \leq f(y - \chi^u + \chi^v) - f(y) \qquad \forall (u, v) \in C^{aux}(x, y). \tag{3}$$

We refer to (2,3) as the potential function conditions. They are identical to the potential function characterization in Theorem 3.1. In particular, inequality (2) is equivalent to $p(v) - p(u) = 0$ for all $(u, v) \in A$, since for each $(u, v) \in A$, we have $(u, v) \in A^{aux}(x, y)$ and $(v, u) \in B^{aux}(x, y)$. The latter equality is also equivalent to Theorem 3.1 (3a), since the arc capacities are infinite. The inequalities in (3) are equivalent to those in Theorem 3.1 (3b) by Theorem 6.26 in [Murota 2003] (and M-convexity of $f(y) - p \cdot y$; see

Theorems 6.13(3), and 6.15 in [Murota 2003]). We conclude that the set of potentials satisfying (2,3) is equivalent to the potentials implied by Theorem 3.1.

From the discussion above we get that $p(u) = p(v)$ for all $(u, v) \in A$, which is our candidate price for the trade (u, v). Furthermore, given a potential function $p : V \to \mathbb{R}$ satisfying the conditions of Theorem 3.1 (3a – 3b), $p'(u) = p(u) + c$ for all $u \in V$ gives another potential satisfying these conditions. Recall, all special vertices are adjacent to each other. This means they all have a potential equal to p_0. By setting $c = -p_0$ we ensure that a potential function always takes value zero on the special vertices.

Potentials are defined on vertices. However, in our construction of flow networks each vertex corresponds to a particular trade-agent pair and the potentials of two adjacent vertices (associated with the same trade) are equal. Theorem 3.1 (3b) implies that if these potentials can be interpreted as prices, and the set of trades y^i chosen for some agent i are modified (through a choice of different in/outflow $\hat{y}^i$ for V^i nodes), then the surplus of agent i cannot be improved. Thus, it follows that an optimal solution (x, y) of the MSFP and the prices that correspond to a potential function satisfying Theorem 3.1 (3b) constitute a competitive equilibrium.

Conversely, given a competitive equilibrium, the prices for trades defines a potential function on all vertices where the potential of a vertex is the price of the corresponding trade and the special vertices get a potential value of zero. The equilibrium conditions imply that the equilibrium prices and outcome satisfy Theorem 3.1 (3a – 3b). Hence, the flow associated with this outcome solves the MSFP. Thus, the equivalence of optimal solutions of the MSFP and efficient outcomes, as well as potential functions and competitive prices follows.

4.1. Immediate Consequences

THEOREM 4.1. *(Theorem 1 in [Hatfield et al. 2013]) There exists a competitive equilibrium.*

PROOF. Given a trading network $G = (N, E)$, we map it to the associated flow problem on the flow network (V, A). The MSFP on (V, A) has an optimal solution (x^*, y^*), since it is a discrete problem and "no flow" is a feasible solution. Theorem 3.1 implies that there exists a potential p^*. The feasible solution (x^*, y^*), along with its potential function p^* is a competitive equilibrium. This completes the proof. □

The outcome associated with a competitive equilibrium is efficient.

THEOREM 4.2. *(First Welfare Theorem, Theorem 2 in [Hatfield et al. 2013]) Suppose that (X, p) is a competitive equilibrium. Then, X is an efficient outcome.*

PROOF. Let (x, y) be a feasible flow associated with the outcome X. The competitive prices imply a potential function for the flow (x, y). By Theorem 3.1, (x, y) is optimal, therefore, the outcome X is efficient. □

Next, we show that competitive prices support all efficient outcomes, i.e., these prices with any efficient outcome constitute a competitive equilibrium.

THEOREM 4.3. *(Second Welfare Theorem (strong version), Theorem 3 in [Hatfield et al. 2013]) For any competitive equilibrium (X, p) and efficient outcome X', (X', p) is also a competitive equilibrium.*

PROOF. The outcomes X, X' correspond to optimal flows (x, y) and (x', y') respectively. The prices imply a potential function associated with the optimal flow (x, y). The second part of Theorem 3.1 in [Murota 1999] states that the potential function satisfies the conditions of the potential criterion for the flow (x', y'). We conclude that (X', p) is a competitive equilibrium. □

The set of competitive prices enjoys a nice structure.

THEOREM 4.4. *(Theorem 4 in [Hatfield et al. 2013]) The set of competitive price vectors is a lattice.*

PROOF. Immediate from the fact that the feasible region of a system of difference constraints (2, 3) is a lattice. [3] □

In this model one can interpret a trade as the sale of goods from a seller to a buyer. The trade specifies the identity of the good (edge) as well as its quantity (flow). The buyer pays the price for given quantity of the product. [Hatfield et al. 2013] gives a sufficient condition for the existence of a competitive equilibrium, where uniform pricing over "identical" trades is realized. The connection to the MSFP allows us to extend this sufficient condition. We define what it means for two trades to be perfect substitutes for each other.

Definition 4.5. Agent i sees trades $e, e' \in \delta(i)$ as perfect substitutes for each if $w_i(X \cup \{e\}) = w_i(X \cup \{e'\})$ for all $X \subset \delta(i) \setminus \{e, e'\}$.

This definition immediately implies that the valuation function of agent i depends only on the **number** of trades chosen in an equivalence class of perfectly substitutable trades Y associated with him, i.e. $w_i(X \cup S) = w_i(X \cup S')$ for all $S, S' \subset Y$ such that $|S| = |S'|$ and for all $X \subset \delta(i) \setminus Y$.

In [Hatfield et al. 2013] it was established that there exists competitive equilibrium where trades that are perfect substitutes receive the same price, provided that these trades are also *mutually incompatible*, i.e., accepting more than two such trades leads to a payoff of $-\infty$. Our next result (see appendix for a proof) shows that such an equilibrium still exists, when the mutual incompatibility assumption is relaxed. Importantly, this relaxation allows the seller to produce and sell **multiple** identical goods.

THEOREM 4.6. *Suppose that for agent i, any pair of trades in $Y \subset \delta(i)$ are perfect substitutes for each other. Then, there exists a competitive equilibrium, such that $p_e = p_{e'}$ for all $e, e' \in Y$.*

5. STABLE OUTCOMES

In this section we list various notions of stability for trading networks that have been proposed in [Hatfield et al. 2013]. Informally, a stable outcome has the property that no subset of agents has incentive to deviate from it. Given a set of trades X, with a slight abuse of notation, we denote the prices of the corresponding trades by p^X, and the set of trades agent i demands once she is restricted to the trades in X by $D_i(p^X) \subset X \cap \delta(i)$. Call an outcome X, along with its prices p^X **individually rational** if

$$X \cap \delta(i) \in \arg\max_{Y \subset X \cap \delta(i)} w_i(Y) + \sum_{e \in Y \cap \delta_+(i)} p_e^X - \sum_{e \in Y \cap \delta_-(i)} p_e^X \quad \forall i \in N.$$

Definition 5.1. An outcome X, along with its prices p^X, is stable if it is individually rational and is unblocked:

There is no feasible nonempty blocking set $Z \subset E$, along with its prices p^Z such that

(1) $Z \cap X = \emptyset$, and

[3] Theorem 9.15 in [Murota 2003] presents a more sophisticated version of the result, i.e. the set of optimal potentials is an L-convex polyhedron. This means that the set of competitive prices, which is a restriction of the potentials to the coordinate plane, is an $L^\natural$-convex polyhedron.

(2) for all agents i involved in Z, for all $Y^i \in D_i(p^{Z\cup X})$, we have $Z \cap \delta(i) \subset Y^i$.

The closely related notion of strongly stable outcome is defined next.

Definition 5.2. An outcome X, along with its prices p^X, is strongly stable if it is individually rational and is strongly unblocked:

There is no feasible nonempty strongly blocking set $Z \subset E$, along with its prices p^Z such that

(1) $Z \cap X = \emptyset$, and
(2) for all agents i involved in Z, there exists a $Y^i \subset \{Z\cup X\}\cap\delta(i)$ such that $Z\cap\delta(i) \subset Y^i$ and

$$w_i(Y^i)+\sum_{e\in Y^i\cap\delta_+(i)} p_e^{Z\cup X}-\sum_{e\in Y^i\cap\delta_-(i)} p_e^{Z\cup X} > w_i(X\cap\delta(i))+\sum_{e\in X\cap\delta_+(i)} p_e^{X}-\sum_{e\in X\cap\delta_-(i)} p_e^{X}.$$

Clearly, a strongly stable outcome is stable.

The next notion of stability is analogous to pairwise stability in bipartite matching. Call a set of consecutive arcs in a graph G, i.e., a set of m arcs $S = \{e_1, \ldots, e_m\}$, such that $e_i^- = e_{i+1}^+$ for all $i = 1, \ldots, m-1$, a chain.

Definition 5.3. An outcome X, along with its prices p^X, is chain stable if it is individually rational and is unblocked by a chain:

There is no feasible nonempty blocking chain $Z \subset E$, along with its prices p^Z such that

(1) $Z \cap X = \emptyset$, and
(2) for all agents i involved in Z, for all $Y^i \in D_i(p^{Z\cup X})$, we have $Z \cap \delta(i) \subset Y^i$.

The related notion of strong chain stability is defined below.

Definition 5.4. An outcome X, along with its prices p^X, is strongly chain stable if it is individually rational and is strongly unblocked by a chain:

There is no feasible nonempty strongly blocking chain $Z \subset E$, along with its prices p^Z such that

(1) $Z \cap X = \emptyset$, and
(2) for all agents i involved in Z, there exists a $Y^i \subset \{Z\cup X\}\cap\delta(i)$ such that $Z\cap\delta(i) \subset Y^i$ and

$$w_i(Y^i)+\sum_{e\in Y^i\cap\delta_+(i)} p_e^{Z\cup X}-\sum_{e\in Y^i\cap\delta_-(i)} p_e^{Z\cup X} > w_i(X\cap\delta(i))+\sum_{e\in X\cap\delta_+(i)} p_e^{X}-\sum_{e\in X\cap\delta_-(i)} p_e^{X}.$$

Clearly, a strongly chain stable outcome is chain stable. Definitions 5.1-5.4 also imply that a (strongly) stable outcome is (strongly) chain stable, since if there exists no (strongly) blocking set, there exists no such set with a chain structure.

Before we show the equivalence of these stability concepts, we focus on the case when the 'no trade' outcome is inefficient. In this case we show it is always possible to find a chain that improves welfare. Intuitively, this preliminary result implies that it may be possible to restrict attention to chains when searching for a blocking set. We subsequently formalize this intuition in Corollary 5.6 for outcomes where no trade is executed.

LEMMA 5.5. *Consider a trading network $G = (N, E)$. Assume that the no trade outcome is inefficient. Then, there exists a chain of trades that improve welfare.*

PROOF. Consider the MSFP formulation of the welfare maximization problem in G, and let (x, y) denote a feasible solution of the MSFP associated with flow network $G' = (V, A)$ that corresponds to the no trade outcome, i.e., that associates zero flow with all arcs in A_1, and hence guarantees $y = 0$. Since outcome $\emptyset$ is not optimal in G, according to Theorem 3.1 there exists a negative cycle in the auxiliary graph $G^{aux}(x, y)$. Pick a negative cycle K with the fewest number of arcs.

We claim that there exists such a cycle K which satisfies the following conditions:

(1) $0 > \sum_{a \in K} c_a^{aux}(x, y) = \sum_{(u,v) \in K \cap C^{aux}(x,y)} [f(y - \chi^u + \chi^v) - f(y)]$,
(2) It contains at most one special vertex **OR** two incident special vertices,
(3) $K \cap B^{aux}(x, y) = \emptyset$, and if $e \in K \cap C^{aux}(x, y)$, then there exists $h_1, h_2 \in K \cap A^{aux}(x, y)$ such that $h_1 - e - h_2$ is a sequence of edges on cycle K.

The first condition follows since K is a negative cycle, and arc costs are nonzero only for arcs in $C^{aux}(x, y)$. Suppose that the second condition is violated. Pick two special vertices and the arc with zero cost between them. Then, we get two smaller cycles, such that at least one is negative, which contradicts our assumption that K is the negative cycle with the fewest number of arcs.

To see the third one, first observe that $C^{aux}(x, y) = \{(u, v) | u, v \in V, f(y - \chi^u + \chi^v) < +\infty\}$ only consists of edges (u, v) where (i) $u, v \in V^i$ for some agent i, (ii) $u \in \{v_e^i | e \in \{0\} \cup \delta_-(i)\}$ and $v \in \{v_e^i | e \in \{0\} \cup \delta_+(i)\}$. To see (i) note that $y = 0$, and by construction $f(z) < \infty$ only when the total net flow into nodes in V^i is zero for all i. Thus, if this claim does not hold, then $f(y - \chi^u + \chi^v) = \infty$, indicating that $(u, v) \notin C^{aux}(x, y)$. Similarly, property (ii) follows since by construction $f = -\sum_i w_i'$, and the definition of w^i implies that $f(y - \chi^u + \chi^v) = \infty$ unless this property holds.

Assume that $(u, v) \in K \cap B^{aux}(x, y)$, then $(v, u) \in A = A_0 \cup A_1$. If $(v, u) \in A_0$, then by construction we also have $(u, v) \in A_0$, and hence $(u, v) \in A^{aux}(x, y)$. Thus, another negative cycle with the same number of arcs and total weight can be obtained by replacing $(u, v) \in K \cap B^{aux}(x, y)$ with the parallel edge between the same nodes that belongs to $A^{aux}(x, y)$. Conversely, if $(v, u) \in A_1$, then $v \in \{v_e^i | e \in \delta_+(i)\}$, hence the next arc (v, v') of K, cannot belong to $C^{aux}(x, y)$ (as this violates (ii)). Since the edges in A_1 are disjoint, this arc is given by $(v, v') = (v, u)$. By omitting both (u, v) and (v, u), a shorter negative cycle can be obtained, thereby leading to a contradiction. Thus, K can be chosen such that $K \cap B^{aux}(x, y) = \emptyset$.

Finally, note that if $(u, v) \in K \cap C^{aux}(x, y)$, then there is no $(v, v') \in K \cap C^{aux}(x, y)$, as otherwise we obtain a contradiction to (ii). Since, $K \cap B^{aux}(x, y) = \emptyset$, this observation implies that any arc $(u, v) \in K \cap C^{aux}(x, y)$ is followed in K by arcs that belong to $A^{aux}(x, y)$, and the third condition follows.

Proposition 9.25 in [Murota 2003] implies:

$$0 > \sum_{(u,v) \in K \cap C^{aux}(x,y)} [f(y - \chi^u + \chi^v) - f(y)] = f\left(y + \sum_{(u,v) \in K \cap C^{aux}(x,y)} (-\chi^u + \chi^v)\right) - f(y),$$

since K is a negative cycle with fewest arcs. Moreover, it can be seen that the first term in the right hand side is the cost of flow obtained after modifying the original flow by sending one unit of flow on edges $A \cap K$ (recall that $K \cap B^{aux}(x, y) = \emptyset$). Thus, we conclude that executing the set of trades associated with edges $A \cap K$ improves welfare. We complete the proof by showing that this set of trades constitute a chain in the trading network.

First observe that if K involves two consecutive arcs in $A^{aux}(x, y) \subset A_0 \cup A_1 = A$, then these arcs are between adjacent special nodes, since by construction all edges in

A_1 correspond to disconnected components of (V, A). This contradicts the second condition above, thus it follows from the third condition that arcs of K alternate between $A^{aux}(x, y)$ and $C^{aux}(x, y)$.

Assume that K does not involve an arc between any special vertices (i.e., $K \cap A^{aux}(x, y) \subset A_1$). Consider an arc $e \in K \cap C^{aux}(x, y)$, and recall that both end points of this arc belong to V^i for some agent i. The next arc in K belongs to A_1, and hence connects a non-special vertex in V^i to a non-special vertex in V^j for some $i \neq j$, thereby capturing a trade between i and j. Since arcs of K alternate between $A^{aux}(x, y)$ and $C^{aux}(x, y)$, it follows that the next arc's endpoints belong to V^j. Hence, proceeding iteratively it can be seen that the arc after this one suggests a trade relation between j and some other agent k. Thus, as claimed the set of trades associated with arcs $A \cap K$ constitute a chain[4] in G.

Assume instead that K involves an arc between special vertices. Since K involves at most two special vertices, there can be only one such arc. Starting with such an arc, and proceeding as before, it can be shown that the remaining arcs of K suggest a chain of trades that correspond to the arcs of $A \cap K$.

Hence, we conclude that the trades identified by the smallest negative cycle induce a chain of welfare-improving trades, as claimed. □

The optimality conditions for the MSFP and the structure of the flow network play a key role in the proof of Lemma 5.5. Lemma 5.5 also has a straightforward corollary that characterizes blocking chains in terms of a *minimal* set T of trades that improve welfare, i.e., T such that no subset of T improves welfare when compared to the no trade outcome.

COROLLARY 5.6. *Consider a trading network $G = (N, E)$. Assume the no trade outcome is inefficient. Then,*

(i) any minimal set of trades that improve welfare constitutes a chain,
(ii) there exist prices which together with these trades constitute a blocking chain.

PROOF. (i) Assume $\emptyset$ is not efficient in G, and let $T \subset E$ be a minimal set of trades that strictly improve welfare. Consider a trading network $\hat{G} = (N, T)$, obtained by restricting the original set of trades to T. Observe that the outcome $\emptyset$ is also not welfare maximizing in $\hat{G}$. Lemma 5.5 implies that there exists a welfare-improving set of trades that constitute a chain in $\hat{G}$. Since, T is the minimal (and only) set of trades that improves welfare, it follows that T is a chain.

(ii) Since T is a minimal welfare-improving set of trades, it follows that in $\hat{G} = (N, T)$ the unique efficient outcome involves executing all trades in T.

Let $\Delta > 0$ be such that $\sum_i w_i(T \cap \delta(i)) - 2\Delta|T| > \sum_i w_i(X \cap \delta(i)) - 2\Delta|X|$ for any $X \subsetneq T$. It suffices to choose a $\Delta > 0$, such that $2\Delta|T| < \sum_i w_i(T \cap \delta(i)) - \sum_i w_i(\emptyset)$ (recall that $\sum_i w_i(X \cap \delta(i)) \leq \sum_i w_i(\emptyset)$ for any $X \subsetneq T$, since T was a minimal set of trades improving the welfare). Consider another economy with the same network structure $\hat{G} = (N, T)$, but with valuations $\bar{w}_i(Z) = w_i(Z) - \Delta|Z|$ where $Z \subset \delta(i)$.[5] Observe that

[4] In such a case, $A \cap K$ is a cycle in G.

[5] The use of "modified valuations" was employed in [Hatfield et al. 2013] to establish that a stable outcome can be supported with appropriate prices to obtain a competitive equilibrium. We follow a similar construction to show that if the efficient allocation is unique, there exist competitive equilibrium prices under which each agent strictly demands her equilibrium set of trades. Note, that this result is independent of the trading network structure, and is a byproduct of strict complementarity in optimization.

for any outcome $X \subsetneq T$ we have

$$\begin{aligned}\sum_i \bar{w}_i(T \cap \delta(i)) &= \sum_i (w_i(T \cap \delta(i)) - \Delta|T \cap \delta(i)|) = \sum_i w_i(T \cap \delta(i)) - 2\Delta|T| \\ &> \sum_i w_i(X \cap \delta(i)) - 2\Delta|X| = \sum_i \bar{w}_i(X \cap \delta(i)).\end{aligned} \tag{4}$$

Thus, it follows that executing all trades in T is still the unique welfare maximizing outcome in this economy. Denote a competitive equilibrium of this economy by (T, p^T).

We claim that (T, p^T) is a competitive equilibrium of the economy with valuations $\{w_i\}$, where $D_i(p^T) = \{T \cap \delta(i)\}$. This is because, if a set of trades $T \cap \delta(i)$ is demanded in the economy with payoffs $\{\bar{w}_i\}$, for any $S \subsetneq T \cap \delta(i)$ we have

$$u_i(T \cap \delta(i), p^T) - \Delta|T \cap \delta(i)| = \bar{u}_i(T \cap \delta(i), p^T) \quad \geq \bar{u}_i(S, p^T) = u_i(S, p^T) - \Delta|S|, \tag{5}$$

where $\bar{u}_i$ is the utility function associated with $\bar{w}_i$. This implies that the surplus of agent i for the trades in $T \cap \delta(i)$ is strictly greater than her surplus for any set of trades $S \subsetneq T \cap \delta(i)$.

Thus, we conclude that in the economy with valuations $\{w_i\}$ given a set of contracts (T, p^T) we have $D^i(p^T) = \{T \cap \delta(i)\}$. Hence, it follows that (T, p^T) is a blocking chain for the outcome $\emptyset$. □

To study stability of outcomes other than the no trade outcome, we introduce the notion of *the contraction of an economy* (see [Hatfield et al. 2013]). For an outcome X, and associated prices p^X, we define a new trading network $G^X = (N, E \setminus X)$ where agent $i \in N$ has a valuation function $\hat{w}_i : 2^{\delta(i) \cap (E \setminus X)} \to \mathbb{R}$ given as follows:

$$\hat{w}_i(S) = \max_{Y \subset X \cap \delta(i)} [w_i(S \cup Y) + \sum_{e \in Y \cap \delta_+(i)} p_e^X - \sum_{e \in Y \cap \delta_-(i)} p_e^X] \tag{6}$$

It follows from [Murota 2003] (Theorem 6.13(3,4), Theorem 6.15(1)) that $\hat{w}_i$ is $M^\natural$-concave for each $i \in N$. We refer to G^X with these valuation functions as the contraction of economy X, with respect to (X, p^X).

5.1. Equivalence of Solution Concepts

We show that all of the stability notions coincide with the concept of a competitive equilibrium in the following two steps (established in Theorems 5.7 and 5.8 respectively):

(1) A competitive equilibrium outcome is a (strongly) stable outcome.
(2) A chain stable outcome is a competitive equilibrium outcome.

The first result follows from the definition of stability, while the second follows from Lemma 5.5 and Corollary 5.6 which exploit the network flow formulation.

THEOREM 5.7. *(Theorem 5 in [Hatfield et al. 2013]) If (X, p) is a competitive equilibrium and p^X is the restriction of p to the arcs in X, then, (X, p^X) is a (strongly) stable outcome.*

PROOF. Since (X, p) is a competitive equilibrium, it follows that (X, p^X) is individually rational. To complete the proof, it suffices to show that there is no chain that (strongly) blocks (X, p^X). Let G^X be the contraction with respect to (X, p^X). Since (X, p) is a competitive equilibrium in G, $(\emptyset, p^{(E \setminus X)})$ is a competitive equilibrium in G^X. Theorem 4.2 implies that $\emptyset$ is an efficient outcome in G^X. Suppose, for a contradiction, there exist contracts and prices (Z, p^Z) that (strongly) block (X, p^X) in G. This would imply that Z has higher welfare than $\emptyset$ in G^X, which contradicts the efficiency of $\emptyset$. □

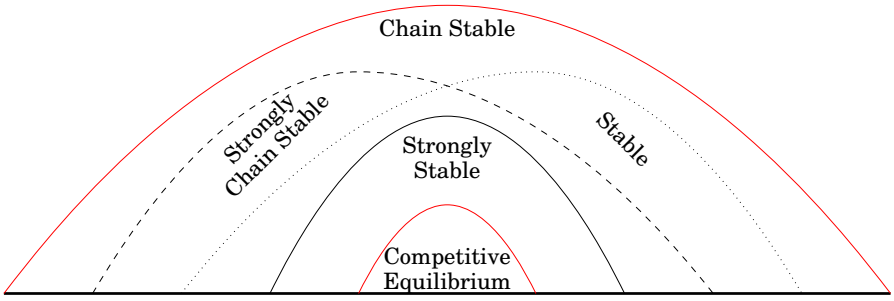

Fig. 3. Outcomes Hierarchy

As any (strongly) stable outcome is (strongly) chain stable, Theorem 5.7 implies the hierarchy displayed in Figure 3.[6]

Next, we establish the equivalence of all the stability notions, by showing that in any chain stable outcome (X, p^X), it is possible to find prices for trades $E \setminus X$ to support X as a competitive equilibrium. Thus, the "weakest" and "strongest" equilibrium/stability notions in Figure 3 are equivalent.

THEOREM 5.8. *Suppose that (X, p^X) is a chain stable outcome. Then, there exists a price vector $p \in \mathbb{R}^E$ such that (X, p) is a competitive equilibrium where $p_e = p_e^X$ for all $e \in X$.*

PROOF. Consider the contraction G^X of the trading network $G = (N, E)$ with respect to (X, p^X). We claim that for some price vector $\hat{p}^{E\setminus X}$, $(\emptyset, \hat{p}^{E\setminus X})$ is a competitive equilibrium in G^X. Assume not, then, it follows from Theorems 4.1 and 4.3 that $\emptyset$ is not welfare maximizing in G^X. Then, Corollary 5.6 implies that this outcome is not chain stable, and there exists a set of trades $T \subset E \setminus X$, and prices p^T that constitute a blocking chain in G^X. This implies that (T, p^T) also blocks (X, p^X) in the original trading network G. Thus, we obtain a contradiction, and it follows that $(\emptyset, \hat{p}^{E\setminus X})$ is a competitive equilibrium in G^X. Since (X, p^X) is chain stable and hence individually rational, this implies that in trading network G, under prices $(p^X, \hat{p}^{E\setminus X})$, each agent i demands $X \cap \delta(i)$. Hence, this outcome corresponds to a competitive equilibrium, and the claim follows. □

5.2. Determining a Blocking Chain

A pseudo-polynomial algorithm for solving the MSFP is given in [Iwata et al. 2005] which corresponds to the Successive Shortest Path (SSP) algorithm on auxiliary networks. In our MSFP reduction, we can assign each arc of the flow network unit capacity. Therefore, the SSP algorithm is strongly polynomial in our case which gives a strongly polynomial algorithm for determining an efficient allocation in a trading network. Supporting prices can be found by applying the Bellman-Ford algorithm to the auxiliary graph associated with the efficient outcome.[7] This is because, as discussed earlier, a solution of (2,3) corresponds to competitive prices, and can be interpreted as the dual to the problem of finding all-pairs shortest paths. Since the number of vertices in the auxiliary network are polynomial in the number of trades in the trading

[6]Notice, a competitive equilibrium lies in a different space (since unlike the notions of stability, at an equilibrium all trades are priced). Thus, in Figure 3, with slight abuse, when we refer to a competitive equilibrium we refer to the equilibrium outcome along with the prices of the trades involved in this outcome.

[7]Optimal potentials, and hence competitive prices, can also be extracted from the SSP algorithm.

network, this approach yields a strongly polynomial algorithm for finding a competitive equilibrium.

We now describe a polynomial time algorithm to determine a blocking chain for a given outcome or to certify that none exists. Clearly, this algorithm can be used to check if an outcome is chain stable. Fix an outcome X along with associated prices p^X. To test chain stability, one should first verify individually rationality, and then examine if the outcome is blocked by a chain.

To verify individual rationality of the outcome $X \cap \delta(i)$ for each agent i it suffices to compare $u_i(X \cap \delta(i), p^X)$ with $u_i((X \cap \delta(i)) \setminus \{e\}, p^X)$ for all $e \in X \cap \delta(i)$, and $u_i((X \cap \delta(i) \setminus \{e_1, e_2\}), p^X)$ for all $e_1 \in X \cap \delta_+(i)$, $e_2 \in X \cap \delta_-(i)$. This follows from the fact that $u_i(X \cap \delta(i), p^X)$ is $M^\natural$-concave and Theorem 6.26 in [Murota 2003]. Thus, individual rationality can be checked by comparing X with polynomially many sets of trades incident to agent i.

Assume that outcome X along with prices p^X is individually rational. Then, Algorithm 1 (see Appendix) can be applied to test if the outcome is unblocked, and provide a blocking chain in case it is blocked.

Algorithm 1 proceeds in two phases. Phase one focuses on the contraction G^X (with respect to (X, p^X), corresponding valuations $\{\hat{w}_i\}_{i \in N}$ (see (6)), and outcome $\emptyset$. It uses the auxiliary graph associated with G^X, and finds a minimal welfare-improving chain T if one exists. If none exists, then there is no blocking chain. Otherwise, the second phase returns the prices p^T which together with T constitute a blocking chain for outcome $\emptyset$ in G^X (and equivalently outcome (X, p^X) in G). In order to do so, it works on the auxiliary graph associated with $\hat{G} = (N, T)$, valuations $\{\bar{w}_i\}_{i \in N}$, and outcome T, as described in Corollary 5.6.

Algorithm 1 does not require computing the valuations $\{\hat{w}_i\}_{i \in N}$ explicitly. It suffices to determine enough information to compute the edge costs of the arcs $C^{aux}(x, y)$ of the auxiliary graphs used in the two steps of the algorithm: $c^{aux}_{(u,v)} = \hat{w}'_i(y^i) - \hat{w}'_i(y^i - \chi^u + \chi^v)$, where $\hat{w}_i{}'$ is the M-concave function associated with $\hat{w}_i$. The definition of $\{\hat{w}_i\}_{i \in N}$ in (6) and M-concavity of w_i implies that $\hat{w}_i(y)$ can be computed via a greedy algorithm in strongly polynomial time ([Shioura 2004]).[8]

The main components of the algorithm is the structure of the auxiliary graph, and three functions $\mathrm{aux.construct}()$, $\mathrm{greedy_X}()$, $\mathrm{BellmanFord}()$. Each auxiliary graph has at most $O(|N| + |E|)$ vertices and $O(|E| + |N|^2)$ arcs. Initially, $\mathrm{aux.construct}(G, (x, y))$ constructs the auxiliary graph associated with graph G and flow (x, y). Next, $\mathrm{greedy_X}()$ is used to assign costs to all arcs $C^{aux}(x, y)$. Recall that obtaining these costs corresponds to solving an M-concave maximization problem, associated with functions $\{\hat{w}_i\}$. Specifically, $\mathrm{greedy_X}(w_i, y^i)$ computes $\{\hat{w}_i\}$ as described in the paragraph above. $\mathrm{BellmanFord}(G^{aux}(x, y), s)$ computes shortest path distances on $G^{aux}(x, y)$ from vertex s, which provide the prices p^T for the blocking chain T.

Phase one of Algorithm 1 finds a chain of arcs improving the total welfare of the agents involved. It works on G^X, and the auxiliary network associated with it, valuations $\{\hat{w}_i\}_{i \in N}$, and outcome $\emptyset$. A negative cycle with the fewest number of arcs on the auxiliary network, implies a blocking chain T (see Lemma 5.5). To find such a negative cycle, it suffices to compute W, where $W[u, v, m]$ stands for the shortest distance from vertex u to vertex v going through at most m arcs. The negative cycle with the fewest number of arcs is given by the diagonal element $W[u, u, m]$ which is negative, for the smallest possible m. Note here that the negative cycle can be found by adding a matrix of predecessors during the computation of W.

[8] In our case this greedy algorithm is strongly polynomial, since $\mathrm{dom} f \subset \{-1, 0, 1\}^V$.

In phase two we know the blocking chain T but the prices p^T have yet to be determined. Following the approach in Corollary 5.6, and focusing on $\hat{G}$, the algorithm first perturbs the valuation functions $\{\hat{w}_i\}_{i\in N}$ by $\Delta > 0$ (specified in the algorithm). As in Corollary 5.6, a competitive equilibrium of this economy, leads to prices p^T such that under these prices in the original economy each agent i strictly demands $T \cap \delta(i)$, thereby establishing that (T, p^T) is a blocking chain for the outcome (X, p^X) in G.

The competitive prices are given by legitimate potentials on $G^{aux}(x, y)$, which can be computed by the Bellman-Ford algorithm. Distances are computed starting from a special vertex in order to guarantee that the special vertices get zero potential. If some vertices are not reachable from the special vertices. Then, the algorithm picks a random vertex and computes shortest distances from such a vertex. Finally, after computing legitimate potentials, the prices are extracted.

Algorithm 1 is strongly polynomial. During the construction of the auxiliary graphs, $\mathrm{greedy_X}()$ is used for each of at most $|V|^2$ arcs in $C^{aux}(x, y)$. Since the M-concave function maximization problem associated with $\mathrm{greedy_X}()$ has complexity, $O(|X|^2)$, the overall complexity of this step is bounded by $O(|V|^2|X|^2)$, which is bounded by $O((|E| + |N|)^2|E|^2)$. Furthermore, in the first step of the algorithm, the computation of matrix W takes $O(|V|^4) = O((|N| + |E|)^4)$ steps. The complexity of this step drops down to $O(|V|^3 \log |V|)$, if binary search on the cycle length is incorporated. Since, $G^{aux}(x, y)$ involves $O(|N| + |E|)$ vertices and $O(|E| + |N|^2)$ arcs, the complexity associated with constructing the prices using Bellman-Ford algorithm is bounded by $O(|N|^3 + |E|^2 + |N|^2|E| + |N||E|)$.

ACKNOWLEDGMENTS

The first author would like to thank Alexandru Nichifor, and Scott Kominers for helpful discussions.

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APPENDIX

A.1. Proof of Theorem 4.6

As before, we introduce a flow network $G' = (V, A)$, associated with the trading network G. Recall that we assign to each agent an M-concave function w_i' with each argument capturing the net flow into a vertex $u \in V^i$. Let $V_Y \subset V^i$ be the vertices associated with the trades in Y. Merge all vertices in V_Y into a vertex v_Y. Make each arc incident to vertex in V_Y incident to v_Y. We obtain a new set of vertices associated with agent i given by $V_m^i = (V^i \setminus V_Y) \cup \{v_Y\}$. We define a new function $\hat{w}_i : \mathbb{Z} \times \mathbb{Z}^{\delta(i)\setminus Y} \times \mathbb{Z} \to \mathbb{R}$ on the vertices V_m^i, and so we do

$$\hat{w}_i(y^i_{(\{0\}\cup\delta(i))\setminus Y}, y_Y) = \sup\{w_i'(y^i_{(\{0\}\cup\delta(i))\setminus Y}, z) | \sum_{e\in Y} z_e = y_Y\}$$

The function $\hat{w}_i$ is said to be generated by aggregation of the original M-concave function w_i'. Aggregation preserves M-concavity (see Theorem 6.13 in [Murota 2003]).

The instance of the MSFP constructed in this way preserves the equivalence between flows and outcomes as argued earlier. This is true because the function w_i' depends only on the net flow into vertices V_y, as the valuation function depends only on the number of trades in Y.

The theorem follows from the equivalence between potentials and competitive prices. Recall that the potentials are defined on vertices, and we imposed an restriction on the potentials on vertices V_Y to be equal, by merging the vertices. As argued before, Theorem 3.1(3b) implies that potentials p on the auxiliary graph associated with the optimal solution are supporting prices. Since there exists a unique potential value $p(v_Y)$ for the vertices V_Y associated with Y, "uniform pricing" follows.

A.2. Algorithm in Section 5.2

ALGORITHM 1: Determining a Blocking Chain

Input: Trading network $G = (N, E)$, valuations $\{w_i\}_{i \in N}$, and outcome X with prices p^X.
Output: Blocking chain T with prices p^T.

$G = G^X$; $(x, y) = \emptyset$; $G^{aux}(x, y) = \text{aux.construct}(G, (x, y))$;
foreach $i \in N$ **do** **foreach** $u \in V^i$ **do** **foreach** $v \in V^i$ **do**
$c^{aux}_{(u,v)}(x, y) = \text{greedy}_{\text{X}}(w_i, y^i) - \text{greedy}_{\text{X}}(w_i, y^i - \chi^u + \chi^v)$;
foreach $i \in V$ **do** **foreach** $j \in V$ **do** $W[u, v, 1] = c^{aux}_{(u,v)}(x, y)$;
for $m = 2 \to |V|$ **do**
  foreach $u \in V$ **do**
    foreach $v \in V$ **do**
      $W[u, v, m] = W[u, v, m - 1]$;
      foreach $t \in V$ **do**
        $W[u, v, m] = \min\{W[u, v, m], W[u, t, m - 1] + c^{aux}_{(t,v)}(x, y)\}$;
      end
    end
  end
end
Find smallest m such that $W[u, u, m] < 0$ for some u and negative cycle K from matrix of predecessors;
if not found **then return** No Blocking Chain;
else $T = K \cap A^{aux}(x, y)$;

$G = (N, T)$; $(x, y) = T$; $G^{aux}(x, y) = \text{aux.construct}(G, (x, y))$;
foreach $i \in N$ **do** **foreach** $u \in V^i$ **do** **foreach** $v \in V^i$ **do**
$c^{aux}_{(u,v)}(x, y) = \text{greedy}_{\text{X}}(w_i, y^i) - \text{greedy}_{\text{X}}(w_i, y^i - \chi^u + \chi^v)$;
$\Delta = \frac{\sum_i \text{greedy}_{\text{X}}(w_i, y^i) - \sum_i \text{greedy}_{\text{X}}(w_i, \emptyset))}{4|T|}$;
foreach $a \in C^{aux}(x, y)$ **do**
  if a involves a special vertex **then**
    $c^{aux}_a(x, y) = c^{aux}_a(x, y) - \Delta$;
  else
    $c^{aux}_a(x, y) = c^{aux}_a(x, y) - 2\Delta$;
  end
end
Pick a random special vertex s in $G^{aux}(x, y)$;
$d = \text{BellmanFord}(G^{aux}(x, y), s)$;
foreach $e \in T$ **do** $p^T[e] = d[v^{e^+}_e]$;
return (T, p^T);

Remark:. In the second phase, the algorithm focuses on network $\hat{G} = (N, T)$, valuations $\{\hat{w}_i\}_{i \in N}$, and outcome T, as described in Corollary 5.6. Let $\{\bar{w}_i(Z) = w_i(Z) - \Delta|Z|\}$ denote valuations obtained after a perturbation by Δ. The perturbation in the payoffs, corresponds to perturbing the cost of arcs in $C^{aux}(x, y)$. The new cost for an arc $(u, v) \in C^{aux}(x, y)$, owned by agent i and involving a special vertex, is given by

$$\begin{aligned} \bar{f}(y - \chi^u + \chi^v) - \bar{f}(y) &= \bar{w}'_i(y^i) - \bar{w}'_i(y^i - \chi^u + \chi^v) \\ &= w'_i(y^i) - \Delta|T| - (w'_i(y^i - \chi^u + \chi^v) - \Delta(|T| - 1)) \\ &= c^{aux}_{(u,v)}(x, y) - \Delta, \end{aligned} \tag{7}$$

where $\bar{w}_i'$ is the M-concave function associated with $\bar{w}_i$, and $\{c^{aux}_{(u,v)}(x,y)\}$ denote the costs associated with C^{aux} before perturbation of the payoffs. An analogous result ($\bar{f}(y-\chi^u+\chi^v)-\bar{f}(y)=c^{aux}_{(u,v)}(x,y)-2\Delta$) follows in the case where the arc does not involve a special vertex. Phase two relies on these edge costs, to compute node potentials (through $\mathrm{BellmanFord}$). These potentials correspond to competitive equilibrium prices in $\hat{G}=(N,T)$ with new valuations, which by Corollary 5.6 are also equivalent to the prices p^T of the blocking chain.