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## Estimation of financial agent-based models with simulated maximum likelihood

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# FinMaP-Working Paper No.63



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PERFORMANCE: EXPECTATIONS, CONSTRAINTS  
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TITLE

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## ABSTRACT

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*Keywords: heterogeneous agent model, simulated maximum likelihood, estimation, intensity of choice, switching.*

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# Estimation of Financial Agent-Based Models with Simulated Maximum Likelihood<sup>☆</sup>

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## Abstract

This paper proposes a general computational framework for empirical estimation of financial agent based models, for which criterion functions do not have known analytical form. For this purpose, we adapt a nonparametric simulated maximum likelihood estimation based on kernel methods. Employing one of the most widely analysed heterogeneous agent models in the literature developed by [Brock and Hommes \(1998\)](#), we extensively test properties of the proposed estimator and its ability to recover parameters consistently and efficiently using simulations. Key empirical findings point us to the statistical insignificance of the switching coefficient but markedly significant belief parameters defining heterogeneous trading regimes with superiority of trend-following over contrarian strategies. In addition, we document slight proportional dominance of fundamentalists over trend following chartists in main world markets.

*JEL: C14, C51, C63, D84, G02, G12*

*Keywords: heterogeneous agent model, simulated maximum likelihood, estimation, intensity of choice, switching*

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## 1. Introduction

After the failure of traditional financial models in the deep financial crisis of 2007–2008, the agent-based approaches in finance have attracted attention of both academicians as well as practitioners and hence gradually replace traditional financial models in the recent literature. This advancement emphasises that although the serious macroeconomic consequences of market fluctuations are worldwide, the essence of problems remains at the level of individual market agents with

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their heterogeneous expectations. The financial agent-based models (FABMs)<sup>1</sup> reflect this well documented and systematic human departure from the representative agent’s full rationality towards reasonably realistic bounded, limited rationality (Simon, 1957). An essential achievement of this field is the ability to replicate the stylised facts of financial data and account for emergence of asset market bubbles followed by sudden crashes. Recently, number of projects propose a courageous attempt to complement or even alternate current mainstream policy making approaches through the use of agent-based approaches, typically at the level of central banks. For this to happen, it is, however, essential to be able to estimate these models on the empirical data.

Although the empirical estimation is an important part of the modelling cycle and is crucial for model validation, one cannot find many attempts on empirical estimation of FABMs. Moreover, looking a decade back in the financial literature, we neither observe any general consensus on the estimation methodology, nor conclusive results. Fagiolo et al. (2007, pg. 202) even emphasise “no consensus at all about how (and if) agent-based models should be empirically validated”. Generally, there are two essential challenges in the estimation. First, a highly nonlinear and complex nature of these systems prohibits researchers of using classical estimation methods as the objective function often has no analytical expression. Second, a possible overparametrisation, high number of degrees of freedom, and optional model settings together with the stochastic dynamics further escalate the complexity of the problem. The emerging properties of these models cannot be analytically deduced, a possible solution via method of moments, “while fine in theory, might be too computationally costly to undertake” (LeBaron and Tesfatsion, 2008, pg. 249), and thus a considerable simulation capacity for the numerical analysis is required.

Literature focusing on estimation of FABMs attempted to use several direct and indirect estimation methods (see Section 2 for detailed review). In terms of direct estimation, the nonlinear least squares and quasi maximum likelihood are applied in most of the cases instead of the classical ordinary least squares or maximum likelihood due to complexity of the models. In these applications, key structural features of agent-based models, e.g. the evolutionary switching between trading strategies, are sometimes restrained to obtain simplified approach which can be estimated directly. However, for many models the aggregation equation, which would contain all parameters of interest, cannot be derived analytically and therefore the application of direct estimation techniques is not feasible. Indirect estimation based on simulating artificial data from the model through which the aggregation concepts such as moments for the method of simulated moments are derived is then used instead. These simulation-based methods “are very applicable and may dramatically open the empirical accessibility of agent-based models in the future” as noted by Chen et al. (2012, pg. 204). Simulation-based methods already used for the estimation include the method of simulated moments, or generally the simulated minimum distance (Grazzini and Richiardi, 2015). All these methods are based on minimising the (weighted) distance between two sets of simulated and observed moments. So far, however, the use of simulation-based methods for validation of agent-based models in economics is relatively rare. Moreover, as pointed out by Fernández-Villaverde and Rubio-Ramírez (2010, pg. 23), the main difficulty remains in choosing set of proper moments characterising the parameters, as choice of different moments may lead to rather different point estimates. Grazzini and Richiardi (2015, pg. 151) further note that although more complex, the simulation-based maximum likelihood estimation could be used instead, which has not been explored by the literature yet.

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<sup>1</sup>For a general overview of the financial agent-based modelling and its development, Chen et al. (2012), Hommes (2006), or LeBaron (2006) provide excellent surveys.

This paper makes a step forward and proposes more general computational framework for empirical validation of full-fledged FABMs based on a recently developed non-parametric simulated maximum likelihood estimator (NPSMLE) by [Kristensen and Shin \(2012\)](#). The main advantage of this framework is that under general conditions met by FABMs, it is able to approximate the conditional density of the data-generating process from numerically simulated observations. Thus, the unknown likelihood function can be replaced by the simulated likelihood in the estimation, and parameters can be recovered in a traditional way. We extensively test capability of the method for the FABMs estimation purposes via large Monte Carlo study. We apply the methodology to the most widely analysed model of [Brock and Hommes \(1998\)](#) for which we customise the estimation. The key feature of the model is an evolutionary switching of agents between simple trading strategies based on past realised profits, so called adaptive belief system, governed by the switching parameter of the intensity of choice. This parameter is responsible for high nonlinearity of the system and even possibly chaotic price motion. We presuppose that if NPSMLE succeeds in estimation of the switching parameter in this generally challenging framework, it is likely to appear more general and useful for other agent-based models in the future.

## 2. A route to empirical estimation of FABMs: Review of existing approaches

We start the discussion with reviewing of the main methods and results of the studies attempting to estimate the FABMs. Design of these models is to a large extent motivated by empirical evidence on behaviour of real financial agents accumulated at the turn of 1980s and 1990s ([Allen and Taylor, 1990](#); [Frankel and Froot, 1990](#)). These studies conclude that interactions of the two main types of expectations govern the dynamics of financial markets. Fundamental traders, who believe that possible mispricing is likely to be corrected over short periods by arbitrageurs and that the market price thus tends to revert to its fundamental value, characterise a stabilising market force. Technical analysts, also called ‘noise traders’ or chartists, who believe that a currently observable trend will continue also in the short-run, constitute a destabilising market force responsible for emergence of speculative bubbles. These types of traders might be rather understood as possible types of available trading strategies as an intelligent market agent is likely to adapt his or her strategy over time based on its relative historical performance. The time-varying evolution of market fraction between these two types of trading strategies is thus an essence of many artificial markets. In the seminal [Brock and Hommes \(1998\)](#) discrete-choice FABM this concept is embodied via switching parameter of the intensity of choice defining the overall willingness of market agents to switch between potential trading strategies. ‘N-type models’ consisting of fundamentalists and few types of chartistic strategies, in which the autonomy of agents is constrained by a predetermined class of beliefs, have been found successful in mimicking many financial stylised facts ([Chen et al., 2012](#)).

By virtue of a relatively simple design, especially the 2-type and 3-type versions have been subject of empirical estimation so far and therefore occur in the spotlight of this paper. Compared to so called autonomous agents models, for which a typical representative is the ‘Santa Fe Artificial Stock Market Model’ ([Holland and Miller, 1991](#); [Palmer et al., 1994](#)) with rich system complexity, advanced individual learning, and idiosyncratic distributions of uncertainty, the heterogeneity in simple ‘N-type models’ is restrained. First, the agents are in fact homogeneous ex ante, but randomness of their stochastic choices gives rise to evolving heterogeneity in time. Second, the evolutionary selection of strategies is based only on social, not individual learning. Especially in case of the 2-type and 3-type models we therefore believe that the phenomenon of supra-imposed

artificial heterogeneity does not play an important role, actually, researchers rather balance on the other side of the problem when designing relatively trivial models that account only for the most robust heterogeneous features of real markets.

Tables 1 and 2 summarise attempts to statistically estimate the parameters of various FABMs in the literature. Looking at the few studies representing circa fifteen years of research, we cannot observe any general consensus on the estimation methodology, nor conclusive results. With this respect, our summary complements the conclusions of Fagiolo et al. (2007, pg. 199, 202). As depicted in the third column of Tables 1 and 2, the three estimation methods: nonlinear least squares, quasi maximum likelihood, and the method of simulated moments, prevail among others. When moving to the fourth column of Tables 1 and 2, we can see how the choice of estimated parameters is affected by various model designs. Nonetheless, we can observe a general tendency to estimate mainly parameters related to behavioural rules of agents: belief coefficients defining individual trading strategies and the intensity of choice or its corresponding concepts such as mutation, herding tendency, and switching thresholds. These parameters are unambiguously important for the economic interpretation of given models.

Note that the estimated intensity of choice from Tables 1 and 2, however, cannot be directly compared across various models, assets, or time periods. It is a unit-free variable and its magnitude is conditional on the specific model design or the specific dataset. On the other hand, the intensity of choice is a crucial and very robust driver of the data generating process behind switching FABMs and to a large extent determines the behaviour of the system in a very consistent manner: zero intensity of choice fixes market fractions and does not allow for any evolutionary switching, high values implicate a wild switching for vast majority of model specifications, assets, or periods. Relatively small positive intensity of choice is associated with a presence of some detectable behavioural switching.

First, as depicted in Tables 1 and 2, we observe a strong dominance of models derived from the adaptive belief system: 28 from 43 models to some extent follow the tradition of the Brock and Hommes (1998) original framework, seven is based on interactive agent hypothesis (Lux, 1995) and only three on the ant type of system (Kirman, 1993). These numbers strongly reflect conclusions of Chen et al. (2012, pg. 207) stating that “the Lux model was rejected, similar to the rejection of the ant type of model” based on its empirical validation in favour of the adaptive belief system that seems feasible, however challenging, for empirical estimation. Indeed, there are only four new empirical studies in Table 2 based on the interactive agent hypothesis and none derived from the ant type. The strong dominance of the adaptive belief system in the recent literature is the principal reason why we analyse the original Brock and Hommes (1998) model later in this study.

Second, we might observe some regularities in the use of particular estimation methods. Ant type models are estimated solely using the method of simulated moments, nonetheless, all three works are considerably related and elaborated by the same group of authors. The three original interactive agent hypothesis models from Table 1 are intentionally designed so that authors are able to theoretically derive the likelihood function and use maximum likelihood, while the four ‘new’ models are estimated solely via simulations based on the method of moments. On the other hand, all adaptive belief system models are estimated via relatively less complicated direct techniques: nonlinear least squares and quasi maximum likelihood or in some specific simplified cases even by ordinary least squares and maximum likelihood.

Third, in terms of estimated parameters, authors dominantly make efforts to minimise the set of estimated coefficients (see the ‘#’ column in Tables 1, 11) to a small number of the most

Table 1: Estimation methods of FABMs I.

Models	Origin	Methods	Parameters estimated	#	Data	Type	Fit	IOC
Alfarano et al. (2005)	IAH	ML	Herding tendency	2	d:5034–9761 o.	s,fx,g	-	-
Alfarano et al. (2006)	IAH	ML	Herding tendency	2	d:5495,6523 o.	s,fx	-	-
Alfarano et al. (2007)	IAH	ML	Herding tendency	2	d:1975–2001	s	-	-
Amilon (2008)	ABS	EMM/ML	Intensity of choice <sup>a</sup>	15	d:1980–2000	s	$p-v=0\%$	1.99(i),1.91(s)
Boswijk et al. (2007)	ABS	NLS	Belief coefficients/Intensity of choice	3	a:132 o.	s	$R^2=.82$	10.29(i),7.54(i)
de Jong et al. (2009b)	ABS	NLS	Belief coefficients/Intensity of choice	5	w:102 o.	fx	$adj R^2=.14$	1.52(i)
de Jong et al. (2010)	ABS	QML	Belief coefficients/Intensity of choice	7	m:238 o.	fx	-	.0007(i)–6.29(s)
Diks and Weide (2005)	ABS	ML	(G)ARCH relations/Sign of MA(1) c.	3	d:3914 o.	fx	-	-
Ecemis et al. (2005)	AA	IEC	Market fractions/Behavioural rules	3	-	s	-	-
Gilli and Winker (2003)	ANT	MSM	Mutation/Conviction rate	3	d:1991–2000	fx	NA	-
Manzan and Westerhoff (2007)	ABS	OLS	Reaction c./Switching threshold	4	m:1/74–12/98	fx	NA	-
Reitz and Westerhoff (2007)	ABS	QML	Behavioural rules/Intensity of choice	6	m:365 o.	c	-	.17(s)–.47(s)
Westerhoff and Reitz (2003)	ABS	QML	Behavioural rules/Intensity of choice	7	d:4431 o.	fx	-	.02(s)–.17(s)
Winker and Gilli (2001)	ANT	MSM	Mutation/Conviction rate	2	d:1991–2000	fx	NA	-
Winker et al. (2007)	ANT	MSM	Mutation/Conviction rate	3	d:1991–2000	fx	$p-v<1\%$ <sup>b</sup>	-

*Note:* The Table is adopted from [Chen et al. \(2012, pg. 203\)](#) and amended by authors. Authors of articles are alphabetised. The full meaning of the acronyms under ‘Origin’: AA stands for autonomous agents, ABS for adaptive belief system, ANT for the ant type of system, and IAH for interactive agent hypothesis. The full meaning of the acronyms under ‘Methods’: ML stands for maximum likelihood, EMM for efficient method of moments, NLS for nonlinear least squares, QML for quasi maximum likelihood, OLS for ordinary least squares, IEC for interactive evolutionary computation, and MSM (SMM) for method of simulated moments. ‘#’ displays total number of estimated parameters; ‘Data’ describes data frequency: ‘d/w/m/q/a’ for daily/weekly/monthly/quarterly/annual, and number of observations (where no specific figure is provided, we report starting and final years); ‘Type’ shows the type of data: ‘s/fx/c/g/re’ for stock markets/FX/commodity markets/gold/real estate; ‘Fit’ reports the statistical fit of the estimation ( $R^2$ , its alternatives, p-value of the J-test of overidentifying restrictions to accept the model as a possible data generating process); and where relevant, ‘|IOC|’ displays the absolute estimated value of the switching parameter of the intensity of choice (see [Equation 17](#)), furthermore ‘s’/‘i’ denotes its statistical significance/insignificance at 5% level. Figures are rounded to 2 decimal digits.

<sup>a</sup>[Chen et al. \(2012\)](#) do not report other important parameters estimated: belief coefficients, intensities of exogenous noises, risk aversion, information costs for fundamentalists, forgetting factors, and memory in the fitness measure.

<sup>b</sup>While p-val for GARCH(1,1) model > 5%.



Table 2: Estimation methods of FABMs II. a).

Models	Origin	Methods	Parameters estimated
Barunik and Vosvrda (2009)	Cusp	ML	Asymmetry and bifurcation factors/Location and scale coefficient
Barunik and Kukacka (2015)	Cusp	RV/ML	Asymmetry and bifurcation factors/Polynomial data approximation
Bolt et al. (2011)	ABS	NLS(?)	Expectations' bias/Discount factor/Belief coefficients/Intensity of choice
Bolt et al. (2014)	ABS	NLS	Belief coefficients/A-synchronous updating ratio/Intensity of choice
Cornea et al. (2013)	ABS	VAR/NLS	Fundamentalists' belief coefficient/Intensity of choice
Chen and Lux (2015)	IAH	MSM	Standard deviation of innovations/Herding tendency
Chiarella et al. (2014)	ABS	QML	Belief & market maker coefficients/Memory decay rate/Intensity of choice
Chiarella et al. (2015)	ABS	QML	Belief coefficients/Variance risk premium/Intensity of choice
de Jong et al. (2009a)	ABS	ML	Belief coefficients/Intensity of choice
ter Ellen and Zwinkels (2010)	ABS	QML	Belief coefficients/Intensity of choice
ter Ellen et al. (2013)	ABS	OLS/NLS	Behavioural rules/Intensity of choice
Franke (2009)	ABS	MSM	Reaction coefficients/Switching threshold
Frijns et al. (2010)	ABS	EMS	Local volatility/Belief coefficients/Intensity of choice
Franke and Westerhoff (2011)	IAH	MSM	Behavioural rules/Flexibility/Predisposition coefficients
Franke and Westerhoff (2012)	ABS/IAH	MSM	Behavioural rules/Wealth/Predisposition/Misalignment coefficients
Ghoshadze and Lux (2015)	IAH	GMM	Standard deviation of innovations/Herding tendency
Grazzini et al. (2013)	Bass (1969)	ML, MSM	Probability of independent adoption/Peer pressure/Population size
Grazzini and Richiardi (2015)	-	SMD	-
Goldbaum and Zwinkels (2014)	ABS	OLS (iterative)	Belief coefficients
Hommes and Veld (2015)	ABS	NLS(?)	Belief coefficients/A-synchronous updating ratio(?)/Intensity of choice
Huisman et al. (2010)	ABS	QML	Belief coefficients/Intensity of choice
Kouwenberg and Zwinkels (2014)	ABS	QML	Belief coefficients/Intensity of choice
Kouwenberg and Zwinkels (2015)	ABS	QML	Price elasticity/Belief coefficients/Intensity of choice
Lof (2012)	ABS	NLS	Belief coefficients/Intensity of choice
Lof (2015)	ABS	VAR/NLS	Discount factors/Belief coefficients/Intensity of choice
Reitz and Slopek (2009)	ABS	QML	GARCH coefficients/Belief coefficients/Transition parameter
Recchioni et al. (2015)	ABS	calibration	Belief coefficient/Intensity of choice/Risk aversion/Fundamental value/Memory
Verschoor and Zwinkels (2013)	ABS	ML	Belief coefficients/Intensity of choice

*Note:* The Table follows the logic of Table 1 and summarises recent research not covered there. Authors of articles are alphabetised. The full meaning of the acronyms under 'Origin': Cusp stands for the cusp catastrophe model, ABS for adaptive belief system, and IAH for interactive agent hypothesis. The full meaning of the acronyms under 'Methods': ML stands for maximum likelihood, RV for realised volatility, NLS for nonlinear least squares, VAR for vector autoregression, MSM (SMM) for method of simulated moments, QML for quasi maximum likelihood, OLS for ordinary least squares, EMS for empirical martingale simulation by Duan and Simonato (1998), GMM for generalized method of moments, and SMD for simulated minimum distance. '?' means that given information is unclear to authors.

relevant ones governing dynamics of the model and determining heterogeneous behaviour of agents: in 22 cases, four or less coefficients are estimated and only nine works consider seven or more parameters. For the adaptive belief system models, belief coefficients are always estimated and where relevant, authors always pay specific attention to the sign and statistical significance of the switching coefficient of the intensity of choice. This is a justified approach and as aptly summarised by [Chen et al. \(2012, pg. 202\)](#) w.r.t. simulation studies: “supposing that we are given the significance of the intensity of choice in generating some stylized facts, then the next legitimate question will be: can this intensity be empirically determined, and if so, how big or how small is it?”.

Another interesting aspect is hidden in the utilised datasets (columns ‘Data’). Almost one-half of studies, 20 out of 43, use daily datasets providing thousands of observations. This is an important aspect both from the statistical point of view, as datasets are long enough for sound statistical inference, as well as from the viewpoint of model stability because number of periods is generally sufficiently long for the model dynamics to stabilise. Especially for simulation-based methods the stabilisation period deserves adequate consideration. For instance in our Monte Carlo study (see [Section 4](#)), we always discard first 100 observations where the model dynamics is being established. Working with quarterly or even annual data, we would be forced to fiercely shorten the stabilisation period, possibly affecting the relevance of the results. Availability of long historical daily-frequency datasets is one of important distinguishing features of financial agent-based models compared to macroeconomic agent-based models. Conversely, surprisingly many studies employ low-frequency data ranging from weekly to annual observations. Regarding the type of the data (columns ‘Type’), stock market (20 studies) and foreign exchange (FX) data (17 studies) largely prevail, followed by housing market data (five studies), various commodities (four studies), and gold (three studies). In the past (see [Table 1](#)), almost all models were estimated using stock market and FX data mainly of daily frequency, more recent estimation attempts (see supplementary [Table 11](#) in [Appendix A](#)) offer considerably richer composition of markets and frequencies.

### *2.1. Performance of OLS, NLS, and MSM*

One of the most intriguing areas of possible analysis is the relative performance of particular models and estimation methods. However, we need to admit that strong mutual heterogeneity of models besides variability of methods make eventual conclusions rather hypothetical and possible candidates for further research. Starting with the performance, in columns ‘Fit’ we report  $R^2$  or its various alternatives for ordinary least squares, nonlinear least squares, and possibly maximum likelihood, and the p-value of the general specification J-test of overidentifying restrictions to accept the structural model as a possible data generating process. Having p-value higher than a chosen significance level, the null hypothesis of the empirical data being possibly generated by the analysed model of the ‘true’ moment-generating process cannot be rejected.

Regarding the  $R^2$  type of fit, the vast majority of models, eight out of ten, exhibit almost suspiciously very good fit ranging between 70–97%. Nonetheless, it is important to add that these results are most often based on low-frequency datasets and therefore relatively straightforward estimation methods might not be very challenged to find a well fitting model for maximum of a few hundreds observations. The exception are cusp models that are, however, estimated via maximum likelihood and only so called pseudo- $R^2$  is reported.

A different situation is observable for various methods of moments: roughly one-half of models are rejected based on the J-test as the data generating processes for given dataset and moments at a usual 5% level. Only the most recent contributions show acceptance of a particular model

but sometimes only for selected datasets from several options (Chen and Lux, 2015; Ghonghadze and Lux, 2015). All these studies are based on daily stock market or foreign exchange data. As a matter of interest, we can compare the performance of models and selected moments based on the S&P500 dataset that is shared by all six stock market models using the method of moments and that covers with a small exception of the older work of Amilon (2008) almost the similar span of data. We can simply follow Franke and Westerhoff (2012, pg. 1208) arguing by a large number of existent FABMs and the ‘wilderness of bounded rationality problem’ and calling for a general guidance and a model contest to “judge which models can mimic the stylized facts, say, ‘fairly well’ and which are even ‘very good’ in this way”. Authors suggest the method of simulated moments as a “superb tool to serve this purpose”. If we compare the six above mentioned models, the two most successful are those by Franke and Westerhoff (2012) based on a rich set of nine moments reaching the J-test p-value 32.6%, and by Ghonghadze and Lux (2015) employing a partly different set of eight moments, with p-value amounting to 50.2% for S&P500.

In summary, application of the methods of moments, mainly the method of simulated moments version, offers a tool for mutual comparison of models and estimation frameworks, however, its application struggles with practical technical issues that require a further development of the method. The most recent contribution of Chen and Lux (2015, pg. 16) explains the main problem prohibiting proper identification that is shared by all these studies: “we have to cope with multiple local minima as well as with relatively flat surfaces in certain regions of the parameter space. Any standard optimisation algorithm could, thus, not be expected to converge to a unique solution from different initial conditions”. Another closely related problem is a very rugged surface of the objective function, further embarrassing standard methods of optimisation search. To tackle all these issues, authors suggest a strategy of a preliminary rough grid search followed by a fine-tuning on a considerably restricted subset of the parameter space. We generally follow these recommendations also within the empirical application of NPSMLE, but opt for a strategy of multiple random starting points. Finally, methods of moments inevitably challenge researchers to partially arbitrarily select a set of moments that must be representative enough to hope in capturing well the most crucial features of the data, but at the same time reasonably bounded to accommodate computational burden and possible clashes in explanatory potential of particular moments. Such a task does not seem to be satisfactorily resolved for FABMs yet, however, the progress in time is evident also within the very limited sample of only several analysed papers.

## 2.2. Performance of ML and QML

Application of methods based on the maximum likelihood principle shares a relatively similar problem with methods of moments: the objective function is often very flat in direction of some parameters, typically the switching parameter of the intensity of choice. Problematic identification of given parameters is then reflected in large standard deviations of estimates preventing from contributive interpretation of results. In the majority, especially older, studies a discussion about the shape of the log-likelihood function is missing and the reader might only guess from insignificant estimates of the switching coefficient. A few most recent studies report that the likelihood is not very informative and the model accuracy is not sensitive for given parameter, and “the other parameters can to a large extent compensate for changes in switching coefficient” (Bolt et al., 2014, pg. 15). However, the shape of the objective function is almost never rigorously studied. An exception is Hommes and Veld (2015), who emphasise a very flat shape of the likelihood function for the intensity of choice selection that hampers validity of the test to reject the null hypothesis of switching, especially for small samples. On the other hand, smoothness of the objective function

does not seem to be an issue for maximum likelihood methods in comparison with methods of moments. This important finding is further confirmed later in this study.

### *2.3. Switching*

Finally, a high importance is devoted to the existence of behavioural switching, that is, to the sign, magnitude, and the statistical significance of the intensity of choice. Following the question of [Chen et al. \(2012, pg. 202\)](#), “how big or how small is it?”, we however need to emphasise once again that the magnitude of this unit-free variable cannot be directly rigorously compared, because its effect is conditional on the particular model and data. Four studies find a very large switching coefficients ([Bolt et al., 2011, 2014](#); [ter Ellen et al., 2013](#); [Frijns et al., 2010](#)), however, statistically insignificant in all cases. In other relevant studies (20 out of 23), the estimated values are mostly found single-digit and often close to zero, that well resembles the economic intuition of some, but realistically low switching frequency between major types of trading strategies. Although the sign of the parameter is of crucial importance, in [Tables 1 and 11](#) we present absolute values because the interpretation of the positive/negative sign depends on the specific design of the model. Almost all studies report the theoretically expected sign of the effect, nonetheless, we do not observe any conclusive results regarding the statistical significance of the intensity of choice, that is, no apparent connection can be observed w.r.t. the ‘#’ number of estimated parameters or the frequency and length of the data. Statistically significant and insignificant findings are reported across these categories without any clear pattern. On the other hand, some, but rather hypothetical, relation might be observed based on the ‘Type’ of the data: statistically significant estimates strongly dominate for commodities and weakly prevail for stock markets; insignificant estimates prevail for real estate markets and dominate for FX markets. However, as the sample of studies is rather small and often problematically mutually comparable, these findings should be interpreted with a high caution.

### *2.4. Concluding remarks*

To sum up, although we claim in the Introduction that so far neither general consensus on the estimation methodology, nor conclusive results are clearly observable, we might, however, conclude the most important findings, issues, and tendencies observed within the field. First, surprisingly no considerable effect of the curse of dimensionality is directly observable across the sample of assessed studies. Models with only few parameters estimated show a balanced result in terms of the estimation performance as well as the ability to reveal the switching parameters of the intensity of choice. Surprisingly, studies where a relatively large number of parameters is estimated reveal mostly favourable results in both aspects. The adaptive belief system constitutes a largely dominant modelling framework and seems relatively successfully estimable via nonlinear least squares and quasi maximum likelihood. But the models need to be specifically designed and often simplified accordingly to allow for taking advantage of these methods. The flatness of the log-likelihood function, especially in the direction of the switching coefficient, seems to be the main handicap of the maximum likelihood based estimation methods that has not been sufficiently studied yet. Simulation-based methods are generally applicable and not constrained by strict theoretical assumptions, but yet not developed enough to bring unambiguous conclusions. Most importantly, they suffer from issues related to flatness and roughness of the surface of the objective function. However, a future progress of simulation-based methods, especially via solving related rather technical issues, is likely to be largely encouraged and fostered by recent rapid development of high-speed computational facilities. With this respect, our paper is a step in this direction.

### 3. Simulation-based estimation of FABMs: The case of Brock and Hommes model

This section proposes a general computational framework for empirical estimation of full-fledged FABMs. As discussed in the previous section, few authors apply simulation-based methods of moments to overcome the problem of unavailability of the criterion functions. We step forward, and adapt the simulated maximum likelihood estimator (MLE) based on nonparametric kernel methods recently suggested by Kristensen and Shin (2012). The methodology was developed for dynamic models where no closed-form representation of the likelihood function exists and thus we cannot derive the usual MLE. As we see later, NPSMLE constitutes an opportune estimation method for general class of FABMs.

#### 3.1. The Brock and Hommes (1998) model

Our modelling framework follows one of the most widely analysed FABMs, the seminal Brock and Hommes (1998) model. The model represents a stylised financial market application of the adaptive belief system, the endogenous evolutionary selection of heterogeneous expectation rules (Lucas, 1978). We consider an asset pricing model with one risky and one risk free asset. The wealth dynamics of each agent is of the following form:

$$W_{t+1} = RW_t + (p_{t+1} + y_{t+1} - Rp_t)z_t, \quad (1)$$

where  $W_{t+1}$  denotes the total wealth at time  $t + 1$ . The risk-free asset is perfectly elastically supplied at a fixed gross rate  $R = 1 + r$ , i.e.  $r$  stands for the constant risk-free interest rate.  $p_t$  and  $\{y_t\}$  denote the ex-dividend price per share of the risky asset at time  $t$  and its stochastic dividend process, respectively. Lastly,  $z_t$  represents the amount of shares of the risky asset purchased at time  $t$ . The utility of each agent is given by an exponential-type constant absolute risk aversion (CARA) utility function  $U(W) = -\exp(-aW)$ , where  $a > 0$  is a risk-aversion parameter. The Walrasian auction scenario for setting the market clearing price is assumed, i.e.  $p_t$  makes demand for the risky asset equal to supply and agents are ‘price takers’.

There are  $H \in \mathbb{N}$  different trading strategies available to agents. Let  $E_t, V_t$  denote the conditional expectation and conditional variance operators.  $E_{h,t}, V_{h,t}$  then represent beliefs of agent class  $1 \leq h \leq H$  that uses trading strategy  $h$  about the conditional expectation and conditional variance of wealth based on a publicly available information set  $\mathcal{F}_t = \{p_t, p_{t-1}, \dots; y_t, y_{t-1}, \dots\}$ . For analytical tractability, beliefs about the conditional variance of excess returns are assumed to be constant, i.e.  $V_{h,t}(p_{t+1} + y_{t+1} - Rp_t) = \sigma^2$ , implying conditional variance of total wealth  $V_{h,t}(W_{t+1}) = z_t^2 \sigma^2$ .

Assuming myopic mean-variance maximisation, the optimal demand for the risky asset  $z_{h,t}$  for each agent class  $h$  solves:

$$\max_{z_t} \left\{ E_{h,t}[W_{t+1}] - \frac{a}{2} V_{h,t}[W_{t+1}] \right\}. \quad (2)$$

Therefore:

$$E_{h,t}[p_{t+1} + y_{t+1} - Rp_t] - a\sigma^2 z_{h,t} = 0, \quad (3)$$

$$z_{h,t} = \frac{E_{h,t}[p_{t+1} + y_{t+1} - Rp_t]}{a\sigma^2}. \quad (4)$$

Let  $n_{h,t}$  denote fractions of agents of classes  $h \in \{1, \dots, H\}$  at time  $t$  satisfying  $\sum_{h=1}^H n_{h,t} = 1$ . Let  $z_{s,t}$  be the overall supply of outside risky shares per all agents. The Walrasian temporary market equilibrium for the risky asset then yields:

$$\sum_{h=1}^H n_{h,t} z_{h,t} = \sum_{h=1}^H n_{h,t} \left\{ \frac{E_{h,t}[p_{t+1} + y_{t+1} - Rp_t]}{a\sigma^2} \right\} = z_{s,t}. \quad (5)$$

In a specific case of zero supply of outside risky shares,  $z_{s,t} = 0 \forall t$ , the market equilibrium satisfies:

$$Rp_t = \sum_{h=1}^H n_{h,t} \{E_{h,t}[p_{t+1} + y_{t+1}]\}. \quad (6)$$

In a completely rational market assuming all agents identical with homogeneous expectations,  $p_t$  is completely determined by fundamentals and Equation 6 thus reduces to  $Rp_t = E_t[p_{t+1} + y_{t+1}]$ . The price of the risky asset is then given by the discounted sum of its future dividend cash flow:

$$p_t^* = \sum_{k=1}^{\infty} \frac{E_t[y_{t+k}]}{(1+r)^k}, \quad (7)$$

where  $p_t^*$  denotes the fundamental price depending upon the stochastic dividend process  $\{y_t\}$ . The fundamental price provides an important benchmark for asset valuation under rational expectations based on economic fundamentals. In a specific case of an independent identically distributed (i.i.d.) process  $\{y_t\}$ ,  $E_t\{y_{t+1}\} = \bar{y}$  is a constant. All agents are then able to derive  $p_t^*$  by the simple formula:

$$p^* = \sum_{k=1}^{\infty} \frac{\bar{y}}{(1+r)^k} = \frac{\bar{y}}{r}. \quad (8)$$

For the subsequent analysis it is convenient to work with the deviation  $x_t$  from  $p_t^*$  instead of price levels:

$$x_t = p_t - p_t^*. \quad (9)$$

### 3.1.1. Heterogeneous beliefs

A key concept of the model is the existence of heterogeneous beliefs about future prices. Beliefs of individual agent classes are assumed in the following form:

$$E_{h,t}(p_{t+1} + y_{t+1}) = E_t(p_{t+1}^* + y_{t+1}) + f_h(x_{t-1}, \dots, x_{t-L}), \quad \text{for all } h, t, \quad (10)$$

where  $E_t(p_{t+1}^* + y_{t+1})$  denotes the conditional expectation of the fundamental price based on the information set  $\mathcal{F}_t$ ,  $f_h$  is a deterministic function which can differ across agent classes  $h$  and represents a ‘ $h$ -type’ trading strategy, and  $L$  indicates the number of lags.

From Equation 10 it follows that beliefs about future dividends flow:

$$E_{h,t}(y_{t+1}) = E_t(y_{t+1}), \quad h = 1, \dots, H, \quad (11)$$

are the same for all agents and equal to the true conditional expectation. On the other hand, agents’ heterogeneous beliefs about future price abandon the perfect rationality paradigm which is crucial step towards the heterogeneous agent modelling.  $f_h$  allows individual agents believe that

the market price will differ from the fundamental value  $p_t^*$  and thus the form of beliefs:

$$E_{h,t}(p_{t+1}) = E_t(p_{t+1}^*) + f_h(x_{t-1}, \dots, x_{t-L}), \quad \text{for all } h, t, \quad (12)$$

allows prices to deviate from their fundamental value.

The heterogeneous market equilibrium from [Equation 6](#) can thus be reformulated in the deviations form and conveniently used in empirical and experimental testing. Combining [Equations 9, 10](#) and  $\sum_{h=1}^H n_{h,t} = 1$  we obtain:

$$Rx_t = \sum_{h=1}^H n_{h,t} E_{h,t}[x_{t+1}] = \sum_{h=1}^H n_{h,t} f_h(x_{t-1}, \dots, x_{t-L}) \equiv \sum_{h=1}^H n_{h,t} f_{h,t}, \quad (13)$$

where  $n_{h,t}$  represent values related to the beginning of period  $t$ , before the equilibrium price deviation  $x_t$  is observed. The actual market clearing price  $p_t$  might then be calculated simply using [Equation 9](#).

### 3.1.2. Selection of strategies

Beliefs of individual agents are updated evolutionary according to the adaptive belief system driven by endogenous market forces ([Brock and Hommes, 1997](#)). The adaptive belief system is an expectation feedback system in which variables depend partly on the known values and partly on the future expectations.

The profitability measures for strategies  $h \in \{1, \dots, H\}$  are derived from past realised profits as:<sup>2</sup>

$$U_{h,t} = (x_t - Rx_{t-1}) \frac{f_{h,t-1} - Rx_{t-1}}{a\sigma^2}. \quad (16)$$

Market fractions  $n_{h,t}$  of agent classes  $h \in \{1, \dots, H\}$  are derived under the discrete choice probability framework using the multinomial logit model:

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^H \exp(\beta U_{h,t-1})}, \quad (17)$$

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<sup>2</sup>Additional memory can be introduced into the profitability measure e.g. as a weighted average of past realised values:

$$U_{M,h,t} = U_{h,t} + \eta U_{M,h,t-1}, \quad (14)$$

where  $0 \leq \eta \leq 1$  denotes the ‘dilution parameter’ of the past memory in the profitability measure. Nonetheless, for the majority of examples, [Brock and Hommes \(1998\)](#) use  $\eta = 0$  to keep derivations analytically tractable and thus they analyse models without memory. Another approach to agents’ memory is suggested by [Barunik et al. \(2009\)](#) where the profitability measure in [Equation 16](#) is extended via memory parameters  $m_h$ :

$$U_{h,t} = \frac{1}{m_h} \sum_{l=0}^{m_h-1} \left[ (x_{t-l} - Rx_{t-1-l}) \frac{f_{h,t-1-l} - Rx_{t-1-l}}{a\sigma^2} \right]. \quad (15)$$

The memory for each individual trading strategy  $h \in \{1, \dots, H\}$  is then randomly generated from the uniform distribution  $U(0, m_h)$ .

where the lagged timing of  $U_{h,t-1}$  ensures that all information for updating of market fractions  $n_{h,t}$  is available at the beginning of period  $t$  and  $\beta \geq 0$  is the intensity of choice parameter measuring how fast are agents willing to switch between different trading strategies.

### 3.1.3. Basic strategy types

The original paper by [Brock and Hommes \(1998\)](#) analyses an artificial market consisting only of few simple trading strategies. Authors argue that only very simple forecasting rules can have a real impact on equilibrium prices because complicated strategies are unlikely to be understood and followed by a sufficient number of traders. Trading strategies thus have a simple linear form:

$$f_{h,t} = g_h x_{t-1} + b_h, \quad (18)$$

where  $g_h$  denotes the trend parameter and  $b_h$  is the bias parameter of the trading strategy  $h$ .

The first agent class are fundamentalists. They believe that the asset prices are determined solely by economic fundamentals according to the efficient market hypothesis ([Fama, 1970](#)). Therefore the price can be simply computed as the present value of the discounted future dividends flow. Fundamentalists also believe that prices always converge to their fundamental values. In the heterogeneous agent model, fundamentalists comprise the special case of [Equation 18](#) with  $g_h = b_h = f_{h,t} = 0$ . The fundamental strategy have complete past market prices and dividends in its information set  $\mathcal{F}_{h,t}$ . The fundamentalists' demand also reflects market actions of other agent classes but fundamental traders are not aware of the fractions  $n_{h,t}$  of other trading strategies. Fundamentalists might pay costs  $C \geq 0$  to obtain market information and to understand how economic fundamentals work. However, [Brock and Hommes \(1998\)](#) themselves mostly set  $C = 0$  to keep simplicity of the analysis. Another agent class represents chartists. They believe that asset prices can be partially predicted taking various patterns observed in the past data into account, e.g. using simple technical trading rules and extrapolation techniques. If  $b_h = 0$ , trader  $h$  is called a pure trend chaser if  $0 < g_h \leq R$  and a strong trend chaser if  $g_h > R$ . Next, the trader  $h$  is called a contrarian if  $-R \leq g_h < 0$  or a strong contrarian if  $g_h < -R$ . If  $g_h = 0$ , trader  $h$  is considered to be purely upward biased if  $b_h > 0$  or purely downward biased if  $b_h < 0$ . Combined trading strategies with  $g_h \neq 0$  and  $b_h \neq 0$  are certainly also possible.

### 3.2. Construction of the NPSMLE

This section introduces the estimation framework for the [Brock and Hommes \(1998\)](#) model based on the simulated MLE proposed by [Kristensen and Shin \(2012\)](#). Let us assume processes  $(x, v)$ ,  $x : t \mapsto \mathbb{R}^k$ ,  $v : t \mapsto \mathcal{V}_t$ ,  $t = 1, \dots, \infty$  with time-varying  $\mathcal{V}_t$ . Suppose that we have  $T$  realisations  $\{(x_t, v_t)\}_{t=1}^T$ . Let us further assume the time series  $\{x_t\}_{t=1}^T$  are generated by a fully parametric model:

$$x_t = q_t(v_t, \varepsilon_t, \theta), \quad t = 1, \dots, T, \quad (19)$$

where a function  $q : \{v_t, \varepsilon_t, \theta\} \mapsto \mathbb{R}^k$ ,  $\theta \in \Theta \subseteq \mathbb{R}^l$  is an unknown parameter vector, and  $\varepsilon_t$  is an i.i.d. sequence with known distribution  $\mathcal{D}_\varepsilon$ , which is (without loss of generality) assumed not to depend on  $t$  or  $\theta$ . In general, the processes  $(x, v)$  can be non-stationary and  $v_t$  is allowed to contain other exogenous variables than lagged  $x_t$ . We also assume the model to have an associated conditional



density  $c_t(x|v; \theta)$ , i.e.

$$C(x \in A|v_t = v) = \int_A c_t(x|v; \theta) dx, \quad t = 1, \dots, T, \quad (20)$$

for any Borel set  $A \subseteq \mathbb{R}^k$ .

Let us now suppose that  $c_t(x|v; \theta)$  from Equation 20 does not have a closed-form representation. In such situation, we are not able to derive the exact likelihood function of the model in Equation 19 and thus a natural estimator of  $\theta$ , the maximiser of the conditional log-likelihood:

$$\tilde{\theta}_{ML} = \operatorname{argmax}_{\theta \in \Theta} \mathcal{L}_T(\theta), \quad \mathcal{L}_T(\theta) = \sum_{t=1}^T \log c_t(x_t|v_t; \theta) \quad (21)$$

is not feasible.

In such situation, however, we are still able to simulate observations from the model in Equation 19 numerically.<sup>3</sup> The method presented allows us to compute a simulated conditional density, which we further use to gain a simulated version of the MLE.

To obtain a simulated version of  $c_t(x_t|v_t; \theta) \forall t \in \langle 1, \dots, T \rangle, x \in \mathbb{R}^k, v \in \mathcal{V}_t$ , and  $\theta \in \Theta$ , we firstly generate  $N \in \mathbb{N}$  i.i.d. draws from  $\mathcal{D}_\varepsilon$ ,  $\{\varepsilon_i\}_{i=1}^N$ , which are used to compute:

$$X_{t,i}^\theta = q_t(v_t, \varepsilon_i, \theta), \quad i = 1, \dots, N. \quad (22)$$

These  $N$  simulated i.i.d. random variables,  $\{X_{t,i}^\theta\}_{i=1}^N$ , follow the target distribution by construction:  $X_{t,i}^\theta \sim c_t(\cdot|v_t; \theta)$ , and therefore can be used to estimate the conditional density  $c_t(x|v; \theta)$  with kernel methods. We define:

$$\hat{c}_t(x_t|v_t; \theta) = \frac{1}{N} \sum_{i=1}^N K_\eta(X_{t,i}^\theta - x_t), \quad (23)$$

where  $K_\eta(\psi) = K(\psi/\eta)/\eta^k$ ,  $K : \mathbb{R}^k \mapsto \mathbb{R}$  is a generic kernel and  $\eta > 0$  is a bandwidth. Under regularity conditions on  $c_t$  and  $K$ , we get:

$$\hat{c}_t(x_t|v_t; \theta) = c_t(x_t|v_t; \theta) + O_P(1/\sqrt{N\eta^k}) + O_P(\eta^2), \quad N \longrightarrow \infty, \quad (24)$$

where the last two terms are  $o_P(1)$  if  $\eta \longrightarrow 0$  and  $N\eta^k \longrightarrow \infty$ .

Having obtained the simulated conditional density  $\hat{c}_t(x_t|v_t; \theta)$  from Equation 23, we can now derive the simulated MLE of  $\theta$ :

$$\hat{\theta}_{NPSMLE} = \operatorname{argmax}_{\theta \in \Theta} \hat{\mathcal{L}}_T(\theta), \quad \hat{\mathcal{L}}_T(\theta) = \sum_{t=1}^T \log \hat{c}_t(x_t|v_t; \theta). \quad (25)$$

The same draws are used for all values of  $\theta$  and we may also use the same set of draws from  $\mathcal{D}_\varepsilon(\cdot)$ ,  $\{\varepsilon_i\}_i^N$ , across  $t$ . Numerical optimisation is facilitated if  $\hat{\mathcal{L}}_T(\theta)$  is continuous and differentiable in  $\theta$ .

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<sup>3</sup>For cases in which the model in Equation 19 is itself intractable and thus we cannot generate observations from the exact model, Kristensen and Shin (2012) suggest a methodology for approximate simulations and define regularity conditions for the associated approximate NPSMLE  $\hat{\theta}_{APPROX}$  to have the same asymptotic properties as the simulated  $\hat{\theta}_{NPSMLE}$  defined in Equation 25.

Considering Equation 23, if  $K$  and  $\theta \mapsto q_t(v, \varepsilon, \theta)$  are  $s \geq 0$  continuously differentiable, the same holds for  $\widehat{\mathcal{L}}_T(\theta)$ .

Under the regularity conditions, the fact that  $\widehat{c}_t(x_t|v_t; \theta) \xrightarrow{P} c_t(x_t|v_t; \theta)$  implies that also  $\widehat{\mathcal{L}}_T(\theta) \xrightarrow{P} \mathcal{L}_T(\theta)$  as  $N \rightarrow \infty$  for a given  $T \geq 1$ . Thus, the simulated MLE,  $\widehat{\theta}_{NPSMLE}$ , retains the same properties as the infeasible MLE,  $\widetilde{\theta}_{ML}$ , as  $T, N \rightarrow \infty$  under suitable conditions.

### 3.3. Advantages and disadvantages

To quote from Kristensen and Shin (2012, pg. 85), “one of the merits of NPSML is its general applicability”. Authors further provide three examples of application of the methodology. The first one comprises an estimation of the short-term interest rate model of Cox et al. (1985). The second example applies the methodology to a jump-diffusion model of daily S&P500 returns by Andersen et al. (2002). In the third one the general capabilities of the NPSMLE to estimate a generic Markov decision processes are examined.

Kristensen and Shin (2012) report several advantages and disadvantages of the proposed estimator. Starting with the former, the estimator works whether the observations  $x_t$  are i.i.d. or non-stationary because the density estimator based on i.i.d. draws is not affected by the dependence structures in the observed data. Second, the estimator does not suffer from the curse of dimensionality, which is usually associated with kernel estimators. In general, high dimensional models, i.e. with larger  $k \equiv \dim(x_t)$  as we smooth only over  $x_t$  here, require larger number of simulations to control the variance component of the resulting estimator. However, the summation in Equation 25 reveals an additional smoothing effect and the additional variance of  $\widehat{\mathcal{L}}_T(\theta)$  caused by simulations retains the standard parametric rate  $1/N$ .

Conversely, the simulated log-likelihood function is a biased estimate of the actual log-likelihood function for fixed  $N$  and  $\eta > 0$ . To obtain consistency, we need  $N \rightarrow \infty$  and  $\eta \rightarrow 0$ . Thus, the parameter  $\eta$  needs to be properly chosen for given sample and simulation size. In the stationary case, the standard identification assumption is:

$$\mathbb{E}[\log c(x_t|v_t, \theta)] < \mathbb{E}[\log c(x_t|v_t, \theta_0)] \quad \forall \theta \neq \theta_0. \quad (26)$$

Under stronger identification assumptions, the choice of the parameter  $\eta$  might be less important and one can prove the consistency of the estimator for any fixed  $0 < \eta < \bar{\eta}$  for some  $\bar{\eta} > 0$  as  $N \rightarrow \infty$  (Altissimo and Mele, 2009). In practice this still requires us to know the threshold level  $\bar{\eta} > 0$  but from the theoretical viewpoint this ensures that parameters can be well identified in large finite samples after a given  $\bar{\eta} > 0$  is set. Moreover, it suggests that the proposed methodology is fairly robust to the choice of  $\eta$ . In their simulation study, Kristensen and Shin (2012) show that NPSMLE performs indeed well using a broad range of bandwidth choices.

### 3.4. Asymptotic properties

As the theoretical convergence of the simulated conditional density towards the true density is met, we would expect the  $\widehat{\theta}_{NPSMLE}$  to have the same asymptotic properties as the infeasible  $\widetilde{\theta}_{ML}$  for a properly chosen sequence  $N = N(T)$  and  $\eta_N = \eta(N)$ . Kristensen and Shin (2012) show that  $\widehat{\theta}_{NPSMLE}$  is first-order asymptotic equivalent to  $\theta_{ML}$  under set a general conditions, allowing even for non-stationary and mixed discrete and continuous distribution of the response variable. Further, using additional assumptions, including stationarity, they provide results regarding the higher-order asymptotic properties of  $\widehat{\theta}_{NPSMLE}$  and derive expressions of the bias and variance components of the  $\widehat{\theta}_{NPSMLE}$  compared to the actual MLE due to kernel approximation and simulations.

Therefore, a set of general conditions, satisfied by most models, need to be verified so that  $\hat{c} \rightarrow c$  sufficiently fast to ensure asymptotic equivalence of  $\hat{\theta}_{NPSMLE}$  and  $\tilde{\theta}_{ML}$ . [Kristensen and Shin \(2012\)](#) define a set of regularity conditions on the model and its associated conditional density that satisfy these general conditions for uniform rates of kernel estimators defined in [Kristensen \(2009\)](#).

The kernel  $K$  from [Equation 23](#) has to belong to a broad class of so-called bias high-order or bias reducing kernels. E.g. the Gaussian kernel, which we use in [Section 4](#), satisfies this condition if  $r \geq 2$ , where  $r$  is the number of derivatives of  $c$ . Higher  $r$  causes faster rate of convergence and determines the degree of  $\hat{c}$  bias reduction. Moreover, general versions of conditions usually required for consistency and well-defined asymptotic distribution (asymptotic normality) of MLEs in stationary and ergodic models are imposed on actual log-likelihood function and the associated MLE to ensure the actual  $\tilde{\theta}_{ML}$  in [Equation 21](#) is asymptotically well-behaved.

#### 4. Simulation study

This section analyses capability of the NPSMLE to recover the parameters of [Brock and Hommes \(1998\)](#) model, and mainly evaluates small sample properties of the estimator via an extensive Monte Carlo study. We simulate data from the [Brock and Hommes \(1998\)](#) to analyse how well and under what conditions is the estimation able to recover true values of parameters in the controlled environment. For its conceptual importance, a detailed focus is devoted to the switching parameter of the intensity of choice  $\beta$ .

##### 4.1. Simulation setup for the [Brock and Hommes \(1998\)](#) model

In the simulation setup, we follow the previous works of [Barunik et al. \(2009\)](#); [Vacha et al. \(2012\)](#); [Kukacka and Barunik \(2013\)](#). The joint setup for the basic model (see [Subsection 3.1](#)) is used for all (if not explicitly stated otherwise) conducted simulations in this section and is defined as follows. The model we use to generate observations is a very stylised simple version compactly described in [Hommes \(2006, pg. 1169\)](#) and consisting of three mutually dependent equations:

$$Rx_t = \sum_{h=1}^H n_{h,t} f_{h,t} + \epsilon_t \equiv \sum_{h=1}^H n_{h,t} (g_h x_{t-1} + b_h) + \epsilon_t, \quad (27)$$

$$n_{h,t} = \frac{\exp(\beta U_{h,t-1})}{\sum_{h=1}^H \exp(\beta U_{h,t-1})}, \quad (28)$$

$$\begin{aligned} U_{h,t-1} &= (x_{t-1} - Rx_{t-2}) \frac{f_{h,t-2} - Rx_{t-2}}{a\sigma^2} \\ &\equiv (x_{t-1} - Rx_{t-2}) \frac{g_h x_{t-3} + b_h - Rx_{t-2}}{a\sigma^2}, \end{aligned} \quad (29)$$

where  $\epsilon_t$ , which coincides with  $\varepsilon_t$  in [Equation 19](#), is an i.i.d. noise term sequence with given distribution representing the market uncertainty and unpredictable market events.

In order to run the model in various different settings, we inevitably need to fix several variables less important for the dynamics of the model to enable estimation of the key parameters. First, we set the constant gross interest rate  $R = 1 + r = 1.0001$  to resemble real market risk free rate. Assuming 250 trading days per year and daily compounding, this daily value represents circa

2.5% annual risk free interest rate which is a reasonable approximation. Although this figure is not based on any rigorous calibration or taken from a specific study, comparable values are largely used in various financial and macroeconomic works. Moreover, as we describe in further analysis, the model exhibits considerable robustness w.r.t. various reasonable risk free values and thus there is no need for more precise derivation of this parameter. We further fix the linear term  $1/a\sigma^2$ , comprising the risk aversion coefficient  $a > 0$  and the beliefs about the conditional variance of excess returns  $\sigma^2$  to 1, similarly to [Barunik et al. \(2009\)](#); [Vacha et al. \(2012\)](#); [Kukacka and Barunik \(2013\)](#). It is important to note that  $a$  and  $\sigma^2$  are only scale factors for the profitability measure  $U$ . Their magnitudes do not affect relative proportions of  $U_{h,t}$  and thus do not influence the dynamics of the model output, that is on the contrary usually characterised by time-varying variance. In other words, although we assume constant  $\sigma^2$ , the output time series generated by the model does not have constant variance. Strategy-specific  $a_h$  or time-varying  $\sigma_{h,t}^2$  are appealing concepts mainly for simulation analyses of the model (see e.g. [Gaunersdorfer, 2000](#); [Chiarella and He, 2002](#); [Amilon, 2008](#)). Moreover, we intentionally use relatively small number of possible trading strategies following [Kukacka and Barunik \(2013\)](#),  $H = 5$ , for the general model setting or  $H \in \{2, 3\}$  for so called 2-type and 3-type model, respectively ([Chen et al., 2012](#)). Following [Hommes \(2006\)](#) via [Equation 29](#), neither ‘dilution parameter’ of the past memory  $\eta$  nor information costs  $C$  for fundamentalists are implemented into the basic model setup to keep the dynamics of the model and impacts of assessed modification as clear as possible. Indeed, also [Brock and Hommes \(1998\)](#) mostly set  $C = 0$  to keep simplicity of the analysis and work with models without memory, i.e. they set the  $\eta = 0$  (see [Subsection 3.1.2](#)) to keep derivations analytically tractable.

Within the Monte Carlo setup, several parameters are repeatedly randomly generated to obtain statistically valid inference. Following [Barunik et al. \(2009\)](#); [Vacha et al. \(2012\)](#); [Kukacka and Barunik \(2013\)](#), trend parameters  $g_h$  are drawn from the normal distribution  $N(0, 0.4^2)$  and bias parameters  $b_h$  are drawn from the normal distribution  $N(0, 0.3^2)$ . ‘Strict’ fundamental strategy in the sense of the original [Brock and Hommes \(1998, pg. 1245\)](#) article appears in the market by default, i.e. the first strategy is always defined as  $g_1 = b_1 = 0$  and therefore some proportion of fundamentalists are always present in the market.

First, we study the capabilities of the NPSMLE under various levels of the switching parameter of the intensity of choice  $\beta$ . Literature estimating  $\beta$  using real marked data is relatively scarce because of difficulties arising from the nonlinear nature of the model. Thus,  $\beta$  still remains a rather theoretical concept. Larger  $\beta$  implies higher willingness of agents to switch between available trading strategies based on their relative profitability. In other words, the best strategy attracts the most agents at each period. On the one hand, comprising a large variety of possible  $\beta$  values might seem as a dominant simulation strategy, on the other hand, one has to consider computational burden of the simulation process in real time. What is perhaps even more important is to consider intensity of choice  $\beta$  from the economic viewpoint. First, high values give rise to unrealistically high switching frequency, which can be hardly observed among market agents in reality. Next, negative  $\beta$  does not make economic sense in the presented model framework as it causes inverse illogical switching towards less profitable strategies. Although the intensity of choice  $\beta$  cannot be directly rigorously compared across various models, assets, or time periods, we utilise the general knowledge of previous estimation efforts for models sharing similar framework to set meaningful simulation grids in this section. Vast majority of research articles sharing the framework derived from [Brock and Hommes \(1998\)](#) find  $\beta$  single-digit and often close to zero values, that well resembles the economic intuition of some, but realistically low switching frequency between major

types of trading strategies. Thus, we employ relatively rich, but reasonable discrete range of  $\beta$  in our simulations:  $\{0, 0.1, 0.5, 1, 3, 5, 10\}$ . It is far beyond the scope of this work to provide a deep analysis of the model behaviour, e.g. how the intensity of choice  $\beta$  influences the dynamics of the model that can under some setting even generate purely chaotic behaviour. Many studies have been devoted to this generally difficult issue in the past two decades. In this context we refer the interested reader to the original paper of [Brock and Hommes \(1998\)](#) containing comprehensive model dynamics analysis, extensive studies by [Hommes \(2006\)](#), [Hommes and Wagener \(2009\)](#), [Chiarella et al. \(2009\)](#), or a recent book summarising two decades of research on the Heterogeneous Expectations Hypothesis by [Hommes \(2013\)](#).

Next, as discussed by [Amilon \(2008\)](#), the magnitude of noise term has to be considered carefully. Noise is an inevitable part of the model as it represents the market uncertainty and unpredictable market events, but it must not overshadow the effect of variables under scrutiny. As mentioned in [Kukacka and Barunik \(2013\)](#), although varying noise variance can cause some minor changes in model outcomes, the analysed model embodies major similarities across various noises. Although theoretically the  $\mathcal{D}_\varepsilon$  from which the  $\{\varepsilon_i\}_{i=1}^N$  are drawn to simulate  $\{X_{t,i}^\theta\}_{i=1}^N$  ([Equation 22](#)) is a generic known distribution, the assumptions about market noise can play crucial role in the NPSMLE application to real world data. Therefore we test the model sensitivity and robustness of proposed methodology using 30 stochastic noise specifications from an extensive range drawn a) from various normal distributions and b) from the uniform distributions

1. that cover the same intervals as are covered by their respective normal counterparts by the 99.74% of the probability mass;
2. with the same variances as their respective normal counterparts.

Basically, for the normally distributed noise the range extends from a ‘miniscule’ standard deviations  $SD = 10^{-8}, 10^{-7}, 10^{-6}$  [a value used by [Hommes \(2013, pg. 170, 174, 177\)](#) in the similar model setting], or  $10^{-5}$ , followed by ‘small’ standard deviations  $SD = 10^{-4}, 10^{-3}, 0.01$  [another value used by [Hommes \(2013, pg. 171\)](#) in the similar setting], standard normal  $SD = 1$ , and finally a relatively large ‘experimental’ standard deviation  $SD = 2$ .

The sensitivity analysis of the NPSMLE to the stochastic noise specification is based on the normality assumption. The normal distribution of market noise seems reasonably realistic and similar assumption has already been used in related studies, where “the non-linear models are fed with an exogenous stochastic process, but the noise process is ‘nice’, which in this case means that it is normally distributed”, as pointed out by [Amilon \(2008, pg. 344\)](#). We also utilise the favourable theoretical properties of the Gaussian kernel ([Kristensen and Shin, 2012, pg. 81](#)) in [Equation 23](#). To check the robustness of the method, we concur the previous research in [Barunik et al. \(2009\)](#); [Vacha et al. \(2012\)](#); [Kukacka and Barunik \(2013\)](#), where uniform stochastic noise specification is utilised. We compare and contrast the results based on normally distributed noise to the two rather extreme and economically unrealistic uniform variants defined above. We intentionally do not consider any at first sight soliciting heavy-tailed noise distribution. The fact that financial data are heavy-tailed does not suggest any specific distribution of the market noise. In fact, the situation is opposite. The attractiveness of the model is based on its ability to produce heavy-tailed distribution of model output although we input normally distributed stochastic noise. Thus the model explains one of the most important stylised facts of financial time series via endogenous interactions of fundamentalists and boundedly rational chartists, not as an effect of a specific distribution of noise input. Finally, five lengths of the resulting series entering the NPSMLE algorithm are used: 100, 500, 1000, 5000,

and 10000. We always simulate 100 extra observations to be discarded as initial period, where the model dynamic is being established.

#### 4.2. Simulation setup for the NPSMLE

While we adapt the methodology of [Kristensen and Shin \(2012\)](#) originally proposed for the estimation of the [Cox et al. \(1985\)](#) short-term interest rate model, we first need to make several choices about the simulation setup. As discussed earlier, there are two main trade-offs: between the precision of the kernel estimation and the computational burden, and between the smoothness of the kernel estimation and the bias. Hence we compare the simulations of 100, 500, and 1000 runs with three different levels of the kernel estimation precision, namely  $N = 100$ ,  $N = 500$ , and  $N = 1000$ . It is important to note that the same draws  $\{\varepsilon_i\}_{i=1}^N$  are used to generate the simulations  $\{X_{t,i}^\theta\}_{i=1}^N$  over time to maintain comparability of the results. Further, while the numerical algorithm needs to be designed to find an optimum of either unconstrained or constrained multivariable function, we experiment with various optimisation setups.

Using the pre-estimation step with unconstrained parameter space we can obtain reasonably sufficient preliminary knowledge about the approximate true value of estimated parameters even for computationally feasible setting. This general principle of a preliminary rough search followed by a fine-tuning on a considerably restricted subset of the parameter space is successfully applied e.g. in [Chen and Lux \(2015\)](#) for an estimation using the method of simulated moments. This helps us to constrain the parameter space in the next step in order to arrive at a computationally feasible setup with precise results. In the subsequent simulations, we opt for a wide enough parameter space as  $\langle -\beta, 3\beta \rangle$  for  $\beta > 0$  and  $\langle -0.5, 0.5 \rangle$  for  $\beta = 0$ . This extended range of the parameter space bounds covering also economically irrelevant negative values is used to ensure the robustness of the method and not imposing excessive demands on the precision of the unconstrained pre-estimation. Moreover, not allowing for negative values might naturally lead to an upward bias of the simulated estimator especially for  $\beta$  close to 0. For the 2-type model simulation study (see [Subsection 4.4](#)), we use even wider and off-centred interval for the bounds of the parameter space for belief coefficients set as  $\langle -3|g_2|, 3|g_2| \rangle$  and  $\langle -3|b_2|, 3|b_2| \rangle$ , respectively, to allow for possible negative values. For the 3-type model simulation estimation study (see [Subsection 4.5](#)), it is, however, important to limit bounds for belief coefficients by zero from one side, i.e.  $\langle 0, 3|g_2| \rangle$  and  $\langle -3|g_3|, 0 \rangle$ , to avoid problems with insufficient specification of the model leading to ambiguous bimodal distributions of estimated parameters. The same intervals are used for a random draw of a single<sup>4</sup> starting point of the optimisation search procedure which is drawn from the uniform distributions.

To estimate the conditional density  $c_t(x|v; \theta)$ , the Gaussian kernel and the [Silverman's \(1986\)](#) rule of thumb for finding the optimal size of the bandwidth is used:

$$\eta = \left( \frac{4}{3N} \right)^{1/5} \hat{\sigma}, \quad (30)$$

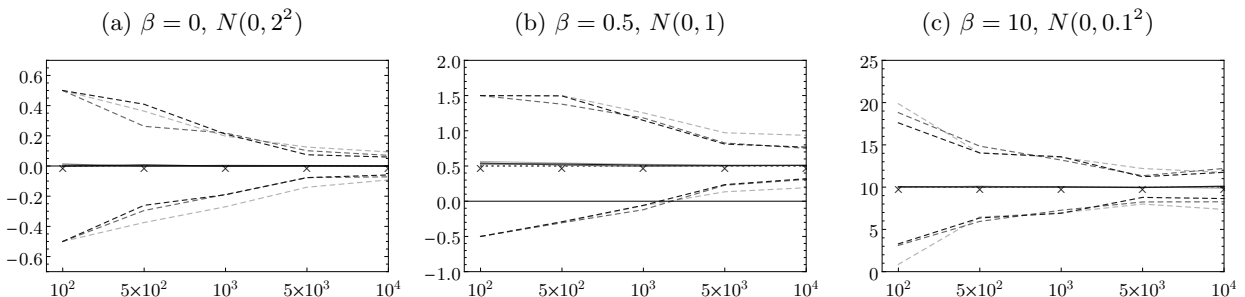
where  $\hat{\sigma}$  denotes the standard deviation of  $\{X_{t,i}^\theta\}_{i=1}^N$ .

Additionally, [Kristensen and Shin \(2012, pg. 82\)](#) suggest to undersmooth the bandwidth size. Moreover, as found by [Jones et al. \(1996\)](#), smaller bandwidths are better for larger kernel

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<sup>4</sup>[Kristensen and Shin \(2012\)](#) use multiple starting points for the numerical optimisation but for estimation of the [Brock and Hommes \(1998\)](#) model in a simulated environment the single starting point is sufficient bringing the merits of markedly reduced computational time and burden.

Figure 1: Simulation results for estimation of  $\beta \in \{0, 0.5, 10\}$ .



Note: Stochastic noise  $\epsilon_t$  and  $\{\epsilon_i\}_{i=1}^N$  drawn from 1000 random runs from normal distribution. Black dotted lines with  $\times$  depict the true  $\beta$ . Grey full lines depict sample means of estimated  $\beta$ . Grey dashed lines depict 2.5% and 97.5% quantiles. Light grey colour represents results for  $N = 100$ , normal grey for  $N = 500$ , and dark grey for  $N = 1000$ . Horizontal axis shows length of generated time series, vertical axis shows values of  $\beta$ .

approximation precision, as estimator should be local in case of more information is present. However, we do not use the undersmoothing in our numerical algorithm as for the considered model, the methodology is robust in this aspect and various levels of undersmoothing do not change the outcomes. All the results of this preliminary testing are available upon request from authors. We do not include them here to keep the length of already pregnant text under control.

In the following text, and all tables, we report sample medians and means of the estimated values together with standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles based on 1000 random runs with sample size of  $t = 5000$ , and the kernel estimation precision  $N = 1000$  i.i.d. draws from given distribution, while figures complement these results with various different sample lengths.

### 4.3. Estimating intensity of choice $\beta$

First and foremost, we assess the simplest but also the most crucial case of  $\beta$  estimation in the general model with  $H = 5$  possible trading strategies. The intensity of choice  $\beta$  is the most important parameter influencing the dynamics of the system through the multinomial logit model of a continuous adaptive evolution of market fractions in Equation 17. Not only its magnitude between two extreme cases  $\beta = 0$  and  $\beta = \infty$  is important, but  $\beta$  also determines the type of the model equilibrium that can generally take the form of a (multiple) steady state(s), cycles, or even chaotic behaviour. The intensity of choice  $\beta$  is also crucial for its conceptual importance, because it represents the dominant approach how the boundedly rational choices of agents are mathematically modelled in the current literature.

Despite of its relative simplicity, the setting is otherwise very challenging as capturing the effect of the switching coefficient  $\beta$  is generally difficult. Moreover, algorithm with a single starting point for the numerical optimisation and new random draws of the parameters  $g_h$  and  $b_h$ ,  $h \in \{2, 3, 4, 5\}$  for each independent run requires very robust performance of the optimisation algorithm.

#### 4.3.1. Qualitative results

We primarily aim to verify how consistently and efficiently is the estimator able to recover model parameter in small samples. In Figure 1 we depict a ‘snapshot’ of simulation results for

Table 3: Simulation results for  $\beta$  estimation with Gaussian noise.

$\beta$	$\hat{\beta}, \epsilon_t, \{\epsilon_i\}_{i=1}^N \sim N(0, \mu^2)$					
	Median	Mean	SD	LQ	HQ	$\mu$
0	-.00	-.00	.05	-.08	.08	2
.1	.10	.10	.04	.01	.20	2
.5	.50	.51	.14	.23	.81	1
1	1.00	1.01	.23	.66	1.45	1
3	2.99	3.05	1.00	1.48	5.86	.1
5	4.99	5.05	1.21	3.75	6.81	.1
10	9.99	9.99	2.22	7.57	11.64	.1

*Note:* Each sample is based on 1000 random runs,  $R = 1.0001$ ,  $H = 5$ ,  $t = 5000$ , and  $N = 1000$ . SD, LQ, and HQ stand for standard deviation, 2.5%, and 97.5% quantiles, respectively. Figures are rounded to 2 decimal digits.

three interesting values of the intensity of choice  $\beta \in \{0, 0.5, 10\}$  combined with three distinct specifications of the stochastic noise:  $\epsilon_t \sim N(0, 2^2)$ ,  $\epsilon_t \sim N(0, 1)$  and  $\epsilon_t \sim N(0, 0.1^2)$ .

First, we can clearly observe how the method is able to reveal the true value of  $\beta$  demonstrated by black dotted lines with  $\times$  for all sample sizes. Grey full lines depict sample means of estimated  $\beta$  and closely follow the true line. Moreover, with increasing length of generated time series  $t$ , as well as with the precision of the kernel estimation  $N$ , precision of the estimator increases. The precision is demonstrated by different shades of grey dashed lines depicting 2.5% and 97.5% quantiles of estimated parameters  $\beta$ . We also experimented with smaller number of runs, namely 100 and 500, and the results confirm that number of runs 500 performs sufficiently. These results are available upon request from authors.

Figure 1 also reveals an important result from the economic point of view of the  $\beta$  parameter value. Although only the non-negative values of  $\beta$  have an economic interpretation, in simulations we also allow for negative estimated values to test the capability of the method even for such extreme values and to avoid the upward bias of the estimator. However, the most important result is that even for a very small  $\beta = 0.5$  and sample length of  $t \doteq 1000$  or for  $\beta = 10$  and an extremely small sample length of  $t = 100$ , we obtain more than 97.5% non-negative estimates. At the same moment 95% of estimates appear reasonably close to the true value, far from the numerical bounds of the parameter space imposed to make the constrained optimisation computationally feasible.

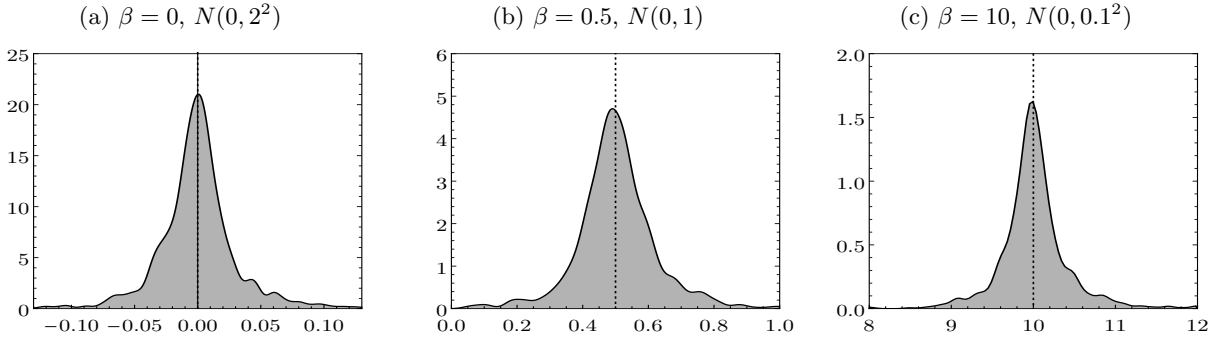
These features have important favourable consequences for application of the method to datasets of various lengths. We should be able to detect even very weak signs of behavioural switching in long-span daily financial data, but also stronger signs of switching in macroeconomic data where typically lower-frequency time series of shorter lengths are available. As for the complexity of the estimation issue with the five repeatedly randomly generated strategies, we consider our results very promising. The most important property of the estimation method in the current setting is the ability to distinguish between statistically significant and insignificant  $\beta$  and this objective is well achieved. Importantly, Figure 1 also allow for comparison between estimation of models with and without switching ( $\beta > 0$  vs.  $\beta = 0$ ).

#### 4.3.2. Quantitative results

Moving from the graphical to the quantitative description of simulation results, we now consider length of generated time series  $t = 5000$ , and the kernel estimation precision  $N = 1000$



Figure 2: Densities for selected  $\hat{\beta}$ .



*Note:* Stochastic noise  $\epsilon_t$  and  $\{\varepsilon_i\}_{i=1}^N$  drawn from given normal distributions,  $R = 1.0001$ . Each sample is based on 1000 random runs,  $H = 5$  possible trading strategies, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . Black dotted lines depict the true  $\beta$ .

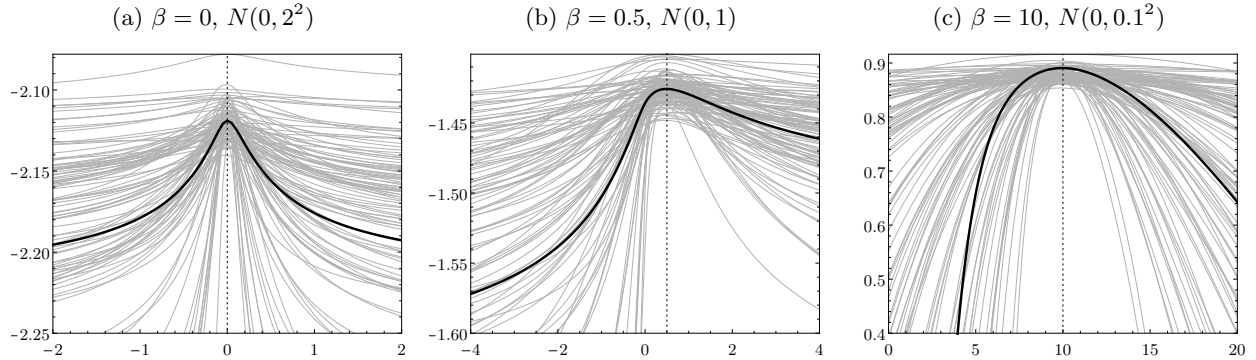
i.i.d. draws from given distribution. Here we focus on the robustness of the method w.r.t. various noise specifications used both for generating the stochastic term  $\epsilon_t$  in Equation 29 as well as for  $N$  i.i.d. draws,  $\{\varepsilon_i\}_{i=1}^N$ , to simulate  $N$  i.i.d. random variables,  $\{X_{t,i}^\theta\}_{i=1}^N$ , used for the kernel estimation of the conditional density. Again, following results offer a direct comparison between estimation of models with and without switching as first rows of all panels in all tables in this section always represent the model setting without switching ( $\beta = 0$ ).

Table 3 confirms that the proposed method is able to recover the true value of  $\beta$  very precisely. Sample means and medians display no bias, standard deviation together with 2.5% and 97.5% empirical quantiles are reasonable, and we can see that the estimation strategy is able to test the null hypothesis  $H_0 : \beta = 0$  even for small values of  $\beta$ . As a robustness check, we repeat the estimation exercise for several different levels of noise with no significant changes in the results. The results for the different noise levels are summarised in Table 12 in Appendix A.

Larger noises seem to stabilise the system but overshadow the effect of switching under low  $\beta$  and therefore favour estimation of higher  $\beta$ . Lower values of  $\beta$  require small noises for the effect of switching to be detectable. These are crucial findings highlighting the necessity of a proper noise specification within the estimation procedure. A puzzling result is then observed in subpart (j) of Table 12 associated with the largest noise intensity  $N(0, 2^2)$  but a high precision of estimates of lowest values of  $\beta$ . Generally, values of the intensity of choice  $\beta$  very close to zero,  $\beta = \{0.1, 0.5\}$ , are the most difficult to estimate. However, these values represent almost the extreme case of no switching of agents among possible strategies in which the dynamics of the model is restrained as there is only a small difference of the model behaviour compared to the agents' absolute inertia case with  $\beta = 0$ .

An important issue emerging from simulations is the occurrence of non-convergent, or exploding, solutions of the model with respect to several noise levels. This occurs mainly for the incrementally small levels of stochastic noise with higher levels of the intensity of choice parameters. Hence, with increasing switching dynamic, a small noise does not always suffice to stabilise the system. While we focus on studying the properties of the estimation methodology, we explore the noise levels which produce stable model solutions. Namely,  $N(0, 0.1^2)$  for  $\beta = \{3, 5, 10\}$ ,  $N(0, 1)$

Figure 3: Shape of the simulated log-likelihood function.



*Note:* Results based on 100 random runs, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . Stochastic noise  $\epsilon_t$  and  $\{\epsilon_i\}_{i=1}^N$  drawn from given normal distribution. Black dotted vertical lines depict the true  $\beta$ . Bold black full lines depict sample averages. Horizontal axis shows values of  $\beta$ , vertical axis shows values of log-likelihood.

for smaller  $\beta = \{0.5, 1\}$  and for extreme cases of  $\beta = \{0, 0.1\}$  noise  $N(0, 2^2)$ . These choices are natural with respect to the intensity of choice parameters and noises stabilising the system. More importantly, these choices are also relevant for empirical applications later in the study. Figure 2 depicts smooth histograms of selected estimated  $\beta$  based on these three noise specifications, complementing Figure 1. One can clearly observe how well the estimation methodology recovers the true parameters.

#### 4.3.3. Behaviour of the simulated log-likelihood function

Kristensen and Shin (2012, pg. 80-81) define a set of regularity conditions A.1-A.4 regarding the model and its associated conditional density that ensure sufficiently fast convergence of  $\hat{c} \rightarrow c$  and thus asymptotic equivalence of  $\hat{\theta}$  and  $\tilde{\theta}$ . These conditions basically impose restrictions on the data-generating functions and the conditional density that is being estimated. With regard to data-generating functions as well as conditional density function, assumptions are quite weak and are satisfied by many models (Kristensen and Shin, 2012). However, for the Brock and Hommes (1998), we are not able to verify these conditions analytically and we must rely on computational tools. Therefore, we explore the smoothness condition, identification of parameters, and existence of unique solution via observing the simulated log-likelihood functions.

For these purposes, we draw the simulated log-likelihood function and verify an existence of a unique maximum. We depict simulated log-likelihood functions for the same three interesting values of the intensity of choice  $\beta \in \{0, 0.5, 10\}$  combined with three specifications of the stochastic noise:  $\epsilon_t \sim N(0, 2^2)$ ,  $\epsilon_t \sim N(0, 1)$  and  $\epsilon_t \sim N(0, 0.1^2)$  as earlier. In Figure 3 we clearly observe very smooth shape of the functions over the entire assessed domain with a unique maximum generally shared for all of 100 random runs. Bold black full lines represent sample averages over these 100 runs. Based on generally smooth shapes and unique optima of the simulated log-likelihood functions we assume that the regularity conditions are met for the model and the identification of parameters is also assured.

Table 4: Results for  $\beta$  estimation with normal noise, fixed  $g_h$  &  $b_h$ .

$\beta$	(a) $\hat{\beta}$ , $N(0, 0.1^2)$					(b) $\hat{\beta}$ , $N(0, 1)$				
	Med.	Mean	SD	LQ	HQ	Med.	Mean	SD	LQ	HQ
0	-.00	.00	.13	-.26	.28	-.00	-.00	.05	-.10	.09
.1	.11	.10	.12	-.10	.30	.10	.10	.06	-.02	.22
.5	.49	.49	.11	.26	.71	.50	.51	.17	.19	.85
1	1.00	1.00	.07	.86	1.14	1.00	1.01	.18	.67	1.40
3	3.01	3.00	.13	2.75	3.23	3.03	3.12	.57	2.49	5.28
5	5.00	5.00	.23	4.56	5.47	5.19	6.42	2.70	4.25	13.65
10	10.00	10.00	.12	9.77	10.23	11.04	12.28	4.64	6.07	24.11

*Note:* Stochastic noise  $\epsilon_t$  and  $\{\epsilon_i\}_{i=1}^N$  drawn from normal distributions of given parameters,  $R = 1.0001$ . Each sample is based on 1000 random runs,  $H = 5$  possible trading strategies, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits.

#### 4.3.4. Robustness checks

One of the most challenging concepts in setting of the estimation method is the repeated random generation of belief parameters  $g_h$  and  $b_h$  for each of the 1000 runs. In Table 4 we abandon this setup feature, fix parameters  $g_h$  and  $b_h$  randomly before the loop and use the very same figures for all 1000 runs. We compute results for the two most useful noise specification based on previous results, namely  $N(0, 0.1^2)$  and  $N(0, 1)$ . Comparing results in Table 4 with respective counterparts in Table 12 [subparts (h) and (i)] in Appendix A, we observe overall significant reduction of standard deviations of  $\beta$  estimates, hence further increase in the estimation precision. Fixing trend a bias parameters thus naturally makes the system more predictable and leads to more efficient estimates.

Next, we focus on ability of the estimation method to recover the true  $\beta$  parameter under belief parameters  $g_h$  and  $b_h$  drawn from the normal distribution with variances  $\{0.1^2, 0.2^2, 0.3^2, 0.4^2\}$ . This is an important robustness check because drawing the belief parameters have a large impact on the model dynamics (Barunik et al., 2009; Vacha et al., 2012; Kukacka and Barunik, 2013). Table 5 reports results from relaxing the assumption on belief parameters with the stochastic noise from the  $N(0, 0.1^2)$  distribution. We also experimented with different levels of noises, but the results are essentially the same, and we make them available upon request.

We observe a general ability of the method to reveal accurately the true value of the intensity of choice  $\beta$  for the vast majority of combinations of the simulation grid. Increasing the variance of beliefs' distribution associated with higher values of randomly generated belief parameters  $g_h$  and  $b_h$ , we generally obtain a richer model dynamics which can be more simply and more efficiently estimated.

Another important question is to what extent is the estimation methodology robust to an assumption of a wrong stochastic noise to perform estimation. This can either be a correct distribution, but with wrong parameters, or a completely different distribution. This question is especially important w.r.t. empirical application because in real world data we are rarely able to ascertain proper assumption about noise. To analyse this issue, we present results of four combinations of different distributions used for random generation of stochastic noises  $\epsilon_t$  and  $\{\epsilon_i\}_{i=1}^N$ . In Table 6 we report the case where stochastic noises  $\epsilon_t$  and  $\{\epsilon_i\}_{i=1}^N$  are drawn from the same distributions with different variances [subparts (a) and (b)], as well as different distributions with same variances [subparts (c) and (d)]. Basically, we use combinations of normal and uniform distributions and for different variances we use specifications with centupled values. Conclusions for this robustness

Table 5: Results for  $\beta$  estimation w.r.t. various distributions of  $g_h$  &  $b_h$ .

$\beta$	(a) $\hat{\beta}$ , $g_h$ & $b_h \sim N(0, 0.1^2)$					(b) $\hat{\beta}$ , $g_h$ & $b_h \sim N(0, 0.2^2)$				
	Median	Mean	SD	LQ	HQ	Med.	Mean	SD	LQ	HQ
0	.02	.01	.29	-.47	.48	-.02	-.01	.28	-.48	.48
.1	.10	.10	.12	-.09	.29	.10	.10	.12	-.10	.30
.5	.50	.51	.57	-.42	1.44	.51	.51	.48	-.40	1.39
1	.87	.94	1.15	-.95	2.91	.99	.99	.80	-.73	2.73
3	2.95	2.95	3.25	-2.67	8.69	3.00	2.97	1.81	-1.65	7.43
5	4.99	5.07	5.04	-4.26	14.17	4.97	4.77	2.60	-3.23	10.94
10	10.02	10.15	9.03	-8.22	27.82	9.98	9.92	3.65	.31	18.84
	(c) $\hat{\beta}$ , $g_h$ & $b_h \sim N(0, 0.3^2)$					(d) $\hat{\beta}$ , $g_h$ & $b_h \sim N(0, 0.4^2)$				
0	.00	.00	.23	-.47	.47	-.01	-.00	.17	-.38	.43
.1	.10	.10	.12	-.10	.30	.10	.10	.11	-.10	.30
.5	.50	.50	.34	-.34	1.27	.50	.50	.24	-.09	1.17
1	1.01	1.01	.54	-.40	2.33	1.00	1.00	.37	.20	1.85
3	3.00	3.01	.83	1.57	4.93	2.99	2.95	.78	1.24	3.76
5	5.00	5.00	1.35	3.02	6.84	5.00	5.00	.85	4.17	5.65
10	9.99	10.01	1.87	8.19	11.55	10.00	10.02	1.33	9.11	10.77

*Note:* Belief parameters  $g_h$  and  $b_h$  drawn from various normal distributions of given parameter, stochastic noise  $\epsilon_t$  and  $\{\varepsilon_i\}_{i=1}^N$  drawn from normal distribution  $N(0, 0.1^2)$ ,  $R = 1.0001$ . Each sample is based on 1000 random runs,  $H = 5$  possible trading strategies, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits.

check are very clear and can be summarised into several following points.

When a distribution with lower variance is used [subparts (a) and (b)] for generating the stochastic noise  $\epsilon_t$  then for  $\{\varepsilon_i\}_{i=1}^N$  defining the kernel approximation precision, the NPSMLE works but it produces statistically insignificant estimates and very uniformly distributed estimated values. When different distributions with the same variances are used [subparts (c) and (d)], we obtain considerably more precise estimates with markedly lower standard deviations. These findings strongly confirm the need of a proper specification of the magnitude of the stochastic noise but also relaxed requirements for accurate specification of the noise distribution for the empirical application of NPSMLE.

To assess the robustness of our general model setting, we further contrast the results in Table 12 with different setup modifications. First, we consider ten times higher gross interest rate  $R = 1 + r = 1.001$ . This daily value unrealistically represents circa 28.4% annual risk free interest rate but can nonetheless serve as a useful robustness check. Repeating the exercise for this unrealistic interest rate level delivers essentially the same results, hence we do not report them here to keep the length of the text under control. Still, this is an important finding, as the robustness of the estimation methodology w.r.t. assumption of the real market risk free rate therefore relaxes the need of a very precise derivation of this parameter for various countries and historical periods and the reasonable approximation  $R = 1 + r = 1.0001$  representing circa 2.5% annual risk free interest rate can be generally used in the empirical estimation later.

Next, we test the ability of the estimation method to provide unbiased estimates even if bounds of the parameter space are off-centred, more specifically shifted up by 50% of actual  $\beta$  to  $\langle -0.5\beta, 3.5\beta \rangle$  for  $\beta > 0$  and to  $\langle -0.375, 625 \rangle$  for  $\beta = 0$ . The ability of the NPSMLE to reveal true parameter with high precision remains unaffected and the standard deviations are similar to the original settings. We therefore verify that there is no need of an excessive precision of the

Table 6: Results for  $\beta$  estimation with various combined noises.

$\beta$	(a) $\hat{\beta}$ , $\epsilon_t \sim N(0, 0.1^2)$ , $\{\varepsilon_i\}_{i=1}^N \sim N(0, 1)$					(b) $\hat{\beta}$ , $\epsilon_t \sim U(-\frac{\sqrt{12}}{2} \times 10^{-1}, \frac{\sqrt{12}}{2} \times 10^{-1})$ , $\{\varepsilon_i\}_{i=1}^N \sim U(-\frac{\sqrt{12}}{2}, \frac{\sqrt{12}}{2})$				
	Median	Mean	SD	LQ	HQ	Med.	Mean	SD	LQ	HQ
0	.00	.00	.29	-.47	.47	-.01	-.00	.29	-.48	.47
.1	.10	.10	.11	-.09	.29	.10	.10	.11	-.09	.29
.5	.50	.50	.58	-.47	1.45	.51	.51	.58	-.47	1.46
1	.97	.99	1.13	-.86	2.89	.99	.98	1.12	-.90	2.87
3	2.87	2.84	3.30	-2.67	8.60	2.87	2.86	3.43	-2.77	8.72
5	4.89	4.87	5.31	-4.48	14.35	5.24	5.32	5.70	-4.69	14.63
10	10.96	11.42	10.56	-8.40	28.99	11.19	10.21	11.65	-9.95	29.17
$\beta$	(c) $\hat{\beta}$ , $\epsilon_t \sim N(0, 0.1^2)$ , $\{\varepsilon_i\}_{i=1}^N \sim U(-\frac{\sqrt{12}}{2} \times 10^{-1}, \frac{\sqrt{12}}{2} \times 10^{-1})$					(d) $\hat{\beta}$ , $\epsilon_t \sim U(-\frac{\sqrt{12}}{2} \times 10^{-1}, \frac{\sqrt{12}}{2} \times 10^{-1})$ , $\{\varepsilon_i\}_{i=1}^N \sim N(0, 0.1^2)$				
	Median	Mean	SD	LQ	HQ	Med.	Mean	SD	LQ	HQ
0	.02	.01	.27	-.50	.50	-.00	-.00	.22	-.47	.46
.1	.10	.10	.14	-.10	.30	.10	.10	.12	-.10	.30
.5	.50	.50	.40	-.43	1.44	.50	.50	.33	-.27	1.27
1	1.01	1.01	.53	-.21	2.39	1.00	1.03	.52	-.31	2.45
3	3.02	3.03	.86	1.46	4.70	3.00	3.03	1.06	.64	6.23
5	5.01	5.01	1.18	3.05	6.76	5.00	5.00	1.26	3.32	6.79
10	10.00	10.01	1.43	8.13	11.88	10.00	10.08	1.67	8.83	11.70

*Note:* Stochastic noise  $\epsilon_t$  and  $\{\varepsilon_i\}_{i=1}^N$  drawn from the same distributions with different variances in (a) and (b), and different distributions with the same variances in (c) and (d),  $R = 1.0001$ . Each sample is based on 1000 random runs,  $H = 5$  possible trading strategies, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits.

unconstrained pre-estimation via which we define bounds of the parameter space for the constrained optimisation.

We further test how the performance is affected by assumption of different stochastic noises  $\epsilon_t$ . For that purpose we select uniform distribution both for its simplicity as well as for its feature of being the maximum entropy probability distribution of its family of symmetric probability distributions. Contrasting original results in [Table 12](#) with results based on the uniform distribution of the stochastic noise, we basically verify the assumption of [Kristensen and Shin \(2012\)](#) that  $\mathcal{D}_\epsilon$  can be any known distribution. The overall result are largely similar and the observed differences can be attributed mainly to different shapes of the normal and uniform distributions.

#### 4.4. 2-type model estimation

An important advantage of FABMs is that their dynamics is mostly driven by few crucial parameters. As a result, we might promisingly attempt to estimate all essential coefficients simultaneously and thus we do not need any rigorous criteria for selection. In the [Brock and Hommes \(1998\)](#) setting we select estimated parameters consistently with the current literature (see [Tables 1 and 2](#)), i.e. the switching parameter  $\beta$  and the behavioural belief coefficients. The other coefficients, e.g. the risk aversion  $a$ , the conditional variance of excess returns  $\sigma^2$ , or the risk free rate  $R$  are simplified already in the original model as constants and shared by all agent classes. The model is then theoretically derived based on these assumptions. Given parameters only influence the absolute values of the profitability measures  $U_h$  but not their relative proportions,  $R$  additionally a little bit adjusts the model output  $x_t$ . Thus we can naturally consider them not influencing dynamics of the model as described in [Subsection 4.1](#).

A natural subsequent step of the NPSMLE method testing is thus a multiple parameter estimation in which we simultaneously estimate the intensity of choice  $\beta$  and agents' belief coefficients  $g_h$  and  $b_h$  defining individual trading strategies in the 2-type and the 3-type models for which both theoretical as well as empirical rationale exists in the current literature as indicated in [Chen et al. \(2012, pg. 191, 207\)](#). With reference to [Biondi et al. \(2012, pg. 5534\)](#), "it has been advocated that the two broad categories of chartism and fundamentalism account for most of possible investment strategies". The aim of this analysis is to assess the performance of the NPSMLE method in estimating other model parameters then solely the intensity of choice  $\beta$ .

First, we study the most simple system consisting of two trading strategies, where fundamental strategy again appears in the market by default, i.e.  $g_1 = b_1 = 0$ . Based on the knowledge gained in [Subsection 4.3](#), we define a discrete grid of combinations of the true intensity of choice  $\beta$  and the chartistic beliefs  $g_2$  and  $b_2$  representing the second-type trading strategy to cover a purposeful range of values w.r.t. issues studied in the previous sections. To keep a reasonable number of combinations and lucidity of results, we opt for  $\beta = \{0, 0.5, 3, 10\}$ . In defining a grid of chartistic beliefs, we also cover various combinations of trend following ( $g_2 > 0$ ), contrarian ( $g_2 < 0$ ), upward-biased ( $b_2 > 0$ ), and downward-biased ( $b_2 < 0$ ) strategies based on multiples of standard deviations used in previous sections. We refer the reader to the first column of [Table 7](#) for detailed specification. We report only one specification of noise, namely  $\epsilon_t \sim N(0, 0.1^2)$ , as the results for  $\epsilon_t \sim N(0, 1)$  essentially do not change.

#### 4.4.1. Quantitative results

In [Table 7](#) we summarise the simulation results. Basically, we are able to confirm all main findings from the simulation analysis of the single parameter  $\beta$  estimation. First, the method is generally able to reveal accurately the true values of estimated parameters also in the 3-parameter simultaneous estimation case. Especially belief coefficients  $g_2$  and  $b_2$  [subparts (b) and (c)], that are of central importance in this section, are estimated overall significant and with almost surprisingly high precision. The estimation precision of the  $\beta$  parameter is not directly comparable to previous results as the setting of the 2-type model is different from the previous model. Notwithstanding, we still get generally conformable figures. We again clearly observe the relative estimation inefficiency in case of setting combinations with small values of the intensity of choice  $\beta = 0, 0.5$ , especially when combined with small values of belief coefficients  $g_2$  and  $b_2$  (the upper half of [Table 7](#)). On the other hand, for high values of belief coefficients  $g_2$  and  $b_2$  (the bottom half of [Table 7](#)) we generally confirm findings from  $\beta$  estimation exercise w.r.t. various distributions of beliefs parameters ([Table 5](#)). We further observe a small upward bias tendency in  $\beta$  estimates for smallest values of belief coefficients  $g_2$  and  $b_2$ . When it comes to efficiency of  $g_2$  and  $b_2$  estimates, the setup produces markedly precise estimates.

Although we report all combination of trend following ( $g_2 > 0$ ), contrarian ( $g_2 < 0$ ), upward-biased ( $b_2 > 0$ ), and downward-biased ( $b_2 < 0$ ) strategy specifications mainly for technical reasons and the analysis of the model dynamics goes beyond the scope of this paper, we can observe some patterns regarding the beliefs' combination. For instance the efficiency of the estimation is considerably higher for trend following ( $g_2 > 0$ ) beliefs than for contrarians ( $g_2 < 0$ ).

The favourable results presented above give promise for the function of the method also in more complex settings.

Table 7: Results of 3-parameter estimation of a 2-type model.

$\beta, g_2, b_2$	(a) $\hat{\beta}$			(b) $\hat{g}_2$		(c) $\hat{b}_2$		(d) $LL$		
	Median	Mean	SD	Mean	SD	Mean	SD	L-rat	$2\Delta LL$	p-v
.0, .2, .15	-.01	-.00	.29	.20	.03	.15	.01	1	0	1
.5, .2, .15	.49	.50	.57	.20	.03	.15	.01	1	0	1
3, .2, .15	3.01	2.99	3.44	.20	.05	.15	.01	1	0	1
10, .2, .15	11.60	12.19	9.46	.19	.10	.15	.01	1	0	1
.0, -.2, -.15	.02	-.00	.28	-.20	.03	-.15	.01	1	0	1
.5, -.2, -.15	.50	.48	.56	-.20	.03	-.15	.01	1	0	1
3, -.2, -.15	3.11	2.97	3.39	-.20	.05	-.15	.01	1	0	1
10, -.2, -.15	10.46	9.93	11.30	-.19	.10	-.15	.01	1	0	1
.0, .2, -.15	.03	.01	.28	.20	.03	-.15	.01	1	0	1
.5, .2, -.15	.56	.52	.56	.20	.03	-.15	.01	1	0	1
3, .2, -.15	3.31	3.12	3.40	.20	.05	-.15	.01	1	0	1
10, .2, -.15	11.97	12.23	9.60	.19	.10	-.15	.01	1	0	1
.0, -.2, .15	-.00	-.01	.30	-.20	.03	.15	.01	1	0	1
.5, -.2, .15	.53	.50	.60	-.20	.03	.15	.01	1	0	1
3, -.2, .15	3.31	3.08	3.53	-.20	.04	.15	.01	1	0	1
10, -.2, .15	11.37	10.62	11.69	-.19	.10	.15	.01	1	0	1
.0, .4, .3	.00	.01	.33	.40	.03	.30	.01	1	0	1
.5, .4, .3	.50	.49	.48	.40	.04	.30	.01	1	0	1
3, .4, .3	2.99	3.04	.46	.40	.04	.30	.01	.99	.01	.92
10, .4, .3	10.00	10.01	.61	.40	.03	.30	.01	.94	.13	.62
.0, -.4, -.3	-.08	-.06	.30	-.40	.03	-.30	.01	1	0	1
.5, -.4, -.3	.31	.35	.58	-.39	.04	-.30	.01	1	0	1
3, -.4, -.3	2.96	2.84	1.47	-.39	.06	-.30	.01	1	0	1
10, -.4, -.3	10.08	10.11	1.24	-.40	.04	-.30	.01	.99	.03	.86
.0, .4, -.3	-.01	.00	.31	.40	.03	-.30	.01	1	0	1
.5, .4, -.3	.51	.48	.48	.40	.04	-.30	.01	1	0	1
3, .4, -.3	3.03	3.04	.46	.40	.04	-.30	.01	.99	.01	.92
10, .4, -.3	9.98	10.02	.63	.40	.03	-.30	.01	.94	.13	.62
.0, -.4, .3	-.06	-.05	.30	-.40	.03	.30	.01	1	0	1
.5, -.4, .3	.23	.32	.58	-.40	.04	.30	.01	1	0	1
3, -.4, .3	2.98	2.87	1.38	-.40	.06	.30	.01	1	0	1
10, -.4, .3	9.98	10.00	1.13	-.40	.04	.30	.01	.99	.03	.86

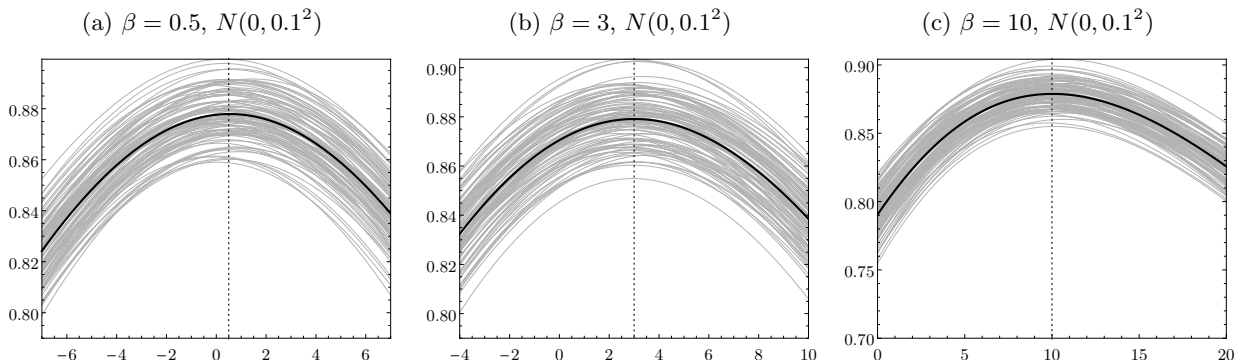
Note: Stochastic noise  $\epsilon_t$  and  $\{\epsilon_i\}_{i=1}^N$  drawn from normal distribution  $N(0, 0.1^2)$ ,  $R = 1.0001$ . Each sample is based on 1000 random runs, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . Sample medians, means, and standard deviations (SD) are reported. Figures are rounded to 2 decimal digits. 'L-rat' denotes the likelihood ratio of the null static (i.e. restricted) model vs. the alternative switching model, ' $2\Delta LL$ ' is the test statistics of the log-likelihood ratio test being approximately  $\chi^2$  distributed with 1 degree of freedom, and 'p-v' is related p-value of the test.

#### 4.4.2. Behaviour of the simulated log-likelihood function

To verify smoothness conditions and identification of parameters in the 2-type model estimation case, we again depict simulated log-likelihood functions. As we can hardly visualise the 4D shape of the resulting simulated log-likelihood function, we depict sub-log-likelihood functions in 2D and 3D providing the global shape when combined in a 4D object. One can think of these sub-log-likelihood functions as transversal cuts or profiles of the likelihood function in the planes of given parameters.

Figure 4 demonstrates simulated 2D sub-log-likelihood functions for estimation of the single parameters of the intensity of choice  $\beta \in \{0.5, 3, 10\}$ , trend coefficient  $g_2 = 0.4$ , and bias coefficient  $b_2 = 0.3$ , keeping the two others fixed. We again clearly observe very smooth shapes and unique

Figure 4: Simulated sub-log-likelihood functions for  $\beta$  estimation.



Note: Results based on 100 random runs,  $g_2 = 0.4$ ,  $b_2 = 0.3$ , number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . Stochastic noise  $\epsilon_t$  and  $\{\epsilon_i\}_{i=1}^N$  drawn from given normal distribution. Black dotted vertical lines depict the true  $\beta$ . Bold black full lines depict sample averages. Horizontal axis shows values of  $\beta$ , vertical axis shows values of log-likelihood.

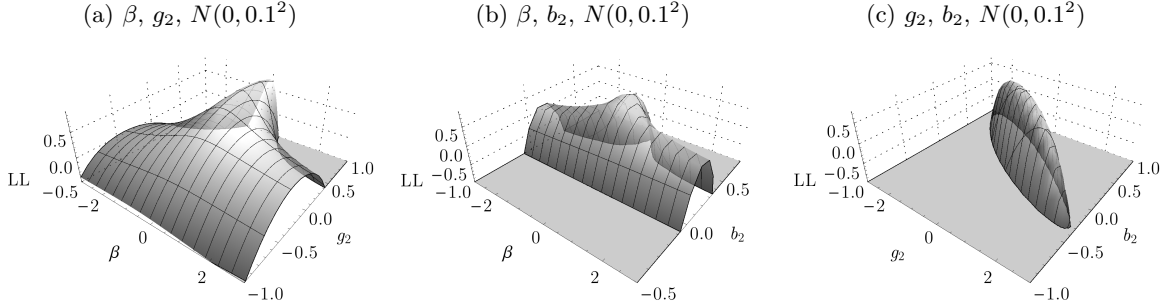
maxima generally shared for all random runs. In Figure 5 we visualise three dimensional simulated sub-log-likelihood functions based on all possible combinations of three parameters of interest, keeping one of them fixed, for the model setting with:  $\beta = 0.5$ ,  $g_2 = 0.4$ ,  $b_2 = 0.3$ . The smoothness of the surface generally keeps retained also in the 3D visualisations and regions of possible maxima are well detectable although the 3D depiction cannot provide such detailed and ‘zoomed’ view as the 2D visualisations in Figure 4. Parameters  $g_2$  and  $b_2$  seem to be relatively well identified which confirms the previous results. As expected, the most challenging part is revelation of the  $\beta$  coefficient in which direction the surface is very flat for a large interval of the domain. These findings are largely in accord with conclusions of Bolt et al. (2014, pg. 15) and Hommes and Veld (2015) who claim that “the other parameters can to a large extent compensate for changes in  $\beta$ ” and report very flat shape of the likelihood function for the intensity of choice selection. In any case, based on these results we again generally assume that the regularity conditions are met and the identification of parameters is assured also for the 2-type model estimation.

#### 4.4.3. Likelihood-ratio test

Previous sections show that the non-parametric simulated maximum likelihood estimation method does a fairly good job in distinguish between various  $\beta$ . As a next step we might be interested how capable the estimation method is in a rigorous statistical comparison between static and switching models. As the static version of the model with  $\beta = 0$  and the switching version are nested, i.e. the less complex static model is derived via a restriction on  $\beta$  from the switching model, we can apply the usual likelihood ratio test to assess the relative goodness of fit between models with and without switching. For this purpose Table 7 further displays information about tests of model fit in subpart (d). ‘L-rat’ column denotes the likelihood ratio of the null static (i.e. restricted) model vs. the alternative switching model, ‘ $2\Delta LL$ ’ is the test statistics of the log-likelihood ratio test being approximately  $\chi^2$  distributed with 1 degree of freedom, and ‘p-v’ is related p-value of the test. Application of the likelihood ratio test seems natural in this situation, nevertheless the Monte Carlo simulation framework brings several imperfections. On the



Figure 5: Simulated sub-log-likelihood functions in 3D.



*Note:* Results averaged over 30 random runs, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . The complete set of true parameters:  $\beta = 0.5$ ,  $g_2 = 0.4$ ,  $b_2 = 0.3$ . Stochastic noise  $\epsilon_t$  and  $\{\epsilon_i\}_{i=1}^N$  drawn from given normal distribution.

one hand, simulated data smartly avoid the problem of model misspecification, on the other hand the goodness of fit test is designed rather for comparison based on a single empirical dataset. In our situation, new variables and a different dataset are randomly generated in each of 1000 random runs and the test is then based on the aggregate mean values over all runs. But since the standard deviation of individual maximum log-likelihoods is negligible compared to their value (around 1%), aggregation does not cause any appreciable loss of information. A second imperfection is related to the relative flatness of the log-likelihood function in the dimension of the restricted switching coefficient  $\beta$  for a large interval of the domain. Although the estimation method detects precisely the true intensity of choice especially for combinations of higher values of  $\beta$ , stronger strategies, and lower stochastic noise specification, due to flat likelihood the test exhibits only a moderate capability of distinguishing between the restricted and the unrestricted model. Translated into p-values of the test, in the most distinct cases the values are far from being able to reject the static model. Inspecting subpart (d) of [Table 7](#), we observe expected behaviour but generally low power of the test. In all cases of  $\beta = 0$ , the likelihood ratio is equal to 1 and the p-value remains 100%. Increasing true  $\beta$  and strength of strategies, the likelihood ratio and the p-value naturally decrease along with increase of the test statistics ‘ $2\Delta LL$ ’, but the pace of the progress is low for the selected range of  $\beta$ .

#### 4.5. 3-type model estimation

Results of simultaneous estimation of 5 parameters in the 3-type model including three basic strategies: fundamental represented by  $g_1 = b_1 = 0$  and two chartistic defined in the [Table 14](#), can be found in [Appendix A](#). We keep the strategy of defining a grid of chartistic beliefs from previous analysis based on various combinations of trend following ( $g_2 > 0$ ), contrarian ( $g_2 < 0$ ), upward-biased ( $b_2 > 0$ ), and downward-biased ( $b_2 < 0$ ) strategies and the same multiples of standard deviations. Conclusions are generally in accord with the results of the 2-type model estimation, the difference is mainly in efficiency of estimates that is by nature lower than for the 2-type model. However, combining two chartistic strategies we also gain some new knowledge about the system behaviour. For instance, in case of a combination of two trend following strategies ( $g_2 > 0, g_3 > 0$ ), it is rather complicated for the NPSMLE method to distinguish between impacts

of these two strategies leading to lower statistical significance of both estimates compared to other combinations of trend following and contrarian strategies.

## 5. HAM estimation on empirical data

Equipped with the performance study of the proposed methodology, we broaden the topic with an empirical application and estimate the [Brock and Hommes \(1998\)](#) model using cross section of world stock markets. We analyse S&P500 and NASDAQ for the U.S., DAX and FTSE and for Europe, NIKKEI 225 and HSI for Japan and Hong Kong, respectively.

### 5.1. The estimation setting

Compared to the simulation study, the setting of models estimated using real data is less challenging. It is also less computationally intensive in terms of statistical validity. On the contrary, the estimation algorithm is a bit technically more complicated as the structure of the real world data is far away from regularity of simulated datasets. Concurring findings about the sufficient setting of the NPSMLE, we compute results for 1000 random runs, number of observations  $t = 5000$ , and the kernel approximation precision is set to  $N = 500$ . On the other hand, because of a problematic numerical stability of the model when real data analysis is introduced, we increase the number of randomly generated starting points for the numerical optimisation to eight. This is a sufficient quantity to filter out issues related to possible numerical instability on the system. The other setting remains the same as defined earlier.

Based on the previous findings, the proper intensity of stochastic market noise is crucial for the proper estimation. A wrong stochastic noise specification is likely to influence the behaviour of the model heavily, hence influence validity or results. We do not longer use the grid strategy to ensure the robustness of result because the stochastic noise intensity in various real markets is a hardly anticipated variable. Leaving this idea, we instead add the intensity of the stochastic market noise to the list of estimated parameters. Generally, we thus apply a simultaneous unconstrained multivariable function estimation of all interesting parameters: agents' belief coefficients defining individual trading strategies  $g_h$  and  $b_h$ , the intensity of choice  $\beta$ , and the intensity of market noise, which is defined as a fraction of the standard deviation of the noise term and the standard deviation of the data and denoted as *noise intensity*.

First, we estimate the simple 2-type model including two basic strategies only: fundamental one represented by implicitly defined  $g_1 = b_1 = 0$  and chartistic one which is to be estimated. Within this setting, we simultaneously estimate four parameters of interest:  $\beta$ ,  $g_2$ ,  $b_2$ , and the *noise intensity*. To support the numerical stability of the estimated system, we constrain the intervals for the starting points of random generation to the economically meaningful values which preserve stability of the model, namely  $\langle -0.5, 0.5 \rangle$  for  $\beta$ ,  $\langle 1.3, 2.3 \rangle$  for  $g_2$ ,  $\langle -0.2, 0.2 \rangle$  for  $b_2$ , and  $\langle 0.4, 0.9 \rangle$  for the *noise intensity*. Nonetheless, as the algorithm is designed to find an optimum of an unconstrained multivariable function, it can freely leave these initial intervals during the search procedure.

We then continue with the estimation of the 3-type model including three basic strategies: fundamental and two different chartistic strategies which are to be estimated. Based on results of the 2-type model estimation, we assume zero bias of both the trend following as well as the contrarian strategy, i.e.  $b_2 = b_3 = 0$ . Moreover, the simultaneous estimation of four parameters of our interest technically requires a modification of the algorithm setting to the constrained multivariable function estimation. The reason is that the two different strategies, the trend following

$g_2 > 0$  and the contrarian  $g_3 < 0$ , need to be strictly distinguished using the following constraints:  $g_2 \in \langle 1.8, 2.8 \rangle$ ,  $g_3 \in \langle 0, -0.5 \rangle$ .

## 5.2. Fundamental price approximation

Approximation of the fundamental price is inevitably the most challenging issue of the entire empirical estimation. In this, we follow the existing literature which approximates the fundamental price by a moving average (Winker et al., 2007; ter Ellen and Zwinkels, 2010; Huisman et al., 2010). For example Winker et al. (2007) assume as the proxy for the fundamental price a moving average (MA) over the last 200 observations of the DM/USD exchange rate time series for the period 1991/11/11 to 2000/11/9. ter Ellen and Zwinkels (2010) use the MA of the Brent and WTI Cushing oil monthly USD prices over 24 months, i.e. from 1984/1 to 2009/8. Huisman et al. (2010) employ the MA of European forward electricity daily historical prices over three years for the base-load calendar year 2008 forward contracts and set the MA window to 3 as a calibration result of the optimal length.<sup>5</sup>

Long-term and short-term MAs are also commonly used by practitioners in trading to extrapolate divergence from the fundamental value in the technical analysis. Since the fundamental value of stocks is essentially unknown, market practitioners often tend to at least estimate whether the stock is over or under-valued, whether the possible mispricing is small or large, and whether the gap is going to increase or whether a soon correction is more likely. As the Brock and Hommes (1998) model is also formulated in deviations from the fundamental price, the MA approach seems to be reasonable approximation. The MA filtering is a cornerstone of the technical analysis and therefore widely used by active traders: Allen and Taylor (1990, pg. 50) present empirical evidence on the perceived importance of technical analysis among London foreign exchange dealers and refer to prevalent mechanical indicators such as trend-following rules: “buy when a shorter MA cuts a longer MA from below”. Taylor and Allen (1992) survey chief foreign exchange dealers operating in London and report that 64.3% of organisations use MAs and/or other trend-following analytical techniques. Brock et al. (1992, pg. 1735) refer to MA technical rules as to one of the two simplest and most widely used: “when the short-period MA penetrates the long-period MA, a trend is considered to be initiated”. Lui and Mole (1998, pg. 541, 535) repeat largely analogical survey as Taylor and Allen (1992) among Hong Kong foreign exchange dealers and report the same conclusion for the usefulness of MAs at intraday, intramonth, and  $> 1$  month horizons. Goldbaum (1999, pg. 70, 71) describes the way how in practice the MA trading rules translate into buy-sell indicators, and Sullivan et al. (1999, pg. 1656) summarise that “MA cross-over rules... are among the most popular and common trading rules discussed in the technical analysis literature”. Closely related to our work, Chiarella et al. (2006, pg. 1748) propose a model in which the demand of chartists is determined by the difference between a long-term MA and current market price.

In our empirical estimation, we keep to the strategy of a wide range of possible settings to ensure robustness of our findings. We present results for two specific MA window lengths,

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<sup>5</sup>In contrast, another class of FABMs of FX markets successfully utilises the Purchasing Power Parity between two countries as the approximation of the fundamental value of the currency exchange rate [see e.g. Vigfusson (1997); Westerhoff and Reitz (2003); Manzan and Westerhoff (2007); Wan and Kao (2009); Goldbaum and Zwinkels (2014); Verschoor and Zwinkels (2013)]. Boswijk et al. (2007) and de Jong et al. (2009a) employ the static Gordon growth model for equity valuation proposed by Gordon (1962), which is, however, infeasible for the empirical validation of the original Brock and Hommes (1998) model. Some other papers simply use a random walk formula to drive the fundamental price (De Grauwe and Grimaldi, 2005, 2006; Winker et al., 2007; Franke, 2009).

namely 61 (MA61) and 121 (MA121) days. For the robustness check, we also tested other variants ranging from one month to two years, namely 21, 241, and 481 days, leading to comparable results. These results are available upon request from authors. Instead of usual ‘historical’ MA taking into account only the past information for given time, we use the ‘centred’ MA. Both MA versions were analysed and found to produce to a large extent comparable results. The centred MA is therefore suggested to reduce the delay of the information flow. Moreover, the centred MA incorporates a convenient property that the price converges to it by definition that is exactly a feature one would expect from the fundamental value. Although undoubtedly our fundamental price approximation differs from the true fundamental value, the MA filter produces a series of an anticipated structures to be modelled.

### 5.3. Data description

We use daily close prices of six world stock market indices as the base of our empirical dataset. For S&P500, we retrieve the close prices of the index from the Yahoo Finance covering the period from 1994/02/23 to 2013/12/31, i.e. 5000 observations in total. For other indices, only the starting dates of the dataset vary by reason of different public holidays a calendar configurations around the world, i.e. 1994/04/22 for DAX, 1994/11/02 for FTSE, 1994/02/23 for NASDAQ, 1993/09/03 for NIKKEI 225, and 1994/06/13 HSI. For each index we thus obtain a comparable amount of 5000 observations with the same end date 2013/12/31. The fundamental price is simultaneously calculated as the centred MA and subtracted from the actual price.<sup>6</sup> Thus we obtain deviations  $x_t$  from the fundamental price that are the subject of further estimation. Descriptive statistics of  $x_t$  series for all indices and two MA lengths for the fundamental value approximation are summarised in [Table 13](#) and in [Appendix A](#).

### 5.4. Full sample estimates of the 2-type model

We start with the full sample estimation summarised in [Table 8](#). Generally, we can observe broad similarities across all indices and markedly statistically significant estimates of a positive belief parameter  $g_2$  revealing superiority of trend following over contrarian strategies in markets. In contrast, the estimates of the intensity of choice  $\beta$  and the bias parameter  $b_2$  are largely statistically insignificant. While for the bias this is an expected result as there is no obvious reason why the trend following strategies should be somehow biased in the long-term, the insignificance of  $\hat{\beta}$  is an important and interesting result. We thus contrast a large subpart of the FABM estimation literature (see [Section 2](#)) but confirm the main results of e.g. [Westerhoff and Reitz \(2005\)](#); [Boswijk et al. \(2007\)](#); [de Jong et al. \(2009b\)](#); [ter Ellen et al. \(2013\)](#); [Bolt et al. \(2014\)](#). Since the heterogeneity in trading regimes is confirmed by the significance of  $g_2$ , this might not worrying as discussed in [Boswijk et al. \(2007, pg. 1995\)](#) or [Hommes \(2013, pg. 203\)](#) who emphasise that “this is a common result in non-linear switching regression models, where the parameter in the transition function is difficult to estimate and has a large standard deviation, because relatively large changes in  $\beta^*$  cause only small variation of the fraction  $n_t$ . [Teräsvirta \(1994\)](#) argues that this should not be worrying as long as there is significant heterogeneity in the estimated regimes”. Furthermore, as [Huisman et al. \(2010, pg. 17, 20\)](#) point out, “the significance of the intensity of choice is not a necessary condition for the switching to have added value to the fit of the model” and “the non-significant

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<sup>6</sup>For the calculation of the fundamental price we need some extra data points preceding and succeeding the defined period, the complete dataset retrieved and used therefore consists of ‘4999 + MA window length’ observations.

Table 8: Empirical results of the 2-type  $\beta$  model estimation.

Data, MA period	(a) $\hat{\beta}$			(b) $\hat{g}_2$			(c) $\hat{b}_2$			(d) $\widehat{\text{noise intensity}}$			(e) $LL$		
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD
SP500, 61	.015	.040	.122	1.567	1.587	.233	.009	.003	.121	.653	.656	.108	-1.486	-1.491	.074
NASDAQ, 61	.002	.006	.146	1.717	1.715	.166	-.005	-.003	.092	.609	.609	.079	.117	.115	.038
DAX, 61	.001	.018	.112	1.640	1.646	.215	.008	.001	.117	.590	.601	.099	-1.259	-1.264	.081
FTSE, 61	-.000	.008	.113	1.671	1.668	.201	.001	.004	.117	.597	.602	.092	-.988	-.995	.059
HSI, 61	-.004	-.002	.150	1.724	1.727	.164	-.003	-.001	.098	.566	.570	.069	-.145	-.147	.036
N225, 61	.008	.021	.100	1.601	1.619	.210	.011	.007	.121	.593	.601	.103	-1.544	-1.546	.085
SP500, 241	-.021	-.009	.195	1.878	1.875	.127	.007	.001	.111	.388	.397	.064	.029	.019	.074
NASDAQ, 241	.038	.047	.189	1.882	1.869	.152	-.009	-.003	.105	.423	.419	.078	.422	.424	.111
DAX, 241	.014	.012	.247	1.932	1.928	.109	.007	.003	.103	.292	.315	.070	.328	.309	.096
FTSE, 241	.010	.012	.162	1.871	1.858	.143	.008	.004	.114	.408	.409	.071	-.153	-.157	.085
HSI, 241	.008	.006	.204	1.914	1.908	.130	.004	.002	.103	.351	.366	.080	.369	.363	.118
N225, 241	.003	.012	.147	1.865	1.850	.165	-.000	-.000	.125	.394	.392	.101	-.793	-.793	.160
Robustness check															
SP500 monthly, 13	-.016	-.031	.272	.886	.891	.238	.006	.002	.121	.863	.869	.104	-3.660	-3.666	.054
SP500 weekly, 13	-.085	-.103	.228	.953	.971	.224	-.003	-.001	.121	.865	.869	.098	-2.336	-2.349	.057
SP500 weekly, 49	.044	.089	.146	1.119	1.168	.241	.005	.002	.118	.724	.720	.103	-2.626	-2.615	.073
SP500 R=1.001, 61	.016	.035	.112	1.585	1.607	.241	.004	.004	.121	.656	.654	.110	-1.480	-1.485	.073
SP500 R=1.001, 241	.001	-.001	.196	1.889	1.880	.134	.005	.002	.111	.393	.397	.062	.029	.019	.073
SP500 $m_h=40$ , 61	-.032	-.041	.149	1.659	1.673	.173	-.002	-.002	.114	.584	.587	.075	-1.393	-1.402	.038
SP500 $m_h=40$ , 241	-.028	-.020	.298	1.908	1.905	.107	-.004	-.003	.099	.338	.355	.056	.100	.086	.054
SP500 $m_h=80$ , 61	-.047	-.054	.177	1.644	1.664	.182	.003	.003	.115	.573	.576	.065	-1.385	-1.392	.032
SP500 $m_h=80$ , 241	-.036	-.014	.309	1.908	1.906	.105	-.002	-.004	.097	.335	.352	.054	.106	.091	.056

*Note:* Results are based on 1000 random runs, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 500$  i.i.d. draws from normal distribution. Sample medians, means, and standard deviations (SD) are reported. ‘LL’ denotes log-likelihoods of estimated models representing statistical fits. Figures are rounded to 3 decimal digits.

intensity of choice... indicates that the switching does not occur systematically". The magnitudes of trend parameter estimates  $\hat{g}_2$ , that keep roughly between 1.6 and 1.9, might seem large, but it is important to note that they influence the price change only from circa 50% implied by the insignificance of the intensity of choice  $\beta$  keeping the population ratio of the two strategies stable around 0.5/0.5.

In this situation, it is useful to revisit the results from simulations in previous sections, where we find the estimation methodology able to recover even very small  $\beta = 0.5$  with a high precision when similar computational setting is used for number of observations, runs, and kernel approximations. Thus, if there is some behavioural switching present in the empirical data, it should have been detectable under similarly robust estimation.

Differences across markets can be partly seen in the (d) column of Table 8 between well efficiently estimated values of the *noise intensity*. Although the difference are often on the border of statistical significance, it might be worth mentioning that the highest stock market noise intensity is estimated for the U.S. indices, specifically S&P500 in case of MA61 based fundamental value and NASDAQ in case of MA241 based fundamental value. Conversely, the lowest values are estimated for DAX and the difference is circa 30% in case of the MA241 based fundamental value.

The level differences in values between the upper part of Table 8 depicting results for the MA61 fundamental price approximation and the middle part with results for the MA241 is perhaps mainly the technical feature of different MA windows. It is therefore important to consider absolute values of estimated coefficient with this respect and compare both versions. Nevertheless, the main results concerning the positive sign and statistical significance of  $\hat{g}_2$  and insignificance of  $\hat{\beta}$  and  $\hat{b}_2$  keep similar as well as the main detected relative relationships between values of the *noise intensity*. Our most important results thus demonstrate *robustness* w.r.t. the choice of the fundamental value specification. The lower values of the *noise intensity* might be explained by reason of a better fundamental value approximation using bigger MA window.

#### 5.4.1. Behaviour of the simulated log-likelihood function

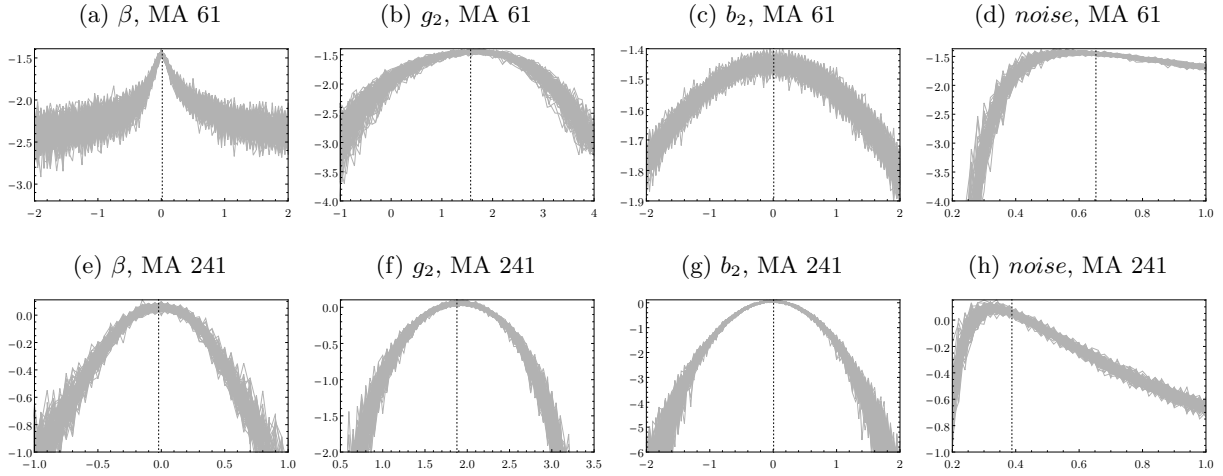
We verify the smoothness conditions and unique maxima presence of the simulated log-likelihood functions that are crucial for the parameter detection and identification also within the empirical application. However, the 5D surface resulting from simultaneous estimation of four parameters makes the graphical demonstration bit complicated. To see the behaviour of log-likelihoods, we therefore draw sub-log-likelihood functions in 2D, assuming other parameters fixed at estimated values from Table 8. This graphical depiction shows how the parameters are identified.

Figure 6 demonstrates partial 2D shapes of the simulated sub-log-likelihood function in direction of individual parameters. Generally we observe a bit rough shape in detail, but very consistent performance of the estimation method over all 100 random runs leading to unique maxima consistent with the full sample estimates in all cases. The shape is affected by the structure of the real world data which is far away from regularity of simulated datasets. For this reason, we also adapt the computational algorithm by increasing the number of initial points as mentioned earlier in the estimation setup.

#### 5.4.2. Robustness check of the 2-type model

For the robustness check of the validity of estimated values (results are reported in Table 8, bottom part), we not only use more than single MA specification of the fundamental value, but also consider several modification of the setup and even different frequency of the data. Equipped with the knowledge from previous analysis, we again only compute results for S&P500. Aside utilisation

Figure 6: Simulated sub-log-likelihood fcn. for single parameters.



*Note:* Results based on 100 random runs, S&P500 data, and given MA fundamental price approximation.  $\{\varepsilon_i\}_{i=1}^N$  drawn from normal distribution. Black dotted vertical lines depict estimated parameters (see Table 8). Horizontal axis shows values of estimated parameters, vertical axis shows values of log-likelihood.

of weekly and monthly data, we also follow the robustness testing from Section 4 and estimate the model using:

1. 10 times higher assumed market risk free rate  $R = 1.001$ ;
2. nontrivial memory of agents defined via parameters  $m_h = \{40, 80\} \forall h \in \{1, \dots, H\}$  from Equation 15. The average memory length thus resembles circa the one- or two-month period, i.e. 20 or 40 days.

Three new datasets cover S&P500 weekly data from 1994/02/28 to 2013/12/30 (i.e. 1035 observations) and monthly data from 1994/03/01 to 2013/12/02 (i.e. 238 observations). Selected periods cover the same span as the original daily dataset. The MA lengths are selected so that they resemble most closely the 61 and 241 days for the fundamental value specification, i.e. 13 and 49 weeks in case of weekly data and 13 months for monthly data. The assumed market risk free rate is adjusted to reflect the modified data periodicity, namely to  $R = 1.0005$  for weekly data and  $R = 1.002$  for monthly data.

The most important findings of the preceding empirical analysis remain unaffected under the robustness burden. The  $\hat{\beta}$  parameter still reveals evident statistical insignificance, the same does the bias parameter  $\hat{b}_2$ . Differences are basically observable at the level of trend parameters  $\hat{g}_2$  and *noise intensity*, but the behaviour keeps patterns uncovered within the original analysis:  $\hat{g}_2$  slightly increases and *noise intensity* decreases moving from MA61 to MA241 fundamental value approximation. Results based on monthly and weekly data show considerably lower  $\hat{g}_2$ , the values fall even under 1 for the MA13 fundamental value specification, that is, instead of a strong trend chasing strategy only a weak trend chasing strategy is detected. However, this seems to be again an implied technical side-effect of a small MA window that produces more average dynamics of the price deviations series. Memory only slightly increases the model fit and as the interconnected effect it simultaneously decreases the *noise intensity*. Nothing surprising is therefore found within the

Table 9: Empirical results of the 3-type  $\beta$  model estimation.

Data, MA p.	(a) $\hat{\beta}$		(b) $\hat{g}_2$		(c) $\hat{g}_3$		(d) $\widehat{\text{noise } i.}$		(e) $LL$	
	Med.	SD	Med.	SD	Med.	SD	Med.	SD	Med.	SD
SP500, 61	-.003	.082	2.502	.175	-.123	.111	.550	.047	-.127	.022
SP500, 241	.007	.050	2.674	.217	-.032	.142	.403	.045	-.289	.094

*Note:* Results are based on 500 random runs, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 500$  i.i.d. draws from normal distribution. Sample medians and standard deviations (SD) are reported. ‘LL’ denotes log-likelihoods of estimated models representing statistical fits. Figures are rounded to 3 decimal digits.

validity check of original results that approve robustness to various data frequency specifications as well as modifications of interesting parameters in the model.

As another robustness check, we estimate the model on a yearly rolling windows to uncover dynamics in time.  $\hat{\beta}$  and  $\hat{b}_2$  keep to zero and are statistically insignificant in all the windows. In general, the rolling approach in particular strongly supports the stability of model behaviour over time and thus confirms the validity of full sample estimation results. This can be expected, as values of the intensity of choice close to zero do not allow rich dynamics in the model.

### 5.5. Full sample estimates of the 3-type model

Estimation of a more-flexible 3-type model reveals markedly similar big picture as the estimation of the 2-type model. For the matter of computational time, based on our knowledge from the 2-type model estimation revealing large similarities across all estimated stock market indices we again only compute results for S&P500. Estimated parameters are reported in Table 9. The only new conclusion is a statistical insignificance of the contrarian strategy represented by coefficient  $\hat{g}_3$ , which is now exactly specified in the model and defined via the constraint  $g_3 < 0$ . Although point estimates reported via median and mean values are negative, this is only an effect of the enforced  $g_3 < 0$  constraint. The distribution mass of estimates from all 500 runs concentrates close to 0. The optimised function is likely to be very flat in the dimension of the  $g_3$  parameter because the effect of a very weak contrarian strategy is overshadowed if combined with a very strong trend following strategy. The estimate of the intensity of choice  $\hat{\beta}$  keeps its statistical insignificance and the trend following strategy coefficient  $\hat{g}_2$  retains its positive sign as well as high statistical significance. The absolute value of  $\hat{g}_2$  is naturally higher because the trend following strategy impacts the price via only the 1/3 weight in the 3-type model compared to 1/2 weight in the 2-type model, in both cases conditional on insignificant  $\hat{\beta}$ . Taking those weights into account, we obtain very similar impact of the trend following strategy in both models. Comparing results for the MA61 and MA241 fundamental price approximation shows very same effects as within the 2-type estimation, under MA241 we reveal somewhat stronger trend following strategy and lower intensity of stochastic noise.

### 5.6. Estimation of market fractions

Statistically insignificant intensity of choice parameter  $\hat{\beta}$  implies that any systematic evolutionary switching between trading strategies can not be expected. It also implies stable population ratio of trading strategies  $n_{1,t}/n_{2,t} \doteq 0.5/0.5$  in time, which means that the population of fundamentalists is forced to be of almost the same magnitude as the population of chartists throughout the entire span of the dataset. Thus the model boils down to a simple weighted AR(1) process and different types of trading strategies cannot be identified because agents do not switch over time.



In such a case the trend  $\widehat{g}_2$  and bias  $\widehat{b}_2$  parameters can be viewed as nuisance parameters, i.e. they to a large extent lose the original model interpretation and we cannot fully trust the estimated magnitudes of these parameters.

Moreover, contrarians in the 3-type model, who technically account for 1/3 of the population size as the  $\widehat{\beta}$  coefficient is insignificant, behave as fundamentalists in terms of their price impact because the size of  $\widehat{g}_3$  coefficient is small and statistically insignificant. These findings from the analysis of the 2-type and 3-type model estimation imply two important conclusions. First, the 3-type model does not really help us to capture additional features of the data-generating process and rather deviates the implied market fraction. Second, it suggests there might be more fundamentalists than chartists in real markets and therefore the almost fixed population ratio of trading strategies  $n_{1,t}/n_{2,t} \doteq 0.5/0.5$  as the result of the 2-type model estimation is likely not capturing the real market population proportions.

As a consequence of these findings, we trivialise the simulated model (Equations 27, 28, and 29) by disabling the evolutionary switching behaviour and fixing the population ratio of trading strategies to  $n_{1,t}/n_{2,t} = \text{const.}$  Equations 28 and 29 are now replaced by Equation 32 and the coefficient  $n_1$ , which we further call percentage *fraction* of fundamentalists, is to be estimated instead of the switching coefficient  $\beta$ :

$$Rx_t = \sum_{h=1}^H n_h f_{h,t} + \epsilon_t \equiv \sum_{h=1}^H n_h (g_h x_{t-1} + b_h) + \epsilon_t, \quad (31)$$

$$n_1 = 1 - n_2, \quad (32)$$

where  $H = 2$  in the 2-type model. Interval for the starting points random generation is constrained to  $\langle 0.3, 0.9 \rangle$  for *fraction* and to  $\langle 1.5, 2.5 \rangle$  for  $g_2$ , the other setting remains the same as in the preceding analysis. The modified setup keeps the logic of aforementioned findings and does not distract the structure of the original model. On the other hand, the population ratio of trading strategies  $n_1/n_2$  and implied percentage *fraction* of fundamentalists in the market is now a direct subject of estimation.

### 5.6.1. Full sample estimates of the 2-type fraction model

Outcomes of the full sample estimation of all six stock market indices are reported in Table 10. The main interest lies in the behaviour of the new variable *fraction* representing the percentage market fraction of fundamentalists ( $g_1 = b_1 = 0$ ). All other variables behave at average very similarly as in the 2-type  $\beta$  model estimation, moreover, we do not longer observe considerable distinctions caused by the MA window length for the fundamental value approximation.

The  $\widehat{\text{fraction}}$  coefficient is strongly statistically significant with value closely around 0.56, leaving 44% of the market population to chartistic strategies. The model therefore suggests overall proportional dominance of the fundamental strategy in investigated world stock markets. Estimates of the trend following coefficient  $g_2$  are generally higher compared to values for the 2-type switching model estimation but one must realise that within the 2-type *fraction* model the trend following strategy is relatively weaker in terms of impact to the market price (see Equation 13) because the proportion of these strong trend chasers is lower than 0.5. If we consider market proportions incorrectly implied by the 2-type  $\widehat{\beta}$  coefficient and related  $\widehat{g}_2$  and compare it to  $\widehat{\text{fraction}}$  and related  $\widehat{g}_2$  estimated in this section according to Equation 13, we deduce almost similar impact.

Table 10: Empirical results of the 2-type *fraction* model estimation.

Data, MA period	(a) $\widehat{fraction}$			(b) $\widehat{g}_2$			(c) $\widehat{b}_2$			(d) $\widehat{noise\ intensity}$			(e) $LL$		
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD
SP500, 61	.569	.555	.088	1.963	1.986	.362	.006	.003	.121	.559	.558	.044	-4.022	-4.023	.013
NASDAQ, 61	.555	.541	.086	1.972	1.985	.344	-.001	-.001	.117	.565	.566	.054	-5.118	-5.119	.019
DAX, 61	.559	.547	.090	2.006	2.021	.387	.008	.005	.123	.489	.490	.035	-5.733	-5.733	.011
FTSE, 61	.556	.545	.085	1.980	1.997	.357	-.009	-.007	.121	.519	.520	.037	-5.507	-5.507	.011
HSI, 61	.559	.549	.082	1.993	2.020	.344	-.006	-.001	.120	.531	.531	.045	-6.999	-7.000	.016
N225, 61	.562	.553	.084	1.995	2.023	.361	.005	.001	.124	.482	.484	.032	-6.700	-6.701	.010
SP500, 241	.562	.556	.084	2.161	2.228	.425	.002	-.001	.133	.331	.348	.053	-4.073	-4.088	.049
NASDAQ, 241	.562	.556	.086	2.191	2.244	.434	-.002	-.001	.133	.311	.345	.077	-5.195	-5.231	.089
DAX, 241	.565	.559	.088	2.219	2.275	.455	.013	.003	.137	.256	.279	.069	-5.782	-5.814	.086
FTSE, 241	.562	.554	.087	2.179	2.227	.445	-.002	.003	.133	.322	.339	.053	-5.547	-5.563	.045
HSI, 241	.562	.554	.084	2.204	2.253	.446	-.009	-.006	.137	.276	.304	.070	-7.057	-7.095	.091
N225, 241	.570	.562	.087	2.232	2.281	.460	-.012	-.005	.138	.255	.272	.058	-6.744	-6.767	.069
Robustness check															
SP500 monthly, 13	.606	.592	.114	1.429	1.451	.306	-.000	.000	.116	.806	.801	.064	-5.150	-5.153	.014
SP500 weekly, 13	.635	.616	.128	1.404	1.422	.309	-.011	-.004	.115	.853	.848	.063	-4.598	-4.603	.028
SP500 weekly, 49	.485	.483	.093	1.591	1.624	.292	.012	.006	.117	.606	.606	.057	-4.721	-4.722	.017
SP500 R=1.001, 61	.566	.553	.089	1.967	1.984	.368	-.004	-.004	.121	.557	.557	.044	-4.022	-4.022	.013
SP500 R=1.001, 241	.570	.557	.085	2.220	2.239	.430	.003	-.001	.132	.332	.349	.054	-4.074	-4.089	.047

*Note:* Results are based on 1000 random runs, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 500$ . Sample medians, means, and standard deviations (SD) are reported. ‘LL’ denotes log-likelihoods of estimated models representing statistical fits. Figures are rounded to 3 decimal digits.

This confirms our suspicion about an improper specification of the model with insignificant  $\widehat{\beta}$  and we correct for this misspecification introducing *fraction* specification via Equation 32. Evolutionary switching between strategies can be now captured via changes in the *fraction* coefficient in its smooth form using the rolling approach as asserted by Teräsvirta (1994, pg. 217): “if one assumes that the agents make only dichotomous decisions or change their behaviour discretely, it is unlikely that they do this simultaneously. Thus if only an aggregated process is observed, then the regime changes in that process may be more accurately described as being smooth rather than discrete.” Nonetheless, the rolling approach does not reveal any significant dynamics in the behaviour of *fraction* which again only confirms the validity of full sample estimation results. We make the rolling sample estimates available upon request.

### 5.6.2. Robustness check of the 2-type fraction model

For the *fraction* model we employ an identical (except for now irrelevant effect of memory) robustness check as in the previous case for the switching model. Results of weekly and monthly data estimation and the model assuming higher market risk free rate are reported in Table 10, bottom part. Basic conclusions for the robustness and validity checks hold identically to previous findings. The *fraction*,  $\widehat{g}_2$ , and *noise intensity* generally reveal strong statistical significance, the opposite behaviour shows the bias parameter  $\widehat{b}_2$ . Differences are again observable at the level of trend parameters  $\widehat{g}_2$  and *noise intensity* based on monthly and weekly data, where results show lower  $\widehat{g}_2$  and higher *noise intensity* compared to daily data. These findings are once again likely to be an implied technical side-effect of a small MA window. Simulated log-likelihood functions are of similar shapes as in previous models.

## 6. Conclusion

This paper proposes innovative computational framework for empirical estimation of FABMs. Motivated by the lack of general consensus on the estimation methodology, not many examples on structural estimation of FABMs, and inconclusive results in recent FABM literature, we aim at developing and testing more general methods for estimation of FABMs that significantly reduce the importance of restrictive theoretical assumptions.

In a large simulation as well as estimation exercise, we show that the NPSMLE generally estimates the parameters in various versions of the Brock and Hommes (1998) model precisely. We confirm that simulated MLE constitutes a flexible method to estimate complicated nonlinear models for which traditional estimation approaches cannot be used and that historically remained to a large extent inestimable. Employing NPSMLE, we are generally able to estimate models for which the closed-form solution or theoretical approximation of the objective function does not exist. We also prove that using simulation-based non-parametric methods the parameters of such systems can be recovered reasonably well. Together with quickly increasing computational capabilities of personal computers, server clusters, and super-computers, we anticipate a bright future and rapid development of simulation-based methods in next years.

The crucial result of the empirical analysis is the statistical insignificance of the switching coefficient across main world market indices. This is a common result in the existing literature, but on the other hand we contrast another part of the estimation literature on FABMs reporting significant switching coefficients for various specific markets. In contrary, our estimation results of the 2-type model reveal markedly statistically significant belief parameters defining heterogeneous

trading regimes with an absolute superiority of trend-following over contrarian strategies. Our findings further indicate robustness w.r.t. the fundamental value specification and remain largely unaffected under the robustness burden of different than daily data frequency, jumps in market risk free rate, or introducing of agents' memory. Graphical inspection of simulated log-likelihood functions reveals a bit rough surface, but very consistent performance of the estimation method over all random runs leading to a unique maxima. The adapted computational algorithm is, however, able to deal well with the not-completely-smooth surface of the simulated log-likelihood function and the important identification feature is thus verified also for the empirical application.

- Alfarano, S., T. Lux, and F. Wagner (2005). Estimation of agent-based models: The case of an asymmetric herding models. *Computational Economics* 26, 19–49.
- Alfarano, S., T. Lux, and F. Wagner (2006). Estimation of a simple agent-based model of financial markets: An application to australian stock and foreign exchange data. *Physica A: Statistical Mechanics and its Applications* 370(1), 38–42. Econophysics Colloquium Proceedings of the International Conference.
- Alfarano, S., T. Lux, and F. Wagner (2007). Empirical validation of stochastic models of interacting agents. *The European Physical Journal B* 55(2), 183–187.
- Allen, H. and M. P. Taylor (1990). Charts, noise and fundamentals in the london foreign exchange market. *The Economic Journal* 100(400), 49–59.
- Altissimo, F. and A. Mele (2009). Simulated non-parametric estimation of dynamic models. *The Review of Economic Studies* 76(2), 413–450.
- Amilon, H. (2008). Estimation of an adaptive stock market model with heterogeneous agents. *Journal of Empirical Finance* 15(2), 342–362.
- Andersen, T. G., L. Benzoni, and J. Lund (2002). An empirical investigation of continuous-time equity return models. *The Journal of Finance* 57(3), 1239–1284.
- Barunik, J. and J. Kukacka (2015). Realizing stock market crashes: stochastic cusp catastrophe model of returns under time-varying volatility. *Quantitative Finance* 15(6), 959–973.
- Barunik, J., L. Vacha, and M. Vosvrda (2009). Smart predictors in the heterogeneous agent model. *Journal of Economic Interaction and Coordination* 4, 163–172.
- Barunik, J. and M. Vosvrda (2009). Can a stochastic cusp catastrophe model explain stock market crashes? *Journal of Economic Dynamics & Control* 33, 1824–1836.
- Bass, F. (1969). A new product growth for model consumer durables. *Management Science* 15, 215–227.
- Biondi, Y., P. Giannoccolo, and S. Galam (2012). Formation of share market prices under heterogeneous beliefs and common knowledge. *Physica A: Statistical Mechanics and its Applications* 391(22), 5532–5545.
- Bolt, W., M. Demertzis, C. Diks, and M. van der Leij (2011). Complex methods in economics: an example of behavioral heterogeneity in house prices. Technical report, De Nederlandsche Bank.
- Bolt, W., M. Demertzis, C. G. H. Diks, C. H. Hommes, and M. van der Leij (2014). Identifying booms and busts in house prices under heterogeneous expectations. Technical report, De Nederlandsche Bank Working Paper No. 450.
- Boswijk, H. P., C. H. Hommes, and S. Manzan (2007, February). Behavioral heterogeneity in stock prices. *Journal of Economic Dynamics & Control* 31(2), 1938–1970.
- Brock, W., J. Lakonishok, and B. LeBaron (1992). Simple technical trading rules and the stochastic properties of stock returns. *The Journal of Finance* 47(5), 1731–1764.
- Brock, W. A. and C. H. Hommes (1997). A rational route to randomness. *Econometrica* 65(5), 1059–1095.
- Brock, W. A. and C. H. Hommes (1998). Heterogeneous beliefs and routes to chaos in a simple asset pricing model. *Journal of Economic Dynamics & Control* 22, 1235–1274.
- Chen, S.-H., C.-L. Chang, and Y.-R. Du (2012, 6). Agent-based economic models and econometrics. *The Knowledge Engineering Review* 27, 187–219.
- Chen, Z. and T. Lux (2015). Estimation of sentiment effects in financial markets: A simulated method of moments approach. FinMaP-Working Paper 37, University of Kiel, Department of Economics, Kiel.
- Chiarella, C., R. Dieci, and X.-Z. He (2009). *Handbook of Financial Markets: Dynamics and Evolution*, Chapter 5: Heterogeneity, Market Mechanisms and Asset Price Dynamics, pp. 277–344. North-Holland, Elsevier, Inc., Amsterdam.
- Chiarella, C. and X.-Z. He (2002). An adaptive model on asset pricing and wealth dynamics with heterogeneous trading strategies. Technical report, University of Technology, Sydney, Australia.
- Chiarella, C., X.-Z. He, and C. Hommes (2006). A dynamic analysis of moving average rules. *Journal of Economic*

- Dynamics and Control* 30(9-10), 1729–1753. Computing in economics and finance 10th Annual Conference on Computing in Economics and Finance.
- Chiarella, C., X.-Z. He, and R. C. Zwickels (2014). Heterogeneous expectations in asset pricing: Empirical evidence from the SP500. *Journal of Economic Behavior & Organization* 105, 1–16.
- Chiarella, C., S. ter Ellen, X.-Z. He, and E. Wu (2015). Fear or fundamentals? heterogeneous beliefs in the european sovereign CDS markets. *Journal of Empirical Finance* 32, 19–34.
- Cornea, A., C. Hommes, and D. Massaro (2013). Behavioral heterogeneity in U.S. inflation dynamics. Tinbergen Institute Discussion Paper 13-015/II, Tinbergen Institute, Amsterdam and Rotterdam.
- Cox, J. C., J. Ingersoll, Jonathan E., and S. A. Ross (1985). A theory of the term structure of interest rates. *Econometrica* 53(2), 385–407.
- De Grauwe, P. and M. Grimaldi (2005). Heterogeneity of agents, transactions costs and the exchange rate. *Journal of Economic Dynamics & Control* 29, 691–719.
- De Grauwe, P. and M. Grimaldi (2006). Exchange rate puzzles: A tale of switching attractors. *European Economic Review* 50, 1–33.
- de Jong, E., W. F. C. Verschoor, and R. C. J. Zwickels (2009a). Behavioural heterogeneity and shift-contagion: Evidence from the asian crisis. *Journal of Economic Dynamics & Control* 33, 1929–1944.
- de Jong, E., W. F. C. Verschoor, and R. C. J. Zwickels (2009b). A heterogeneous route to the european monetary system crisis. *Applied Economic Letters* 16, 929–932.
- de Jong, E., W. F. C. Verschoor, and R. C. J. Zwickels (2010). Heterogeneity of agents and exchange rate dynamics: Evidence from the EMS. *Journal of International Money and Finance* 29, 1652–1669.
- Diks, C. and R. Weide (2005). Herding, a-synchronous updating and heterogeneity in memory in a CBS. *Journal of Economic Dynamics & Control* 29, 741–763.
- Duan, J.-C. and J.-G. Simonato (1998). Empirical martingale simulation for asset prices. *Management Science* 44(9), 1218–1233.
- Ecemis, I., E. Bonabeau, and T. Ashburn (2005). Interactive estimation of agent-based financial markets models: Modularity and learning. In *Proceedings of the 7th Annual Conference on Genetic and Evolutionary Computation, GECCO '05*, New York, NY, USA, pp. 1897–1904. ACM.
- Fagiolo, G., A. Moneta, and P. Windrum (2007). A critical guide to empirical validation of agent-based models in economics: Methodologies, procedures, and open problems. *Computational Economics* 30(3), 195–226.
- Fama, E. F. (1970). Efficient capital markets: A review of theory and empirical work. *The Journal of Finance* 25(2), 383–417.
- Fernández-Villaverde, J. and J. F. Rubio-Ramírez (2010, December). Macroeconomics and volatility: Data, models, and estimation. Working Paper 16618, National Bureau of Economic Research.
- Franke, R. (2009). Applying the method of simulated moments to estimate a small agent-based asset pricing model. *Journal of Empirical Finance* 16, 804–815.
- Franke, R. and F. Westerhoff (2011). Estimation of a structural stochastic volatility model of asset pricing. *Computational Economics* 38(1), 53–83.
- Franke, R. and F. Westerhoff (2012). Structural stochastic volatility in asset pricing dynamics: Estimation and model contest. *Journal of Economic Dynamics and Control* 36(8), 1193–1211. Quantifying and Understanding Dysfunctions in Financial Markets.
- Frankel, J. A. and K. A. Froot (1990, earlier versions 1986, 1988). Chartists, fundamentalists, and trading in the foreign exchange market. *AEA Papers and Proceedings* 80(2), 181–185.
- Frijns, B., T. Lehnert, and R. C. J. Zwickels (2010). Behavioral heterogeneity in the option market. *Journal of Economic Dynamics & Control* 34, 2273–2287.
- Gaunersdorfer, A. (2000). Adaptive beliefs and the volatility of asset prices. Technical report, Vienna University of Economics, Austria, Working Paper No. 74.
- Ghonghadze, J. and T. Lux (2015). Bringing an elementary agent-based model to the data: Estimation via GMM and an application to forecasting of asset price volatility. FinMaP-Working Paper 38, University of Kiel, Department of Economics, Kiel.
- Gilli, M. and P. Winker (2003). A global optimization heuristic for estimating agent based models. *Computational Statistics & Data Analysis* 42, 299–312.
- Goldbaum, D. (1999). A nonparametric examination of market information: application to technical trading rules. *Journal of Empirical Finance* 6(1), 59–85.
- Goldbaum, D. and R. C. Zwickels (2014). An empirical examination of heterogeneity and switching in foreign exchange markets. *Journal of Economic Behavior & Organization* 107, Part B, 667–684. Empirical Behavioral Finance.

- Gordon, M. J. (1962). The savings investment and valuation of a corporation. *The Review of Economics and Statistics* 44(1), 37–51.
- Grazzini, J. and M. Richiardi (2015). Estimation of ergodic agent-based models by simulated minimum distance. *Journal of Economic Dynamics and Control* 51, 148–165.
- Grazzini, J., M. Richiardi, and L. Sella (2013). Indirect estimation of agent-based models. an application to a simple diffusion model. *Complexity Economics* 1(2), 25–40.
- Holland, J. H. and J. H. Miller (1991). Artificial adaptive agents in economic theory. *The American Economic Review* 81(2), 365–370.
- Hommes, C. (2013). *Behavioral Rationality and Heterogeneous Expectations in Complex Economic Systems*. Cambridge University Press. Cambridge Books Online.
- Hommes, C. and D. Veld (2015). Booms, busts and behavioral heterogeneity in stock prices. Technical report, University of Amsterdam, CeNDEF Working Paper.
- Hommes, C. and F. O. O. Wagener (2009). *Handbook of Financial Markets: Dynamics and Evolution*, Chapter 4: Complex Evolutionary Systems in Behavioral Finance, pp. 217–276. North-Holland, Elsevier, Inc., Amsterdam.
- Hommes, C. H. (2006). Chapter 23: Heterogeneous agent models in economics and finance. In L. Tesfatsion and K. Judd (Eds.), *Handbook of Computational Economics*, Volume 2 of *Handbook of Computational Economics*, pp. 1109–1186. Elsevier.
- Huisman, R., R. A. F. Maliepaard, and R. C. J. Zwinkels (2010). Heterogeneous agents in electricity forward markets. Technical report, Erasmus University Rotterdam.
- Jones, M. C., J. S. Marron, and S. J. Sheather (1996). A brief survey of bandwidth selection for density estimation. *Journal of the American Statistical Association* 91(433), 401–407.
- Kirman, A. (1993). Ants, rationality, and recruitment. *The Quarterly Journal of Economics* 108(1), 137–156.
- Kouwenberg, R. and R. Zwinkels (2014). Forecasting the US housing market. *International Journal of Forecasting* 30(3), 415–425.
- Kouwenberg, R. and R. C. J. Zwinkels (2015, 06). Endogenous price bubbles in a multi-agent system of the housing market. *PLoS ONE* 10(6), e0129070.
- Kristensen, D. (2009, 10). Uniform convergence rates of kernel estimators with heterogeneous dependent data. *Econometric Theory* 25, 1433–1445.
- Kristensen, D. and Y. Shin (2012). Estimation of dynamic models with nonparametric simulated maximum likelihood. *Journal of Econometrics* 167(1), 76–94.
- Kukacka, J. and J. Barunik (2013). Behavioural breaks in the heterogeneous agent model: The impact of herding, overconfidence, and market sentiment. *Physica A: Statistical Mechanics and its Applications* 392(23), 5920–5938.
- LeBaron, B. (2006). Chapter 24: Agent-based computational finance. In L. Tesfatsion and K. Judd (Eds.), *Handbook of Computational Economics*, Volume 2 of *Handbook of Computational Economics*, pp. 1187–1233. Elsevier.
- LeBaron, B. and L. Tesfatsion (2008). Modeling macroeconomics as open-ended dynamic systems of interacting agents. *The American Economic Review* 98(2), 246–250.
- Lof, M. (2012). Heterogeneity in stock prices: A STAR model with multivariate transition function. *Journal of Economic Dynamics and Control* 36(12), 1845–1854.
- Lof, M. (2015). Rational speculators, contrarians, and excess volatility. *Management Science* 61(8), 1889–1901.
- Lucas, R. E. J. (1978). Asset prices in an exchange economy. *Econometrica* 46, 1429–1445.
- Lui, Y.-H. and D. Mole (1998). The use of fundamental and technical analyses by foreign exchange dealers: Hong kong evidence. *Journal of International Money and Finance* 17(3), 535–545.
- Lux, T. (1995). Herd behaviour, bubbles and crashes. *The Economic Journal* 105(431), 881–896.
- Manzan, S. and F. H. Westerhoff (2007). Heterogeneous expectations, exchange rate dynamics and predictability. *Journal of Economic Behavior & Organization* 64, 111–128.
- Palmer, R., W. B. Arthur, J. H. Holland, B. LeBaron, and P. Tayler (1994). Artificial economic life: a simple model of a stockmarket. *Physica D: Nonlinear Phenomena* 75(1-3), 264–274.
- Recchioni, M. C., G. Tedeschi, and M. Gallegati (2015). A calibration procedure for analyzing stock price dynamics in an agent-based framework. *Journal of Economic Dynamics & Control* 60, 1–25.
- Reitz, S. and U. Slopek (2009). Non-linear oil price dynamics: A tale of heterogeneous speculators? *German Economic Review* 10(3), 270–283.
- Reitz, S. and F. H. Westerhoff (2007). Commodity price cycles and heterogeneous speculators: A STAR-GARCH model. *Empirical Economics* 33, 231–244.
- Silverman, B. W. (1986). *Density Estimation for Statistics and Data Analysis*. London: Chapman and Hall.
- Simon, H. A. (1957). *Models of Man*. New York: Wiley.
- Sullivan, R., A. Timmermann, and H. White (1999). Data-snooping, technical trading rule performance, and the

- bootstrap. *The Journal of Finance* 54(5), 1647–1691.
- Taylor, M. P. and H. Allen (1992). The use of technical analysis in the foreign exchange market. *Journal of International Money and Finance* 11(3), 304–314.
- ter Ellen, S., W. F. Verschoor, and R. C. Zwinkels (2013). Dynamic expectation formation in the foreign exchange market. *Journal of International Money and Finance* 37, 75–97.
- ter Ellen, S. and R. C. Zwinkels (2010). Oil price dynamics: A behavioral finance approach with heterogeneous agents. *Energy Economics* 32(6), 1427–1434.
- Teräsvirta, T. (1994). Specification, estimation, and evaluation of smooth transition autoregressive models. *Journal of the American Statistical Association* 89(425), 208–218.
- Vacha, L., J. Barunik, and M. Vosvrda (2012). How do skilled traders change the structure of the market. *International Review of Financial Analysis* 23(0), 66–71. Complexity and Non-Linearities in Financial Markets: Perspectives from Econophysics.
- Verschoor, W. F. and R. C. Zwinkels (2013). Do foreign exchange fund managers behave like heterogeneous agents? *Quantitative Finance* 13(7), 1125–1134.
- Vigfusson, R. (1997). Switching between chartists and fundamentalists: A markov regime-switching approach. *International Journal of Finance and Economics* 2, 291–305.
- Wan, J.-Y. and C.-W. Kao (2009). Evidence on the contrarian trading in foreign exchange markets. *Economic Modelling* 26, 1420–1431.
- Westerhoff, F. H. and S. Reitz (2003). Nonlinearities and cyclical behavior: The role of chartists and fundamentalists. *Studies in Nonlinear Dynamics & Econometrics* 7(4), 1–13.
- Westerhoff, F. H. and S. Reitz (2005). Commodity price dynamics and the nonlinear market impact of technical traders: Empirical evidence for the us corn market. *Physica A* 349, 641–648.
- Winker, P. and M. Gilli (2001). Indirect estimation of the parameters of agent based models of financial markets. Technical report, International University in Germany and University of Geneva, Switzerland.
- Winker, P., M. Gilli, and V. Jeleskovic (2007). An objective function for simulation based inference on exchange rate data. *Journal of Economic Interaction and Coordination* 2(2), 125–145.

## A. Appendix: Supplementary Tables and Figures

Table 11: Estimation methods of FABMs II. b).

Models	#	Data	Type	Fit	IOC
Barunik and Vosvrda (2009)	8,17	d:1987–1988,2001–2002	s	<i>pseudo-R</i> <sup>2</sup> up to .8	-
Barunik and Kukacka (2015)	10	d:6739,409 o.	s	<i>pseudo-R</i> <sup>2</sup> =.8, .86	-
Bolt et al. (2011)	5	q:164 o.	re	NA	2716(i),12420(i)
Bolt et al. (2014)	4	q:178 o.	re	NA	795(i)–26333(i)
Cornea et al. (2013)	2	q:204 o.	U.S. inflation	<i>R</i> <sup>2</sup> =.78, .94	4.78(s)
Chen and Lux (2015)	3	d:1/1980–12/2010	s/fx/g	<i>p-v</i> ∈ {4.6%, 45.5%}	-
Chiarella et al. (2014)	6	m:502,251 o.	s	-	.44(s),.54(s),.69(s)
Chiarella et al. (2015)	5	w:2007–4/2013	CDS spreads	-	.74(i)–6.84(s)
de Jong et al. (2009a)	10	q:112 o.	s	-	1.03(s),2.87(s)
ter Ellen and Zwinkels (2010)	7	m:295,319	c (crude oil)	-	1.19(s),1.36(s)
ter Ellen et al. (2013)	2–5	w:1/2003–2/2008	fx	<i>adjR</i> <sup>2</sup> up to .7	7.72(i)–454.4(i)
Franke (2009)	6	d:4115–6867 o.	s,fx	<i>p-v</i> ∈ {0%, 2%}	-
Frijns et al. (2010)	5	d:01–12/2000	s (index options)	-	107.34(i)
Franke and Westerhoff (2011)	6	d:6866,6861 o.	s,fx	<i>p-v</i> =12.8%, 27.7%	-
Franke and Westerhoff (2012)	9	d:6866 o.	s	<i>p-v</i> =12.7%–32.6%	-
Ghoshadze and Lux (2015)	3	d:1/1980–12/2009	s/fx/g	<i>p-v</i> ∈ { .3%, 67%}	-
Grazzini et al. (2013)	3	-	-	-	-
Grazzini and Richiardi (2015)	1	d:400 o.	s	-	-
Goldbaum and Zwinkels (2014)	4	m:2825–2941 o.	fx (experts' forecasts)	<i>adjR</i> <sup>2</sup> =.55–.79	-
Hommel and Veld (2015)	4	q:252 o.	re	<i>R</i> <sup>2</sup> =.95	2.44(i)
Huisman et al. (2010)	4	d:694,753,1038 o.	c (electricity futures)	-	1.06(s),1.77(s),15.87(i)
Kouwenberg and Zwinkels (2014)	4	q:127,198 o.	re	-	2.98(s),1.36(s)
Kouwenberg and Zwinkels (2015)	5	q:204 o.	re	-	2.18(s)
Lof (2012)	7	q:208 o.	s	<i>R</i> <sup>2</sup> =.97	7.45(s), 4.74(s)
Lof (2015)	5	a:140 o.	s	<i>R</i> <sup>2</sup> =.55	type-specific: .8(i),1.13(s),5.18(i)
Reitz and Slopek (2009)	6	m:252 o.	c (crude oil)	-	-
Recchioni et al. (2015)	4	d:245 o.	s	-	2.14(s),.59(i),.03(s),.36(i)
Verschoor and Zwinkels (2013)	5	m:107 o.	fx	-	2.64(i),14.51(i)

*Note:* The Table complements information in Table 2 following the logic of Table 1. Authors of articles are alphabetised. ‘#’ displays total number of estimated parameters; ‘Data’ describes data frequency: ‘d/w/m/q/a’ for daily/weekly/monthly/quarterly/annual, and number of observations (where no specific figure is provided, we report starting and final years); ‘Type’ shows the type of data: ‘s/fx/c/g/re’ for stock markets/FX/commodity markets/gold/real estate; ‘Fit’ reports the statistical fit of the estimation (*R*<sup>2</sup>, its alternatives, *p*-value of the J-test of overidentifying restrictions to accept the model as a possible data generating process); and where relevant, ‘|IOC|’ displays the absolute estimated value of the switching parameter of the intensity of choice (see Equation 17), furthermore ‘s’/‘i’ denotes its statistical significance/insignificance at 5% level. Figures are rounded to 2 decimal digits.



Table 12: Simulation results for  $\beta$  estimation with Gaussian noise.

$\beta$	(a) $\hat{\beta}$ , $N(0, 10^{-16})$					(b) $\hat{\beta}$ , $N(0, 10^{-14})$				
	Median	Mean	SD	LQ	HQ	Median	Mean	SD	LQ	HQ
0	-.00	.00	.03	-.02	.04	-.00	-.00	.07	-.15	.11
.1	.10	.10	.02	.08	.12	.10	.10	.05	-.03	.21
.5	.50	.50	.05	.47	.54	.50	.50	.11	.40	.67
1	1.00	.99	.08	.91	1.05	1.00	1.00	.18	.84	1.15
3	3.00	3.02	.17	2.88	3.38	3.00	3.01	.35	2.77	3.35
5	5.00	5.08	.90	4.88	5.20	5.00	4.99	.29	4.74	5.16
10	10.00	9.98	.07	9.78	10.10	10.00	9.99	.29	9.57	10.54
$\beta$	(c) $\hat{\beta}$ , $N(0, 10^{-12})$					(d) $\hat{\beta}$ , $N(0, 10^{-10})$				
	Median	Mean	SD	LQ	HQ	Median	Mean	SD	LQ	HQ
0	.00	-.01	.17	-.46	.41	.00	.01	.24	-.49	.49
.1	.10	.10	.11	-.10	.30	.09	.09	.13	-.10	.30
.5	.50	.49	.26	-.19	1.15	.50	.49	.36	-.37	1.29
1	1.00	.99	.33	.28	1.78	1.00	1.00	.50	-.22	2.25
3	3.00	3.01	.60	2.21	3.78	3.01	3.02	1.07	.39	5.31
5	5.00	4.97	.62	4.12	5.59	5.01	4.97	1.39	2.51	6.83
10	10.00	9.96	1.04	8.91	11.00	10.01	9.80	2.10	5.93	11.29
$\beta$	(e) $\hat{\beta}$ , $N(0, 10^{-8})$					(f) $\hat{\beta}$ , $N(0, 10^{-6})$				
	Median	Mean	SD	LQ	HQ	Median	Mean	SD	LQ	HQ
0	-.00	-.00	.23	-.48	.48	-.00	-.01	.23	-.46	.46
.1	.11	.11	.12	-.10	.30	.10	.10	.12	-.10	.30
.5	.50	.49	.33	-.30	1.21	.49	.47	.35	-.37	1.26
1	1.01	1.04	.50	-.09	2.36	1.01	1.04	.51	-.10	2.43
3	3.01	3.03	.91	1.33	5.07	2.99	3.00	.95	.93	5.03
5	4.99	5.01	1.27	3.13	6.90	5.00	4.98	1.19	2.52	6.52
10	10.00	10.02	2.28	7.85	12.48	9.99	9.94	2.00	6.83	11.61
$\beta$	(g) $\hat{\beta}$ , $N(0, 0.01^2)$					(h) $\hat{\beta}$ , $N(0, 0.1^2)$				
	Median	Mean	SD	LQ	HQ	Median	Mean	SD	LQ	HQ
0	.01	-.00	.23	-.49	.45	.01	.01	.22	-.45	.46
.1	.09	.10	.12	-.10	.30	.11	.11	.12	-.10	.30
.5	.50	.49	.34	-.32	1.26	.50	.50	.35	-.35	1.30
1	.99	.99	.52	-.35	2.31	.99	1.00	.50	-.19	2.46
3	3.01	3.00	.89	1.04	4.74	2.99	3.05	1.00	1.48	5.86
5	5.01	5.01	1.26	2.39	7.11	4.99	5.05	1.21	3.75	6.81
10	10.00	9.85	2.42	5.97	11.90	9.99	9.99	2.22	7.57	11.64
$\beta$	(i) $\hat{\beta}$ , $N(0, 1)$					(j) $\hat{\beta}$ , $N(0, 2^2)$				
	Median	Mean	SD	LQ	HQ	Median	Mean	SD	LQ	HQ
0	.00	.00	.11	-.24	.23	-.00	-.00	.05	-.08	.08
.1	.11	.11	.08	-.09	.30	.10	.10	.04	.01	.20
.5	.50	.51	.14	.23	.81	.50	.51	.11	.33	.72
1	1.00	1.01	.23	.66	1.45	1.01	1.05	.27	.71	1.76
3	3.07	3.59	1.41	2.35	7.93	3.34	4.01	1.69	2.14	8.49
5	5.61	7.23	3.30	3.82	14.41	4.96	5.01	1.64	2.57	8.44
10	11.20	13.43	6.31	5.16	28.13	7.77	5.63	5.87	-9.53	10.64

*Note:* Stochastic noise  $\epsilon_t$  and  $\{\varepsilon_i\}_{i=1}^N$  drawn from normal distributions of given parameters,  $R = 1.0001$ . Each sample is based on 1000 random runs,  $H = 5$  possible trading strategies, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . Sample medians, means, standard deviations (SD), 2.5% (LQ), and 97.5% (HQ) quantiles are reported. Figures are rounded to 2 decimal digits.

Table 13: Descriptive statistics of deviations from fundamental prices.

Data, MA period	Mean	Median	Min.	Max.	SD	Skew.	Kurt.	LQ	HQ	AC	AC $x_t^2$
SP500, 61	-.03	2.0	-145.2	113.0	26.5	-.65	5.5	-61.4	50.7	.87	.73
NASDAQ, 61	-.10	2.4	-753.5	639.0	81.8	.03	13.6	-170.7	147.9	.89	.81
DAX, 61	-.28	10.1	-939.6	716.1	163.2	-.50	5.4	-361.8	311.6	.89	.80
FTSE, 61	-.12	7.5	-702.5	404.6	122.7	-.58	5.1	-275.4	235.7	.88	.73
HSI, 61	.35	24.3	-4253.2	3463.5	562.2	-.29	6.7	-1186.7	1130.0	.90	.74
NIKKEI 225, 61	.16	6.8	-2249.8	1900.2	434.7	-.30	4.2	-953.8	799.4	.89	.77
SP500, 241	-.60	2.8	-252.9	160.1	48.4	-.64	4.5	-112.7	85.7	.96	.90
NASDAQ, 241	-1.78	1.6	-756.4	1253.7	168.1	1.01	11.3	-350.9	322.0	.97	.96
DAX, 241	-3.35	1.4	-1531.3	1242.6	330.4	-.21	4.5	-728.0	669.1	.97	.94
FTSE, 241	-.73	11.9	-1072.4	721.0	210.6	-.56	4.4	-479.4	388.5	.96	.90
HSI, 241	2.26	-5.5	-6505.5	7099.5	1177.6	.18	5.7	-2424.1	2282.0	.98	.94
NIKKEI 225, 241	-6.38	-22.6	-3497.8	2872.4	860.0	-.20	3.3	-1821.4	1581.3	.97	.92

*Note:* Sample means, medians, minima, maxima, standard deviations (SD), skewnesses, kurtoses, 2.5% (LQ) and 97.5% (HQ) quantiles, and autocorrelations (AC) are reported. Figures are rounded to 1 or 2 decimal digits.

Table 14: Results of 5-parameter estimation of a 3-type model.

$\beta, g_2, b_2, g_3, b_3$	(a) $\hat{\beta}$			(b) $\hat{g}_2$			(c) $\hat{b}_2$			(d) $\hat{g}_3$			(e) $\hat{b}_3$		
	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD	Med.	Mean	SD
.5, .4, .3, .2, .15	.39	.45	.53	.37	.38	.27	.25	.28	.19	.24	.22	.27	.20	.17	.19
3, .4, .3, .2, .15	2.53	2.77	1.66	.37	.38	.30	.25	.28	.19	.23	.21	.30	.20	.17	.19
10, .4, .3, .2, .15	9.23	8.70	4.23	.34	.35	.25	.24	.26	.16	.25	.24	.25	.22	.19	.16
.5, -.4, -.3, .2, .15	.46	.53	.50	-.53	-.53	.17	-.26	-.28	.11	.34	.33	.18	.11	.13	.11
3, -.4, -.3, .2, .15	2.87	2.92	3.22	-.39	-.40	.16	-.24	-.27	.12	.23	.26	.19	.09	.12	.11
10, -.4, -.3, .2, .15	11.16	11.10	10.62	-.35	-.35	.17	-.25	-.27	.12	.21	.28	.19	.10	.13	.11
.5, -.4, .3, .2, -.15	.50	.55	.50	-.53	-.53	.17	.26	.28	.11	.34	.34	.17	-.11	-.13	.11
3, -.4, .3, .2, -.15	2.76	2.84	3.32	-.38	-.40	.17	.24	.26	.11	.24	.27	.20	-.09	-.11	.11
10, -.4, .3, .2, -.15	12.14	12.27	10.45	-.35	-.36	.17	.25	.28	.12	.20	.26	.18	-.10	-.14	.11
.5, .8, .6, .4, .3	.51	.48	.19	.75	.78	.45	.45	.50	.33	.45	.42	.45	.45	.40	.33
3, .8, .6, .4, .3	2.47	2.32	.90	.59	.65	.41	.44	.48	.29	.62	.55	.41	.45	.41	.29
10, .8, .6, .4, .3	9.97	9.89	.75	.79	.63	.20	.59	.47	.15	.42	.58	.21	.31	.43	.15
.5, -.8, -.6, .4, .3	.33	.43	.34	-1.00	-1.01	.30	-.69	-.71	.22	.61	.62	.31	.39	.41	.22
3, -.8, -.6, .4, .3	2.29	3.15	2.49	-.85	-.91	.26	-.64	-.66	.22	.49	.54	.30	.34	.37	.21
10, -.8, -.6, .4, .3	10.06	10.12	2.09	-.80	-.80	.06	-.60	-.60	.05	.40	.41	.09	.30	.30	.04
.5, -.8, .6, .4, -.3	.34	.43	.34	-1.02	-1.01	.32	.69	.73	.23	.62	.61	.33	-.39	-.43	.23
3, -.8, .6, .4, -.3	2.49	3.32	2.51	-.83	-.89	.26	.62	.66	.21	.45	.51	.28	-.33	-.36	.21
10, -.8, .6, .4, -.3	10.03	10.05	2.67	-.80	-.79	.07	.60	.59	.06	.40	.41	.12	-.30	-.29	.05

Note: Stochastic noise  $\epsilon_t$  and  $\{\epsilon_i\}_{i=1}^N$  drawn from normal distribution  $N(0, 0.1^2)$ ,  $R = 1.0001$ . Each sample is based on 1000 random runs, number of observations  $t = 5000$ , and the kernel estimation precision  $N = 1000$ . Sample medians, means, standard deviations (SD) are reported. Figures are rounded to 2 decimal digits.